RIGIDITY OF A CLASS OF SPECIAL LAGRANGIAN FIBRATIONS SINGULARITY *

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1. Introduction. In [8], Strominger, Yau and Zaslow conjectured that the mirror pairs in mirror conjecture are pairs of dual special Lagrangian fibrations. Since then there has been a lot of research on special Lagrangian submanifolds or special Lagrangian fibrations. In this note, we will look at the possible singularities that can arise from a special Lagrangian fibrations. We will show that a fiber of a special Lagrangian fibration that has only isolated singularity of homogeneous type are essentially of the type given by Harvey-Lawson.

We let \( \mathbb{C}^3 \) be the complex 3-space endowed with the standard Kaehler metric with the associated Kaehler form \( \omega_0 \) and the \((3,0)\) form \( \Omega_0 = dz_1 \wedge dz_2 \wedge dz_3 \). A submanifold \( L \subset \mathbb{C}^3 \) is called a special Lagrangian submanifold (in short SL-submanifold) if \( \omega_0|_L = 0 \) and \( \text{Im}(\Omega_0)|_L = 0 \).

We let \( S^5 \subset \mathbb{C}^3 \) be the unit sphere. For \( p = (z_1, z_2, z_3) \in \mathbb{C}^3 \) and \( t \in \mathbb{R} \) we use \( tp \) to denote the point \( (tz_1, tz_2, tz_3) \in \mathbb{C}^3 \). For any subset \( \Sigma \subset S^5 \) we define the cone supposed on \( \Sigma \) to be

\[
C(\Sigma) = \{ tp \mid t \in \mathbb{R}^+, p \in \Sigma \}.
\]

We say \( C(\Sigma) \) is an SL-cone if the smooth locus of \( C(\Sigma) \) is dense in \( C(\Sigma) \) and is an SL-submanifold of \( \mathbb{C}^3 \).

Now we introduce the notion of homogeneous SL-fibration of \( \mathbb{C}^3 \).

**Definition 1.** Let \( F : \mathbb{C}^3 \rightarrow \mathbb{R}^3 \) be a smooth surjective map. We say \( F \) is an SL-fibration if the components \( f_1, f_2 \) and \( f_3 \) of \( F \) are real valued functions in \( x_1, x_2, x_3, y_1, y_2, y_3 \), where \( z_k = x_k + iy_k \), so that all Poisson brackets \( \{ f_i, f_j \} = 0 \) and the real part \( \text{Re}(\text{det}_C((\partial f_i/\partial z_j))) = 0 \). We say the fibration is homogeneous if all \( f_i \) are homogeneous polynomials and we say the fiber \( L_0 = F^{-1}(0) \) is a regular cone if \( L_0 \) has only isolated singularity 0 and all \( f_i \) are irreducible.

Recall that the smooth locus of any fibers of \( F \) as in the Definition are automatically SL-submanifolds [3]. For convenience we denote the punctured cone \( L_0 - \{ 0 \} \) by \( L_0^* \). We first observe that in case \( L_0 \) is a regular cone, then \( \deg f_k \geq 2 \) for all \( k \). Indeed, let \( T_0 \) be the linear combination of all tangent spaces of points in \( L_0^* \), after translating to the origin 0. We now show that \( \dim T_0 = 6 \). First, in case \( \dim T_0 = 4 \), then there are two points \( p, q \in L_0^* \) so that \( \dim(T_pL_0^* \cap T_qL_0^*) = 2 \). Because \( T_pL_0^* \) and \( T_qL_0^* \) are special Lagrangian subspaces in \( \mathbb{C}^3 \), we must have \( T_pL_0^* = T_qL_0^* \), a contradiction. Now assume \( \dim T_0 = 5 \). Then there is an unit vector \( Jv \in T_0 \). This implies that \( Jv \) is normal to \( L_0^* \) everywhere and hence \( v \) is a vector field of \( L_0^* \). So we can write \( L_0 = tv \times \Gamma \), where \( \Gamma \) is a cone of dimension 2 with singularity 0. Thus \( L_0 \) has at least singularity \( \mathbb{R} \), a contradiction. Now from \( \dim T_0 = 6 \), we can easily obtain \( \deg f_k \geq 2 \) for all \( k \). Moreover we can obtain that \( \Sigma = L_0 \cap S^5 \) is full in \( \mathbb{C}^3 \).

The prototype of SL-fibration in \( \mathbb{C}^3 \) with homogeneous isolated singularity is the example of Harvey and Lawson [3].

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Example 2. Let \( F = (f_1, f_2, f_3) \) be defined by
\[
  f_1 = |z_1|^2 - |z_2|^2, \quad f_2 = |z_1|^2 - |z_3|^2 \quad \text{and} \quad f_3 = \text{Im}(z_1 z_2 z_3).
\]
Then \( F \) is a homogeneous SL-fibration of \( \mathbb{C}^3 \) and \( L_0 = F^{-1}(0) \) is a regular SL-cone.

So far this is the only known example of homogeneous SL-fibration of \( \mathbb{C}^3 \) whose central fiber is a regular cone. The fibration given by \( (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3 \) with
\[
  f_1 = x_1 y_2 - x_2 y_1, \quad f_2 = x_1 y_1 + x_2 y_2 \quad \text{and} \quad f_3 = y_3.
\]
is a homogeneous SL-fibration but its central fiber is not regular.

In this note, we will prove the following uniqueness result on homogeneous SL-fibrations with singular central fibers.

Theorem 3. Let \( F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3 \) be a homogeneous SL-fibration so that its central fiber \( L_0 \) is a regular cone. We let \( n_i = \deg f_i \), so arranged that \( n_1 \leq n_2 \leq n_3 \). Then we must have \( (n_1, n_2, n_3) = (2, 2, 3) \). Furthermore, there is a unitary matrix \( S \) so that if we make the Darboux coordinates change
\[
  (x_1, x_2, x_3, y_1, y_2, y_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T
\]
and let \( w_k = p_k + iq_k \). Then \( (f_1, f_2, f_3) \) is linearly equivalent to
\[
  \tilde{f}_1 = |w_1|^2 - |w_2|^2, \quad \tilde{f}_2 = |w_1|^2 - |w_3|^2 \quad \text{and} \quad \tilde{f}_3 = \text{Im}(w_1 w_2 w_3).
\]

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2. Harmonic 1-forms on SL cone. In this section, we collect a few facts concerning harmonic 1-forms on special Lagrangian cones in \( \mathbb{C}^3 \).

Let \( L_0 \) be an SL cone of \( \mathbb{C}^3 \) with isolated singularity 0. The question we will address in section is whether there is a family of smooth proper SL-submanifolds \( L_s \) of \( \mathbb{C}^3 \) such that \( L_s \to L_0 \) as \( s \to 0 \). In case such families exist, then on \( L_0^* \) we have the associated normal vector field \( W(x) \) and the associated 1-form \( \theta(x) = W(x) \cdot \omega_0 \). In the following, we will call such 1-form the deformation 1-form associated to the family \( L_s \). By a result of McLean, \( \theta \) is a harmonic 1-form on \( L_0^* \). Further since \( L_s \) are smooth \( \theta \) is singular at 0.

Lemma 4. Let \( L_0 = C(\Sigma) \) be an SL-cone with isolated singularity 0 and let \( t \) be the distance function \( t(x) = \text{dist}(x, 0) \) on \( \mathbb{C}^3 \). Then the space of harmonic 1-forms on \( L_0^* \) is spanned by
\[
  t^{-1} \eta, \quad d(t^{-1}), \quad d(t^{-\mu_i} \phi_i) \quad \text{and} \quad d(t^{\mu_i} \phi_i)
\]
where \( \eta \) are harmonic 1-forms on \( \Sigma \), \( \phi_i \) are eigenfunctions on \( \Sigma \) with eigenvalues \( \lambda_i > 0 \) with \( \mu_i = (1 + \sqrt{1 + 4\lambda_i})/2 \) and \( \mu_i' = (-1 + \sqrt{1 + 4\lambda_i})/2 \).

Proof. Let \( \tau = \frac{\partial}{\partial t} \) be the unit tangent vector field on \( L_0^* \). Let \( m_0 \in \Sigma \) be any point. We pick an orthonormal vector fields \( (e_1, e_2) \) on \( \Sigma \) near \( m_0 \) that is covariantly constant at \( m_0 \) with respect to the connection on \( \Sigma \). We then extend the vector fields \( e_k \), considered as vector fields of \( \Sigma \subset S^5 \subset \mathbb{C}^3 \), to the cone \( L_0^* \) by parallel translation.
along rays in the cone. Combined with \( \tau \), we obtain an orthonormal frame of \( L_0^* \) near \( \mathbb{R}m_0 \). Now let \( l : \mathbb{R}^+ \times \Sigma \to C(\Sigma) \subset \mathbb{C}^3 \) be defined via \( l(t, m) = tm \). We give \( \mathbb{R}^+ \times \Sigma \) the metric \( \bar{g} = dt^2 + t^2 d\Sigma^2 \), where \( d\Sigma^2 \) is the metric on \( \Sigma \). We let \( \tau : \mathbb{R}^+ \times \Sigma \to \Sigma \) be the project. Then \( (\tau, E_1(t, m), E_2(t, m)) \) forms an orthonormal frame on \( \mathbb{R}^+ \times \Sigma \) with \( \tau = \frac{\partial}{\partial t}, E_1(t, m) = \frac{1}{t} \tau^* e_1(m) \) and \( E_2(t, m) = \frac{1}{t} \tau^* e_2(m) \). Its dual frame is given by \( dt, \omega_1(t, m) = \tau^* \omega_1(m) \) and \( \omega_2(t, m) = \tau^* \omega_2(m) \). Clearly, we have

\[
l_* E_1(t, m) = e_i(tm) = e_i(m) \quad \text{and} \quad l^* \omega_i(tm) = l^* \omega_i(m) = \omega_i(t, m)
\]

On \( L_0^* \) we have the structure equation

\[
d\omega_i(tm) = -\omega_{ij}(tm) \wedge \omega_j(tm) - t^{-1} \omega_i(tm) \wedge dt
\]

Our convention is that we use \((t, m)\) to denote the point in \( \mathbb{R}^+ \times \Sigma \) while we use \( tm \) to denote the corresponding point in \( C(\Sigma) \). Note \( l(t, m) = tm \). Over \( \mathbb{R}^+ \times \Sigma \) we have the structure equation

\[
d\omega_i(t, m) = d(l^* \omega_i(tm)) = -l^*(\omega_{ij}(tm)) \wedge \omega_j(t, m) - t^{-1} \omega_i(t, m) \wedge dt
\]

Let harmonic 1-form

\[
\theta = f(tm)dt + \omega(tm) = f(tm)dt + \sum \alpha_i(tm)\omega_i(tm)
\]

then

\[
l^* \theta = f(t, m)dt + \sum \alpha_i(t, m)\omega_i(t, m).
\]

From \( dl^* \theta = 0 \), we obtain

\[
t \nabla E_i(t, m) f(t, m) - \frac{\partial}{\partial t}(t\alpha_i(t, m)) = 0, \text{ for } i = 1, 2
\]

Using the above equation, a straightforward computation shows that

\[
d\left( \int_1^t f(r, m)dr \right) = l^* \theta - \tau^* \eta(m),
\]

where \( \eta(m) = \sum_{i=1}^2 \alpha_i(1, m)\omega_i(m) \), a 1-form on \( \Sigma \). Because \( l^* \theta \) is harmonic on \( \mathbb{R}^+ \times \Sigma \), \( \tau^* \eta \) is closed on \( \mathbb{R}^+ \times \Sigma \) and hence \( \eta \) must be closed on \( \Sigma \). Now let \( \eta_h \) be the harmonic part of \( \eta \), namely, \( \eta = \eta_h + dk \). Then

\[
l^* \theta = d\left( \int_1^t f(r, \cdot)dr + \tau^* k \right) + \tau^* \eta_h = dF + \tau^* \eta_h,
\]

where \( F = \int_1^t f(r, \cdot)dr + \tau^* k \). Clearly, \( \tau^* \eta_h \) is harmonic on \( \mathbb{R}^+ \times \Sigma \). Since \( l^* \theta \) is harmonic, \( dF \) must be harmonic. Hence by \([7, \text{page 98}]\)

\[
\Delta_\Sigma(F(t, m)) + 2t \frac{\partial}{\partial t} F(t, m) + t^2 \frac{\partial^2}{\partial t^2} F(t, m) = 0.
\]

Now let \( \phi_i \) be the eigenfunctions on \( \Sigma \) with eigenvalues \( \lambda_i \). Then

\[
F(t, m) = \sum_{i=0}^{\infty} f_i(t)\phi_i(m)
\]
for some functions \( f_i(t) \). From this we obtain

\[
\sum_{i=0}^{\infty} (-\lambda_i f_i(t) + 2t f_i'(t) + t^2 f_i''(t))\phi_i(m) = 0.
\]

Therefore \( f_i \) satisfies the equation

\[-\lambda_i f_i(t) + 2t f_i'(t) + t^2 f_i''(t) = 0,
\]

whose general solutions are \( f_i = C_i t^{\mu_i} + C_i t^{-\mu_i} \) with \( \mu_i = (-1 + \sqrt{1 + 4\lambda_i})/2 \) and \( \mu_i = (1 + \sqrt{1 + 4\lambda_i})/2 \).

When \( i = 0 \) then \( \mu_0 = 1 \) and \( \phi_0 \) is a constant. In this case \( d(t^{-1}\phi_0) \) reduces to the 1-form \( d(t^1) \).

We now compute the deformation 1-forms of two examples of smoothing of SL-cones. We begin with the deformation 1-forms of Harvey-Lawson's example. Let \( \theta_k \) be the deformation 1-form on \( \mathbb{C}^3 \) associated to the family \( L^{(k)}_s \) defined by \( f_k = s \) and \( f_j = 0 \) for \( j \neq k \). Here \( (f_1, f_2, f_3) \) is the defining equation in Example 2. Then by a direct computation we have \( \theta_1 = \frac{3}{2}(d\alpha_1 - 2d\alpha_2 + d\alpha_3) \), \( \theta_2 = \frac{3}{2}(d\alpha_1 + d\alpha_2 - 2d\alpha_3) \) and \( \theta_3 = \frac{1}{2}d(\frac{1}{t}) \). Here \( \alpha_k \) is the function on \( \Sigma \) defined by \( x_k = r \cos \alpha_k \), \( y_k = r \sin \alpha_k \). Note that \( \theta_1|_\Sigma \) and \( \theta_2|_\Sigma \) are harmonic 1-forms on the 2-torus \( \Sigma \).

Now we consider the case of a homogeneous SL-fibration \( F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3 \) whose central fiber is a regular cone, as defined in Definition 1. We let \( L^{(i)}_s \) be the family \( \{f_i = s, f_j = 0 \text{ for } j \neq i\} \) and let \( W_i \) be the deformation vector field associated to the family \( L^{(i)}_s \).

**Lemma 5.** Let \( W_1 = \sum c_{i1}\partial_{x_i} + c_{i1}+3\partial_{y_i} \) be the normal deformation vector field of the family \( L^{(i)}_s \) at \( L_0 \). Then \( c_{i1} \) can be written as \( c_{i1} = h_i/s_i \), where \( h_i \) and \( s_i \) are homogeneous polynomials with \( \deg h_i - \deg s_i = 1 - n_1 \) for \( i = 1, \ldots, 6 \).

**Proof.** Normal deformation vector field \( W_1 \) satisfies the following equation:

\[
A \cdot (c_{11}, c_{12}, \ldots, c_{16})^T = (1, 0, \ldots, 0)^T
\]

where

\[
A = \begin{bmatrix}
\frac{\partial f_i}{\partial x_i} & \frac{\partial f_i}{\partial y_i} \\
-\frac{\partial f_i}{\partial x_i} & \frac{\partial f_i}{\partial y_i}
\end{bmatrix}_{1 \leq i, j \leq 3}
\]

So,

\[
(c_{11}, c_{12}, \ldots, c_{16})^T = A^{-1}(1, 0, \ldots, 0)^T = (\det A)^{-1}(A_{11}, A_{12}, \ldots, A_{16})^T
\]

Because \( \deg(\det A) = 2(n_1 + n_2 + n_3) - 6 \) and \( \deg(A_{1i}) = 2(n_2 + n_3) - 5 \), we can write \( c_{1i} = h_i/s_i \) with \( h_i \) and \( s_i \) are homogeneous and \( \deg h_i - \deg s_i = 1 - n_1 \).

In this case we say the deformation 1-form \( \theta_1 = W_1|_{L_0} \) has order \( n_1 - 1 \).

The previous examples show that deformation 1-forms associated to the families \( L_s \) are spanned by 1-forms \( t^{-1}\eta \) or \( d(t^{-1}) \). In following, we will study a class of SL-submanifolds that has similar property.

**Definition 6.** Let \( L \) be a smooth SL-submanifold in \( \mathbb{C}^3 \) and let \( L_0 \) be an SL-cone in \( \mathbb{C}^3 \) with isolated singularity 0. We say \( L \) is asymptotically conical (in short AC) to
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Let $L_0$ and call $L_0$ the asymptotic cone of $L$ if $L$ is asymptotic to $L_0$ to order $O(t^{-1})$ as $t \to \infty$.

Clearly, when $L$ is AC to $L_0$ than $sL$ is also AC to $L_0$ for $s \in \mathbb{R}^+$ and $sL \to L_0$ as $s \to 0$. To obtain a deformation 1-form on $L_0^*$ we need certain convergence condition on $sL$.

**Definition 7.** Let $L$ be a smooth SL-submanifold in $\mathbb{C}^3$ that is AC to $L_0$. We assume $L_0 = C(\Sigma)$ for a smooth $\Sigma \subset S^5$. We say $L$ is strongly AC to $L_0$ if there is a constant $q > 0$ and a smooth maps $\Phi(\cdot, s) : \Sigma \times [0,1] \to S^5$ such that

1. $\Phi(\Sigma, s) = (s^{1/1+q}L) \cap S^5$ and $\Phi(\cdot, s)$ is a diffeomorphism from $\Sigma$ to its image;
2. The family of maps $H_s(p, t) = t\Phi(p, s/t^{1+q})$ from $\Sigma \times [1, \infty) \to \mathbb{C}^3$ $C^1$-converges to the standard map $\Sigma \times [1, \infty) \to L_0 - B_1$ when $s \to 0$, where $B_1$ is the unit ball in $\mathbb{C}^3$;
3. The vector field $v(p) = \frac{d}{ds} |_{s=0} \Phi(p, s)$ is a non-trivial vector field on each connected component of $\Sigma \subset S^5$.

The notion of SL-submanifolds AC to a cone was introduced in [5].

**Example.** [3] Let $L_s = \{(x, y) \in \mathbb{C}^3 \mid |x|y = |y|x$ and $\text{Im}(|x| + i|y|)^3 = s\}$

Then $L_s$ is a SL-submanifold strongly AC to a cone $L_0$ that is the union of two linear subspaces singular at 0. The associated $q$ is 2 in this case.

**Example.** Let $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ be a homogeneous SL-fibration whose central fiber is a regular cone. We let $n_i = \deg f_i$ with $f_i$ so arranged that $n_1, n_2, n_3$. By Sard theorem, there is a point $\xi \neq 0 \in \mathbb{R}^3$ such that $L = F^{-1}(\xi)$ is a smooth complete SL-submanifold. Then $L$ is strongly AC to the cone $L_0 = F^{-1}(0)$. The associated $q$ is $1/n_i$ where $i$ is the smallest index so that $\xi_i \neq 0$.

Let $L$ be an SL-submanifold which is strongly AC to an SL-cone $L_0$ with the associated constant $q$. It follows from the definition that the deformation 1-form on $L_0^*$ associated to the family $L_s = s^{1/1+q}L$ is a smooth non-trivial 1-form on $L_0^*$.

**Lemma 8.** Let $L, L_0$ and $q$ be as before and let $\theta$ be the deformation 1-form on $L_0^*$ associated to the family $L_s$. Then $\theta$ can only take values $1, 2$ or $\mu_i + 1$, where $\mu_i = (1 + \sqrt{1 + 4\lambda_i})/2$. Further, when $q = 1$ (resp. $q = 2$; resp. $q = \mu_i + 1$) the form $\theta = t^{-1}\eta$ (resp. $\theta = dt^{-1}$; resp. $\theta = d(t^{-\mu_i}\phi_i)$), where $\eta$ and $\phi_i$ are as in Lemma 4.

**Proof.** Let $v(p)$ be the vector field on $L_0^*$ that is the limit of $\frac{d}{ds}L_s$ when $s \to 0$. Then by definition, $v(p)$ is non-trivial on each component of $L_0^*$. By the definition of $H_s$, it is direct to check that $v(tp) = t^{-q}v(p)$. Hence the deformation 1-form $\theta$ on $L_0$ is also homogeneous, which must be of the forms $\frac{1}{t}\eta$, $d(\frac{1}{t})$, or $d(t^{-\mu_i}\phi_i)$.

**3. Harmonic 1-forms on SL-submanifolds.** Let $L$ be an SL-submanifold that is strongly AC to an SL-cone $L_0$. We first give $L$ a new metric that is quasi-isometric to its induced metric $g$.

**Lemma 9.** Let the notation be as before. Then there is a metric $\tilde{g}$ on $L$ that is quasi-isometric to $(L, g)$ so that $\tilde{g}$ is isomorphic to a cone metric away from a compact subset of $L$.

**Proof.** The proof is standard. It follows from the $C^1$ convergence of $H_t$ in Definition 7.
Let $M$ be a complete non-compact Riemannian manifold. We recall several groups associated to $M$. Here we use $\|\theta\|_2$ to mean the $L^2$ norm $\int_M |\theta|^2$ and use $L^2$ to denote the space of $L^2$ finite functions or forms. We define

$$H_D(M) = \{ f | \Delta f = 0, \; df \in L^2 \} \quad \text{and} \quad H_D^\infty(M) = \{ f \in H_D(M) | f \in L^\infty \}$$

and

$$H^1(M) = \{ \theta | d\theta = 0, \; \theta \in L^2 \} \quad \text{and} \quad H^1_0(M) = \{ df | \Delta f = 0, \; df \in L^2 \}.$$

Here $f$ are functions and $\theta$ are one forms on $M$. We also define

$$H^1_{(2)}(M) = \{ \theta | d\theta = 0, \; \theta \in L^2 \}/\{ df | f \in L^2, \; df \in L^2 \}.$$

First, recall that we have the Hodge decomposition $H^1(M) \cong H^1_{(2)}(M)$ [4]. In the following, we will apply a theorem P.Li and L-H.Tam [6, Thm 4.2] to prove the following fact.

**Lemma 10.** Let $L$ be an SL-submanifold strongly AC to an SL-cone $L_0$ endowed with the induced metric $g$ and let $\tilde{L}$ be $L$ endowed with the metric $\tilde{g}$ given in Lemma 9. Then we have

$$\dim H_D^\infty(\tilde{L}) = \dim H_D^\infty(L) = \#(\text{ends of } L).$$

*Proof.* Let $K(R)$ be $B(R) \cap L$. By the construction, for large enough $R$ the compliment $L - K(R)$ with metric $\tilde{g}$ is a union of cones. Our strategy is to apply [6, Theorem 4.2] to $(\tilde{L}, \tilde{g})$.

We now check that this theorem can be applied in our situation. First, $\tilde{L}$ is large because $L$ is strongly AC to the cone $L_0$. Now let $E$ be an end of $L - K(R)$. To proceed, we need to check that there is a constant $C$ so that the Ricci curvature of $(E, \tilde{g})$ satisfies

$$\text{Ric}(x) \geq -\frac{2C}{(1 + r(x))^2}$$

where $r(x) = \text{dist}(p, x)$ for a $p$ in $E$. Because $C(\Sigma)$ is minimal, we have

$$R_{ij} = \sum_{\alpha,j} (h^\alpha_{ij} h^\alpha_{ij} \alpha - (h^\alpha_{ij})^2) = -\sum_{\alpha,j} (h^\alpha_{ij})^2$$

where $h^\alpha_{ij} = h^\alpha_{ij}(t, m)$ is the second fundamental form of $C(\Sigma)$. Now let $h^\alpha_{ij}(m)$ be the second fundamental form of $\Sigma$ in $S^5$. Because $h^\alpha_{ij}(t, m) = \frac{1}{t} h^\alpha_{ij}(m)$ (on $C(\Sigma)$), we have

$$R_{ij}(m, t) = -t^{-2} \sum_{\alpha,j} (h^\alpha_{ij}(m))^2 \geq t^{-2}C_1$$

for $C_1 = \sup_{i,m} \sum_{j,\alpha} (h^\alpha_{ij}(m))^2$. Because $t^2 \sim r^2(x)$, there is a constant $C_2$ so that $R_{ij}(x) \geq -\frac{C_2}{r(x)^2}$ on $E$. This shows that there is a constant $C$ so that (3.1) holds.

Finally, we need to check that $E$ satisfies the condition (VC) in [6, p.282]. Namely, there is a constant $\zeta > 0$ such that for all $r$ and all $x \in \partial B_p(r) \cap E$, we have $V_{p,E}(r) < \zeta V_{x,E}(\frac{r}{2})$. First, it is clear that it suffices to check this condition for $r \geq R'$
for some constant $R'$. We choose an $R'$ so that $\partial B_p(R') \subset E$ and that for any $x \in \partial B_p(R')$, $B_x(R'/2) \cap E \subset \bar{L} - K(R)$. Then when $r \geq R'$, we have

$$V_{x,E}(r) = V_{r \cdot x,E}(R'/2) \cdot \frac{r^3}{R^3}.$$ 

Let $\zeta_i = \min_{x \in \partial B_p(R') \cap E} V_{x,E}(R'/2)$. Then $V_{x,E}(r/2) > \zeta_i R'^{-3} r^3$. Since $V_{p,E}(r) \approx \text{Area}(\Sigma)(r^3 - (2R)^3)$, therefore there is a constant $\zeta$ so that $V_{p,E}(r) < \zeta V_{x,E}(r/2)$.

This shows that we can apply [6, Theorem 4.2] to $\bar{L}$ to conclude that the dimension of $\mathcal{H}^\infty_0(\bar{L})$ is equal to the number of ends of $\bar{L}$. Since $L$ is quasi-isometric to $\bar{L}$, $\dim \mathcal{H}^\infty_0(L) = \dim \mathcal{H}^\infty_0(\bar{L})$ [2]. Therefore $\dim \mathcal{H}^\infty_0(L)$ is equal to the number of ends on $L$. \(\Box\)

**Lemma 11.** Let $L$ be a SL submanifold which is strongly AC to SL cone $L_0$, then $\dim \mathcal{H}^0_0(\bar{L}) = \#(\text{ends of } L) - 1$.

**Proof.** From Lemma 4, we obtain $\mathcal{H}_D(\bar{L}) = \mathcal{H}^\infty_0(\bar{L})$. Then the Lemma follows from $\dim \mathcal{H}^0_0(\bar{L}) = \dim \mathcal{H}_D(\bar{L}) - 1$ and Lemma 10. \(\Box\)

Let

$$H^1_c(L) = \{ \theta \mid d\theta = 0, \theta \text{ has compact support} \}/\{ df \mid f \text{ has compact support} \}$$

and let $H^1(L)$ be the first de Rham cohomology group. Consider the natural map $i^*_1 : H^1_c(L) \to H^1(L)$. Then

$$\ker(i^*_1) = \{ dg \mid dg \text{ has compact support} \}/\{ df \mid f \text{ has compact support} \}$$

We continue to assume that $L$ is an SL-submanifold strongly AC to an SL-cone $L_0$.

**Lemma 12.** $\ker(i^*_1) = \#(\text{ends of } L) - 1$.

**Proof.** Let $[dg] \neq 0 \in \ker(i^*_1)$. Because $L$ is strongly AC to $L_0$, without loss of generality, we can assume that the compact subset $K \subset L$ of $dg$ is so large that $L - K$ is diffeomorphic to the disjoint union of $\Sigma_i \times (0, \infty)$. Hence when restricted to the ends $L - K$ is locally constant but not constant. Now we see that $\ker(i^*_1) = \#(\text{ends of } L) - 1$. \(\Box\)

Now we consider $\text{Im}(i^*_1)$. From [4, Page 9], any compactly supported cohomology class on a complete Riemannian manifold that defines a non-trivial de Rham cohomology class is automatically represented by an $L^2$-harmonic form. This defines a natural Hodge projective

$$\pi: \text{Im}(i^*_1) \to \mathcal{H}^1(\bar{L}).$$

Define

$$[\pi]: \text{Im}(i^*_1) \to \mathcal{H}^1(\bar{L})/\mathcal{H}^0_0(\bar{L}).$$

**Lemma 13.** Let $L$ be an SL-submanifold strongly AC to an SL-cone $L_0$, then

$$\text{Im}(i^*_1) \cong \mathcal{H}^1(\bar{L})/\mathcal{H}^0_0(\bar{L}).$$
and the isomorphism is induced by above Hodge projection.

Proof. Clearly, $[\pi]$ is injective. And we must prove $[\pi]$ is surjective.

Let $[\omega] \in \mathcal{H}^1(\bar{L})/\mathcal{H}^1_0(\bar{L})$. Because $L$ is strongly AC to $L_0$, we can pick a sufficiently large compact subset $K(2r_0) \subset L$ so that $L - K(2r_0)$ is diffeomorphic to the disjoint union of $\Sigma_i \times (0, \infty)$. We let $\Lambda_i$ be the $i$th component of $\bar{L} - K(2r_0)$. Then from lemma 4, we can write

$$\omega |_{\Lambda_i} = C_i \cdot \frac{1}{t} \eta_i + df_i,$$

where $C_i$ is some constant and $\eta_i$ is some harmonic 1-form on $\Sigma_i$, $f_i$ is a harmonic function on $\Lambda_i$. If $C_i \neq 0$ for some $i$, then

$$\int_{\Lambda_i} |C_i \cdot \frac{1}{t} \eta_i|^2 = +\infty$$

and

$$\int_{\Lambda_i} <C_i \cdot \frac{1}{t} \eta_i, df_i >= 0.$$

Thus $\omega \notin \mathcal{H}^1(\bar{L})$. So we can write $\omega |_{L - K(2r_0)} = df$, for some harmonic function $f$ on $L - K(2r_0)$. Then we can write $\omega = (\omega - d(\rho f)) + d(\rho f)$, where function $\rho$ with takes values between 0 and 1 and such that

$$\rho(x) = 1, \text{ for } x \in L - K(2r_0) \quad \text{and} \quad \rho(x) = 0, \text{ for } x \in K(r_0).$$

Thus, $\theta = \omega - d(\rho f)$ has compact support $K(2r_0)$. But from the Hodge projection $\pi$, we can write $\theta = \pi \theta + d\varphi$ for some function $\varphi$ with $\int |d\varphi|^2 < +\infty$. So we have $\omega = \pi \theta + d\varphi + d(\rho f) = \pi \theta + d(\varphi + \rho f)$ with $\int |d(\varphi + \rho f)|^2 < +\infty$ and thus $d(\varphi + \rho f) \in \mathcal{H}^1_0(L)$. So $[\pi] \theta = [\omega]$. □

**Theorem 14.** Let $L$ be an SL submanifold which is strongly AC to an SL cone, then $\dim \mathcal{H}^1(L) = \dim \mathcal{H}^1_0(L)$.

**Proof.** From lemma 11, 12 and 13, we have $\dim \mathcal{H}^1(\bar{L}) = \dim \mathcal{H}^1_0(L)$. Combined with $\mathcal{H}^1(L) \cong \mathcal{H}^1(\bar{L})$, we prove the Lemma. □

**4. Proof of the main result.**

**Lemma 15.** Let $F = (f_1, f_2, f_3) : C^3 \to \mathbb{R}^3$ be a homogeneous SL fibration so that its central fiber $L_0$ is a regular cone. Then every connected component $L$ of regular fiber is diffeomorphic to $\mathbb{R} \times T^2, \mathbb{R}^2 \times S^1$ or $\mathbb{R}^3$.

**Proof.** Let

$$g_i = \frac{f_i^2}{1 + f_i^2}, \text{ for } i = 1, 2, 3,$$

then

$$\{g_i, g_j\} = < \text{grad } g_i, \text{ grad } g_j >$$

$$= \frac{2f_i f_j}{(1 + f_i^2)^2(1 + f_j^2)^2} < \text{grad } f_i, \text{ grad } f_j >= 0.$$
So \( G = (g_1, g_2, g_3) : C^3 \rightarrow \mathbb{R}^3 \) defines a 3-degree of freedom Liouville integrable Hamiltonian system. Because \( f_i \) are homogeneous polynomials, Hamiltonian vector fields \( X_{g_i} \) are bounded on \( C^3 \). So from a theorem in [10, Cor 2, p.17] solutions of Cauchy problem: 
\[
\frac{d}{dt} = X_{g_i}, z(t_0) = z_0 \text{ are complete, i.e., } X_{g_i} \text{ are complete.}
\]
Thus by the generalized Liouville theorem, every connected component of regular fiber \( G^{-1}(v_1, v_2, v_3) \) is differential homeomorphic to \( R \times T^2, R^2 \times S^1, \mathbb{R}^3 \) or \( T^3 \). Now every connected component \( L \) of regular fiber \( F^{-1}(v_1, v_2, v_3) \) is a connected component of \( G^{-1}(v_1, v_2, v_3) \), where \( v_i = \frac{s^2}{1+s^4} \). So \( L \) is differential homeomorphic to \( R \times T^2, R^2 \times S^1, \mathbb{R}^3 \) or \( T^3 \). But from Example in section 2, we know that \( L \) is strongly AC to SL cone and is not compact, so \( L \) is not diffeomorphic to \( T^3 \). □

Now we can prove the following

**Proposition 16.** Let \( F = (f_1, f_2, f_3) : C^3 \rightarrow \mathbb{R}^3 \) be a homogeneous SL-fibration so that its central fiber \( L_0 \) is a regular cone. We let \( n_i = \deg f_i \), so arranged that \( n_1 \leq n_2 \leq n_3 \). Then we must have \( (n_1, n_2, n_3) = (2, 2, 3) \).

**Proof.** By observation in section 1, we have \( \deg f_k \geq 2 \) for all \( k \). Let \( \theta_i \) denote the deformation 1-form associated to the family \( L_{s_i} = \{ f_i = s_i ; f_j = 0 \text{ for } j \neq i \} \). Because we assume that \( f_i \) are irreducible, \( \theta_1, \theta_2 \) and \( \theta_3 \) are linearly independent at any point \( p \in L_0^* \). Note that \( dt \) is 1-form in \( L_0^* \).

Now we prove the proposition by studying case by case:

**Case 1:** \( \deg F = (2, 2, n_3) \) with \( n_3 \neq 3 \).

In this case, \( \theta_1 \) and \( \theta_2 \) have order 1. By Lemma 4, we can let \( \theta_1 = \frac{1}{l} \eta_1 \) and \( \theta_2 = \frac{1}{l} \eta_2 \), where \( \eta_1 \) and \( \eta_2 \) are harmonic 1-forms on \( \Sigma = L_0 \cap S^5 \). Now if \( n_3 = 2 \), we can also let \( \theta_3 = \frac{1}{l} \eta_3 \), where \( \eta_3 \) is the harmonic form on \( \Sigma \). Thus \( \theta_i, dt \geq 0 \) for \( i = 1, 2, 3 \) and \( \theta_1, \theta_2 \) and \( \theta_3 \) are linearly dependent on \( L_0^* \). This is impossible. If \( n_3 \geq 4 \), by Lemma 4 we can let \( \theta_3 = d(t^{-\mu_i} \phi_i) \), where \( \phi_i \) is an eigenfunction of \( \Sigma \) which is not constant. Since \( \Sigma \) is compact, we know that there is a point \( p \) of \( \Sigma \) such that \( \phi_i(p) = 0 \). So \( \theta_3(p) \) doesn’t contain \( dt \) as component at point \( p \) and thus \( \theta_1, \theta_2 \) and \( \theta_3 \) are linearly dependent at point \( p \).

**Case 2:** \( \deg F = (2, 3, n_3) \).

In this case, the deformation 1-form \( \theta_2 \) of \( L_{s_2} \) has order 2. Hence \( \theta_2 = C_2 d(t^{\frac{1}{2}}) \) and therefore \( n_3 \) cannot be 3. The 1-form associated to the family \( L_{s_3} \) can be written \( \theta_3 = d(t^{-\mu_i} \phi_i) \), where \( \phi_i \) is a non-constant eigenfunction on \( \Sigma \). Because \( \Sigma \) is compact, then there is a point \( m \in \Sigma \) such that \( \phi_i \) attains maximum at point \( m \). So \( \theta_3(m) = \phi_i(m)/\mu_i \), and \( \theta_2, \theta_3 \) is linearly dependent at point \( m \). This is a contradiction.

**Case 3:** \( \deg F = (n_1, n_2, n_3) \) with \( n_3 \geq n_2 \geq 4 \).

If there exists such \( F \), then there is a point \( s_0 = (s_{10}, s_{20}, s_{30}) \in \mathbb{R}^3 \), such that \( L_{s_0} = F^{-1}(s_{10}, s_{20}, s_{30}) \) is a smooth SL submanifold. We have proven that \( L_{s_0} \) is strongly AC to an SL cone \( L_0 \) in the example before Lemma 8. Let \( L \) be a connected component of \( L_{s_0} \). By Theorem 14, we have \( \dim H^1(L) = \dim H^1_c(L) \). By Poincare Lemma and Lemma 15, we have \( \dim H^1_c(L) \leq 1 \). Thus \( \dim H^1(L) \leq 1 \).

Because \( L \) is smooth, we can get three deformation 1-forms \( \theta_1, \theta_2 \) and \( \theta_3 \). Certainly these forms are harmonic on \( L \). But we assume that \( n_3 \geq n_2 \geq 4 \), so \( \theta_2 \) and \( \theta_3 \) has order at least 3. Now by the homogeneous, we can get \( \theta_2, \theta_3 \in \mathcal{H}^1(L) \), which contradicts to \( \dim H^1(L) \leq 1 \). □
In order to discuss the case of \((n_1, n_2, n_3) = (2, 2, 3)\), we need the following Lemma. In the following we denote by \(\text{diag}(a_1, a_2, \ldots, a_n)\) the \(n \times n\) diagonal matrix with diagonal entries \(a_1, a_2, \ldots, a_n\).

**Lemma 17.** Let \(\tilde{A} = \text{diag}(1,-k,0,1,-k,0)\) with \(k > 0\) and let matrix \(\tilde{B}\) be symmetric such that (1) \(\tilde{A}\tilde{B} = \tilde{B}\tilde{A}\), where matrix \(E\) has the form
\[
E = \begin{pmatrix}
0 & -I_{n \times n} \\
I_{n \times n} & 0 
\end{pmatrix}
\]
and (2) there is a \(s_0 \in \mathbb{R}\) such that \(\exp(\tilde{E}\tilde{B}s_0) = I\), then there is a symplectic matrix \(Q\) such that \(Q^T\tilde{A}Q = \tilde{A}\) and \(Q^T\tilde{B}Q\) has the form \(\text{diag}(\alpha, \beta, \gamma, \alpha, \beta, \gamma)\).

**Proof.** From \(\tilde{A}\tilde{B} = \tilde{B}\tilde{A}\), we find that \(\tilde{B}\) has the following form:
\[
\tilde{B} = \begin{pmatrix}
a & c & 0 & 0 & e & 0 \\
c & b & 0 & e & 0 & 0 \\
0 & 0 & b_{33} & 0 & 0 & b_{36} \\
0 & e & 0 & a & -c & 0 \\
e & 0 & 0 & -c & b & 0 \\
0 & 0 & b_{63} & 0 & 0 & b_{66}
\end{pmatrix}
\]
and furthermore,

\[(4.1) \quad c = e = 0 \quad \text{when} \quad k \neq 1.\]

Let
\[
B_1 = \begin{pmatrix}
a & c & 0 & e \\
c & b & e & 0 \\
0 & e & a & -c \\
e & 0 & -c & b
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
b_{33} & b_{36} \\
b_{36} & b_{66}
\end{pmatrix}.
\]

From \(\exp(\tilde{E}\tilde{B}s_0) = I\), we can obtain \(\exp(EB_1s_0) = I\) and \(\exp(EB_2s_0) = I\). Now we first consider the matrix \(B_2\). Let \(\lambda\) be an eigenvalue of \(EB_2\) with the corresponding eigenvector \(\xi\), (i.e., \((EB_2)\xi = \lambda\xi\)) then \(\exp(EB_2s_0)\xi = e^{\lambda s_0}\xi\). Since \(\exp(EB_2s_0) = I\), we have \(e^{\lambda s_0} = 1\). Thus \(\lambda\) must be \(\pm \mu i\) for \(\mu \in \mathbb{R}\). If \(b_{33} = 0\), then \(b_{36} = 0\). Thus \(B_2\) has the diagonal form and we are done. If \(b_{33} \neq 0\), without loss of generality, we can assume \(b_{33} > 0\). Take \(w = (\frac{b_{36} - b_{33}^2}{b_{33}})^{\frac{1}{2}}\) and consider the symplectic matrix
\[
Q_1 = \begin{pmatrix}
w & \frac{b_{36}}{b_{33}} \\
0 & \frac{1}{w}
\end{pmatrix},
\]
then
\[
Q_1^TB_2Q_1 = \begin{pmatrix}
\sqrt{b_{33}b_{66} - b_{36}^2} & 0 \\
0 & \sqrt{b_{33}b_{66} - b_{36}^2}
\end{pmatrix}.
\]

Next we consider the matrix \(B_1\). From \(\exp(EB_1s_0) = I\), the eigenvalues of \(EB_1\) are \(\pm \nu i\) for \(\nu \in \mathbb{R}\). But by direct calculation, the eigenvalues of \(EB_1\) are
\[
\lambda = \pm (a - b)i \pm \sqrt{4(e^2 + c^2) - (a + b)^2}.\]
So we obtain

\[(4.2) \quad 4(e^2 + c^2) \leq (a + b)^2.\]

If \(a + b = 0\), then \(e = c = 0\) and \(B_1\) is of the diagonal form. So without loss of generality, we can assume \(a + b > 0\). From (4.2), we have 
\((a + b)^2 - 4c^2 \geq 0\). If \(c = 0\), we take \(u = 0\). If \(c \neq 0\), we take \(u = \frac{1}{2c}[(a + b) - \sqrt{(a + b)^2 - 4c^2}]\). Notice that \(1 - u^2 \geq 0\). If \(1 - u^2 = 0\), then \(a + b = \pm 2c\) and \(e = 0\). It is easy to prove that in this case there isn’t any \(s_0\) such that \(\exp(EB_1s_0) = I\). Thus \(1 - v^2 > 0\). So we can take the symplectic matrix

\[
Q_2 = \frac{1}{\sqrt{1 - u^2}} \begin{pmatrix}
1 & u & 0 & 0 \\
u & 1 & 0 & 0 \\
0 & 0 & 1 & -u \\
0 & 0 & -u & 1
\end{pmatrix}.
\]

One easily checks that

\[
B_3 = Q_2^T B_1 Q_2 = \begin{pmatrix}
a_1 & 0 & 0 & e \\
0 & b_1 & e & 0 \\
0 & e & a_1 & 0 \\
e & 0 & 0 & b_1
\end{pmatrix},
\]

where \(a_1 = \frac{1}{1-u^2}(bu^2 + 2cu + a)\) and \(b_1 = \frac{1}{1-u^2}(au^2 + 2cu + b)\). One verifies

\[(4.3) \quad (a_1 + b_1)^2 = (a + b)^2 - 4e^2.\]

Now from (4.2) and (4.3) we have 
\((a_1 + b_1)^2 - 4e^2 \geq 0\). If \(e = 0\), we take \(v = 0\). If \(e \neq 0\), we take \(v = \frac{1}{2c}(a_1 + b_1 - \sqrt{(a_1 + b_1)^2 - 4e^2})\). Again we have \(1 - v^2 \geq 0\).

If \(1 - v^2 = 0\), then \(a_1 + b_1 = \pm 2e\). If we take the symplectic matrix

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

then we have

\[
R^T B_3 R = \begin{pmatrix}
a_1 & e & 0 & 0 \\
e & b_1 & 0 & 0 \\
0 & 0 & a_1 & -e \\
0 & 0 & -e & b_1
\end{pmatrix}.
\]

As before, we can prove that \(1 - v^2 = 0\) is impossible.

Now we are reduced to the case \(1 - v^2 > 0\). We take symplectic matrix

\[
Q_3 = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix}
1 & 0 & 0 & v \\
0 & 1 & v & 0 \\
0 & v & 1 & 0 \\
v & 0 & 0 & 1
\end{pmatrix}.
\]

Then

\[
B_4 = Q_3^T B_3 Q_3 = \begin{pmatrix}
a_2 & 0 & 0 & 0 \\
0 & b_2 & 0 & 0 \\
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & b_2
\end{pmatrix}.
\]
We let

\[ Q = \frac{1}{\sqrt{1-u^2}} \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & u & 0 & uv & v & 0 \\ u & 1 & 0 & v & uw & 0 \\ 0 & 0 & w & 0 & 0 & -\frac{2\omega_3}{\omega_3} \frac{1}{w} \\ -uv & v & 0 & 1 & -u & 0 \\ v & -uv & 0 & -u & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \end{pmatrix} \]

where \( u, v \) and \( w \) are taken as before. Then we can check that \( Q^T B Q \) has diagonal form. If \( k = 1 \), we have \( Q^T A Q = \tilde{A} \). If \( k \neq 1 \), from (4.1) we can take \( u = v = 0 \). Thus \( Q^T A Q = \tilde{A} \). This proves the Lemma. \( \square \)

Now we discuss the case of \((2, 2, 3)\).

**Theorem 18.** Let \( F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3 \) be a homogeneous SL-fibration with \((n_1, n_2, n_3) = (2, 2, 3)\) so that its central fiber \( L_0 \) is a regular cone. Then there is a unitary matrix \( S \) so that if we make the Darboux coordinates change

\[ (p_1, p_2, p_3, q_1, q_2, q_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T \]

and let \( w_k = p_k + iq_k \). Then \((f_1, f_2, f_3)\) is linearly equivalent to

\[ \tilde{f}_1 = |w_1|^2 - |w_2|^2, \quad \tilde{f}_2 = |w_1|^2 - |w_3|^2 \quad \text{and} \quad \tilde{f}_3 = \text{Im}(w_1 w_2 w_3). \]

**Proof.** Let \( L_0 = F^{-1}(0) \) and let \( x_0 \in L_0 \). Let \( \Sigma \) be a connected component of \( L_0 \cap S^5 \) containing \( x_0 \). Let \( f_1(x) = x^T A x \) and \( f_2(x) = x^T B x \), where \( x^T = (x_1, x_2, x_3, y_1, y_2, y_3) \). Because \( A \) and \( B \) are symmetric matrices, \( EA, EB \in \text{sp}(6, \mathbb{R}) \), the later is the Lie algebra of symplectic group \( Sp(6, \mathbb{R}) \). So

\[ G = \{ \exp(EA t), \exp(EB s) | t, s \in \mathbb{R} \} \subset Sp(6, \mathbb{R}) \]

is a Lie subgroup of the symplectic group. From \( \{ f_1, f_2 \} = 0 \), we know that \( AEB = BEA \) and thus \( (EA)(EB) = (EB)(EA) \). Thus \( G \) is a commutative Lie subgroup of \( Sp(6, R) \). On \( \mathbb{R}^6 \) we define the distribution

\[ D = \{ (EA)x, (EB)x | x \in \mathbb{R}^6 \}. \]

Then the distribution \( D \) is completely integrable because

\[ [(EA)x, (EB)x] = \nabla_{(EA)x}(EB)x - \nabla_{(EB)x}(EA)x = (EB)(EA)x - (EA)(EB)x = 0. \]

Thus the orbit \( G \cdot x_0 \) is the maximal connected integral submanifold of \( D \) through \( x_0 \).

On the other hand, we will prove \( \Sigma \) is also the maximal connected integral submanifold of \( D \) through \( x_0 \). Let \( W_3 \) be the normal deformation vector field of \( L_0 \) associated to the family \( L_{s_1} = \{ f_1 = f_2 = 0, f_3 = s_3 \} \). As in the proof of Lemma 5, we have

\[ (4.4) \]

\[ < W_3, \nabla f_1 >= < W_3, \nabla f_2 >= 0. \]

Because \( \text{deg} f_3 = 3 \), from Lemma 4, we have \( \theta_3 = W_3|\omega_0 = C_3 t^{-2} d t \), where \( C_3 \) is a constant on \( \Sigma \). Thus \( W_3(x) = C_3 |x|^{-3} J x \) for any \( x \in \Sigma \). So from (4.4) we have equations

\[ (4.5) \]

\[ < x, (EA)x >= < x, (EB)x >= 0. \]
On the other hand $J_{\text{grad}} f_1 = 2EAx$ and $J_{\text{grad}} f_2 = 2EBx$ are vector fields on $L^*_0$. Then (4.5) says that $(EA)x$ and $(EB)x$ are 2 linearly independent vector fields on $\Sigma$. So $\Sigma$ is also the integral submanifold of $D$ through $x_0$. Thus we have $\Sigma = G \cdot x_0$.

On $\Sigma$, there are 2 commuting linearly independent vector fields $(EA)x$ and $(EB)x$. From [1, Lemma 2, p.274] and its proof, we know that $\Sigma$ is a 2-torus and that $\exp(\epsilon A t_0) \cdot x_0$ is a circle on $\Sigma$. Let $t_0$ be the first $t$ such that $\exp(\epsilon A t_0) x_0 = x_0$. Thus for any $x = \exp(\epsilon A t) \exp(\epsilon B s) \cdot x_0 \in \Sigma$, we have $\exp(\epsilon A t_0) x = x$, namely, $(\exp(\epsilon A t_0) - I)x = 0$. But $\Sigma$ is full as the submanifold of $\mathbb{R}^6$ as we observed in section 1, so we must have $\exp(\epsilon A t_0) = I$ and $\exp(\epsilon A t)$ is a circle on $G$. For the same reason, $\exp(\epsilon B s)$ is a circle on $G$. We let $s_0$ be the first $s$ such that $\exp(\epsilon B s_0) = I$.

Now by Williamson's theorem [9], we can reduce $A$ to normal forms by means of a real symplectic transformation. In [1, Appendix 6] we can find the list of normal forms. From $\exp(\epsilon A t_0) = I$, eigenvalues of $EA$ are of the form $0$ or $\pm i \mu$ for $\mu \in \mathbb{R}$. Thus we only need to discuss the case with eigenvalues $0$ or $\pm i \mu$. In other words, we need to check which symmetric matrices $C$ with eigenvalues $0$ or $\pm i \mu$ satisfy the equation $\exp(\epsilon C t_0) = I$. After that we can find a symplectic matrix $P_1$ such that

$$A' = P_1^T A P_1 = \text{diag}(\pm i \mu_1, \pm i \mu_2, \pm i \mu_3, \pm \delta_1, \pm \delta_2, \pm \delta_3),$$

where $\pm i \mu_j (j = 1, 2, 3)$ are eigenvalues of $EA$ and $\delta_j = 1$ if $\mu_j \neq 0$; $\delta_j = 0$ if $\mu_j = 0$. Certainly there is another symplectic matrix $P_2$ such that

$$\tilde{A} = (P_1 P_2)^T A (P_1 P_2) = \text{diag}(r_1, r_2, r_3, r_1, r_2, r_3),$$

where $r_j = \pm \mu_j (j = 1, 2, 3)$.

Without loss of generality, we can assume that $\det(\epsilon A) = 0$ and $\pm i$ are eigenvalues of $EA$. This is because if not, we can take $A_1 = (u A - v B)$ for $u, v \in \mathbb{R}$ such that $\det(\epsilon A) = 0$ and $\pm i$ are eigenvalues of $(\epsilon A_1)$. If we take $\tilde{f}_1 = u f_1 - v f_2 = u s_1 - v s_2$, $\tilde{f}_2 = f_2 = s_2$ and $\tilde{f}_3 = f_3 = s_3$, then $\tilde{F} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ also defines homogeneous SL fibration. This SL fibration has the same geometric structure as the SL fibration $F$. This is the meaning of "linearly equivalent" in the theorem.

Let $P = P_1 P_2$, then $\tilde{A} = P^T A P = \text{diag}(1, -k, 0, 1, -k, 0)$. We note that $k > 0$. Because if $k \leq 0$, then $C(\Sigma) = \{0\}$. Let $\tilde{B} = P^T B P$. First we have $\exp(\epsilon \tilde{B} s_0) = I$. On the other hand, from $AEB = BEA$, we have

$$\tilde{A} \tilde{B} = (P^T A P) E (P^T B P) = P^T AEB P = P^T BEAP = \tilde{B} \tilde{E} \tilde{A}.$$

So from Lemma 17, there is a symplectic matrix $Q$ such that $Q^T \tilde{A} Q = \tilde{A}$ and $Q^T \tilde{B} Q = \text{diag}(\alpha, \beta, \gamma, \alpha, \beta, \gamma)$.

Thus if we take $S = PQ$, then $S^T AS = \text{diag}(1, -k, 0, 1, -k, 0)$ and $S^T BS = \text{diag}(\alpha, \beta, \gamma, \alpha, \beta, \gamma)$. If we take suitable linearly transformation, we can take $S^T BS = \text{diag}(1, 0, -l, 1, 0, -l)$ with $l > 0$.

Now if we take the Darboux coordinates

$$(p_1, p_2, p_3, q_1, q_2, q_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T,$$

we have proven that

$$\tilde{f}_1 = (p_1^2 + q_1^2) - k(p_2^2 + q_2^2)$$

$$\tilde{f}_2 = (p_1^2 + q_1^2) - l(p_3^2 + q_3^2).$$
Because Poisson bracket is preserved by the symplectic transformation, we still have \( \{ f_i, f_j \} = 0 \) at the Darboux coordinates \((p_j, q_j)\). From \( \{ f_j, f_3 \} = 0 \) for \( j = 1, 2 \), we have

\[
p_1 \frac{\partial f_3}{\partial q_1} - q_1 \frac{\partial f_3}{\partial p_1} = k(p_2 \frac{\partial f_3}{\partial q_2} - q_2 \frac{\partial f_3}{\partial p_2}) = l(p_3 \frac{\partial f_3}{\partial q_3} - q_3 \frac{\partial f_3}{\partial p_3}).
\]

Using above equations, by observation, \( \tilde{f}_3 \) can not contain the following items:

\[
p_i q_i, p_i q_i^2, p_i^2 q_i, p_i^2 p_j, p_i^2 q_j, q_i q_j^2, q_i p_j q_i, p_i q_i q_j (i \neq j).
\]

So \( \tilde{f}_3 \) only contains following items:

\[
p_1 p_2 p_3, p_1 p_2 q_3, p_1 q_2 p_3, p_1 q_2 q_3, q_1 p_2 p_3, q_1 p_2 q_3, q_1 q_2 p_3, q_1 q_2 q_3.
\]

Using \( \{ \tilde{f}_i, \tilde{f}_j \} = 0 \), a straight forward computation shows that \( k = l = 1 \) and

\[
\tilde{f}_3 = a(p_1 p_2 p_3 - p_1 q_2 q_3 - q_1 p_2 q_3 - q_1 q_2 p_3) + b(p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 - q_1 q_2 q_3)
\]

\[
= a Re(p_1 + i q_1)(p_2 + i q_2)(p_3 + i q_3) + b Im(p_1 + i q_1)(p_2 + i q_2)(p_3 + i q_3),
\]

where \( a \) and \( b \) are constants. So if let \( \sin \theta = \frac{a}{\sqrt{a^2 + b^2}} \), \( \cos \theta = \frac{b}{\sqrt{a^2 + b^2}} \), then

\[
\tilde{f}_3 = \sqrt{a^2 + b^2} Im[e^{i \theta}(p_1 + i q_1)(p_2 + i q_2)(p_3 + i q_3)].
\]

So we can assume that

\[
\tilde{f}_3 = Im(p_1 + i q_1)(p_2 + i q_2)(p_3 + i q_3) = p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 - q_1 q_2 q_3
\]

by some unitary translation and linear translation.

Now we must prove \( S \in O(6, \mathbb{R}) \). Let

\[
U = \text{diag}(1, -1, 0, 1, -1, 0)
\]

and let

\[
V = \text{diag}(1, 0, -1, 1, 0, -1).
\]

We have proven that \( S^TAS = U \) and \( S^TBS = V \). Then

\[
S^{-1} \exp(EAt)S = \exp(S^{-1}EAST) = \exp(ES^TAST) = \exp(EUt) = \text{diag}(e^{it}, e^{-it}, 1)
\]

and

\[
S^{-1} \exp(EBs)S = \exp(EVs) = \text{diag}(e^{is}, 1, e^{-is}).
\]

So we have \( S^{-1}GS = T^2 = \{ \text{diag}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})|t_1, t_2 \in \mathbb{R} \} \) or \( G = STS^{-1} \). Let \( C(\Sigma') = \{ f_1(p, q) = f_2(p, q) = f_3(p, q) = 0 \} \) and \( p_0 = \frac{1}{\sqrt{3}}(1, 1, 1, 0, 0, 0) \) \( T \in C(\Sigma') \), then there is a point \( x_0 \in \Sigma \) such that \( S^{-1}x_0 = cp_0 \), where \( c \) is a constant. From
\(G \cdot x_0 = \Sigma \subset S^5(1)\), we have \(\langle gx_0, gx_0 \rangle = 1\) for any \(g \in G\). So for any \(\tau \in T^2\), we have

\[\langle S\tau S^{-1}x_0, S\tau S^{-1}x_0 \rangle = c^2 < S\tau p_0, S\tau p_0 >= 1,\]

or

\[(4.6) \quad (\tau p_0)^T (S^T S)(\tau p_0) = c^{-2}.\]

Let \(\tau = \text{diag}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})\) and let \(u = \cos t_1, v = \cos t_2\). Let \(S^T S = (m_{ij})\) and let

\[h(u,v) = m_{33} + m_{44} + m_{55} - 2m_{46}v - 2m_{56}u + 2m_{12}uv + 2(m_{13} + m_{46})u^2v + 2(m_{25} + m_{56})uv^2 + (m_{11} - m_{44})u^2 + (m_{22} - m_{55})v^2 + (m_{33} - m_{66})(2u^2v^2 - u^2 - v^2) + \{ -2m_{35} + 2(m_{14} + m_{36})u + 2m_{24}v + 2(m_{34} - m_{16})uv + 2(m_{35} - m_{26})v^2 - 4m_{36}uv^2 \} \sqrt{1-u^2} + \{ -2m_{34} + 2(m_{25} + m_{36})v + 2m_{15}u + 2(m_{34} - m_{16})u^2 + 2(m_{35} - m_{26})uv - 4m_{36}uv^2 \} \sqrt{1-v^2} + \{ 2m_{45} - 2(m_{13} + m_{46})u - 2(m_{23} + m_{56})v \}

Then (4.6) can be rewritten

\[(4.7) \quad h(u,v) = 3c^{-2}\]

for any \(-1 \leq u, v \leq 1\). So we have \(\frac{\partial^2 h}{\partial u \partial v} |_{u=v=0} = 0\). By direct calculation, we can get

\[\frac{\partial^6 h}{\partial u^3 \partial v^3} |_{u=v=0} = m_{33} - m_{66} = 0\]
\[\frac{\partial^6 h}{\partial u^3 \partial v^3} |_{u=0} = -18(m_{13} + m_{46})v(1-v^2)^{-\frac{3}{2}} = 0\]
\[\frac{\partial^6 h}{\partial u^3 \partial v^3} |_{v=0} = -18(m_{23} + m_{56})u(1-u^2)^{-\frac{3}{2}} = 0\]
\[\frac{\partial^6 h}{\partial u^3 \partial v^3} = 18m_{45}uv(1-u^2)^{-\frac{3}{2}}(1-v^2)^{-\frac{3}{2}} = 0\]

From above equations, we can obtain \(m_{45} = 0, m_{33} = m_{66}, m_{13} = -m_{46}\) and \(m_{23} = -m_{56}\). Thus \(h\) can be write in the following form:

\[h(u,v) = m_{33} + m_{44} + m_{55} + 2m_{13}v + 2m_{23}u + 2m_{12}uv + 2(m_{11} - m_{44})u^2 + (m_{22} - m_{55})v^2 + (m_{33} - m_{66})(2u^2v^2 - u^2 - v^2) + \{ -2m_{35} + 2(m_{14} + m_{36})u + 2m_{24}v + 2(m_{34} - m_{16})uv + 2(m_{35} - m_{26})v^2 - 4m_{36}uv^2 \} \sqrt{1-u^2} + \{ -2m_{34} + 2(m_{25} + m_{36})v + 2m_{15}u + 2(m_{34} - m_{16})u^2 + 2(m_{35} - m_{26})uv - 4m_{36}uv^2 \} \sqrt{1-v^2}\]
Using the same method, at last, we can obtain

\begin{equation}
S^T S = \text{diag}(m_{11}, m_{22}, m_{33}, m_{11}, m_{22}, m_{33})
\end{equation}

and

\[ m_{11} + m_{22} + m_{33} = 3c^{-2}. \]

But \( S \) is the symplectic matrix, so \( S^{-1} \) and \( S^T \) are symplectic matrices too. Thus we have

\begin{equation}
S^T S E S = S^T E S = E
\end{equation}

Now from (4.8) and (4.9), we easily can get \( m_{11} = m_{22} = m_{33} = 1 \). From above discussion, we have \( S^T S = I \) and \( S \in O(6, \mathbb{R}) \). Thus we have proven \( S \in U(3) \) and this completes the proof of theorem 18. \( \square \)

REFERENCES