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HOMOGENEOUS BÄCKLUND TRANSFORMATIONS OF HYPERBOLIC MONGE-AMPÈRE SYSTEMS *

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Abstract. A Bäcklund transformation between two hyperbolic Monge-Ampère systems may be described as a certain type of exterior differential system on a 6-dimensional manifold $\mathcal{B}$. The transformation is *homogeneous* if the group of symmetries of the system acts transitively on $\mathcal{B}$. We give a complete classification of homogeneous Bäcklund transformations between hyperbolic Monge-Ampère systems.

1. Introduction. In this paper we will study Bäcklund transformations between two hyperbolic Monge-Ampère equations. A *Monge-Ampère equation* is a partial differential equation of the form

$$A(z_{xx}z_{yy} - z_{xy}^2) + Bz_{xx} + 2Cz_{xy} + Dz_{yy} + E = 0$$

where the coefficients A, B, C, D, E are functions of the variables x, y, z, z_x, z_y. The equation is *hyperbolic* if it has distinct, real characteristics at each point, i.e., if $AE - BD + C^2 > 0$.

There are many definitions of Bäcklund transformations given in the literature. Rather than attempting to give an all-encompassing definition, we will use Bäcklund's original notion. In [1] he posed the following general problem: Let $M^5 = \bar{M}^5 = \mathbb{R}^5$, with coordinates (x, y, z, p, q) on M and $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q})$ on $\bar{M}$. Given four equations

$$F_i(x, y, z, p, q, \bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}) = 0, \quad i = 1, \ldots, 4, \tag{1.1}$$

find parametrized surfaces $X : U \to M$, $\bar{X} : U \to \bar{M}$ and a one-to-one correspondence between them (which may be established by using the same parameter domain U for both surfaces) such that the coordinate functions $x(u, v), \ldots, \bar{q}(u, v)$ of the two surfaces satisfy the conditions

$$\begin{aligned} F_i(x(u, v), \ldots, \bar{q}(u, v)) &= 0, \quad i = 1, \ldots, 4 \\ dz &= p\,dx + q\,dy \\ d\bar{z} &= \bar{p}\,d\bar{x} + \bar{q}\,d\bar{y}. \end{aligned}$$

(The last two equations imply that the coordinates $p, q, \bar{p}, \bar{q}$ should be regarded as the partial derivatives $z_x, z_y, \bar{z}_{\bar{x}}, \bar{z}_{\bar{y}}$, respectively.)

Bäcklund's approach to this problem was to assume that X is a graph of the form

$$(x, y, z, p, q) = (x, y, z(x, y), z_x(x, y), z_y(x, y))$$

for some known function $z(x, y)$. Two of the equations (1.1) can be solved for the variables x and y, and substituting these expressions into the remaining two equations yields equations of the form

$$\begin{aligned} f(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}) &= 0 \\ g(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}) &= 0. \end{aligned} \tag{1.2}$$

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This may be regarded as an overdetermined first-order PDE system for the function $\bar{z}(\bar{x},\bar{y})$, and the compatibility conditions for this system take the form of partial differential equations that must be satisfied by the function $z(x,y)$. If $z(x,y)$ satisfies these conditions, then the system (1.2) has a 1-parameter family of solutions $\bar{z}(\bar{x},\bar{y})$ which can be found by solving ordinary differential equations. In this case, the four equations (1.1) may be regarded as a transformation of the surface $z = z(x,y)$ into the surface $\bar{z} = \bar{z}(\bar{x},\bar{y})$.

EXAMPLE. The classical Bäcklund transformation for the sine-Gordon equation

$$z_{xy} = \tfrac{1}{2}\sin(2z) \tag{1.3}$$

is usually described by the two equations

$$\begin{aligned} z_x + \bar{z}_x &= \lambda \sin(z - \bar{z}) \\ z_y - \bar{z}_y &= \frac{1}{\lambda} \sin(z + \bar{z}) \end{aligned} \tag{1.4}$$

where λ is a nonzero constant. In Bäcklund's notation we would write equations (1.4) as

$$\begin{aligned} p + \bar{p} &= \lambda \sin(z - \bar{z}) \\ q - \bar{q} &= \frac{1}{\lambda} \sin(z + \bar{z}), \end{aligned}$$

together with the two additional equations $\bar{x} = x$, $\bar{y} = y$. It is straightforward to show that if $z(x,y), \bar{z}(x,y)$ satisfy equations (1.4), then both must be solutions of the sine-Gordon equation (1.3). Moreover, if $z(x,y)$ is any known solution of (1.3), then (1.4) is a compatible, overdetermined system for the unknown function $\bar{z}(x,y)$, whose solution depends only on solving ordinary differential equations. For instance, taking $z(x,y) = 0$ yields

$$\bar{z}(x,y) = \tan^{-1}(e^{-(\lambda x + \frac{1}{\lambda}y + c)}).$$

These are the *1-soliton* solutions of the sine-Gordon equation.

The problem that arises in Bäcklund's approach is that for a generic choice of equations (1.1), the compatibility conditions for (1.2) cannot be written as separate PDEs for the functions z and $\bar{z}$; rather they involve z and $\bar{z}$ together, and so equations (1.1) do not give a transformation of the desired form. This raises the question: for what sets of equations (1.1) can the compatibility conditions for (1.2) be written as separate PDEs for z and $\bar{z}$? If this is the case, then the system (1.1) is called a *Bäcklund transformation* between the two PDEs. Thus the question raised above becomes: what PDEs (or pairs of PDEs) have Bäcklund transformations? This is an open problem which has attracted much attention over the past century. Its difficulty is attested to by the extensive work that analysts such as Goursat [9, 10] put into investigating special cases, such as equations of the form

$$z_{xy} = \rho z_x z_y + a z_x + b z_y + c$$

where a, b, c, ρ are functions of x, y and z. More recently, McLaughlin and Scott [12] classified auto-Bäcklund transformations (i.e., transformations for which z and $\bar{z}$ satisfy the same PDE) of equations of the form

$$z_{xy} + a z_x + b z_y = F(z)$$

where a and b are constants, and Byrnes [6] generalized this work by allowing F to depend on x and y as well as z. Zvyagin [15, 16], following Goursat's approach, has studied a certain type of Bäcklund transformation which he calls *harmonic*; he has also given a classification [17] of Bäcklund transformations of the wave equation $z_{xy} = 0$, although the descriptions of the systems on his list are somewhat unsatisfying and his paper contains no proof. These references represent only a small sample of the work that has been done on this problem; it would be impossible to give a complete list.

Although Goursat's foundational work appears to be highly dependent on working in coordinates, he was the first to focus on the geometric structures underlying Bäcklund transformations. This approach has since proven quite fruitful, and these structures are best decribed in terms of exterior differential systems.

An *exterior differential system* on a manifold M is a differentially closed ideal $\mathcal{I}$ in the algebra of differential forms on M. Any system of partial differential equations can be formulated as an exterior differential system $\mathcal{I}$, and solutions of the PDE system correspond to *integral manifolds* of $\mathcal{I}$, i.e., submanifolds $N \subset M$ which satisfy the condition that all the forms in $\mathcal{I}$ vanish when restricted to N. A *Monge-Ampère system* $\mathcal{I}$ is an exterior differential system on a 5-dimensional manifold M that is locally generated by a contact form θ (i.e., a 1-form θ with the property that $\theta \wedge d\theta \wedge d\theta \neq 0$), the 2-form $\Theta = d\theta$, and another 2-form Ψ. A Monge-Ampère system $\mathcal{I}$ is *hyperbolic* if the quadratic equation

$$(\lambda\,\Theta + \mu\,\Psi) \wedge (\lambda\,\Theta + \mu\,\Psi) \equiv 0 \mod \theta$$

has distinct, real roots. This condition agrees with the traditional definition of hyperbolicity, and it implies that there are two independent linear combinations $\lambda\,\Theta + \mu\,\Psi$ which are *decomposable* 2-forms (i.e., 2-forms which can be written as $\omega^1 \wedge \omega^2$ for some 1-forms ω^1, ω^2) modulo θ. (See [5] for a discussion of hyperbolic exterior differential systems.)

EXAMPLE (CONT'D). The sine-Gordon equation (1.3) may be described as a hyperbolic Monge-Ampère system on $\mathbb{R}^5$ (with coordinates (x, y, z, p, q)) generated by the forms

$$\begin{aligned} \theta &= dz - p\,dx - q\,dy \\ \Theta &= -dp \wedge dx - dq \wedge dy \\ \Psi &= [dp - \tfrac{1}{2}\sin(2z)\,dy] \wedge dx. \end{aligned}$$

Note that Ψ is decomposable; the other decomposable linear combination of Ψ and Θ is $-(\Psi + \Theta) = [dq - \frac{1}{2}\sin(2z)\,dx] \wedge dy$. Two-dimensional integral manifolds of this system that satisfy the independence condition $dx \wedge dy \neq 0$ are naturally in one-to-one correspondence with solutions of (1.3).

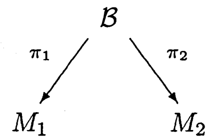
Bäcklund's original notion may be expressed in this context as follows. Suppose that $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$ are hyperbolic Monge-Ampère systems, with

$$\mathcal{I}_1 = \{\theta_1, \Theta_1, \Psi_1\}$$

$$\mathcal{I}_2 = \{\theta_2, \Theta_2, \Psi_2\}.$$

A *Bäcklund transformation* between $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$ is a 6-dimensional submanifold $\mathcal{B} \subset M_1 \times M_2$ which has the following properties:

1. The natural projections $\pi_1 : \mathcal{B} \to M_1$ and $\pi_2 : \mathcal{B} \to M_2$ are submersions.



2. The pullbacks to $\mathcal{B}$ of the forms $\Theta_1, \Theta_2, \Psi_1, \Psi_2$ satisfy the condition that

$$\{\Psi_1, \Psi_2\} \equiv \{\Theta_1, \Theta_2\} \mod \{\theta_1, \theta_2\}.$$

Since Θ_1, Ψ_1 are linearly independent forms (as are Θ_2, Ψ_2), this condition implies that

$$\{\Theta_1, \Psi_1\} \equiv \{\Theta_2, \Psi_2\} \mod \{\theta_1, \theta_2\}.$$

This second equation is really the desired property; the first equation ensures that, in addition, the forms Θ_1, Θ_2 are linearly independent.

That this definition captures the desired behavior may be seen as follows: suppose that $N \hookrightarrow M_1$ is a 2-dimensional integral manifold of $\mathcal{I}_1$. The inverse image $\pi_1^{-1}(N)$ is a 3-dimensional submanifold of $\mathcal{B}$. Now consider the restriction of $\pi_2^*(\mathcal{I}_2)$ to $\pi_1^{-1}(N)$. By Property (2) above, the restriction of $\pi_2^*(\mathcal{I}_2)$ is a Frobenius system (i.e., an exterior differential system which is generated *algebraically* by its 1-forms) on $\pi_1^{-1}(N)$. By the Frobenius Theorem, $\pi_1^{-1}(N)$ is foliated by 2-dimensional integral manifolds of $\pi_2^*(\mathcal{I}_2)$, each of which projects to an integral manifold of $(M_2, \mathcal{I}_2)$; moreover, these integral manifolds can be constructed by solving ODEs.

From the point of view of Bäcklund's original problem, any 2-dimensional integral manifold $\tilde{S} \subset \mathcal{B}$ of the ideal $\mathcal{J} = \{\theta_1, \theta_2, \Theta_1, \Theta_2\}$ projects to surfaces $S_1 \subset M_1$, $S_2 \subset M_2$ which are integral manifolds of $\mathcal{I}_1, \mathcal{I}_2$ respectively. The condition that $\tilde{S}$ be an integral manifold of $\mathcal{J}$ is exactly the requirement that the compatibility conditions for the equations (1.2) be satisfied.

Our primary tool for classifying such structures will be Cartan's method of equivalence; this is a method for computing local invariants of exterior differential systems and deciding when two systems are equivalent under some natural class of diffeomorphisms. In principle, it should be possible to completely classify all Bäcklund transformations of hyperbolic Monge-Ampère systems using this method. Unfortunately, in practice it is rarely possible to carry out this process in full generality. In this paper we will perform the somewhat simpler task of classifying the *homogeneous* Bäcklund transformations, i.e., those transformations for which the group of symmetries of the structure $(\mathcal{B}, \mathcal{I}_1, \mathcal{I}_2)$ acts transitively on $\mathcal{B}$. The main result is the following theorem.

THEOREM (cf. Theorem 12.1). *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation. Then $\mathcal{B}$ is locally contact equivalent to one of the following:*

1. *A Bäcklund transformation between solutions of the wave equation $z_{xy} = 0$*
2. *A holonomic Bäcklund transformation of the form described in Theorem 6.1*
3. *The classical Bäcklund transformation between the wave equation $z_{xy} = 0$ and Liouville's equation $z_{xy} = e^z$*
4. *A Bäcklund transformation between surfaces of constant negative Gauss curvature in $\mathbb{E}^3$*
5. *A Bäcklund transformation between surfaces of constant Gauss curvature $0 < K < 1$ in S^3*

6. *A Bäcklund transformation between surfaces of constant Gauss curvature* $-\infty < K < -1$ *in* $\mathbb{H}^3$
7. *A Bäcklund transformation between spacelike surfaces of constant positive Gauss curvature in* $\mathbb{E}^{2,1}$
8. *A Bäcklund transformation between timelike surfaces of constant positive Gauss curvature, or equivalently, constant nonzero mean curvature, in* $\mathbb{E}^{2,1}$
9. *A Bäcklund transformation between timelike minimal surfaces in* $\mathbb{E}^{2,1}$
10. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $1 < K < \infty$ *in* $S^{2,1}$
11. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $1 < K < \infty$, *or equivalently, constant mean curvature* $H \in \mathbb{R}$, *in* $S^{2,1}$
12. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $-1 < K < \infty$, $K \neq 0$ *in* $\mathbb{H}^{2,1}$
13. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $-1 < K < \infty$, $K \neq 0$, *or equivalently, constant mean curvature* $|H| > 1$, *in* $\mathbb{H}^{2,1}$
14. *A Bäcklund transformation between timelike surfaces of constant mean curvature* $|H| \leq 1$ *in* $\mathbb{H}^{2,1}$.
15. *A Bäcklund transformation between certain surfaces in a 5-dimensional quotient space of* $SO^*(4)$.

Throughout this paper we will work locally. Statements such as "assume that $C \neq 0$" should be interpreted as "assume that C is not identically zero and restrict to the open set where $C \neq 0$".

2. The equivalence problem. Suppose that $\mathcal{B}$ is a Bäcklund transformation between two hyperbolic Monge-Ampère systems $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$. Let $\mathcal{J}$ be the ideal on $\mathcal{B}$ generated by the pullbacks of $\mathcal{I}_1$ and $\mathcal{I}_2$; according to our definition of a Bäcklund transformation, $\mathcal{J}$ is generated algebraically by the forms $\{\theta_1, \theta_2, \Theta_1, \Theta_2\}$.

Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are hyperbolic, locally there exist 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$ on $\mathcal{B}$ such that $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$ is a coframing of $\mathcal{B}$ (i.e., a basis for the space of 1-forms on $\mathcal{B}$) and

$$\mathcal{J} = \{\theta_1, \theta_2, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\}.$$

(It is important to note that θ_1 and θ_2 are each *separately* determined up to a scalar multiple, since θ_i determines the contact structure on M_i.) Any such coframing has the property that

$$d\theta_1 \equiv A_1\,\omega^1 \wedge \omega^2 + A_2\,\omega^3 \wedge \omega^4 \mod \{\theta_1, \theta_2\}$$
$$d\theta_2 \equiv A_3\,\omega^1 \wedge \omega^2 + A_4\,\omega^3 \wedge \omega^4 \mod \{\theta_1, \theta_2\}$$

for some nonvanishing functions A_1, A_2, A_3, A_4. Since $d\theta_1$, $d\theta_2$ are required to be linearly independent 2-forms at each point of $\mathcal{B}$, we must have $A_1A_4 - A_2A_3 \neq 0$.

By rescaling the ω^i and adding multiples of θ_1 and θ_2 to the ω^i if necessary, we can arrange that

$$\begin{aligned} d\theta_1 &\equiv A_1\,\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \mod \theta_1 \\ d\theta_2 &\equiv \omega^1 \wedge \omega^2 + A_2\,\omega^3 \wedge \omega^4 \mod \theta_2 \end{aligned} \tag{2.1}$$

for some nonvanishing functions A_1, A_2 on $\mathcal{B}$ with $A_1A_2 \neq 1$. This coframing is not unique; any other such coframing $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4\}$ has the form

$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^1 \\ \tilde{\omega}^2 \\ \tilde{\omega}^3 \\ \tilde{\omega}^4 \end{bmatrix} = \begin{bmatrix} b_{11}b_{22}-b_{12}b_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} \tag{2.2}$$

where $b_{11}b_{22} - b_{12}b_{21} \neq 0$, $a_{11}a_{22} - a_{12}a_{21} \neq 0$. (The inverse is included for greater ease of computation in what follows.) A coframing satisfying (2.1) is called *0-adapted*, and the group G_0 of matrices of the above form is called the *structure group* of the equivalence problem. (In fact, the most general choice of structure group would include a discrete component interchanging the distributions $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$. However, this freedom does not contribute anything crucial to the structure group, and it is easier to work with a connected group.)

Now consider the exterior derivatives of the ω^i. Because θ_1 is well-defined (up to scalar multiples) on M_1, its Cartan system $\mathcal{C} = \{\theta_1, \omega^1, \omega^2, \omega^3, \omega^4\}$ is well-defined on M_1. (The *Cartan system* of a 1-form θ may be thought of as the span of a minimal set of 1-forms required to express θ and $d\theta$. It is always a Frobenius system; see [3] for details.) In fact, M_1 is (locally) the quotient of $\mathcal{B}$ by the leaves of the foliation defined by $\mathcal{C}$. Let $\{\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\omega^1}, \frac{\partial}{\partial\omega^2}, \frac{\partial}{\partial\omega^3}, \frac{\partial}{\partial\omega^4}\}$ denote the basis for the tangent space of $\mathcal{B}$ which is dual to the coframing $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$. The ideal

$$\mathcal{I}_1 = \{\theta_1, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\}$$

is well-defined on M_1, as are its characteristic systems

$$\mathcal{K}_{11} = \{\theta_1, \omega^1 \wedge \omega^2\}, \qquad \mathcal{K}_{12} = \{\theta_1, \omega^3 \wedge \omega^4\}.$$

Therefore the Lie derivative $\mathcal{L}_{\frac{\partial}{\partial\theta_2}}(\omega^1 \wedge \omega^2)$ must satisfy

$$\begin{aligned} 0 &\equiv \mathcal{L}_{\frac{\partial}{\partial\theta_2}}(\omega^1 \wedge \omega^2) \mod \{\theta_1, \omega^1 \wedge \omega^2\} \\ &\equiv \frac{\partial}{\partial\theta_2} \lrcorner (d\omega^1 \wedge \omega^2 - \omega^1 \wedge d\omega^2) \mod \{\theta_1, \omega^1 \wedge \omega^2\}. \end{aligned} \tag{2.3}$$

Reducing equation (2.3) modulo ω^1 yields

$$\left(\frac{\partial}{\partial\theta_2} \lrcorner d\omega^1\right) \wedge \omega^2 \equiv 0 \mod \{\theta_1, \omega^1\},$$

and therefore

$$\frac{\partial}{\partial\theta_2} \lrcorner d\omega^1 \equiv 0 \mod \{\theta_1, \omega^1, \omega^2\}.$$

Consequently, $d\omega^1$ cannot contain any terms involving the 2-forms $\theta_2 \wedge \omega^3$ or $\theta_2 \wedge \omega^4$. Similarly, reducing equation (2.3) modulo ω^2 shows that $d\omega^2$ cannot contain any terms involving the 2-forms $\theta_2 \wedge \omega^3$ or $\theta_2 \wedge \omega^4$. An analogous argument using the equation

$$\mathcal{L}_{\frac{\partial}{\partial\theta_2}}(\omega^3 \wedge \omega^4) \equiv 0 \mod \{\theta_1, \omega^3 \wedge \omega^4\}$$

shows that $d\omega^3$ and $d\omega^4$ cannot contain any terms involving the 2-forms $\theta_2 \wedge \omega^1$ or $\theta_2 \wedge \omega^2$.

This argument can be repeated for the characteristic systems

$$\mathcal{K}_{21} = \{\theta_2, \omega^1 \wedge \omega^2\}, \qquad \mathcal{K}_{22} = \{\theta_2, \omega^3 \wedge \omega^4\}$$

of $(M_2, \mathcal{I}_2)$; this shows that $d\omega^1$ and $d\omega^2$ cannot contain any terms involving the 2-forms $\theta_1 \wedge \omega^3$ or $\theta_1 \wedge \omega^4$, and $d\omega^3$ and $d\omega^4$ cannot contain any terms involving the 2-forms $\theta_1 \wedge \omega^1$ or $\theta_1 \wedge \omega^2$. It follows that that

$$\left.\begin{aligned} d\omega^1 &\equiv B_1\,\theta_1 \wedge \theta_2 + C_1\,\omega^3 \wedge \omega^4 \\ d\omega^2 &\equiv B_2\,\theta_1 \wedge \theta_2 + C_2\,\omega^3 \wedge \omega^4 \end{aligned}\right\} \quad \text{mod } \{\omega^1, \omega^2\}$$

$$\left.\begin{aligned} d\omega^3 &\equiv B_3\,\theta_1 \wedge \theta_2 + C_3\,\omega^1 \wedge \omega^2 \\ d\omega^4 &\equiv B_4\,\theta_1 \wedge \theta_2 + C_4\,\omega^1 \wedge \omega^2 \end{aligned}\right\} \quad \text{mod } \{\omega^3, \omega^4\}$$

for some functions B_i, C_i on $\mathcal{B}$.

These equations, taken together with equations (2.1), form the *structure equations*

$$\begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\omega^1 \\ d\omega^2 \\ d\omega^3 \\ d\omega^4 \end{bmatrix} = - \begin{bmatrix} \beta_1+\beta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1+\alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 0 & \beta_3 & \beta_4 \end{bmatrix} \wedge \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} + \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{bmatrix} \tag{2.4}$$

where the α_i, β_i are 1-forms on $\mathcal{B}$ and

$$\begin{aligned} \Theta_1 &= \gamma \wedge \theta_1 + A_1\,\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\ \Theta_2 &= \delta \wedge \theta_2 + \omega^1 \wedge \omega^2 + A_2\,\omega^3 \wedge \omega^4 \\ \Omega^1 &= B_1\,\theta_1 \wedge \theta_2 + C_1\,\omega^3 \wedge \omega^4 \\ \Omega^2 &= B_2\,\theta_1 \wedge \theta_2 + C_2\,\omega^3 \wedge \omega^4 \\ \Omega^3 &= B_3\,\theta_1 \wedge \theta_2 + C_3\,\omega^1 \wedge \omega^2 \\ \Omega^4 &= B_4\,\theta_1 \wedge \theta_2 + C_4\,\omega^1 \wedge \omega^2 \end{aligned}$$

for some 1-forms γ, δ on $\mathcal{B}$. These equations are chosen so that the matrix in (2.4) takes values in the Lie algebra $\mathfrak{g}_0$ of G_0; this is in accordance with the method of equivalence. (See [8] for details.) The functional coefficients of the terms appearing in Θ_i, Ω^i are called *torsion* terms.

We can modify the α_i, β_i if necessary to arrange that

$$\begin{aligned} \gamma &= E_1\,\theta_2 + F_1\,\omega^1 + F_2\,\omega^2 \\ \delta &= E_2\,\theta_1 + F_3\,\omega^3 + F_4\,\omega^4 \end{aligned}$$

for some functions E_i, F_i on $\mathcal{B}$. The forms α_i, β_i are still not uniquely determined; they are determined only up to transformations of the form

$$\begin{aligned} &\alpha_1 \mapsto \alpha_1 + r_1\,\omega^1 + r_2\,\omega^2 && \beta_1 \mapsto \beta_1 + s_1\,\omega^3 + s_2\,\omega^4 \\ &\alpha_2 \mapsto \alpha_2 + r_2\,\omega^1 + r_3\,\omega^2 && \beta_2 \mapsto \beta_2 + s_2\,\omega^3 + s_3\,\omega^4 \\ &\alpha_3 \mapsto \alpha_3 + r_4\,\omega^1 - r_1\,\omega^2 && \beta_3 \mapsto \beta_3 + s_4\,\omega^3 - s_1\,\omega^4 \\ &\alpha_4 \mapsto \alpha_4 - r_1\,\omega^1 - r_2\,\omega^2 && \beta_4 \mapsto \beta_4 - s_1\,\omega^3 - s_2\,\omega^4. \end{aligned} \tag{2.5}$$

Differentiating the structure equations yields

$$\begin{aligned} 0 &\equiv d(d\theta_1) \mod \{\theta_1, \omega^1, \omega^2\} \\ &\equiv -E_1\,\theta_2 \wedge \omega^3 \wedge \omega^4 && \Rightarrow E_1 = 0. \\ 0 &\equiv d(d\theta_2) \mod \{\theta_2, \omega^3, \omega^4\} \\ &\equiv -E_2\,\theta_1 \wedge \omega^1 \wedge \omega^2 && \Rightarrow E_2 = 0. \\ 0 &\equiv d(d\theta_1) \mod \{\theta_1, \omega^2\} \\ &\equiv -(F_1 + A_1C_2)\,\omega^1 \wedge \omega^3 \wedge \omega^4 && \Rightarrow F_1 = -A_1C_2. \\ 0 &\equiv d(d\theta_1) \mod \{\theta_1, \omega^1\} \\ &\equiv (-F_2 + A_1C_1)\,\omega^2 \wedge \omega^3 \wedge \omega^4 && \Rightarrow F_2 = A_1C_1. \\ 0 &\equiv d(d\theta_2) \mod \{\theta_2, \omega^4\} \\ &\equiv -(F_3 + A_2C_4)\,\omega^1 \wedge \omega^2 \wedge \omega^3 && \Rightarrow F_3 = -A_2C_4. \\ 0 &\equiv d(d\theta_2) \mod \{\theta_2, \omega^3\} \\ &\equiv (-F_4 + A_2C_3)\,\omega^1 \wedge \omega^2 \wedge \omega^4 && \Rightarrow F_4 = A_2C_3. \end{aligned}$$

Next we examine how the functions A_i, B_i, C_i vary if we change from one 0-adapted coframing to another. A computation shows that under a transformation of the form (2.2), we have

$$\tilde{A}_1 = \frac{(a_{11}a_{22} - a_{12}a_{21})}{(b_{11}b_{22} - b_{12}b_{21})} A_1$$

$$\tilde{A}_2 = \frac{(b_{11}b_{22} - b_{12}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} A_2$$

$$\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = (b_{11}b_{22} - b_{12}b_{21}) \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{B}_3 \\ \tilde{B}_4 \end{bmatrix} = (a_{11}a_{22} - a_{12}a_{21}) \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix} \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} = \frac{(b_{11}b_{22} - b_{12}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{C}_3 \\ \tilde{C}_4 \end{bmatrix} = \frac{(a_{11}a_{22} - a_{12}a_{21})}{(b_{11}b_{22} - b_{12}b_{21})} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}.$$

From this we see that the functions A_1, A_2 and the vectors

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}, \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$$

are *relative invariants*: if they vanish for any 0-adapted coframing, then they vanish for every 0-adapted coframing.

The general procedure in the method of equivalence is to choose a 0-adapted coframing that normalizes the torsion terms as much as possible. This has the effect of reducing the structure group to a subgroup $G_1 \subset G_0$ which preserves the normalized torsion terms. This in turn introduces new torsion terms, which can then be further normalized, etc. Ideally, this process eventually leads to a uniquely determined coframing whose torsion terms are invariants of the system $\mathcal{J}$ on $\mathcal{B}$. Even in those cases where a unique coframing is not obtained, it may be possible to reduce the structure group to the point that some of the torsion terms are uniquely determined. Our hypothesis that $\mathcal{B}$ is homogeneous implies that once the structure group has been reduced to the point that it acts trivially on a torsion term, that term must be constant on $\mathcal{B}$.

In order to proceed with the method of equivalence, we will divide into cases depending on whether certain of these invariants are zero or nonzero.

3. Case 1: $[C_1 \;\; C_2] = [C_3 \;\; C_4] = [0 \;\; 0]$. Suppose that $C_1 = C_2 = C_3 = C_4 = 0$. Differentiating the structure equations yields

$$\begin{aligned}
0 &\equiv d(d\omega^1) \mod \{\theta_1, \omega^1, \omega^2\} \\
&\equiv B_1\, \theta_2 \wedge \omega^3 \wedge \omega^4 && \Rightarrow B_1 = 0. \\
0 &\equiv d(d\omega^2) \mod \{\theta_1, \omega^1, \omega^2\} \\
&\equiv B_2\, \theta_2 \wedge \omega^3 \wedge \omega^4 && \Rightarrow B_2 = 0. \\
0 &\equiv d(d\omega^3) \mod \{\theta_2, \omega^3, \omega^4\} \\
&\equiv B_3\, \theta_1 \wedge \omega^1 \wedge \omega^2 && \Rightarrow B_3 = 0. \\
0 &\equiv d(d\omega^4) \mod \{\theta_2, \omega^3, \omega^4\} \\
&\equiv B_4\, \theta_1 \wedge \omega^1 \wedge \omega^2 && \Rightarrow B_4 = 0.
\end{aligned}$$

Now we see from the structure equations that

$$\left.\begin{aligned} d\omega^1 &\equiv 0 \\ d\omega^2 &\equiv 0 \end{aligned}\right\} \mod \{\omega^1, \omega^2\}$$

$$\left.\begin{aligned} d\omega^3 &\equiv 0 \\ d\omega^4 &\equiv 0 \end{aligned}\right\} \mod \{\omega^3, \omega^4\}.$$

Therefore the systems $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$ are completely integrable; this implies that there exist functions X, Y, P, Q on $\mathcal{B}$ (in fact, these functions are well-defined on M_1 and M_2) such that

$$\begin{aligned} \{\omega^1, \omega^2\} &= \{dX, dP\} \\ \{\omega^3, \omega^4\} &= \{dY, dQ\}. \end{aligned}$$

We can choose a 0-adapted coframing with

$$\omega^1 = dX, \qquad \omega^2 = dP, \qquad \omega^3 = dY, \qquad \omega^4 = dQ.$$

Then we see from the structure equations for the $d\omega^i$ that, taking advantage of the ambiguity (2.5) in α_i, β_i, we can assume that

$$\begin{gathered} \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 0 \\ \alpha_1 = G_1\,dX, \qquad \alpha_4 = G_2\,dP, \\ \beta_1 = G_3\,dY, \qquad \beta_4 = G_4\,dQ \end{gathered}$$

for some functions G_1, G_2, G_3, G_4 on $\mathcal{B}$. Differentiating the structure equations for the $d\theta_i$ then shows that

$$\begin{aligned} dA_1 &= A_{11}\,dX + A_{12}\,dP - A_1G_3\,dY - A_1G_4\,dQ \\ dA_2 &= -A_2G_1\,dX - A_2G_2\,dP + A_{23}\,dY + A_{24}\,dQ \\ dG_1 &= G_{10}\,\theta_2 + G_{11}\,dX + G_{12}\,dP \\ dG_2 &= G_{20}\,\theta_2 + G_{12}\,dX + G_{22}\,dP \\ dG_3 &= G_{30}\,\theta_1 + G_{33}\,dY + G_{34}\,dQ \\ dG_4 &= G_{40}\,\theta_1 + G_{34}\,dY + G_{44}\,dQ \end{aligned}$$

for some functions A_{ij}, G_{ij} on $\mathcal{B}$, and differentiating the equations for the dG_i in turn shows that

$$G_{10} = G_{20} = G_{30} = G_{40} = 0.$$

Therefore, G_1 and G_2 are functions of X and P alone, while G_3 and G_4 are functions of Y and Q alone. Moreover, we have

$$d(G_1\,dX + G_2\,dP) = d(G_3\,dY + G_4\,dQ) = 0;$$

therefore there exist nonvanishing functions $\lambda(X, P)$, $\mu(Y, Q)$ such that

$$\begin{aligned} \lambda^{-1}\,d\lambda &= G_1\,dX + G_2\,dP \\ \mu^{-1}\,d\mu &= G_3\,dY + G_4\,dQ. \end{aligned}$$

Let $\tilde{\theta}_1 = \mu\theta_1$, $\tilde{\theta}_2 = \lambda\theta_2$. Then a computation shows that

$$\begin{aligned} d\tilde{\theta}_1 &= A_1\mu\,dX \wedge dP + \mu\,dY \wedge dQ \\ d\tilde{\theta}_2 &= \lambda\,dX \wedge dP + A_2\lambda\,dY \wedge dQ, \end{aligned}$$

and that, moreover,

$$d(A_1\mu\,dX \wedge dP) = d(\mu\,dY \wedge dQ) = d(\lambda\,dX \wedge dP) = d(A_2\lambda\,dY \wedge dQ) = 0.$$

Therefore, by Darboux's Theorem there exist functions

$$\begin{aligned} x_1 &= x_1(X, P) \qquad & x_2 &= x_2(X, P) \\ p_1 &= p_1(X, P) \qquad & p_2 &= p_2(X, P) \\ y_1 &= y_1(Y, Q) \qquad & y_2 &= y_2(Y, Q) \\ q_1 &= q_1(Y, Q) \qquad & q_2 &= q_2(Y, Q) \end{aligned}$$

such that

$$\begin{aligned} A_1\mu\, dX \wedge dP &= dx_1 \wedge dp_1 \\ \mu\, dY \wedge dQ &= dy_1 \wedge dq_1 \\ \lambda\, dX \wedge dP &= dx_2 \wedge dp_2 \\ A_2\lambda\, dY \wedge dQ &= dy_2 \wedge dq_2. \end{aligned}$$

It follows that

$$\begin{aligned} d\tilde{\theta}_1 &= dx_1 \wedge dp_1 + dy_1 \wedge dq_1 \\ d\tilde{\theta}_2 &= dx_2 \wedge dp_2 + dy_2 \wedge dq_2, \end{aligned}$$

and by Pfaff's Theorem, there must exist functions z_1 on M_1 and z_2 on M_2 such that

$$\begin{aligned} \tilde{\theta}_1 &= dz_1 - p_1\, dx_1 - q_1\, dy_1 \\ \tilde{\theta}_2 &= dz_2 - p_2\, dx_2 - q_2\, dy_2. \end{aligned}$$

Finally, we see that the ideals

$$\begin{aligned} \mathcal{I}_1 &= \{\tilde{\theta}_1,\, dx_1 \wedge dp_1,\, dy_1 \wedge dq_1\} \\ \mathcal{I}_2 &= \{\tilde{\theta}_2,\, dx_2 \wedge dp_2,\, dy_2 \wedge dq_2\} \end{aligned}$$

both represent the wave equation

$$z_{xy} = 0,$$

and that the Bäcklund transformation is given by equations of the form

$$\begin{aligned} x_2 &= x_2(x_1, p_1) \\ p_2 &= p_2(x_1, p_1) \\ y_2 &= y_2(y_1, q_1) \\ q_2 &= q_2(y_1, q_1). \end{aligned}$$

These may be written in PDE notation as

$$\begin{aligned} \bar{x} &= \bar{x}(x, z_x) \\ \bar{z}_{\bar{x}} &= \bar{z}_{\bar{x}}(x, z_x) \\ \bar{y} &= \bar{y}(y, z_y) \\ \bar{z}_{\bar{y}} &= \bar{z}_{\bar{y}}(y, z_y), \end{aligned}$$

and the nondegeneracy conditions imply that

$$0 \neq \frac{\partial p_2}{\partial p_1}\frac{\partial x_2}{\partial x_1} - \frac{\partial p_2}{\partial x_1}\frac{\partial x_2}{\partial p_1} \neq \frac{\partial q_2}{\partial q_1}\frac{\partial y_2}{\partial y_1} - \frac{\partial q_2}{\partial y_1}\frac{\partial y_2}{\partial q_1} \neq 0.$$

These transformations are more general than typical point transformations (or even gauge transformations), in that they do not necessarily preserve the space of independent variables.

Thus we have the following theorem.

THEOREM 3.1. *Let $\mathcal{B} \subset M_1 \times M_2$ be a Bäcklund transformation with $C_1 = C_2 = C_3 = C_4 = 0$. Then $\mathcal{B}$ is locally contact equivalent to a transformation between solutions of the wave equation*

$$z_{xy} = 0$$

with the property that given any solution, the new solutions given by the transformation may be obtained by quadrature.

Note that in this case the assumption of homogeneity was not necessary. In the remaining cases, however, homogeneity will play a crucial role in the analysis.

4. Case 2: $[C_1 \ \ C_2] = [0 \ \ 0]$, $[C_3 \ \ C_4] \neq [0 \ \ 0]$**.** Suppose that $C_1 = C_2 = 0$, but C_3 and C_4 are not both zero. By a transformation of the form (2.2), we can arrange that

$$C_3 = 0, \quad C_4 = 1, \quad A_1 = 1.$$

A coframing satisfying this condition will be called *1-adapted.* If $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$ is a 1-adapted coframing, then any other 1-adapted coframing $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4\}$ has the form

(4.1)
$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^1 \\ \tilde{\omega}^2 \\ \tilde{\omega}^3 \\ \tilde{\omega}^4 \end{bmatrix} = \begin{bmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_{21} & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}.$$

The same computation as in the previous section shows that $B_1 = B_2 = 0$. Furthermore, we have

$$\begin{aligned} 0 &\equiv d(d\omega^3) \mod \{\theta_1, \omega^3, \omega^4\} \\ &\equiv (\beta_2 + B_3\,\theta_2) \wedge \omega^1 \wedge \omega^2. \\ 0 &\equiv d(d\omega^3) \mod \{\theta_2, \omega^3, \omega^4\} \\ &\equiv (\beta_2 - B_3\,\theta_1) \wedge \omega^1 \wedge \omega^2. \end{aligned}$$

Together, these equations imply that

$$\beta_2 = B_3\,\theta_1 - B_3\,\theta_2 + H_1\,\omega^1 + H_2\,\omega^2 + H_3\,\omega^3 + H_4\,\omega^4$$

for some functions H_1, H_2, H_3, H_4 on $\mathcal{B}$. Similarly, computing $d(d\omega^4) \equiv 0$ modulo $\{\theta_1, \omega^3, \omega^4\}$ and $\{\theta_2, \omega^3, \omega^4\}$ shows that

$$\beta_4 = \alpha_1 + \alpha_4 + B_4\,\theta_1 - B_4\,\theta_2 + J_1\,\omega^1 + J_2\,\omega^2 + J_3\,\omega^3 + J_4\,\omega^4$$

for some functions J_1, J_2, J_3, J_4 on $\mathcal{B}$. By taking advantage of the ambiguity (2.5) in the forms β_i, we can assume that

$$H_3 = H_4 = J_3 = 0.$$

Now computing $d(d\theta_1) \equiv 0 \mod \theta_1$ shows that

$$\beta_1 = K_0\,\theta_1 + B_4\,\theta_2 + K_1\,\omega^1 + K_2\,\omega^2 + \omega^3 - J_4\,\omega^4$$

for some functions K_0, K_1, K_2 on $\mathcal{B}$.

Under a transformation of the form (4.1), the function A_2 remains unchanged; therefore by our assumption of homogeneity it must be constant. Moreover, the nondegeneracy assumptions imply that $A_2 \neq 0, 1$. So we have

$$\begin{aligned} 0 &\equiv d(d\theta_2) \mod \theta_2 \\ &\equiv -A_2[(K_0 + B_4)\,\theta_1 + (K_1 + J_1)\,\omega^1 + (K_2 + J_2)\,\omega^2] \wedge \omega^3 \wedge \omega^4, \end{aligned}$$

which implies that

$$\begin{aligned} K_0 &= -B_4 \\ K_1 &= -J_1 \\ K_2 &= -J_2. \end{aligned}$$

Now we have

$$\begin{aligned} 0 = d(d\theta_1) &= \Upsilon_1 \wedge \theta_1 \\ 0 = d(d\theta_2) &= \Upsilon_2 \wedge \theta_2 \end{aligned}$$

where

$$\begin{aligned} \Upsilon_1 &= d\alpha_1 + d\alpha_4 + (J_1\,\omega^1 + J_2\,\omega^2) \wedge \omega^3 - (H_1\,\omega^1 + H_2\,\omega^2) \wedge \omega^4 - J_4\,\omega^3 \wedge \omega^4 \\ \Upsilon_2 &= d\alpha_1 + d\alpha_4 + A_2[(J_1\,\omega^1 + J_2\,\omega^2) \wedge \omega^3 - (H_1\,\omega^1 + H_2\,\omega^2) \wedge \omega^4 - J_4\,\omega^3 \wedge \omega^4]. \end{aligned}$$

These equations imply that Υ_1 must be a multiple of θ_1 and Υ_2 must be a multiple of θ_2, so

$$\begin{aligned} 0 &\equiv \Upsilon_2 - \Upsilon_1 \mod \{\theta_1, \theta_2\} \\ &\equiv (A_2 - 1)[(J_1\,\omega^1 + J_2\,\omega^2) \wedge \omega^3 - (H_1\,\omega^1 + H_2\,\omega^2) \wedge \omega^4 - J_4\,\omega^3 \wedge \omega^4]. \end{aligned}$$

Therefore, since $A_2 \neq 1$ we have

$$H_1 = H_2 = J_1 = J_2 = J_4 = 0.$$

The structure equations for a 1-adapted coframing now take the form

$$(4.2)\quad \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\omega^1 \\ d\omega^2 \\ d\omega^3 \\ d\omega^4 \end{bmatrix} = - \begin{bmatrix} \alpha_1 + \alpha_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + \alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 & \alpha_1 + \alpha_4 \end{bmatrix} \wedge \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} + \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{bmatrix}$$

where

$$\begin{aligned}
\Theta_1 &= \omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 - \theta_1) \\
\Theta_2 &= \omega^1 \wedge \omega^2 + A_2\, \omega^3 \wedge (\omega^4 - \theta_2) \\
\Omega^1 &= 0 \\
\Omega^2 &= 0 \\
\Omega^3 &= B_3\, (\theta_1 - \omega^4) \wedge (\theta_2 - \omega^4) + B_4\, (\theta_1 - \theta_2) \wedge \omega^3 \\
\Omega^4 &= B_4\, (\theta_1 - \omega^4) \wedge (\theta_2 - \omega^4) + \omega^1 \wedge \omega^2.
\end{aligned}$$

Now

$$0 = d(d\theta_1) = -d(\alpha_1 + \alpha_4) \wedge \theta_1,$$

and so

$$d(\alpha_1 + \alpha_4) = \psi \wedge \theta_1$$

for some 1-form ψ. Differentiating this equation and reducing modulo θ_1 yields

$$-\psi \wedge (\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4) \equiv 0 \mod \theta_1.$$

But since $\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4$ is nondecomposable, this implies that

$$\psi \equiv 0 \mod \theta_1$$

and hence that

$$d(\alpha_1 + \alpha_4) = 0.$$

Therefore, there exists a nonvanishing function λ on $\mathcal{B}$ such that

$$\lambda^{-1}\, d\lambda = \alpha_1 + \alpha_4.$$

We can choose a new 1-adapted coframing in which θ_1 is replaced by $\lambda\, \theta_1$. This coframing will have the property that

$$\alpha_1 + \alpha_4 = 0.$$

Now we have

$$d(\omega^1 \wedge \omega^2) = 0,$$

and so by Darboux's Theorem there exist functions x, p on $\mathcal{B}$ (which are also well-defined on M_1 and M_2) such that

$$\omega^1 \wedge \omega^2 = dx \wedge dp.$$

Therefore

$$\begin{aligned}
0 = d(d\theta_1) &= d(\omega^3 \wedge (\omega^4 - \theta_1)) \\
0 = d(d\theta_2) &= d(A_2\, \omega^3 \wedge (\omega^4 - \theta_2)).
\end{aligned}$$

Again by Darboux's Theorem, there exist functions y_1, q_1 on M_1 and y_2, q_2 on M_2 such that

$$\omega^3 \wedge (\omega^4 - \theta_1) = dy_1 \wedge dq_1$$
$$A_2\, \omega^3 \wedge (\omega^4 - \theta_2) = dy_2 \wedge dq_2.$$

By Pfaff's Theorem there exist functions z_1 on M_1 and z_2 on M_2 such that

$$\theta_1 = dz_1 - p\, dx - q_1\, dy_1$$
$$\theta_2 = dz_2 - p\, dx - q_2\, dy_2.$$

The ideals $\mathcal{I}_1, \mathcal{I}_2$ now take the form

$$\mathcal{I}_1 = \{\theta_1,\, \omega^1 \wedge \omega^2,\, \omega^3 \wedge (\omega^4 - \theta_1)\} = \{dz_1 - p\, dx - q_1\, dy_1,\, dx \wedge dp,\, dy_1 \wedge dq_1\}$$
$$\mathcal{I}_2 = \{\theta_2,\, \omega^1 \wedge \omega^2,\, A_2\, \omega^3 \wedge (\omega^4 - \theta_2)\} = \{dz_2 - p\, dx - q_2\, dy_2,\, dx \wedge dp,\, dy_2 \wedge dq_2\}.$$

Both represent the wave equation

$$z_{xy} = 0,$$

and the Bäcklund transformation is given by equations of the form

$$\begin{aligned} x_2 &= x_1 \\ y_2 &= y_2(x_1, y_1, z_1, z_2, p_1, q_1) \\ p_2 &= p_1 \\ q_2 &= q_2(x_1, y_1, z_1, z_2, p_1, q_1)), \end{aligned}$$

or, in PDE notation,

$$\begin{aligned} \bar{x} &= x \\ \bar{y} &= \bar{y}(x, y, z, \bar{z}, z_x, z_y) \\ \bar{z}_{\bar{x}} &= z_x \\ \bar{z}_{\bar{y}} &= \bar{z}_{\bar{y}}(x, y, z, \bar{z}, z_x, z_y). \end{aligned}$$

As in the previous case, these transformations do not in general preserve the space of independent variables.

Thus we have the following theorem.

THEOREM 4.1. *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation with one of the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$ identically zero and the other nonzero. Then $\mathcal{B}$ is locally contact equivalent to a transformation between solutions of the wave equation*

$$z_{xy} = 0.$$

5. Case 3: $[C_1\ \ C_2]$, $[C_3\ \ C_4] \neq [0\ \ 0]$. Suppose that the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$ are both nonzero. By a transformation of the form (2.2), we can arrange that

$$C_1 = C_3 = 0, \qquad C_2 = C_4 = 1.$$

A coframing satisfying this condition is called *1-adapted.* If $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$ is a 1-adapted coframing, then any other 1-adapted coframing $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4\}$ has the form

$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^1 \\ \tilde{\omega}^2 \\ \tilde{\omega}^3 \\ \tilde{\omega}^4 \end{bmatrix} = \begin{bmatrix} a_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b_{22}}{a_{22}} & 0 & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a_{22}}{b_{22}} & 0 \\ 0 & 0 & 0 & 0 & b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}. \tag{5.1}$$

Similarly to the previous case, computing

$$\begin{aligned} d(d\omega^1) &\equiv 0 \mod \{\theta_1, \omega^1, \omega^2\} \\ d(d\omega^1) &\equiv 0 \mod \{\theta_2, \omega^1, \omega^2\} \\ d(d\omega^3) &\equiv 0 \mod \{\theta_1, \omega^3, \omega^4\} \\ d(d\omega^3) &\equiv 0 \mod \{\theta_2, \omega^3, \omega^4\} \end{aligned}$$

shows that

$$\begin{aligned} \alpha_2 &= A_2 B_1\, \theta_1 - B_1\, \theta_2 + G_1\, \omega^1 + G_2\, \omega^2 + G_3\, \omega^3 + G_4\, \omega^4 \\ \beta_2 &= B_3\, \theta_1 - A_1 B_3\, \theta_2 + H_1\, \omega^1 + H_2\, \omega^2 + H_3\, \omega^3 + H_4\, \omega^4 \end{aligned}$$

for some functions G_i, H_i on $\mathcal{B}$. By taking advantage of the ambiguity (2.5), we can assume that

$$G_1 = G_2 = H_3 = H_4 = 0.$$

Now computing

$$\begin{aligned} d(d\omega^2) &\equiv 0 \mod \{\theta_1, \omega^1, \omega^2\} \\ d(d\omega^2) &\equiv 0 \mod \{\theta_2, \omega^1, \omega^2\} \\ d(d\omega^4) &\equiv 0 \mod \{\theta_1, \omega^3, \omega^4\} \\ d(d\omega^4) &\equiv 0 \mod \{\theta_2, \omega^3, \omega^4\} \end{aligned}$$

shows that

$$\begin{aligned} \alpha_1 &= \beta_4 - \alpha_4 - B_4\, \theta_1 + A_1 B_4\, \theta_2 + J_1\, \omega^1 + J_2\, \omega^2 + J_3\, \omega^3 + J_4\, \omega^4 \\ \beta_1 &= \alpha_4 - \beta_4 - A_2 B_2\, \theta_1 + B_2\, \theta_2 + K_1\, \omega^1 + K_2\, \omega^2 + K_3\, \omega^3 + K_4\, \omega^4 \end{aligned}$$

for some functions J_i, K_i on $\mathcal{B}$. Using some of the remaining ambiguity (2.5), we can assume that

$$J_3 = K_1 = 0.$$

The structure equations for a 1-adapted coframing now take the form

$$
(5.2)\qquad \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\omega^1 \\ d\omega^2 \\ d\omega^3 \\ d\omega^4 \end{bmatrix} = - \begin{bmatrix} \alpha_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_4 - \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_4 - \beta_4 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 & \beta_4 \end{bmatrix} \wedge \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} + \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{bmatrix}
$$

where

$$
\begin{aligned}
\Theta_1 &= \theta_1 \wedge (B_2\,\theta_2 + A_1\,\omega^1 + K_2\,\omega^2 + K_3\,\omega^3 + K_4\,\omega^4) + A_1\,\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
\Theta_2 &= \theta_2 \wedge (-B_4\,\theta_1 + J_1\,\omega^1 + J_2\,\omega^2 + A_2\,\omega^3 + J_4\,\omega^4) + \omega^1 \wedge \omega^2 + A_2\,\omega^3 \wedge \omega^4 \\
\Omega^1 &= \omega^1 \wedge (-B_4\,\theta_1 + A_1 B_4\,\theta_2 + J_2\,\omega^2 + J_4\,\omega^4) \\
&\qquad + \omega^2 \wedge (A_2 B_1\,\theta_1 - B_1\,\theta_2 + G_3\,\omega^3 + G_4\,\omega^4) + B_1\,\theta_1 \wedge \theta_2 \\
\Omega^2 &= B_2\,\theta_1 \wedge \theta_2 + \omega^3 \wedge \omega^4 \\
\Omega^3 &= \omega^3 \wedge (-A_2 B_2\,\theta_1 + B_2\,\theta_2 + K_2\,\omega^2 + K_4\,\omega^4) \\
&\qquad + \omega^4 \wedge (B_3\,\theta_1 - A_1 B_3\,\theta_2 + H_1\,\omega^1 + H_2\,\omega^2) + B_3\,\theta_1 \wedge \theta_2 \\
\Omega^4 &= B_4\,\theta_1 \wedge \theta_2 + \omega^1 \wedge \omega^2.
\end{aligned}
$$

A computation shows that under a transformation of the form (5.1), we have

$$
\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} (a_{22})^2 & 0 \\ -a_{21}a_{22} & b_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
$$

$$
\begin{bmatrix} \tilde{B}_3 \\ \tilde{B}_4 \end{bmatrix} = \begin{bmatrix} (b_{22})^2 & 0 \\ -b_{21}b_{22} & a_{22} \end{bmatrix} \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}.
$$

In particular, the functions B_1, B_3 are now relative invariants. In order to proceed further, we will need to divide into cases depending on the values of the B_i. First we prove the following lemma:

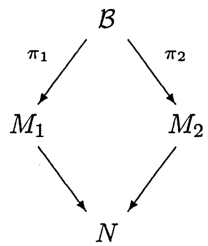
LEMMA 5.1. *For any 1-adapted coframing as above, the vectors* $[B_1 \;\; B_2]$, $[B_3 \;\; B_4]$ *are either both zero or both nonzero.*

Proof. Suppose that $B_1 = B_2 = 0$. Then

$$
\begin{aligned}
0 &\equiv d(d\omega^2) \mod \{\omega^1, \omega^2\} \\
&\equiv \theta_1 \wedge \theta_2 \wedge (B_3\,\omega^4 - B_4\,\omega^3).
\end{aligned}
$$

Therefore, $B_3 = B_4 = 0$. A similar argument demonstrates the converse. □

6. Case 3A: $[B_1 \;\; B_2] = [B_3 \;\; B_4] = [0 \;\; 0]$**.** Suppose that $B_1 = B_2 = B_3 = B_4 = 0$. Then the forms $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ form a Frobenius system, and so locally there exists a 4-manifold N which is a quotient of $\mathcal{B}$ and for which the 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$ are semi-basic for the projection $\mathcal{B} \to N$. (Here "locally" refers to the fact that any point in $\mathcal{B}$ has a neighborhood which possesses such a quotient, and "semi-basic" means that the restrictions of the ω^i to the fibers of the projection vanish identically. See [3] for details.) In fact, this quotient fibers through each of the quotients $\pi_i : \mathcal{B} \to M_i$, as shown.



Let X, Y, P, Q be local coordinates on N, and let Z_1, Z_2 be functions on M_1, M_2, respectively, such that $\{X, Y, P, Q, Z_i\}$ is a local coordinate system on M_i. Then $\{X, Y, P, Q, Z_1, Z_2\}$ may be regarded as a local coordinate system on $\mathcal{B}$, and we can write

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix} = F^{-1} \begin{bmatrix} dX \\ dP \\ dY \\ dQ \end{bmatrix}$$

where $F = (f_{ij})$ is a nonsingular matrix whose entries are functions on $\mathcal{B}$. By relabelling X, Y, P, Q if necessary, we can assume that the 2×2 sub-matrices

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \qquad \begin{bmatrix} f_{33} & f_{34} \\ f_{43} & f_{44} \end{bmatrix}$$

are nonsingular, and we can choose a 0-adapted coframing with

$$F = \begin{bmatrix} 1 & 0 & f_{13} & f_{14} \\ 0 & 1 & f_{23} & f_{24} \\ f_{31} & f_{32} & 1 & 0 \\ f_{41} & f_{42} & 0 & 1 \end{bmatrix}.$$

A messy but straightforward computation shows that the requirements

$$\mathcal{L}_{\frac{\partial}{\partial\theta_2}}(\omega^1 \wedge \omega^2) \equiv 0 \mod \{\theta_1, \omega^1 \wedge \omega^2\}$$
$$\mathcal{L}_{\frac{\partial}{\partial\theta_2}}(\omega^3 \wedge \omega^4) \equiv 0 \mod \{\theta_1, \omega^3 \wedge \omega^4\}$$

are satisfied if and only if

$$\tilde{F} \begin{bmatrix} \frac{\partial f_{13}}{\partial Z_1} \\ \frac{\partial f_{23}}{\partial Z_1} \end{bmatrix} = \tilde{F} \begin{bmatrix} \frac{\partial f_{14}}{\partial Z_1} \\ \frac{\partial f_{24}}{\partial Z_1} \end{bmatrix} = \tilde{F} \begin{bmatrix} \frac{\partial f_{13}}{\partial Z_2} \\ \frac{\partial f_{23}}{\partial Z_2} \end{bmatrix} = \tilde{F} \begin{bmatrix} \frac{\partial f_{14}}{\partial Z_2} \\ \frac{\partial f_{24}}{\partial Z_2} \end{bmatrix} = 0,$$

where

$$\tilde{F} = \begin{bmatrix} f_{23}f_{31} + f_{24}f_{41} & 1 - f_{13}f_{31} - f_{14}f_{41} \\ f_{23}f_{32} + f_{24}f_{42} - 1 & -f_{13}f_{32} - f_{14}f_{42} \end{bmatrix}.$$

Since $\det(\tilde{F}) = \det(F) \neq 0$, this implies that

$$\frac{\partial f_{13}}{\partial Z_1} = \frac{\partial f_{13}}{\partial Z_2} = \frac{\partial f_{23}}{\partial Z_1} = \frac{\partial f_{23}}{\partial Z_2} = \frac{\partial f_{14}}{\partial Z_1} = \frac{\partial f_{14}}{\partial Z_2} = \frac{\partial f_{24}}{\partial Z_1} = \frac{\partial f_{24}}{\partial Z_2} = 0,$$

and so the functions f_{13}, f_{23}, f_{14}, f_{24} are well-defined on N. Therefore ω^1 and ω^2 are well-defined forms on N; in particular, the 2-form $\omega^1 \wedge \omega^2$ is well-defined on N. A similar argument using the conditions

$$\mathcal{L}_{\frac{\partial}{\partial \theta_1}}(\omega^1 \wedge \omega^2) \equiv 0 \mod \{\theta_2, \omega^1 \wedge \omega^2\}$$
$$\mathcal{L}_{\frac{\partial}{\partial \theta_1}}(\omega^3 \wedge \omega^4) \equiv 0 \mod \{\theta_2, \omega^3 \wedge \omega^4\}$$

shows that ω^3 and ω^4 (and hence the 2-form $\omega^3 \wedge \omega^4$) are well-defined on N as well.

Since the ω^i are well-defined on N, we see from the structure equations for the $d\omega^i$ that the forms α_i, β_i are linear combinations of the ω^i. In particular, we have

$$d\theta_1 = -\gamma \wedge \theta_1 + A_1\, \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \tag{6.1}$$
$$d\theta_2 = -\delta \wedge \theta_2 + \omega^1 \wedge \omega^2 + A_2\, \omega^3 \wedge \omega^4 \tag{6.2}$$

for some 1-forms γ, δ which are linear combinations of the ω^i alone. It follows that θ_i is a well-defined 1-form on M_i, from which it follows in turn that A_i is a well-defined function on M_i, i.e.,

$$A_1 = A_1(X, Y, P, Q, Z_1), \qquad A_2 = A_2(X, Y, P, Q, Z_2).$$

Now we use the hypothesis of homogeneity for the first time: because the product $A_1 A_2$ is an invariant independent of the choice of 0-adapted coframing, it must be constant on $\mathcal{B}$. It follows that the functions A_1, A_2 are actually independent of Z_1, Z_2 and so are well-defined functions on N.

Now differentiating equation (6.1) and reducing modulo $\Lambda^3(\{\omega^i\})$ yields

$$0 \equiv -d\gamma \wedge \theta_1 \mod \Lambda^3(\{\omega^i\}),$$

but this implies that in fact

$$d\gamma \wedge \theta_1 = 0.$$

By an argument identical to that given in Case 2 for the form $\alpha_1 + \alpha_4$, it follows that

$$d\gamma = 0.$$

Therefore, there exists a nonvanishing function λ on $\mathcal{B}$ such that

$$\lambda^{-1}\, d\lambda = \gamma.$$

Let $\tilde{\theta}_1 = \lambda\, \theta_1$. Then

$$\begin{aligned} d\tilde{\theta}_1 &= d\lambda \wedge \theta_1 + \lambda\, d\theta_1 \\ &= \lambda(A_1\, \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4). \end{aligned}$$

Since this is a closed form which is semi-basic for the projection $\mathcal{B} \to N$, it is in fact a well-defined form on N. By Darboux's Theorem there exist functions x_1, y_1, p_1, q_1 on N such that

$$d\tilde{\theta}_1 = -dp_1 \wedge dx_1 - dq_1 \wedge dy_1.$$

Then by Pfaff's Theorem there exists a function z_1 on M_1 such that

$$\tilde{\theta}_1 = dz_1 - p_1\, dx_1 - q_1\, dy_1.$$

A similar argument shows that there exist functions x_2, y_2, p_2, q_2 on N and z_2, μ on M_2 such that $\tilde{\theta}_2 = \mu\, \theta_2$ has the form

$$\tilde{\theta}_2 = dz_2 - p_2\, dx_2 - q_2\, dy_2.$$

The ideal $\bar{\mathcal{I}} = \{\omega^1 \wedge \omega^2,\ \omega^3 \wedge \omega^4\}$ on N is now spanned by the forms

$$\{dp_1 \wedge dx_1 + dq_1 \wedge dy_1,\ dp_2 \wedge dx_2 + dq_2 \wedge dy_2\},$$

and the ideals $\mathcal{I}_1, \mathcal{I}_2$ are *integrable extensions* of $\bar{\mathcal{I}}$. (For a definition and discussion of integrable extensions, see [4].) The equations defining the Bäcklund transformation are simply those defining the change of coordinates on N:

$$\begin{aligned} x_2 &= x_2(x_1, y_1, p_1, q_1) \\ y_2 &= y_2(x_1, y_1, p_1, q_1) \\ p_2 &= p_2(x_1, y_1, p_1, q_1) \\ q_2 &= q_2(x_1, y_1, p_1, q_1), \end{aligned} \tag{6.3}$$

or, in PDE notation,

$$\begin{aligned} \bar{x} &= \bar{x}(x, y, z_x, z_y) \\ \bar{y} &= \bar{x}(x, y, z_x, z_y) \\ \bar{z}_{\bar{x}} &= \bar{z}_{\bar{x}}(x, y, z_x, z_y) \\ \bar{z}_{\bar{y}} &= \bar{z}_{\bar{y}}(x, y, z_x, z_y). \end{aligned}$$

Note that if $z(x, y)$ is a known solution of the PDE corresponding to the ideal $(M_1, \mathcal{I}_1)$, the corresponding solution $\bar{z}(\bar{x}, \bar{y})$ of the PDE corresponding to the ideal $(M_2, \mathcal{I}_2)$ can be constructed by quadrature.

We have proved the following theorem:

THEOREM 6.1. *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation with the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$ both nonzero and $B_1 = B_2 = B_3 = B_4 = 0$. Then $\mathcal{B}$ arises in the following way: let $\{x_1, y_1, p_1, q_1\}, \{x_2, y_2, p_2, q_2\}$ be two sets of local coordinates on a 4-manifold N such that the 2-forms*

$$\{dp_1 \wedge dx_1 + dq_1 \wedge dy_1,\ dp_2 \wedge dx_2 + dq_2 \wedge dy_2\}$$

span a hyperbolic pencil (i.e., there exist two distinct linear combinations of these 2-forms which are decomposable) at each point of N. Let

$$\begin{aligned} M_1 &= N \times \mathbb{R} \text{ with coordinate } z_1 \text{ on the } \mathbb{R} \text{ factor} \\ M_2 &= N \times \mathbb{R} \text{ with coordinate } z_2 \text{ on the } \mathbb{R} \text{ factor.} \end{aligned}$$

Let $\mathcal{I}_1$ be the ideal on M_1 generated by the forms

$$\begin{aligned} \theta_1 &= dz_1 - p_1\, dx_1 - q_1\, dy_1 \\ d\theta_1 &= -dp_1 \wedge dx_1 - dq_1 \wedge dy_1 \\ \Upsilon_1 &= dp_2 \wedge dx_2 + dq_2 \wedge dy_2 \end{aligned}$$

and let $\mathcal{I}_2$ *be the ideal on* M_2 *generated by the forms*

$$\begin{aligned}\theta_2 &= dz_2 - p_2\,dx_2 - q_2\,dy_2\\ d\theta_2 &= -dp_2 \wedge dx_2 - dq_2 \wedge dy_2\\ \Upsilon_2 &= dp_1 \wedge dx_1 + dq_1 \wedge dy_1.\end{aligned}$$

Then $\mathcal{B} \subset M_1 \times M_2$ *is defined by the equations* (6.3).

Zvyagin [15] calls Bäcklund transformations with $B_1 = B_2 = B_3 = B_4 = 0$ *holonomic.* Even without the assumption of homogeneity our argument shows that any holonomic Bäcklund transformation arises locally from a hyperbolic system

$$\bar{\mathcal{I}} = \{\omega^1 \wedge \omega^2,\ \omega^3 \wedge \omega^4\}$$

on a 4-manifold N such that $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$ are integrable extensions of $(N, \bar{\mathcal{I}})$. These transformations are generally of limited interest.

7. Case 3B: $B_1 = B_3 = 0$; $B_2, B_4 \neq 0$. First we compute that

$$\begin{aligned}0 &\equiv d(d\omega^1) \mod \{\omega^1, \omega^2\}\\ &\equiv B_2\,\theta_1 \wedge \theta_2 \wedge (G_3\,\omega^3 + G_4\,\omega^4) \qquad \Rightarrow G_3 = G_4 = 0\\ 0 &\equiv d(d\omega^3) \mod \{\omega^3, \omega^4\}\\ &\equiv B_4\,\theta_1 \wedge \theta_2 \wedge (H_1\,\omega^1 + H_2\,\omega^2) \qquad \Rightarrow H_1 = H_2 = 0.\end{aligned}$$

Next we observe that under a transformation of the form (5.1), we have

$$\begin{aligned}\tilde{B}_2 &= b_{22} B_2\\ \tilde{B}_4 &= a_{22} B_4,\end{aligned}$$

so we can choose a coframing with $B_2 = B_4 = 1$. Such a coframing will be called *2-adapted*; any two 2-adapted coframings differ by a transformation of the form

$$\begin{bmatrix}\tilde{\theta}_1\\ \tilde{\theta}_2\\ \tilde{\omega}^1\\ \tilde{\omega}^2\\ \tilde{\omega}^3\\ \tilde{\omega}^4\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&a_{21}&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&b_{21}&1\end{bmatrix}^{-1} \begin{bmatrix}\theta_1\\ \theta_2\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{bmatrix}. \tag{7.1}$$

For a 2-adapted coframing, computing

$$\begin{aligned}d(d\omega^2) &\equiv 0 \mod \{\omega^1, \omega^2\}\\ d(d\omega^4) &\equiv 0 \mod \{\omega^3, \omega^4\}\end{aligned}$$

shows that

$$\begin{aligned}\alpha_4 &= L_1\,\theta_1 + L_2\,\theta_2 - (A_1 + J_1 + 1)\,\omega^1 - (J_2 + K_2)\,\omega^2 + M_3\,\omega^3 + M_4\,\omega^4\\ \beta_4 &= L_3\,\theta_1 + L_4\,\theta_2 + M_1\,\omega^1 + M_2\,\omega^2 - (A_2 + K_3 + 1)\,\omega^3 - (J_4 + K_4)\,\omega^4\end{aligned}$$

for some functions L_i, M_i on $\mathcal{B}$.

It is straightforward to show that under a transformation of the form (7.1), the functions $A_1, A_2, J_2, J_4, K_2, K_4, L_1, L_2, L_3, L_4, M_2, M_4$ remain unchanged. By our assumption of homogeneity, they must therefore be constants. Moreover,

$$\begin{aligned} \tilde{J}_1 &= J_1 + a_{21} J_2 \\ \tilde{K}_3 &= K_3 + b_{21} K_4 \\ \tilde{M}_1 &= M_1 + a_{21} M_2 \\ \tilde{M}_3 &= M_3 + b_{21} M_4. \end{aligned}$$

Now we compute:

$$\begin{aligned}
0 &\equiv d(d\theta_1) \mod \{\theta_1, \theta_2, \omega^3\} \\
&\equiv A_1(M_4 + 2K_4)\, \omega^1 \wedge \omega^2 \wedge \omega^4 && \Rightarrow M_4 = -2K_4 \\
0 &\equiv d(d\theta_2) \mod \{\theta_1, \theta_2, \omega^1\} \\
&\equiv A_2(M_2 + 2J_2)\, \omega^2 \wedge \omega^3 \wedge \omega^4 && \Rightarrow M_2 = -2J_2 \\
0 &\equiv d(d\theta_1) \mod \{\theta_1, \omega^3, \omega^4\} \\
&\equiv A_1(L_2 - L_4 - A_1 + 1)\, \theta_2 \wedge \omega^1 \wedge \omega^1 && \Rightarrow L_4 = L_2 - A_1 + 1 \\
0 &\equiv d(d\theta_2) \mod \{\theta_2, \omega^1, \omega^2\} \\
&\equiv A_2(L_3 - L_1 + A_2 - 1)\, \theta_1 \wedge \omega^3 \wedge \omega^4 && \Rightarrow L_3 = L_1 - A_2 + 1 \\
0 &\equiv d(d\theta_1) \mod \{\theta_1, \theta_2, \omega^4\} \\
&\equiv (A_1 M_3 + 2A_1 K_3 + A_1 A_2 + A_1 - 1)\, \omega^1 \wedge \omega^2 \wedge \omega^3 \\
& && \Rightarrow M_3 = -2K_3 - A_2 - 1 - \frac{1}{A_1} \\
0 &\equiv d(d\theta_2) \mod \{\theta_1, \theta_2, \omega^2\} \\
&\equiv (A_2 M_1 + 2A_2 J_1 + A_1 A_2 + A_2 - 1)\ \omega^1 \wedge \omega^3 \wedge \omega^4 \\
& && \Rightarrow M_1 = -2J_1 - A_1 - 1 - \frac{1}{A_2} \\
0 &\equiv d(d\theta_1) \mod \{\omega^1, \omega^3, \omega^4\} \\
&\equiv J_2\, \theta_1 \wedge \theta_2 \wedge \omega^2 && \Rightarrow J_2 = 0 \\
0 &\equiv d(d\theta_2) \mod \{\omega^1, \omega^2, \omega^3\} \\
&\equiv K_4\, \theta_1 \wedge \theta_2 \wedge \omega^4 && \Rightarrow K_4 = 0.
\end{aligned}$$

Therefore the functions J_1, K_3 in fact remain unchanged under a transformation of the form (7.1), and so they must be constants as well. Next we compute:

$$\begin{aligned}
0 &\equiv d(d\omega^2) \mod \{\omega^1, \omega^3, \omega^4\} \\
&\equiv (-L_1 - A_2 L_2)\,\theta_1 \wedge \theta_2 \wedge \omega^2 && \Rightarrow L_1 = -A_2 L_2 \\
0 &\equiv d(d\omega^1) \mod \{\omega^2, \omega^3, \omega^4\} \\
&\equiv -K_2\,\theta_1 \wedge \theta_2 \wedge \omega^1 && \Rightarrow K_2 = 0 \\
0 &\equiv d(d\theta_1) \mod \{\theta_2, \omega^3, \omega^4\} \\
&\equiv (A_1A_2 - 1)(L_2 + 1)\,\theta_1 \wedge \omega^1 \wedge \omega^2 && \Rightarrow L_2 = -1 \\
0 &\equiv d(d\omega^3) \mod \{\omega^1, \omega^2, \omega^4\} \\
&\equiv -J_4\,\theta_1 \wedge \theta_2 \wedge \omega^3 && \Rightarrow J_4 = 0 \\
0 &\equiv d(d\theta_1) \mod \{\omega^2, \omega^3, \omega^4\} \\
&\equiv -(J_1 + A_1 + 1)\,\theta_1 \wedge \theta_2 \wedge \omega^1 && \Rightarrow J_1 = -A_1 - 1 \\
0 &\equiv d(d\theta_1) \mod \{\omega^1, \omega^2, \omega^4\} \\
&\equiv -A_1(K_3 + A_2 + 1)\,\theta_1 \wedge \theta_2 \wedge \omega^3 && \Rightarrow K_3 = -A_2 - 1 \\
0 &\equiv d(d\omega^1) \mod \{\theta_1, \omega^2, \omega^4\} \\
&\equiv (A_1 + 1)(A_2 + 1)\,\theta_2 \wedge \omega^1 \wedge \omega^3 && \Rightarrow (A_1 + 1)(A_2 + 1) = 0.
\end{aligned}$$

Without loss of generality, we can assume that $A_2 = -1$. Then

$$0 = d(d\theta_2) = \frac{(A_1^2 - 1)}{A_1}\theta_2 \wedge \omega^1 \wedge \omega^3 \qquad \Rightarrow\ A_1^2 = 1.$$

Since $A_1A_2 - 1 \neq 0$, we must have $A_1 = 1$. The structure equations for a 2-adapted coframing may now be written as

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\
d\omega^1 &= \omega^1 \wedge (\theta_1 + \theta_2 - \omega^3) \\
d\omega^2 &= -\alpha_3 \wedge \omega^1 - \omega^2 \wedge (\theta_1 + \theta_2 - \omega^3) + \theta_1 \wedge \theta_2 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= \omega^3 \wedge (\theta_2 - \theta_1 - \omega^1) \\
d\omega^4 &= -\beta_3 \wedge \omega^3 - \omega^4 \wedge (\theta_2 - \theta_1 - \omega^1) + \theta_1 \wedge \theta_2 + \omega^1 \wedge \omega^2
\end{aligned}$$

for some 1-forms α_3, β_3 on $\mathcal{B}$.

Now suppose that $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$ is any 2-adapted coframing. Since

$$\begin{aligned}
d\omega^1 &\equiv 0 \mod \omega^1 \\
d\omega^3 &\equiv 0 \mod \omega^3
\end{aligned}$$

there exist functions x, y, r_1, r_2 on $\mathcal{B}$ and nonzero constants λ_1, λ_2 such that

$$\omega^1 = \lambda_1 e^{r_1}\, dx, \qquad \omega^3 = \lambda_2 e^{r_2}\, dy.$$

Since the systems $\{\theta_1, \omega^1, \omega^3\}$ and $\{\theta_2, \omega^1, \omega^3\}$ are completely integrable, there must exist functions $z_1, z_2, p_1, p_2, q_1, q_2, \rho_1, \rho_2$ on $\mathcal{B}$, with ρ_1, ρ_2 nonvanishing, such that

$$\begin{aligned}
\theta_1 &= \rho_1(dz_1 - p_1\, dx - q_1\, dy) \\
\theta_2 &= \rho_2(dz_2 - p_2\, dx - q_2\, dy).
\end{aligned}$$

Moreover, since

$$d\theta_1, d\theta_2 \equiv 0 \mod \{\omega^1, \omega^3\},$$

ρ_1 must be a function of the variables x, y, z_1 alone and ρ_2 must be a function of the variables x, y, z_2 alone. By making the contact transformation

$$\begin{aligned}
\tilde{x} &= x \\
\tilde{y} &= y \\
\tilde{z}_1 &= -\tfrac{1}{2}\int_0^{z_1} \rho_1(x,y,\tau)\,d\tau \\
\tilde{z}_2 &= -\tfrac{1}{2}\int_0^{z_2} \rho_2(x,y,\tau)\,d\tau \\
\tilde{p}_1 &= -\tfrac{1}{2}\int_0^{z_1} \frac{\partial\rho_1(x,y,\tau)}{\partial x}\,d\tau - \tfrac{1}{2}\rho_1(x,y,z_1)p_1 \\
\tilde{p}_2 &= -\tfrac{1}{2}\int_0^{z_2} \frac{\partial\rho_2(x,y,\tau)}{\partial x}\,d\tau - \tfrac{1}{2}\rho_2(x,y,z_2)p_2 \\
\tilde{q}_1 &= -\tfrac{1}{2}\int_0^{z_1} \frac{\partial\rho_1(x,y,\tau)}{\partial y}\,d\tau - \tfrac{1}{2}\rho_1(x,y,z_1)q_1 \\
\tilde{q}_2 &= -\tfrac{1}{2}\int_0^{z_2} \frac{\partial\rho_2(x,y,\tau)}{\partial y}\,d\tau - \tfrac{1}{2}\rho_2(x,y,z_2)q_2
\end{aligned}$$

we can assume that $\rho_1 = \rho_2 = -\frac{1}{2}$.

Substituting the expressions given above for $\theta_1, \theta_2, \omega^1, \omega^3$ into the equations for $d\omega^1, d\omega^3$ yields

$$\left.\begin{aligned}
dr_1 &\equiv \tfrac{1}{2}(dz_2 + dz_1) \\
dr_2 &\equiv \tfrac{1}{2}(dz_2 - dz_1)
\end{aligned}\right\} \mod \{dx, dy\}.$$

Therefore we have

$$\begin{aligned}
r_1 &= \tfrac{1}{2}(z_2 + z_1) + f(x,y) \\
r_2 &= \tfrac{1}{2}(z_2 - z_1) + g(x,y)
\end{aligned}$$

for some functions f, g. By making the contact transformation

$$\begin{aligned}
\tilde{x} &= x \\
\tilde{y} &= y \\
\tilde{z}_1 &= z_1 + f(x,y) - g(x,y) \\
\tilde{z}_2 &= z_2 + f(x,y) + g(x,y) \\
\tilde{p}_1 &= p_1 + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x} \\
\tilde{p}_2 &= p_2 + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \\
\tilde{q}_1 &= q_1 + \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \\
\tilde{q}_2 &= q_2 + \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}
\end{aligned}$$

we can assume that $f = g = 0$. Now substituting into the equations for $d\omega^1, d\omega^3$ yields

$$p_2 - p_1 = 2\lambda_1 e^{\frac{1}{2}(z_2+z_1)} \tag{7.2}$$
$$q_2 + q_1 = 2\lambda_2 e^{\frac{1}{2}(z_2-z_1)}.$$

The equations for $d\theta_1, d\theta_2$ imply that

$$\begin{aligned}
\omega^2 &\equiv -\frac{1}{2\lambda_1} e^{-\frac{1}{2}(z_2+z_1)} dp_1 + \theta_1 \\
&\equiv -\frac{1}{2\lambda_1} e^{-\frac{1}{2}(z_2+z_1)} (dp_2 - 2\lambda_1\lambda_2 e^{z_2}\, dy) - \theta_2 \quad \bmod \omega^1 \\
\omega^4 &\equiv -\frac{1}{2\lambda_2} e^{-\frac{1}{2}(z_2-z_1)} dq_1 + \theta_1 \\
&\equiv \frac{1}{2\lambda_2} e^{-\frac{1}{2}(z_2-z_1)} (dq_2 - 2\lambda_1\lambda_2 e^{z_2}\, dx) + \theta_2 \quad \bmod \omega^2.
\end{aligned}$$

By scaling x and y if necessary, we can assume that

$$\lambda_1 = \frac{1}{2\lambda_2} = \frac{\lambda}{\sqrt{2}}$$

for some nonzero constant λ. Then equations (7.2) become

$$p_2 - p_1 = \sqrt{2}\lambda\, e^{\frac{1}{2}(z_2+z_1)} \tag{7.3}$$
$$q_2 + q_1 = \frac{\sqrt{2}}{\lambda}\, e^{\frac{1}{2}(z_2-z_1)},$$

or, in PDE notation,

$$\bar{z}_x - z_x = \sqrt{2}\lambda\, e^{\frac{1}{2}(\bar{z}+z)}$$
$$\bar{z}_y + z_y = \frac{\sqrt{2}}{\lambda}\, e^{\frac{1}{2}(\bar{z}-z)}.$$

This is the classical Bäcklund equation between the wave equation

$$z_{xy} = 0$$

and Liouville's equation

$$\bar{z}_{xy} = e^{\bar{z}}.$$

We have proved the following theorem.

THEOREM 7.1. *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation with the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$, $[B_1\ \ B_2]$, $[B_3\ \ B_4]$ all nonzero and the pairs $[C_1\ \ C_2]$, $[B_1\ \ B_2]$ and $[C_3\ \ C_4]$, $[B_3\ \ B_4]$ both linearly dependent. Then $\mathcal{B}$ is locally contact equivalent to the transformation* (7.3) *between the wave equation*

$$z_{xy} = 0$$

and Liouville's equation

$$\bar{z}_{xy} = e^{\bar{z}}.$$

8. Case 3C: Exactly one of B_1, B_3 is nonzero. Without loss of generality, we can assume that $B_1 \neq 0,\ B_3 = 0$. Under a transformation of the form (5.1), we have

$$\begin{aligned}
\tilde{A}_1 &= \frac{b_{22}}{a_{22}} A_1 \\
\tilde{A}_2 &= \frac{a_{22}}{b_{22}} A_2 \\
\tilde{B}_1 &= (a_{22})^2 B_1 \\
\tilde{B}_2 &= -a_{21} a_{22} B_1 + b_{22} B_2 \\
\tilde{B}_4 &= a_{22} B_4.
\end{aligned}$$

By Lemma 5.1, the function B_4 must be nonzero, so we can choose a coframing with $B_2 = 0,\ A_1 = B_4 = 1$. Such a coframing will be called *2-adapted.* By our homogeneity assumption, the functions A_2, B_1 are constant for any 2-adapted coframing. Moreover, any two 2-adapted coframings differ by a transformation of the form

$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^1 \\ \tilde{\omega}^2 \\ \tilde{\omega}^3 \\ \tilde{\omega}^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_{21} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}. \tag{8.1}$$

For a 2-adapted coframing, computing

$$d(d\omega^2) \equiv 0 \mod \{\omega^1, \omega^2\}$$

shows that

$$\alpha_3 = L_1\,\theta_1 + L_2\,\theta_2 + M_1\,\omega^1 + M_2\,\omega^2 + \frac{1}{B_1}\,\omega^3$$

for some functions L_i, M_i on $\mathcal{B}$. Using some of the remaining ambiguity (2.5), we can assume that $M_1 = 0$. Computing

$$\begin{aligned}
d(d\omega^1) &\equiv 0 \mod \{\omega^1, \omega^2\} \\
d(d\omega^4) &\equiv 0 \mod \{\omega^3, \omega^4\}
\end{aligned}$$

shows that

$$\alpha_4 = P_1\,\theta_1 + P_2\,\theta_2 - (J_1 + 1)\,\omega^1 + (B_1 - J_2 - K_2)\,\omega^2 - \tfrac{1}{2}(A_2 + K_3)\,\omega^3 - \tfrac{1}{2}K_4\,\omega^4$$

for some functions P_1, P_2 on $\mathcal{B}$. Computing

$$\begin{aligned}
d(d\theta_1) &\equiv 0 \mod \theta_1 \\
d(d\theta_2) &\equiv 0 \mod \theta_2
\end{aligned}$$

shows that

$$\beta_4 = (P_1 + 1)\,\theta_1 + (P_2 - 1)\,\theta_2 + \frac{(1 - A_2 - 2A_2J_1)}{A_2}\,\omega^1 + (B_1 - 2J_2)\,\omega^2 + \frac{(K_3 - A_2 - 2)}{2}\,\omega^3 + \frac{(K_4 - 2J_4)}{2}\,\omega^4.$$

It is straightforward to show that under a transformation of the form (7.1), the functions $G_4, J_1, J_2, J_4, K_2, K_4, L_1, L_2, M_2, P_1, P_2$ remain unchanged. By our assumption of homogeneity, they must therefore be constants. Now we compute

$$\begin{aligned}
0 &\equiv d(d\omega^3) \mod \{\omega^3, \omega^4\} \\
&\equiv (H_1\,\omega^1 + H_2\,\omega^2) \wedge \theta_1 \wedge \theta_2) && \Rightarrow H_1 = H_2 = 0 \\
0 &\equiv d(d\theta_1) \mod \{\omega^1, \omega^2, \omega^3\} \\
&\equiv \tfrac{1}{2}K_4\,\theta_1 \wedge \theta_2 \wedge \omega^4 && \Rightarrow K_4 = 0 \\
0 &\equiv d(d\theta_1) \mod \{\omega^1, \omega^3, \omega^4\} \\
&\equiv B_1(J_1 + 1)\,\theta_1 \wedge \theta_2 \wedge \omega^2 && \Rightarrow J_1 = -1 \\
0 &\equiv d(d\theta_1) \mod \{\omega^1, \omega^3\} \\
&\equiv -G_4\,\theta_1 \wedge \omega^2 \wedge \omega^4 && \Rightarrow G_4 = 0 \\
0 &\equiv d(d\theta_1) \mod \{\omega^2, \omega^3\} \\
&\equiv \frac{(A_2L_2(B_1 - J_2) - P_2)}{A_2}\,\theta_1 \wedge \theta_2 \wedge \omega^1 && \Rightarrow P_2 = A_2L_2(B_1 - J_2) \\
0 &\equiv d(d\theta_2) \mod \{\omega^2, \omega^3\} \\
&\equiv (-L_1(B_1 - J_2) + P_1)\,\theta_1 \wedge \theta_2 \wedge \omega^1 && \Rightarrow P_1 = L_1(B_1 - J_2).
\end{aligned}$$

Next we compute

$$\begin{aligned}
0 &\equiv d(d\theta_1) \mod \omega^3 \\
&\equiv [(L_1 + A_2L_2 + M_2)(J_2 - B_1) - K_2]\,\theta_1 \wedge \omega^1 \wedge \omega^2 \\
0 &\equiv d(d\theta_2) \mod \omega^3 \\
&\equiv \left[(L_1 + A_2L_2 + M_2)(J_2 - B_1) - \frac{K_2}{A_2}\right]\,\theta_2 \wedge \omega^1 \wedge \omega^2.
\end{aligned}$$

Since $A_2 \neq 1$, these equations imply that $K_2 = 0$. Now

$$\begin{aligned}
0 &\equiv d(d\omega^4) \mod \{\theta_2, \omega^2, \omega^3\} \\
&\equiv -(L_1J_2 + 1)\,\theta_1 \wedge \omega^1 \wedge \omega^4.
\end{aligned}$$

Therefore L_1, J_2 are both nonzero, and

$$J_2 = -\frac{1}{L_1}.$$

Next we compute

$$\begin{aligned}
0 &\equiv d(d\omega^4) \mod \{\theta_2, \omega^1, \omega^3\} \\
&\equiv B_1(A_2+1)\,\theta_1 \wedge \omega^2 \wedge \omega^4 && \Rightarrow A_2 = -1 \\
0 &\equiv d(d\omega^4) \mod \{\theta_1, \omega^2, \omega^3\} \\
&\equiv \frac{(L_2 - L_1)}{L_1}\,\theta_2 \wedge \omega^1 \wedge \omega^4 && \Rightarrow L_2 = L_1 \\
0 &\equiv d(d\omega^2) \mod \omega^3 \\
&\equiv B_1 M_2\,\theta_1 \wedge \theta_2 \wedge \omega^2 && \Rightarrow M_2 = 0.
\end{aligned}$$

Now it is straightforward to check that under a transformation of the form (7.1), the functions G_3, K_3 remain unchanged; therefore they must be constants. Continuing, we have

$$\begin{aligned}
0 &\equiv d(d\omega^3) \mod \{\theta_1, \omega^4\} \\
&\equiv -J_4\,\omega^1 \wedge \omega^2 \wedge \omega^3 && \Rightarrow J_4 = 0 \\
0 &\equiv d(d\omega^1) \mod \{\theta_1, \theta_2\} \\
&\equiv \frac{(1 - K_3)}{L_1}\,\omega^1 \wedge \omega^2 \wedge \omega^3 && \Rightarrow K_3 = 1 \\
0 &\equiv d(d\theta_1) \mod \{\theta_2, \omega^2, \omega^4\} \\
&\equiv -\frac{(L_1 B_1 + 1)}{L_1 B_1}\,\theta_1 \wedge \omega^1 \wedge \omega^3 && \Rightarrow L_1 = -\frac{1}{B_1} \\
0 &= d(d\theta_1) \\
&= (B_1 - G_3)\,\theta_1 \wedge \omega^2 \wedge \omega^3 && \Rightarrow G_3 = B_1.
\end{aligned}$$

In summary, we have now shown that the structure equations of a 2-adapted coframing take the form

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\
d\omega^1 &= B_1(\theta_1 \wedge \theta_2 + \theta_1 \wedge \omega^2 + \theta_2 \wedge \omega^2 + \omega^2 \wedge \omega^3) \\
d\omega^2 &= \frac{1}{B_1}(\theta_1 + \theta_2 - \omega^3) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= (\theta_1 - \theta_2 - B_1\,\omega^2) \wedge \omega^3 \\
d\omega^4 &= -\beta_3 \wedge \omega^3 + \theta_1 \wedge \theta_2 - \theta_1 \wedge \omega^4 + \theta_2 \wedge \omega^4 + B_1\,\omega^2 \wedge \omega^4 + \omega^1 \wedge \omega^2
\end{aligned} \tag{8.2}$$

for some 1-form β_3 on $\mathcal{B}$.

LEMMA 8.1. *We can choose a 2-adapted coframing with* $\beta_3 = 0$.

Proof. We will make liberal use of the fact that β_3 is only well-defined modulo ω^3, so we can add multiples of ω^3 to β_3 at will.

Suppose that $\beta_3 \neq 0$. Differentiating the last equation in (8.2) yields

$$0 = d(d\omega^4) = [-d\beta_3 + -2(\theta_1 - \theta_2 - B_1\,\omega^2) \wedge \beta_3] \wedge \omega^3;$$

therefore

$$d\beta_3 \equiv -2(\theta_1 - \theta_2 - B_1\,\omega^2) \wedge \beta_3 \mod \omega^3. \tag{8.3}$$

By the Frobenius Theorem, there exist functions x, y, λ, μ, ν on $\mathcal{B}$ such that

$$\begin{aligned} \omega^3 &= e^{\lambda}\, dx \\ \beta_3 &= e^{\mu}\, dy + \nu\, dx. \end{aligned}$$

In fact, because of the ω^3-ambiguity in β_3, we can assume that $\nu = 0$.

From the equation for $d\omega^3$ in (8.2) and equation (8.3) we have

$$\begin{aligned} &[d\lambda - (\theta_1 - \theta_2 - B_1\,\omega^2)] \wedge \omega^3 = 0 \\ &[d\mu + 2(\theta_1 - \theta_2 - B_1\,\omega^2)] \wedge \beta_3 \equiv 0 \mod \omega^3. \end{aligned}$$

It follows that

$$\begin{aligned} d\lambda &= \theta_1 - \theta_2 - B_1\,\omega^2 + r_1\, dx \\ d\mu &= -2(\theta_1 - \theta_2 - B_1\,\omega^2) + r_2\, dx + r_3\, dy \end{aligned}$$

for some functions r_1, r_2, r_3 on $\mathcal{B}$. Therefore $d(\mu + 2\lambda)$ is a linear combination of dx and dy. This implies that there exists a function $f(x, y)$ such that

$$\mu + 2\lambda = f(x, y).$$

Thus we have

$$\beta_3 = e^{-2\lambda} e^{f(x,y)} dy,$$

and by replacing the function y by the function $\int e^{f(x,y)} dy$ (and adding multiples of ω^3 to β_3 if necessary) we can assume that

$$\beta_3 = e^{-2\lambda} dy.$$

From the structure equation for $d\omega^4$ in (8.2), it follows that under a change of 2-adapted coframing of the form

$$\tilde{\omega}^4 = \omega^4 - b_{21}\,\omega^3$$

we have

$$\begin{aligned} \tilde{\beta}_3 &= \beta_3 + db_{21} + 2b_{21}\, d\lambda \\ &= e^{-2\lambda} dy + db_{21} + 2b_{21}\, d\lambda. \end{aligned}$$

Taking $b_{21} = -ye^{-2\lambda}$ yields $\tilde{\beta}_3 = 0$, as desired. □

For a 2-adapted coframing as in the lemma, the structure equations (8.2) take the form

$$\begin{aligned} d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\ d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\ d\omega^1 &= B_1(\theta_1 \wedge \theta_2 + \theta_1 \wedge \omega^2 + \theta_2 \wedge \omega^2 + \omega^2 \wedge \omega^3) \\ d\omega^2 &= \frac{1}{B_1}\,(\theta_1 + \theta_2 - \omega^3) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\ d\omega^3 &= (\theta_1 - \theta_2 - B_1\,\omega^2) \wedge \omega^3 \\ d\omega^4 &= \theta_1 \wedge \theta_2 - \theta_1 \wedge \omega^4 + \theta_2 \wedge \omega^4 + B_1\,\omega^2 \wedge \omega^4 + \omega^1 \wedge \omega^2. \end{aligned}$$

The intepretation of these equations requires some preliminaries regarding the geometry of frame bundles, which we will postpone until after the next section.

9. Case 3D: $B_1, B_3 \neq 0$. Under a transformation of the form (5.1), we have

$$\begin{aligned}
\tilde{A}_1 &= \frac{b_{22}}{a_{22}} A_1 \\
\tilde{A}_2 &= \frac{a_{22}}{b_{22}} A_2 \\
\tilde{B}_1 &= (a_{22})^2 B_1 \\
\tilde{B}_2 &= -a_{21} a_{22} B_1 + b_{22} B_2 \\
\tilde{B}_3 &= (b_{22})^2 B_3 \\
\tilde{B}_4 &= -b_{21} b_{22} B_3 + a_{22} B_4.
\end{aligned}$$

Since $B_1, B_3 \neq 0$, we can choose a coframing for which $B_2 = B_4 = 0$ and B_1, B_3 are constants. Such a coframing will be called *2-adapted.* A 2-adapted coframing is uniquely determined by the constants B_1 and B_3, and by the homogeneity assumption, all the other torsion functions are constants as well. For now we will not specify the values of B_1 and B_3; rather we will use this ambiguity to specify the values of other torsion coefficients in what follows.

For a 2-adapted coframing, computing

$$\begin{aligned}
d(d\omega^2) &\equiv 0 \mod \{\omega^1, \omega^2\} \\
d(d\omega^4) &\equiv 0 \mod \{\omega^3, \omega^4\}
\end{aligned}$$

shows that

$$\begin{aligned}
\alpha_3 &= L_1\, \theta_1 + L_2\, \theta_2 + M_1\, \omega^1 + M_2\, \omega^2 - \frac{B_3}{B_1}\, \omega^4 \\
\beta_3 &= L_3\, \theta_1 + L_4\, \theta_2 - \frac{B_1}{B_3}\, \omega^2 + M_3\, \omega^3 - M_4\, \omega^4
\end{aligned}$$

for some functions L_i, M_i on $\mathcal{B}$. Using the remaining ambiguity (2.5), we can assume that $M_1 = M_3 = 0$. Computing

$$\begin{aligned}
d(d\omega^1) &\equiv 0 \mod \{\omega^1, \omega^2\} \\
d(d\omega^3) &\equiv 0 \mod \{\omega^3, \omega^4\}
\end{aligned}$$

shows that

$$\begin{aligned}
\alpha_4 &= P_1\, \theta_1 + P_2\, \theta_2 + Q_1\, \omega^1 + Q_2\, \omega^2 - \tfrac{1}{2}(A_2 + K_3)\, \omega^3 - \tfrac{1}{2} K_4\, \omega^4 \\
\beta_4 &= P_3\, \theta_1 + P_4\, \theta_2 - \tfrac{1}{2}(A_1 + J_1)\, \omega^1 - \tfrac{1}{2} J_2\, \omega^2 + Q_3\, \omega^3 + Q_4\, \omega^4
\end{aligned}$$

for some functions P_i, Q_i on $\mathcal{B}$. Now we compute

$$
\begin{aligned}
0 &\equiv d(d\theta_1) \quad \text{mod } \{\theta_1, \omega^3, \omega^4\} & \\
&\equiv A_1(P_2 - P_4)\,\theta_2 \wedge \omega^1 \wedge \omega^2 & \Rightarrow P_4 = P_2 \\
0 &\equiv d(d\theta_2) \quad \text{mod } \{\theta_2, \omega^1, \omega^2\} & \\
&\equiv A_2(P_3 - P_1)\,\theta_1 \wedge \omega^3 \wedge \omega^4 & \Rightarrow P_3 = P_1 \\
0 &\equiv d(d\theta_1) \quad \text{mod } \{\theta_1, \theta_2, \omega^3\} & \\
&\equiv A_1(\tfrac{1}{2}K_4 - J_4 - Q_4)\,\omega^1 \wedge \omega^2 \wedge \omega^4 & \Rightarrow Q_4 = \tfrac{1}{2}K_4 - J_4 \\
0 &\equiv d(d\theta_2) \quad \text{mod } \{\theta_1, \theta_2, \omega^1\} & \\
&\equiv A_2(\tfrac{1}{2}J_2 - K_2 - Q_2)\,\omega^2 \wedge \omega^3 \wedge \omega^4 & \Rightarrow Q_2 = \tfrac{1}{2}J_2 - K_2 \\
0 &\equiv d(d\theta_1) \quad \text{mod } \{\omega^1, \omega^3, \omega^4\} & \\
&\equiv -B_1Q_1\,\theta_1 \wedge \theta_2 \wedge \omega^2 & \Rightarrow Q_1 = 0 \\
0 &\equiv d(d\theta_2) \quad \text{mod } \{\omega^1, \omega^2, \omega^3\} & \\
&\equiv -B_3Q_3\,\theta_1 \wedge \theta_2 \wedge \omega^4 & \Rightarrow Q_3 = 0 \\
0 &\equiv d(d\theta_1) \quad \text{mod } \{\omega^1, \omega^2, \omega^4\} & \\
&\equiv (\tfrac{1}{2}K_4L_4 - A_2P_2)\,\theta_1 \wedge \theta_2 \wedge \omega^3 & \Rightarrow P_2 = \frac{K_4L_4}{2A_2} \\
0 &\equiv d(d\theta_2) \quad \text{mod } \{\omega^2, \omega^3, \omega^4\} & \\
&\equiv (A_1P_1 - \tfrac{1}{2}J_2L_1)\,\theta_1 \wedge \theta_2 \wedge \omega^1 & \Rightarrow P_1 = \frac{J_2L_1}{2A_1} \\
0 &\equiv d(d\theta_1) \quad \text{mod } \theta_1 & \\
&\equiv (\tfrac{1}{2}A_1K_3 - \tfrac{1}{2}A_1A_2 - 1)\,\omega^1 \wedge \omega^2 \wedge \omega^3 & \Rightarrow K_3 = \frac{A_1A_2 + 2}{A_1} \\
0 &\equiv d(d\theta_2) \quad \text{mod } \theta_2 & \\
&\equiv (\tfrac{1}{2}A_2J_1 - \tfrac{1}{2}A_1A_2 - 1)\,\omega^1 \wedge \omega^3 \wedge \omega^4 & \Rightarrow J_1 = \frac{A_1A_2 + 2}{A_2} \\
0 &\equiv d(d\theta_1) \quad \text{mod } \{\theta_2, \omega^2, \omega^4\} & \\
&\equiv \frac{(1 - (A_1A_2)^2)}{A_1A_2}\,\theta_1 \wedge \omega^1 \wedge \omega^3 & \Rightarrow (A_1A_2)^2 = 1.
\end{aligned}
$$

Since we require $A_1A_2 \neq 1$, we must have $A_1A_2 = -1$, and therefore $A_2 = -\frac{1}{A_1}$.

Continuing, we have

$$
\begin{aligned}
0 &\equiv d(d\theta_1) \mod \{\theta_2, \omega^1, \omega^3\} \\
&\equiv \frac{(H_2 - A_1^2 G_4)}{A_1}\,\theta_1 \wedge \omega^2 \wedge \omega^4 &&\Rightarrow H_2 = A_1^2 G_4 \\
0 &\equiv d(d\omega^1) \mod \{\omega^2, \omega^3, \omega^4\} \\
&\equiv \frac{B_1(L_2 - A_1 L_1)}{A_1}\,\theta_1 \wedge \theta_2 \wedge \omega^1 &&\Rightarrow L_2 = A_1 L_1 \\
0 &\equiv d(d\omega^3) \mod \{\omega^1, \omega^2, \omega^4\} \\
&\equiv -B_3(L_4 + A_1 L_3)\,\theta_1 \wedge \theta_2 \wedge \omega^3 &&\Rightarrow L_4 = -A_1 L_3 \\
0 &\equiv d(d\omega^1) \mod \{\theta_1, \theta_2, \omega^4\} \\
&\equiv \frac{(B_3 G_3 M_2 - B_1 K_4)}{B_3}\,\omega^1 \wedge \omega^2 \wedge \omega^3 &&\Rightarrow K_4 = \frac{B_3 G_3 M_2}{B_1} \\
0 &\equiv d(d\omega^3) \mod \{\theta_1, \theta_2, \omega^2\} \\
&\equiv \frac{(B_1 H_1 M_4 - B_3 J_2)}{B_1}\,\omega^1 \wedge \omega^3 \wedge \omega^4 &&\Rightarrow J_2 = \frac{B_1 H_1 M_4}{B_3} \\
0 &\equiv d(d\omega^2) \mod \{\omega^1, \omega^3, \omega^4\} \\
&\equiv B_1 M_2\,\theta_1 \wedge \theta_2 \wedge \omega^2 &&\Rightarrow M_2 = 0 \\
0 &\equiv d(d\omega^4) \mod \{\omega^1, \omega^2, \omega^3\} \\
&\equiv B_3 M_4\,\theta_1 \wedge \theta_2 \wedge \omega^4 &&\Rightarrow M_4 = 0.
\end{aligned}
$$

Now we have

$$
0 = d(d\theta_1) = [-A_1 K_2\,\omega^1 \wedge \omega^2 + \frac{H_1}{A_1}\,\omega^1 \wedge \omega^4 - A_1 G_3\,\omega^2 \wedge \omega^3 - \frac{J_4}{A_1}\,\omega^3 \wedge \omega^4] \wedge \theta_1
$$
$$
\Rightarrow\ G_3 = H_1 = J_4 = K_2 = 0
$$

$$
\begin{aligned}
0 &\equiv d(d\omega^2) \mod \{\theta_2, \omega^2, \omega^4\} \\
&\equiv \frac{(B_1 L_1 + A_1 B_3 L_3)}{A_1 B_1}\,\theta_1 \wedge \omega^1 \wedge \omega^3 &&\Rightarrow L_1 = -\frac{A_1 B_3 L_3}{B_1} \\
0 &\equiv d(d\omega^2) \mod \{\theta_1, \omega^1, \omega^3\} \\
&\equiv \frac{-B_3(B_1 + A_1^2 G_4 L_3)}{B_1}\,\theta_2 \wedge \omega^2 \wedge \omega^4 &&\Rightarrow B_1 + A_1^2 G_4 L_3 = 0.
\end{aligned}
$$

Since $B_1 \neq 0$, this implies that $G_4, L_3 \neq 0$ and $L_3 = -\frac{B_1}{A_1^2 G_4}$.

The structure equations now take the form

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (A_1\,\omega^1 + \frac{1}{A_1}\,\omega^3) + A_1\,\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (A_1\,\omega^1 + \frac{1}{A_1}\,\omega^3) + \omega^1 \wedge \omega^2 - \frac{1}{A_1}\,\omega^3 \wedge \omega^4 \\
d\omega^1 &= (\frac{B_1}{A_1}\,\theta_1 + B_1\,\theta_2 - G_4\,\omega^4) \wedge \omega^2 + B_1\,\theta_1 \wedge \theta_2 \\
d\omega^2 &= (-\frac{B_3}{A_1 G_4}\,\theta_1 - \frac{B_3}{G_4}\,\theta_2 + \frac{B_3}{B_1}\,\omega^4) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= (-B_3\,\theta_1 + A_1 B_3\,\theta_2 - A_1^2 G_4\,\omega^2) \wedge \omega^4 + B_3\,\theta_1 \wedge \theta_2 \\
d\omega^4 &= (\frac{B_1}{A_1^2 G_4}\,\theta_1 - \frac{B_1}{A_1 G_4}\,\theta_2 + \frac{B_1}{B_3}\,\omega^2) \wedge \omega^3 + \omega^1 \wedge \omega^2.
\end{aligned}$$

By a transformation of the form (5.1) with

$$a_{22} = \frac{1}{\sqrt{|G_4|}}, \qquad b_{22} = \frac{1}{A_1\sqrt{|G_4|}}$$

we can arrange that $A_1 = 1, G_4 = \pm 1$. Let $\varepsilon = G_4 = \pm 1$; then the structure equations take the form

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\
d\omega^1 &= (B_1\,\theta_1 + B_1\,\theta_2 - \varepsilon\,\omega^4) \wedge \omega^2 + B_1\,\theta_1 \wedge \theta_2 \\
d\omega^2 &= (-\varepsilon B_3\,\theta_1 - \varepsilon B_3\,\theta_2 + \frac{B_3}{B_1}\,\omega^4) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= (-B_3\,\theta_1 + B_3\,\theta_2 - \varepsilon\,\omega^2) \wedge \omega^4 + B_3\,\theta_1 \wedge \theta_2 \\
d\omega^4 &= (\varepsilon B_1\,\theta_1 - \varepsilon B_1\,\theta_2 + \frac{B_1}{B_3}\,\omega^2) \wedge \omega^3 + \omega^1 \wedge \omega^2.
\end{aligned}$$

The next section contains a discussion of frame bundles which will be necessary in order to interpret these structure equations and those of the previous section.

10. Local geometry of surfaces in 3-dimensional Riemannian and Lorentzian space forms. First we will discuss the familiar geometry of surfaces in 3-dimensional Euclidean space; then we can examine what changes when the curvature and/or the signature of the underlying space form is allowed to vary.

Let $\mathbb{E}^3$ denote the vector space $\mathbb{R}^3$ with the Euclidean inner product

$$\langle x, y\rangle = x^1y^1 + x^2y^2 + x^3y^3.$$

An *orthonormal frame* at a point $x \in \mathbb{E}^3$ an orthonormal basis $\{e_1, e_2, e_3\}$ for the tangent space $T_x\mathbb{E}^3$. The set of all orthonormal frames at all points of $\mathbb{E}^3$ is called the *frame bundle* of $\mathbb{E}^3$, denoted $\mathcal{F}(\mathbb{E}^3)$; it is a principal bundle over $\mathbb{E}^3$ whose fiber over each point $x \in \mathbb{E}^3$ is naturally isomorphic to the Lie group $O(3)$ (or, if we require our frames to be positively oriented, $SO(3)$).

The frame bundle $\mathcal{F}(\mathbb{E}^3)$ is in fact naturally isomorphic to the Lie group $E(3)$, the group of isometries of $\mathbb{E}^3$. Recall that

$$E(3) = \left\{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} : A \in O(3), b \in \mathbb{E}^3 \right\}.$$

If we represent a vector $y \in \mathbb{E}^3$ by the 4-dimensional vector $\begin{bmatrix} y \\ 1 \end{bmatrix}$, then elements of $E(3)$ act on y by matrix multiplication:

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} Ay + b \\ 1 \end{bmatrix}.$$

An orthonormal frame $\{e_1, e_2, e_3\}$ at $x \in \mathbb{E}^3$ may be regarded as an element of $E(3)$ by letting A be the matrix whose columns are the vectors e_1, e_2, e_3 and letting b be the vector x.

The vectors x, e_1, e_2, e_3 may all be thought of as $\mathbb{E}^3$-valued functions on $\mathcal{F}(\mathbb{E}^3)$. Thus their exterior derivatives dx, de_i are $T\mathbb{E}^3$-valued 1-forms on $\mathcal{F}(\mathbb{E}^3)$. Since $\{e_1, e_2, e_3\}$ is a basis for the tangent space to $\mathbb{E}^3$ at each point, we can express dx, de_i as linear combinations of e_1, e_2, e_3 whose coefficients are ordinary scalar-valued 1-forms on $\mathcal{F}(\mathbb{E}^3)$. Hence we can define 1-forms η^i, η_i^j, $1 \leq i, j \leq 3$, on $\mathcal{F}(\mathbb{E}^3)$ by the equations

$$dx = \sum_{i=1}^{3} e_i \, \eta^i \qquad (10.1)$$
$$de_i = \sum_{j=1}^{3} e_j \, \eta_i^j.$$

The 1-forms η^1, η^2, η^3 are semi-basic for the natural projection $\pi : \mathcal{F}(\mathbb{E}^3) \to \mathbb{E}^3$. They have the property that if $\sigma : \mathbb{E}^3 \to \mathcal{F}(\mathbb{E}^3)$ is a section of the frame bundle defined by

$$\sigma(x) = (e_1(x), e_2(x), e_3(x)),$$

then the pullbacks $\sigma^*(\eta^i)$ are dual to the basis $\{e_1(x), e_2(x), e_3(x)\}$ for the tangent space $T_x\mathbb{E}^3$ at each point $x \in \mathbb{E}^3$. Thus the forms $\{\sigma^*(\eta^1), \sigma^*(\eta^2), \sigma^*(\eta^3)\}$ are a basis for the 1-forms on $\mathbb{E}^3$. For this reason, the η^i are called the *dual forms* on $\mathcal{F}(\mathbb{E}^3)$. The η_j^i, on the other hand, form a basis for the 1-forms on each fiber of π. If σ is a section as above, then the pullbacks $\sigma^*(\eta_j^i)$ are the Levi-Civita connection forms of the Euclidean metric on $\mathbb{E}^3$ for the frame defined by σ. For this reason, the η_j^i are called the *connection forms* on $\mathcal{F}(\mathbb{E}^3)$. Together, the forms $\{\eta^i, \eta_j^i\}$ form a basis for the left-invariant forms on the group $E(3)$, and hence for the Lie algebra $\mathfrak{e}(3)$.

Differentiating equations (10.1) shows that the forms η^i, η_i^j satisfy the *structure equations*

$$d\eta^i = -\sum_{j=1}^{3} \eta_j^i \wedge \eta^j \qquad (10.2)$$
$$d\eta_j^i = -\sum_{k=1}^{3} \eta_k^i \wedge \eta_j^k.$$

(These equations are equivalent to the structure equations for the Lie algebra $\mathfrak{e}(3)$.) Differentiating the equations

$$\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

shows that $\eta^i_j = -\eta^j_i$; in particular, $\eta^i_i = 0$ for $i = 1, 2, 3$.

Now let $U \subset \mathbb{R}^2$ be open and let $X : U \to \mathbb{E}^3$ be a regular surface. An *adapted orthonormal frame field* along X is a choice, for each $x \in X$, of an orthonormal frame $\{e_1, e_2, e_3\}$ at x such that the vectors e_1, e_2 form a basis for the tangent space T_xX (and hence e_3 is a unit normal vector to X at x.) If $\{e_1, e_2, e_3\}$ is an adapted orthonormal frame field, then any other adapted orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ has the form

$$\begin{aligned}
\tilde{e}_1 &= \pm[(\cos\varphi)\, e_1 - (\sin\varphi)\, e_2] \\
\tilde{e}_2 &= \pm[(\sin\varphi)\, e_1 + (\cos\varphi)\, e_2] \\
\tilde{e}_3 &= \pm e_3
\end{aligned}$$

for some function φ on X. (The ambiguities of sign can be removed by specifying a choice of unit normal and requiring that the frame field be positively oriented.)

A choice of an adapted orthonormal frame field may be thought of as a lifting $\tilde{X} : U \to \mathcal{F}(\mathbb{E}^3)$. Now consider the pullbacks of the forms η^i, η^i_j via $\tilde{X}$ to the surface X. (The pullback notation will be omitted for simplicity.) Since e_1, e_2 form a basis for T_xX at each point $x \in X$, the 1-form $dx = \sum e_i\, \eta^i$ must be a linear combination of e_1 and e_2; therefore, $\eta^3 = 0$. Moreover, the 1-forms η^1, η^2 are linearly independent and so form a basis for the 1-forms on X. Differentiating the equation $\eta^3 = 0$ yields

$$0 = d\eta^3 = -\eta^3_1 \wedge \eta^1 - \eta^3_2 \wedge \eta^2.$$

By Cartan's Lemma, there exist functions h_{11}, h_{12}, h_{22} on X such that

$$\begin{aligned}
\eta^3_1 &= h_{11}\, \eta^1 + h_{12}\, \eta^2 \\
\eta^3_2 &= h_{12}\, \eta^1 + h_{22}\, \eta^2.
\end{aligned}$$

The structure equations for the dual forms can now be written in the form

$$\begin{aligned}
d\eta^1 &= -\eta^1_2 \wedge \eta^2 \\
d\eta^2 &= \eta^1_2 \wedge \eta^1
\end{aligned} \tag{10.3}$$

where η^1_2 is the Levi-Civita connection form for the induced metric on X. The first and second fundamental forms of X are

$$\begin{aligned}
I &= \langle dX, dX \rangle = (\eta^1)^2 + (\eta^2)^2 \\
II &= \langle dX, de_3 \rangle = h_{11}\, (\eta^1)^2 + 2h_{12}\, \eta^1\, \eta^2 + h_{22}\, (\eta^2)^2.
\end{aligned}$$

The *Gauss curvature* K of X is defined to be the determinant of II, i.e.,

$$K = h_{11}h_{22} - h_{12}^2,$$

and the structure equation for the Levi-Civita form η^1_2 can now be written in the form

$$d\eta^1_2 = K\, \eta^1 \wedge \eta^2.$$

(This is called the *Gauss equation.*) The *mean curvature* H of X is defined to be one-half of the trace of II with respect to the metric defined by I, i.e.,

$$H = \tfrac{1}{2}(h_{11} + h_{22}).$$

The quantities K and H are independent (up to the sign of H) of the choice of adapted orthonormal frame field on X. Note that

$$\eta_1^3 \wedge \eta_2^3 = (h_{11}h_{22} - h_{12}^2)\,\eta^1 \wedge \eta^2 = K\,\eta^1 \wedge \eta^2$$
$$\eta_1^3 \wedge \eta^2 + \eta^1 \wedge \eta_2^3 = (h_{11} + h_{22})\,\eta^1 \wedge \eta^2 = 2H\,\eta^1 \wedge \eta^2.$$

So for instance, let $X : U \to \mathbb{E}^3$ be any surface whose Gauss curvature K satisfies $K \equiv -1$. If $\tilde{X} : U \to \mathcal{F}(\mathbb{E}^3)$ is any choice of adapted orthonormal coframing along X, then the image of $\tilde{X}$ is an integral manifold of the exterior differential system

$$\tilde{\mathcal{I}} = \{\eta^3,\ d\eta^3,\ \eta_1^3 \wedge \eta_2^3 + \eta^1 \wedge \eta^2\}$$

on $\mathcal{F}(\mathbb{E}^3)$. But there is one further wrinkle to consider. Generally our objects of interest are surfaces, and while the unit normal vector e_3 is determined up to sign by the surface, in general there is no canonical choice of basis $\{e_1, e_2\}$ for the tangent spaces T_xX. Rather than lifting X to the entire frame bundle $\mathcal{F}(\mathbb{E}^3)$, it is more natural to consider liftings of X to the space M of *contact elements* of $\mathbb{E}^3$. This is the space of tangent planes to points of $\mathbb{E}^3$, and if we allow these planes to be oriented by a choice of unit normal vector, M may be described as

$$M = \{(x, e_3) : x \in \mathbb{E}^3, e_3 \in T_x\mathbb{E}^3, \langle e_3, e_3\rangle = 1\}.$$

This is a 5-dimensional manifold, and it is naturally the quotient of $\mathcal{F}(\mathbb{E}^3)$ by the circle action consisting of rotations between e_1 and e_2 at each point.

The 1-form η^3 is well-defined on M, and in fact it is a contact form on M. The forms $\eta^1, \eta^2, \eta_1^3, \eta_2^3$ are semi-basic for the natural projection $\mathcal{F}(\mathbb{E}^3) \to M$, and the form η_2^1 spans the cotangent space of each fiber of the projection. While the forms $\eta^1, \eta^2, \eta_1^3, \eta_2^3$ are not well-defined on M, certain combinations of them are. In particular, since η^3 is well-defined on M, so is the form

$$d\eta^3 = -\eta_1^3 \wedge \eta^1 - \eta_2^3 \wedge\ \eta^2.$$

In addition, the area form $\eta^1 \wedge \eta^2$ is well-defined on M, as are the 2-forms $\eta_1^3 \wedge \eta_2^3$ and $\eta_1^3 \wedge \eta^2 + \eta^1 \wedge \eta_2^3$ which describe Gauss and mean curvature. So in the example above, the ideal $\tilde{\mathcal{I}}$ on $\mathcal{F}(\mathbb{E}^3)$ actually projects to a well-defined ideal

$$\mathcal{I} = \{\eta^3,\ d\eta^3,\ \eta_1^3 \wedge \eta_2^3 + \eta^1 \wedge \eta^2\}$$

on M. Integral manifolds of this ideal are the canonical liftings to M of surfaces in $\mathbb{E}^3$ with constant Gauss curvature $K \equiv -1$. Furthermore, in this case the pair $(M, \mathcal{I})$ is a hyperbolic Monge-Ampère system.

These constructions can all be carried out when $\mathbb{E}^3$ is replaced by the space forms $S^3, \mathbb{H}^3$, by flat Lorentzian space (which we will denote $\mathbb{E}^{2,1}$), or by Lorentzian space forms of constant sectional curvature 1 or -1 (which we will denote $S^{2,1}$ and $\mathbb{H}^{2,1}$, respectively). In each case the frame bundle will be isomorphic to the Lie group of isometries of the underlying space form, and the structure equations will vary depending on the group. In addition, in the Lorentzian case there will be variations depending on whether we are considering spacelike or timelike surfaces. In either case we choose orthonormal frames along the surface with e_1 and e_2 tangent to the surface; in the spacelike case we choose frames with

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1, \quad \langle e_3, e_3\rangle = -1,$$

and in the timelike case we choose frames with

$$\langle e_1, e_1\rangle = \langle e_3, e_3\rangle = 1, \quad \langle e_2, e_2\rangle = -1.$$

For spacelike surfaces in either Riemannian or Lorentzian space forms, the Gauss equation

$$d\eta_2^1 = K\,\eta^1 \wedge \eta^2$$

is taken as a definition of the Gauss curvature of the surface. For timelike surfaces in Lorentzian space forms, the analog of the Gauss equation is

$$d\eta_2^1 = -K\,\eta^1 \wedge \eta^2.$$

(See [13] for a discussion of curvature in Lorentzian spaces.) Moreover, whenever the underlying space form has nonzero sectional curvature K_0, the relationship of between the Gauss curvature K of a surface and the second fundamental form of the surface is

$$K = K_0 + (h_{11}h_{22} - h_{12}^2)$$

when the underlying space form is Riemannian and

$$K = K_0 - (h_{11}h_{22} - h_{12}^2)$$

for either spacelike or timelike surfaces when the underlying space form is Lorentzian. Thus we have

$$\eta_1^3 \wedge \eta_2^3 = (K - K_0)\,\eta^1 \wedge \eta^2$$

for surfaces in Riemannian space forms and

$$\eta_1^3 \wedge \eta_2^3 = (K_0 - K)\,\eta^1 \wedge \eta^2$$

for either spacelike or timelike surfaces in Lorentzian space forms. Finally, for timelike surfaces in Lorentzian space forms the mean curvature is defined to be one-half of the trace of II with repect to the Lorentzian metric I, so

$$H = \tfrac{1}{2}(h_{11} - h_{22}).$$

In this case we have

$$\eta_1^3 \wedge \eta^2 - \eta^1 \wedge \eta_2^3 = (h_{11} - h_{22})\,\eta^1 \wedge \eta^2 = 2H\,\eta^1 \wedge \eta^2.$$

The structure equations in the various cases are:

- Surfaces in $\mathbb{E}^3$: the frame bundle is isomorphic to $E(3)$, and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta_j^i \wedge \eta^j$$
$$d\eta_j^i = -\sum_{k=1}^{3} \eta_k^i \wedge \eta_j^k,$$

  with $\eta_j^i = -\eta_i^j$.

- Surfaces in S^3: the frame bundle is isomorphic to $O(4)$, and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j$$
$$d\eta^i_j = -\sum_{k=1}^{3} \eta^i_k \wedge \eta^k_j + \eta^i \wedge \eta^j,$$

with $\eta^i_j = -\eta^j_i$.

- Surfaces in $\mathbb{H}^3$: the frame bundle is isomorphic to $O(3,1)$, and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j$$
$$d\eta^i_j = -\sum_{k=1}^{3} \eta^i_k \wedge \eta^k_j - \eta^i \wedge \eta^j,$$

with $\eta^i_j = -\eta^j_i$.

- Spacelike surfaces in $\mathbb{E}^{2,1}$: the frame bundle is isomorphic to the Lorentzian group $E(2,1)$ (i.e., the Lorentzian analog of $E(3)$), and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j$$
$$d\eta^i_j = -\sum_{k=1}^{3} \eta^i_k \wedge \eta^k_j,$$

with $\eta^i_i = 0,\ \eta^2_1 = -\eta^1_2,\ \eta^1_3 = \eta^3_1,\ \eta^2_3 = \eta^3_2$.

- Timelike surfaces in $\mathbb{E}^{2,1}$: the frame bundle is isomorphic to the Lorentzian group $E(2,1)$, and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j$$
$$d\eta^i_j = -\sum_{k=1}^{3} \eta^i_k \wedge \eta^k_j,$$

with $\eta^i_i = 0,\ \eta^2_1 = \eta^1_2,\ \eta^1_3 = -\eta^3_1,\ \eta^2_3 = \eta^3_2$.

- Spacelike surfaces in $S^{2,1}$: the frame bundle is isomorphic to $O(3,1)$, and the structure equations are

$$d\eta^i = -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j$$
$$d\eta^1_2 = -\eta^1_3 \wedge \eta^3_2 + \eta^1 \wedge \eta^2$$
$$d\eta^3_1 = -\eta^3_2 \wedge \eta^2_1 + \eta^3 \wedge \eta^1$$
$$d\eta^3_2 = -\eta^3_1 \wedge \eta^1_2 + \eta^3 \wedge \eta^2$$

with $\eta^i_i = 0,\ \eta^2_1 = -\eta^1_2,\ \eta^1_3 = \eta^3_1,\ \eta^2_3 = \eta^3_2$. It is straightforward to show that this case is actually isomorphic to the case of surfaces in $\mathbb{H}^3$ via the change of basis

$$\{\eta^1, \eta^2, \eta^3, \eta^1_2, \eta^3_1, \eta^3_2\} \rightarrow \{-\eta^3_2, \eta^3_1, \eta^3, \eta^1_2, \eta^2, -\eta^1\}.$$

This correspondence sends surfaces of Gauss curvature $K \neq -1$ in $\mathbb{H}^3$ to their Gauss images, which are spacelike surfaces of Gauss curvature $K \neq 1$ in $S^{2,1}$.

- Timelike surfaces in $S^{2,1}$: the frame bundle is isomorphic to $O(3,1)$, and the structure equations are

$$\begin{aligned} d\eta^i &= -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j \\ d\eta^1_2 &= -\eta^1_3 \wedge \eta^3_2 - \eta^1 \wedge \eta^2 \\ d\eta^3_1 &= -\eta^3_2 \wedge \eta^2_1 + \eta^3 \wedge \eta^1 \\ d\eta^3_2 &= -\eta^3_1 \wedge \eta^1_2 - \eta^3 \wedge \eta^2 \end{aligned}$$

with $\eta^i_i = 0,\ \eta^2_1 = \eta^1_2,\ \eta^1_3 = -\eta^3_1,\ \eta^2_3 = \eta^3_2$.

- Spacelike surfaces in $\mathbb{H}^{2,1}$: the frame bundle is isomorphic to $O(2,2)$, and the structure equations are

$$\begin{aligned} d\eta^i &= -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j \\ d\eta^1_2 &= -\eta^1_3 \wedge \eta^3_2 - \eta^1 \wedge \eta^2 \\ d\eta^3_1 &= -\eta^3_2 \wedge \eta^2_1 - \eta^3 \wedge \eta^1 \\ d\eta^3_2 &= -\eta^3_1 \wedge \eta^1_2 - \eta^3 \wedge \eta^2 \end{aligned}$$

with $\eta^i_i = 0,\ \eta^2_1 = -\eta^1_2,\ \eta^1_3 = \eta^3_1,\ \eta^2_3 = \eta^3_2$.

- Timelike surfaces in $\mathbb{H}^{2,1}$: the frame bundle is isomorphic to $O(2,2)$, and the structure equations are

$$\begin{aligned} d\eta^i &= -\sum_{j=1}^{3} \eta^i_j \wedge \eta^j \\ d\eta^1_2 &= -\eta^1_3 \wedge \eta^3_2 + \eta^1 \wedge \eta^2 \\ d\eta^3_1 &= -\eta^3_2 \wedge \eta^2_1 - \eta^3 \wedge \eta^1 \\ d\eta^3_2 &= -\eta^3_1 \wedge \eta^1_2 + \eta^3 \wedge \eta^2 \end{aligned}$$

with $\eta^i_i = 0,\ \eta^2_1 = \eta^1_2,\ \eta^1_3 = -\eta^3_1,\ \eta^2_3 = \eta^3_2$.

11. Interpretation of Cases 3C and 3D. In Cases 3C and 3D, we found a coframing $\{\theta_1, \theta_2, \omega^1, \omega^2, \omega^3, \omega^4\}$ whose structure equations have constant coefficients. This implies that the forms in the coframing form a Lie algebra. This in turn gives the manifold $\mathcal{B}$ a Lie group structure (at least locally) by regarding the forms in the coframing as the left-invariant forms on $\mathcal{B}$. The first step in interpreting the structure equations is to identify the Lie algebra that they define, and in all but one case it turns out to be one of those described in the previous section. Then because the contact forms θ_1, θ_2 are each determined up to scalar multiples, we must find two distinct bases for the Lie algebra: a basis $\{\eta^i, \eta^i_j\}$ for which η^3 is a multiple of θ_1, and a basis $\{\zeta^i, \zeta^i_j\}$ for which ζ^3 is a multiple of θ_2. The Bäcklund transformation is then given by the transformation relating these two bases for the Lie algebra. These transformations can all be described by geodesic congruences of some sort, in the same way that the classical Bäcklund transformation between pseudospherical surfaces is given by line congruences.

These computations were carried out using the algorithm in [14] with the assistance of Maple. The algorithm divides into several cases depending on the value of B_1 in case 3C and the values of B_1, B_3, ε in case 3D. The change-of-basis matrices are rather complicated and not very enlightening, so they will be omitted here.

11.1. Case 3C. Recall that the structure equations in this case are

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\
d\omega^1 &= B_1(\theta_1 \wedge \theta_2 + \theta_1 \wedge \omega^2 + \theta_2 \wedge \omega^2 + \omega^2 \wedge \omega^3) \\
d\omega^2 &= \frac{1}{B_1}(\theta_1 + \theta_2 - \omega^3) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= (\theta_1 - \theta_2 - B_1 \omega^2) \wedge \omega^3 \\
d\omega^4 &= \theta_1 \wedge \theta_2 - \theta_1 \wedge \omega^4 + \theta_2 \wedge \omega^4 + B_1 \omega^2 \wedge \omega^4 + \omega^1 \wedge \omega^2
\end{aligned}$$

with $B_1 \neq 0$. Carrying out the algorithm described above shows that:

- If $B_1 \neq 2$, then the Lie algebra is $\mathfrak{so}(2,2)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{H}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form
$$\begin{aligned}
\mathcal{I}_1 &= \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2 - 2\eta^1 \wedge \eta^2\} \\
\mathcal{I}_2 &= \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2 - 2\zeta^1 \wedge \zeta^2\}.
\end{aligned}$$
So up to contact equivalence, $\mathcal{B}$ represents a transformation between timelike surfaces of constant mean curvature equal to 1 in $\mathbb{H}^{2,1}$. We note that the change-of-basis matrices have different expressions for B_1 in the ranges $B_1 < 0,\ 0 < B_1 < 2$, and $B_1 > 2$.
- If $B_1 = 2$, then the Lie algebra is $\mathfrak{e}(2,1)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{E}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form
$$\begin{aligned}
\mathcal{I}_1 &= \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2\} \\
\mathcal{I}_2 &= \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2\}.
\end{aligned}$$
So up to contact equivalence, $\mathcal{B}$ represents a transformation between timelike minimal surfaces in $\mathbb{E}^{2,1}$. This transformation is explored in detail in [7].

Thus we have the following theorem.

THEOREM 11.1. *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation with the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$, $[B_1\ \ B_2]$, $[B_3\ \ B_4]$ all nonzero, the pair $[C_1\ \ C_2]$, $[B_1\ \ B_2]$ linearly independent, and the pair $[C_3\ \ C_4]$, $[B_3\ \ B_4]$ linearly dependent. Then $\mathcal{B}$ is locally contact equivalent to either*

1. *A Bäcklund transformation between timelike minimal surfaces in $\mathbb{E}^{2,1}$, or*
2. *A Bäcklund transformation between timelike surfaces of constant mean curvature equal to 1 in $\mathbb{H}^{2,1}$.*

In both cases, the transformation may be described in terms of geodesic congruences.

11.2. Case 3D. Recall that the structure equations in this case are

$$\begin{aligned}
d\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\
d\theta_2 &= -\theta_2 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \\
d\omega^1 &= (B_1 \theta_1 + B_1 \theta_2 - \varepsilon \omega^4) \wedge \omega^2 + B_1 \theta_1 \wedge \theta_2 \\
d\omega^2 &= (-\varepsilon B_3 \theta_1 - \varepsilon B_3 \theta_2 + \frac{B_3}{B_1} \omega^4) \wedge \omega^1 + \omega^3 \wedge \omega^4 \\
d\omega^3 &= (-B_3 \theta_1 + B_3 \theta_2 - \varepsilon \omega^2) \wedge \omega^4 + B_3 \theta_1 \wedge \theta_2 \\
d\omega^4 &= (\varepsilon B_1 \theta_1 - \varepsilon B_1 \theta_2 + \frac{B_1}{B_3} \omega^2) \wedge \omega^3 + \omega^1 \wedge \omega^2
\end{aligned}$$

with $B_1, B_3 \neq 0$ and $\varepsilon = \pm 1$. The algorithm described above divides into many cases depending on the values of these parameters.

When $\varepsilon = -1$, the $B_1 B_3$ plane divides into regions as shown in Figure 1. The curve in this graph is defined by the equation

$$4B_1^2 B_3^2 - 4B_1^2 B_3 + 4B_1 B_3^2 + B_1^2 + 2B_1 B_3 + B_3^2 = 0,$$

and it may be parametrized by

$$B_1 = -\frac{1}{2}(t+1)^2, \qquad B_3 = \frac{1}{2}\left(\frac{1}{t} + 1\right)^2$$

for $t \neq 0$. (The point corresponding to $t = -1$ is (0,0) and so is not included in our parameter space.) For convenience, we define

$$Q^- = 4B_1^2 B_3^2 - 4B_1^2 B_3 + 4B_1 B_3^2 + B_1^2 + 2B_1 B_3 + B_3^2.$$

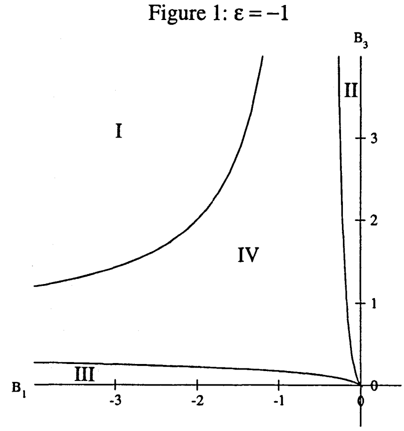
When the point (B_1, B_3) is in the second or fourth quadrant, Q^- can be factored as

$$Q^- = q_1^- q_2^-$$

with

$$\begin{aligned}
q_1^- &= 2B_1 B_3 - B_1 + B_3 + 2\sqrt{-B_1 B_3} \\
q_2^- &= 2B_1 B_3 - B_1 + B_3 - 2\sqrt{-B_1 B_3}.
\end{aligned}$$

Figure 1: ε = −1



- If $Q^- = 0$ and $t > 0$ (so that $B_1 < -\frac{1}{2}$ and $B_3 > \frac{1}{2}$), then the Lie algebra is $\mathfrak{e}(3)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for surfaces in $\mathbb{E}^3$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ represents a transformation between surfaces of constant Gauss curvature $K = -1$ in $\mathbb{E}^3$. This is the classical Bäcklund transformation between pseudospherical surfaces, and the parameter t along the curve $Q^- = 0$ is a function of the usual parameter appearing in this transformation.

- If $Q^- = 0$ and $t < 0$ (so that either $B_1 > -\frac{1}{2}$ or $B_3 < \frac{1}{2}$), then the Lie algebra is $\mathfrak{e}(2,1)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for spacelike surfaces in $\mathbb{E}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ represents a transformation between spacelike surfaces of constant Gauss curvature $K = 1$ in $\mathbb{E}^{2,1}$.

- In Region I of Quadrant 2, the Lie algebra is $\mathfrak{so}(4)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for surfaces in S^3; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \left(\frac{q_1^-}{q_2^-}\right) \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \left(\frac{q_1^-}{q_2^-}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ represents a transformation between surfaces of constant Gauss curvature

$$K = 1 - \frac{q_1^-}{q_2^-}$$

in S^3. As (B_1, B_3) ranges over Region I, K takes values in the interval $(0, 1)$.

- In Regions II and III of Quadrant 2 and in Quadrant 4, the Lie algebra is $\mathfrak{so}(2,2)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for spacelike surfaces in $\mathbb{H}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \left(\frac{q_2^-}{q_1^-}\right) \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \left(\frac{q_2^-}{q_1^-}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ represents a transformation between spacelike surfaces of constant Gauss curvature

$$K = \frac{q_2^-}{q_1^-} - 1$$

in $\mathbb{H}^{2,1}$. As (B_1, B_3) ranges over these regions, K takes values in the interval $(-1, 0)$ in Regions II and III of Quadrant 2 and in the interval $(0, \infty)$ in Quadrant 4. We note that the change-of-basis matrices have different expressions in each of the three regions.

- In Region IV of Quadrant 2, the Lie algebra is $\mathfrak{so}(3,1)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for surfaces in $\mathbb{H}^3$, or equivalently, for spacelike surfaces in $S^{2,1}$. Regarded as surfaces in $\mathbb{H}^3$, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 - \left(\frac{q_2^-}{q_1^-}\right) \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 - \left(\frac{q_2^-}{q_1^-}\right) \zeta^1 \wedge \zeta^2\}.$$

Regarded as spacelike surfaces in $S^{2,1}$, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 - \left(\frac{q_1^-}{q_2^-}\right) \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 - \left(\frac{q_1^-}{q_2^-}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ may be regarded as representing either a transformation between surfaces of constant Gauss curvature

$$K = \frac{q_2^-}{q_1^-} - 1$$

in $\mathbb{H}^3$, or a transformation between spacelike surfaces of constant Gauss curvature

$$K = 1 - \frac{q_1^-}{q_2^-}$$

in $S^{2,1}$. In the first case K takes values in the interval $(-\infty, -1)$ as (B_1, B_3) ranges over Region IV, and in the second case K takes values in the interval $(1, \infty)$ as (B_1, B_3) ranges over Region IV.

- In Quadrants 1 and 3, the Lie algebra is $\mathfrak{so}(2,2)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{H}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ take the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2 - 2\frac{2B_1B_3 - B_1 + B_3}{\sqrt{Q^-}}\eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2 - 2\frac{2B_1B_3 - B_1 + B_3}{\sqrt{Q^-}}\zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ represents a transformation between timelike surfaces of constant mean curvature

$$H = \frac{2B_1B_3 - B_1 + B_3}{\sqrt{Q^-}}$$

in $\mathbb{H}^{2,1}$. As (B_1, B_3) ranges over these regions, H takes values in the interval $(-1, 1)$. We note that the change-of-basis matrices have different expressions in each quadrant.

When $\varepsilon = 1$, the B_1B_3 plane divides into regions as shown in Figure 2. The curve in this graph is defined by the equation

$$4B_1^2B_3^2 + 4B_1^2B_3 - 4B_1B_3^2 + B_1^2 + 2B_1B_3 + B_3^2 = 0,$$

and it may be parametrized by

$$B_1 = \frac{1}{2}(t+1)^2, \qquad B_3 = -\frac{1}{2}\left(\frac{1}{t}+1\right)^2$$

for $t \neq 0$. (The point corresponding to $t = -1$ is (0,0) and so is not included in our parameter space.) For convenience, we define

$$Q^+ = 4B_1^2B_3^2 + 4B_1^2B_3 - 4B_1B_3^2 + B_1^2 + 2B_1B_3 + B_3^2.$$

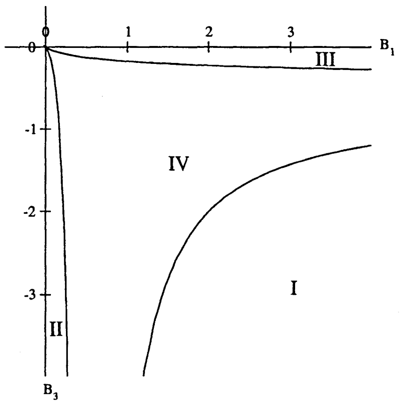
When the point (B_1, B_3) is in the second or fourth quadrant, Q^+ can be factored as

$$Q^+ = q_1^+ q_2^+$$

with

$$q_1^+ = 2B_1B_3 + B_1 - B_3 + 2\sqrt{-B_1B_3}$$
$$q_2^+ = 2B_1B_3 + B_1 - B_3 - 2\sqrt{-B_1B_3}.$$

Figure 2: ε = 1



- If $Q^+ = 0$ then the Lie algebra is $\mathfrak{e}(2,1)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{E}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ can be written either in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta_1^3 \wedge \eta_2^3 + \eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta_1^3 \wedge \zeta_2^3 + \zeta^1 \wedge \zeta^2\}$$

or in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta_1^3 \wedge \eta^2 - \eta^1 \wedge \eta_2^3 - 2\eta^1 \wedge \eta^2\}$$
$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta_1^3 \wedge \zeta^2 - \zeta^1 \wedge \zeta_2^3 - 2\zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ may be regarded as representing either a transformation between timelike surfaces of constant Gauss curvature $K = 1$ or a transformation between timelike surfaces of constant mean curvature $H = 1$ in $\mathbb{E}^{2,1}$.

- In Region I of Quadrant 4, the Lie algebra is $\mathfrak{so}(2,2)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{H}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ can be written either in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \left(\frac{q_1^+}{q_2^+}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \left(\frac{q_1^+}{q_2^+}\right) \zeta^1 \wedge \zeta^2\}$$

or in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ may be regarded as representing either a transformation between timelike surfaces of constant Gauss curvature

$$K = \frac{q_1^+}{q_2^+} - 1$$

or a transformation between timelike surfaces of constant mean curvature

$$H = \frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}$$

in $\mathbb{H}^{2,1}$. As (B_1, B_3) ranges over Region I, K takes values in the interval $(0,1)$; meanwhile, H takes values in the interval $(-\infty, -1)$.

- In Regions II and III of Quadrant 4 and in Quadrant 2, the Lie algebra is $\mathfrak{so}(2,2)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $\mathbb{H}^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ can be written either in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 + \left(\frac{q_2^+}{q_1^+}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 + \left(\frac{q_2^+}{q_1^+}\right) \zeta^1 \wedge \zeta^2\}$$

or in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ may be regarded as representing either a transformation between timelike surfaces of constant Gauss curvature

$$K = \frac{q_2^+}{q_1^+} - 1$$

or a transformation between timelike surfaces of constant mean curvature

$$H = \frac{2B_1B_3 + B_1 - B_3}{\sqrt{Q^+}}$$

in $\mathbb{H}^{2,1}$. As (B_1, B_3) ranges over these regions, K takes values in the interval $(-1, 0)$ in Regions II and III of Quadrant 4 and in the interval $(0, \infty)$ in Quadrant 2; meanwhile, H takes values in the interval $(1, \infty)$ in Regions II and III of Quadrant 4 and in the interval $(-\infty, -1)$ in Quadrant 2. We note that the change-of-basis matrices have different expressions in each of the three regions.

- In Region IV of Quadrant 4, the Lie algebra is $\mathfrak{so}(3,1)$. For each of the two bases computed by the algorithm, the structure equations coincide with those for timelike surfaces in $S^{2,1}$; moreover, the ideals $\mathcal{I}_1, \mathcal{I}_2$ can be written either in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^3_2 - \left(\frac{q_2^+}{q_1^+}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^3_2 - \left(\frac{q_2^+}{q_1^+}\right) \zeta^1 \wedge \zeta^2\}$$

or in the form

$$\mathcal{I}_1 = \{\eta^3,\ d\eta^3,\ \eta^3_1 \wedge \eta^2 - \eta^1 \wedge \eta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{-Q^+}}\right) \eta^1 \wedge \eta^2\}$$

$$\mathcal{I}_2 = \{\zeta^3,\ d\zeta^3,\ \zeta^3_1 \wedge \zeta^2 - \zeta^1 \wedge \zeta^3_2 - 2\left(\frac{2B_1B_3 + B_1 - B_3}{\sqrt{-Q^+}}\right) \zeta^1 \wedge \zeta^2\}.$$

So up to contact equivalence, $\mathcal{B}$ may be regarded as representing either a transformation between timelike surfaces of constant Gauss curvature

$$K = 1 - \frac{q_2^+}{q_1^+}$$

or a transformation between timelike surfaces of constant mean curvature

$$H = \frac{2B_1B_3 + B_1 - B_3}{\sqrt{-Q^+}}$$

in $S^{2,1}$. As (B_1, B_3) ranges over Region IV, K takes values in the interval $(1, \infty)$; meanwhile, H ranges over all real numbers.

- In Quadrants 1 and 3, the Lie algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(2,1)$. The corresponding Lie group is denoted $SO^*(4)$ in Cartan's list of Lie groups as described in [11]. This group has no natural 3-dimensional quotients compatible with the contact structures given by θ_1 and θ_2, and so there is no natural way to regard these examples as transformations of surfaces in any 3-dimensional space. They may naturally be regarded as transformations of certain surfaces in a 5-dimensional quotient space of $SO^*(4)$.

Putting all these cases together yields the following theorem.

THEOREM 11.2. *Let $\mathcal{B} \subset M_1 \times M_2$ be a homogeneous Bäcklund transformation with the vectors $[C_1\ \ C_2]$, $[C_3\ \ C_4]$, $[B_1\ \ B_2]$, $[B_3\ \ B_4]$ all nonzero and the pairs $[C_1\ \ C_2]$, $[B_1\ \ B_2]$ and $[C_3\ \ C_4]$, $[B_3\ \ B_4]$ both linearly independent. Then $\mathcal{B}$ is locally contact equivalent to one of the following:*

1. *A Bäcklund transformation between surfaces of constant negative Gauss curvature in* $\mathbb{E}^3$
2. *A Bäcklund transformation between surfaces of constant Gauss curvature* $0 < K < 1$ *in* S^3
3. *A Bäcklund transformation between surfaces of constant Gauss curvature* $-\infty < K < -1$ *in* $\mathbb{H}^3$
4. *A Bäcklund transformation between spacelike surfaces of constant positive Gauss curvature in* $\mathbb{E}^{2,1}$
5. *A Bäcklund transformation between timelike surfaces of constant positive Gauss curvature, or equivalently, constant nonzero mean curvature, in* $\mathbb{E}^{2,1}$
6. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $1 < K < \infty$ *in* $S^{2,1}$
7. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $1 < K < \infty$, *or equivalently, constant mean curvature* $H \in \mathbb{R}$, *in* $S^{2,1}$
8. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $-1 < K < \infty$, $K \neq 0$ *in* $\mathbb{H}^{2,1}$
9. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $-1 < K < \infty$, $K \neq 0$, *or equivalently, constant mean curvature* $|H| > 1$, *in* $\mathbb{H}^{2,1}$
10. *A Bäcklund transformation between timelike surfaces of constant mean curvature* $|H| < 1$ *in* $\mathbb{H}^{2,1}$.
11. *A Bäcklund transformation between certain surfaces in a 5-dimensional quotient space of* $SO^*(4)$.

In all cases, the transformation may be described in terms of geodesic congruences.

12. Conclusion. Theorems 3.1, 4.1, 6.1, 7.1, 11.1, and 11.2 may be combined to yield the following result.

THEOREM 12.1. *Let* $\mathcal{B} \subset M_1 \times M_2$ *be a homogeneous Bäcklund transformation. Then* $\mathcal{B}$ *is locally contact equivalent to one of the following:*

1. *A Bäcklund transformation between solutions of the wave equation* $z_{xy} = 0$
2. *A holonomic Bäcklund transformation of the form described in Theorem 6.1*
3. *The classical Bäcklund transformation between the wave equation* $z_{xy} = 0$ *and Liouville's equation* $z_{xy} = e^z$
4. *A Bäcklund transformation between surfaces of constant negative Gauss curvature in* $\mathbb{E}^3$
5. *A Bäcklund transformation between surfaces of constant Gauss curvature* $0 < K < 1$ *in* S^3
6. *A Bäcklund transformation between surfaces of constant Gauss curvature* $-\infty < K < -1$ *in* $\mathbb{H}^3$
7. *A Bäcklund transformation between spacelike surfaces of constant positive Gauss curvature in* $\mathbb{E}^{2,1}$
8. *A Bäcklund transformation between timelike surfaces of constant positive Gauss curvature, or equivalently, constant nonzero mean curvature, in* $\mathbb{E}^{2,1}$
9. *A Bäcklund transformation between timelike minimal surfaces in* $\mathbb{E}^{2,1}$
10. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $1 < K < \infty$ *in* $S^{2,1}$
11. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $1 < K < \infty$, *or equivalently, constant mean curvature* $H \in \mathbb{R}$, *in* $S^{2,1}$

12. *A Bäcklund transformation between spacelike surfaces of constant Gauss curvature* $-1 < K < \infty,\ K \neq 0$ *in* $\mathbb{H}^{2,1}$
13. *A Bäcklund transformation between timelike surfaces of constant Gauss curvature* $-1 < K < \infty,\ K \neq 0$*, or equivalently, constant mean curvature* $|H| > 1$*, in* $\mathbb{H}^{2,1}$
14. *A Bäcklund transformation between timelike surfaces of constant mean curvature* $|H| \leq 1$ *in* $\mathbb{H}^{2,1}$.
15. *A Bäcklund transformation between certain surfaces in a 5-dimensional quotient space of* $SO^*(4)$.

Now this is certainly not the end of the story. There are interesting Bäcklund transformations which are not homogeneous; in particular, the classical Bäcklund transformation for the sine-Gordon equation does not appear on this list. Moreover, the notion of Bäcklund transformation used here does not take into account the presence of the arbitrary parameter λ that plays such an important role in the theory of Bäcklund transformations of integrable systems such as the sine-Gordon equation. We hope to address these and other issues in future papers.

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