GENERALIZED LAGRANGIAN MEAN CURVATURE FLOWS IN SYMPLECTIC MANIFOLDS[*]

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Abstract. An almost Kähler structure on a symplectic manifold (N, ω) consists of a Riemannian metric g and an almost complex structure J such that the symplectic form ω satisfies $\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$. Any symplectic manifold admits an almost Kähler structure and we refer to (N, ω, g, J) as an almost Kähler manifold. In this article, we propose a natural evolution equation to investigate the deformation of Lagrangian submanifolds in almost Kähler manifolds. A metric and complex connection $\widehat{\nabla}$ on TN defines a generalized mean curvature vector field along any Lagrangian submanifold M of N. We study the evolution of M along this vector field, which turns out to be a Lagrangian deformation, as long as the connection $\widehat{\nabla}$ satisfies an Einstein condition. This can be viewed as a generalization of the classical Lagrangian mean curvature flow in Kähler-Einstein manifolds where the connection $\widehat{\nabla}$ is the Levi-Civita connection of g. Our result applies to the important case of Lagrangian submanifolds in a cotangent bundle equipped with the canonical almost Kähler structure and to other generalization of Lagrangian mean curvature flows, such as the flow considered by Behrndt [B] in Kähler manifolds that are almost Einstein.

Key words. Lagrangian mean curvature flow, almost Kähler structure, symplectic manifold, cotangent bundle.

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1. Introduction. Special Lagrangian submanifolds [HL] and Lagrangian mean curvature flows [TY] attract much attentions due to their relations to the SYZ conjecture [SYZ] on mirror symmetry between Calabi-Yau manifolds. A Calabi-Yau, or in general a Kähler-Einstein manifold, is a great place to study the mean curvature flow as this process provides a Lagrangian deformation [SM1]. This important property no longer holds if the ambient space is a general symplectic manifold. However, there are important conjectures (see for example [FSS]) concerning the Lagrangian isotopy problem in general symplectic manifolds such as cotangent bundles which do not carry Kähler-Einstein structures.

In this article, we aim at defining a generalized Lagrangian mean curvature flow in general almost Kähler manifolds N. We consider generalized mean curvature vector fields $\overrightarrow{\mathrm{H}}$ (see Definition 3 in §4) along Lagrangian or more generally almost Lagrangian submanifolds, i.e. submanifolds M for which $J(TM) \cap TM = \{0\}$. The definition of the generalized mean curvature vector $\overrightarrow{\mathrm{H}}$ relies on a choice of a complex and metric connection $\widehat{\nabla}$ on TN that could carry non-trivial torsion $\widehat{T}$. We then say that a smooth family of almost Lagrangian immersions

$$F : M \times [0, T) \to N$$

satisfies the generalized mean curvature flow, if

$$(1) \qquad \frac{\partial F}{\partial t}(p, t) = \overrightarrow{\mathrm{H}}(p, t), \quad \text{and} \quad F(M, 0) = M_0$$

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where $\overrightarrow{\widehat{\mathrm{H}}}(p,t)$ is the generalized mean curvature vector of the almost Lagrangian submanifold $M_t = F(M,t)$ at $F(p,t)$. This flow is uniquely defined up to tangential diffeomorphisms of M.

Recall that the Riemannian metric g on N also defines the classical mean curvature vector $\overrightarrow{\mathrm{H}}$ on M through the first variation of volume. $\overrightarrow{\widehat{\mathrm{H}}}$ differs from $\overrightarrow{\mathrm{H}}$ by some lower order terms involving the torsion of $\widehat{\nabla}$. Since the mean curvature flow is a nonlinear parabolic system which is non-degenerate after gauge fixing, the short time existence of the generalized mean curvature flow in the class of almost Lagrangian submanifolds can thus be established.

THEOREM 1. *Suppose (N, ω, g, J) is an almost Kähler manifold and $\widehat{\nabla}$ is a complex and metric connection on TN. For any initial smooth compact almost Lagrangian submanifold M_0, there exists a maximal time $T \in (0, \infty]$ so that the generalized mean curvature flow (1) exists smoothly on $[0, T)$ in the class of almost Lagrangian submanifolds.*

We will show in Lemma 3 that the generalized mean curvature vector $\overrightarrow{\widehat{\mathrm{H}}}$ is related to a 1-form $\widehat{H}$ on M that can be seen as a generalization of the classical mean curvature form (or Maslov form) of Lagrangian submanifolds of Kähler manifolds.

It turns out that $\widehat{H}$ is closed on Lagrangian submanifolds and the flow preserves the Lagrangian condition, if in addition the *Ricci form* of $\widehat{\nabla}$,

$$\widehat{\rho}(V, W) := \frac{1}{2} \operatorname{trace}(\widehat{R}(V, W) \circ J),$$

where $\widehat{R}$ is the curvature operator of $\widehat{\nabla}$, satisfies the following Einstein condition:

DEFINITION 1. *A metric and complex connection $\widehat{\nabla}$ on an almost Kähler manifold (N, ω, g, J) is called Einstein, if the Ricci form of $\widehat{\nabla}$ satisfies*

$$\widehat{\rho} = f\omega$$

for some smooth function f on N.

REMARK 1. *In general Einstein connections, if they exist, are not unique. For example the canonical connection $\widehat{\nabla}$ on the cotangent bundle T^*M of a Riemannian manifold M equipped with the metric of type $I + III$ as defined in [YI] is Einstein (even Ricci flat, i.e. the Ricci form vanishes). Any connection $\tilde{\nabla} := \widehat{\nabla} + c\lambda \otimes J$, where λ is the Liouville form of T^*M and c some constant is also Einstein with $f = -nc$.*

THEOREM 2. *Suppose (N, ω, g, J) is an almost Kähler manifold and $\widehat{\nabla}$ is a complex and metric connection that satisfies the Einstein condition. Suppose M_0 is a closed Lagrangian submanifold of N. Then the generalized mean curvature flow (1) with respect to $\widehat{\nabla}$ preserves the Lagrangian condition.*

This gives a new and large class of symplectic manifolds where Lagrangian mean curvature flows can be defined. Our class includes the classical Lagrangian mean curvature flow in Kähler-Einstein manifolds, the modified Lagrangian mean curvature flows considered by Behrndt [B] for Kähler manifolds that are almost Einstein and the important class of Lagrangian submanifolds in the cotangent bundle of any given Riemannian manifold.

Our strategy for proving that the Lagrangian condition is preserved is similar to that in [SM1]. We will first consider the flow in the larger (and open) class of almost Lagrangian submanifolds in N. After the short time existence is established in this class, we apply the maximum principle to prove that the Lagrangian condition is preserved.

REMARK 2. *Theorems 1 and 2 show that there exists a generalized Lagrangian mean curvature flow on an almost Kähler manifold with an Einstein connection. It is an interesting question to identify which almost Kähler manifold has this property. In §5, we give a list of examples that include all currently known cases to our knowledge.*

The article is organized as follows. In §2, the geometry of almost Kähler manifolds and the space of complex and metric connections are reviewed. In §3 we consider the geometry of Lagrangian and almost Lagrangian submanifolds of almost Kähler manifolds and define generalized second fundamental forms and mean curvature forms with respect to a metric and complex connection $\widehat{\nabla}$. In particular we show that the generalized mean curvature form with respect to an Einstein connection is closed on a Lagrangian submanifold. In §4 we define the generalized mean curvature flow and prove our main Theorems 1 and 2. In §5, we present known examples of almost Kähler manifolds with Einstein connections.

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2. Almost Kähler manifolds. In this section we recall some properties of almost Kähler manifolds.

2.1. Almost Kähler manifolds and Einstein connections.
Suppose (N, ω, g, J) is an almost Kähler manifold. This means that (N, g) is a Riemannian manifold, $J \in \mathrm{End}(TN)$ is an almost complex structure (i.e. an endomorphisms with $J^2 = -\mathrm{Id}$) and in addition the characteristic 2-form $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$ is symplectic, i.e. ω is skew and $d\omega = 0$. J needs not be integrable and thus (N, J) in general is not a complex manifold. An almost Kähler manifold is Kähler if and only if J is parallel with respect to the Levi-Civita connection ∇ of g.

We start with some general properties of almost Kähler manifolds. For an almost Kähler manifold (N, ω, g, J), let us define the class $\mathscr{C}$ of metric and complex connections

$$\mathscr{C} := \{\widehat{\nabla} : \widehat{\nabla}g = 0,\ \widehat{\nabla}J = 0\}.$$

It is well known that this set is non-empty. There exists a "canonical" connection $\widehat{\nabla}$ in $\mathscr{C}$ in the following sense: If ∇ denotes the Levi-Civita connection of g, then $\widehat{\nabla}$ defined by

$$\widehat{\nabla}_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)(JY)$$

lies in $\mathscr{C}$.

The following lemma can be easily proved:

LEMMA 1. *If $\widehat{\nabla} \in \mathscr{C}$ and $\sigma \in \Omega^1(N)$ is an arbitrary 1-form, then $\widetilde{\nabla} := \widehat{\nabla} + \sigma \otimes J \in \mathscr{C}$. The curvature tensors $\widetilde{R}$ and $\widehat{R}$ of $\widetilde{\nabla}$ resp. $\widehat{\nabla}$ are related by*

$$\widetilde{R} = \widehat{R} + d\sigma \otimes J$$

and the torsions $\widetilde{T}$ and $\widehat{T}$ are related by

$$\widetilde{T} = \widehat{T} + \sigma \wedge J\,.$$

Throughout this paper $\widehat{\nabla}$ denotes a metric and complex connection. By the compatibility of ω and J, we see that $\widehat{\nabla}$ is also symplectic, i.e. $\widehat{\nabla}\omega = 0$. Since $\widehat{\nabla}$ is metric, the curvature tensor $\widehat{R}$ of $\widehat{\nabla}$ satisfies

(2) $$\langle \widehat{R}(X,Y)V, W \rangle = -\langle \widehat{R}(Y,X)V, W \rangle = -\langle \widehat{R}(X,Y)W, V \rangle\,.$$

Moreover, the fact that $\widehat{\nabla}$ is complex implies

(3) $$\langle \widehat{R}(X,Y)JV, W \rangle = \langle J\widehat{R}(X,Y)V, W \rangle = \langle \widehat{R}(X,Y)JW, V \rangle\,.$$

As $\langle \widehat{R}(X,Y)JV, W \rangle$ is symmetric in V and W we can take the trace over these two arguments with respect to the Riemannian metric g on N. This gives the Ricci form of $\widehat{\nabla}$

$$\widehat{\rho}(X,Y) := \frac{1}{2}\,\mathrm{trace}(\widehat{R}(X,Y) \circ J)\,.$$

The first Bianchi identity for $\widehat{R}$ is

$$\widehat{R}(X,Y)Z + \widehat{R}(Y,Z)X + \widehat{R}(Z,X)Y$$
$$= \widehat{T}(\widehat{T}(X,Y),Z) + \widehat{T}(\widehat{T}(Y,Z),X) + \widehat{T}(\widehat{T}(Z,X),Y)$$
(4) $$\quad + (\widehat{\nabla}_X\widehat{T})(Y,Z) + (\widehat{\nabla}_Y\widehat{T})(Z,X) + (\widehat{\nabla}_Z\widehat{T})(X,Y)\,,$$

where

$$\widehat{T}(X,Y) = \widehat{\nabla}_X Y - \widehat{\nabla}_Y X - [X,Y]$$

is the torsion tensor of $\widehat{\nabla}$.

Due to this weaker Bianchi identity, the Ricci form $\widehat{\rho}$ in general does not satisfy the equation $\widehat{\rho}(V, JW) = \widehat{Ric}(V,W)$, where $\widehat{Ric}$ denotes the usual Ricci curvature $\widehat{Ric}(V,W) = \mathrm{trace}\langle \widehat{R}(\cdot, V)\cdot, W \rangle$.

3. The geometry of Lagrangian submanifolds of almost Kähler manifolds. Suppose $\widehat{\nabla} \in \mathscr{C}$ is an arbitrary metric and complex connection on the tangent bundle TN of an almost Kähler manifold (N, ω, g, J).

The torsion tensor $\widehat{T}^{\alpha}_{\beta\gamma}$ of $\widehat{\nabla}$ is locally given by

$$\widehat{T}^{\alpha}_{\beta\gamma} := \widehat{\Gamma}^{\alpha}_{\beta\gamma} - \widehat{\Gamma}^{\alpha}_{\gamma\beta}\,,$$

where $\widehat{\Gamma}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of $\widehat{\nabla}$. Locally for a vector field $V = V^{\alpha}\partial/\partial y^{\alpha} \in \Gamma(TN)$ we have

$$\widehat{\nabla}_{\alpha} V^{\beta} = V^{\beta}_{,\alpha} + \widehat{\Gamma}^{\beta}_{\alpha\gamma} V^{\gamma}\,.$$

Suppose now that $F : M \to N$ is a smooth immersion. The differential

$$dF = \frac{\partial F^{\alpha}}{\partial x^i}\frac{\partial}{\partial y^{\alpha}} \otimes dx^i$$

is a smooth section in the bundle $F^{-1}TN \otimes T^*M$. Let ∇ denote the Levi-Civita connection on TM with respect to the induced Riemannian metric $g_{ij} = g_{\alpha\beta}F_i^\alpha F_j^\beta$, where $F_i^\alpha := \partial F^\alpha / \partial x^i$. A connection $\widehat{\nabla}$ on TN induces a connection $\widehat{\nabla}^E$ on the pull-back bundle $E := F^{-1}TN$ along M by

$$\widehat{\nabla}_X^E \sigma := \widehat{\nabla}_{dF(X)}\sigma \,,$$

where $X \in TM$ and $\sigma \in \Gamma(E)$. A product connection $\widehat{\nabla}^E \otimes \nabla$ on the bundle $E \otimes TM$ is then defined by

$$(\widehat{\nabla}^E \otimes \nabla)_X(\sigma \otimes Y) := \widehat{\nabla}_X^E \sigma \otimes Y + \sigma \otimes \nabla_X Y \,.$$

Similarly we obtain product connections on all bundles of the form

$$E \otimes \underbrace{TM \otimes \cdots \otimes TM}_{p-times} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q-times} \,.$$

In some abuse of notation, let us denote all these connections on bundles containing E as a factor by $\widehat{\nabla}^E$. Since on E we also have the Levi-Civita connection induced by g, there are always two different connections $\nabla, \widehat{\nabla}^E$ on bundles containing E. We shall use both.

For example, for a section

$$V_i^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \in \Gamma(F^{-1}TN \otimes T^*M)$$

we get

$$\widehat{\nabla}_i^E V_j^\alpha = V_{j,i}^\alpha - \Gamma_{ij}^k V_k^\alpha + \widehat{\Gamma}_{\beta\gamma}^\alpha F_i^\beta V_j^\gamma$$

and likewise

$$\nabla_i V_j^\alpha = V_{j,i}^\alpha - \Gamma_{ij}^k V_k^\alpha + \Gamma_{\beta\gamma}^\alpha F_i^\beta V_j^\gamma \,,$$

where $\widehat{\Gamma}_{\beta\gamma}^\alpha$ resp. $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of $\widehat{\nabla}$ resp. ∇ on TN.

We derive

$$(5) \qquad \widehat{\nabla}_i^E \widehat{\nabla}_j^E V_k^\alpha - \widehat{\nabla}_j^E \widehat{\nabla}_i^E V_k^\alpha = -R_{kij}^m V_m^\alpha + \widehat{R}_{\delta\beta\epsilon}^\alpha F_i^\beta F_j^\epsilon V_k^\delta \,,$$

where R_{kij}^m denotes the Riemann curvature of the Levi-Civita connection on TM.

Let us define two second fundamental tensors

$$\widehat{A}_{ij}^\alpha := \widehat{\nabla}_i^E F_j^\alpha \,, \quad A_{ij}^\alpha := \nabla_i F_j^\alpha \,.$$

We obtain

$$(6) \qquad \widehat{A}_{ij}^\alpha - \widehat{A}_{ji}^\alpha = (\widehat{\Gamma}_{\beta\gamma}^\alpha - \widehat{\Gamma}_{\gamma\beta}^\alpha)F_i^\beta F_j^\gamma = \widehat{T}_{\beta\gamma}^\alpha F_i^\beta F_j^\gamma$$

and

$$A_{ij}^\alpha = \widehat{A}_{ij}^\alpha + (\Gamma_{\beta\gamma}^\alpha - \widehat{\Gamma}_{\beta\gamma}^\alpha)F_i^\beta F_j^\gamma \,.$$

Applying (5) to the section F_k^α yields the Codazzi equation

$$(7) \qquad \widehat{\nabla}_i^E \widehat{A}_{jk}^\alpha - \widehat{\nabla}_j^E \widehat{A}_{ik}^\alpha = -R_{kij}^m F_m^\alpha + \widehat{R}_{\delta\beta\epsilon}^\alpha F_i^\beta F_j^\epsilon F_k^\delta \,.$$

Note, that (7) does not contain any torsion terms $\widehat{T}$ though the connection $\widehat{\nabla}$ on TN in general has torsion. This is because the part of $\widehat{\nabla}^E$ that acts on the tangent bundle is given by the Levi-Civita connection. All information on $\widehat{\nabla}$ contained in the Codazzi equation is then encoded in the curvature term $\widehat{R}^\alpha_{\delta\beta\epsilon} F_i^\beta F_j^\epsilon F_k^\delta$.

Suppose now that $F : M \to N$ is almost Lagrangian, i.e. $\dim M = n = \frac{1}{2}\dim N$ and the map

$$\phi : TM \to T^\perp M \,, \quad \phi V := (JV)^\perp$$

provides an isomorphism between the tangent and the normal bundle $T^\perp M$ of M. This holds if and only if the symmetric tensor

$$\eta(V, W) := \langle \phi V, \phi W \rangle$$

is invertible everywhere.

In local coordinates $(x^i)_{i=1,\dots,n}$ on M, $F^*\omega$ can be written in the form $F^*\omega = \omega_{ij} dx^i \otimes dx^j$ with

$$\omega_{ij} := \omega_{\alpha\beta} F_i^\alpha F_j^\beta \,.$$

For $\eta = \eta_{ij} dx^i \otimes dx^j$ we obtain

$$\eta_{ij} = g_{ij} - \omega_i{}^m \omega_{jm} \,,$$

where here and in the following indices will be raised and lowered by contraction with the metric tensors g^{ij} and g_{ij} and the Einstein convention always applies. An exception will be the inverse of η_{ij} which we denote by η^{ij} (note that this differs from $g^{ik} g^{jl} \eta_{kl}$). From $g^{ij} = g^{im} \delta_m{}^j$ and $\delta_m{}^j = \eta_{mk} \eta^{kj}$ we observe

$$(8) \qquad\qquad \eta^{ij} = g^{ij} + \omega^{il} \omega_{sl} \eta^{sj} \,.$$

DEFINITION 2. *Denote*

$$\widehat{h}_{kij} := \langle \phi F_k, \widehat{A}_{ij} \rangle,$$

the generalized mean curvature form of M is defined to be $\widehat{H} = \widehat{H}_i dx^i$ where

$$\widehat{H}_i := g^{kj} \widehat{h}_{kij} = g^{kj} \langle (JF_k)^\perp, \widehat{\nabla}_i F_j \rangle$$

We also define two auxiliary tensors

$$\widehat{r}_{kij} := \omega(F_k, \widehat{A}_{ij}) \,, \text{and} \quad \widehat{s}_{kij} := \langle F_k, \widehat{A}_{ij} \rangle.$$

The following relations can be easily verified:

$$(9) \qquad\qquad \widehat{h}_{kij} = \widehat{r}_{kij} - \omega_k{}^m \widehat{s}_{mij} \,, \quad \widehat{H}_i = \widehat{r}_{ki}{}^k + \omega^{mk} \widehat{s}_{mik} \,.$$

In the rest of the section, we compute $d\widehat{H}$. To this end let us first compute $\nabla_l \widehat{r}_{kij}$. Here $\widehat{r}$ is considered as a section in $\Gamma(T^*M \otimes T^*M \otimes T^*M)$ and ∇ denotes the Levi-Civita connection on M. From $\widehat{r}_{kij} = \omega_{\alpha\beta} F_k^\alpha \widehat{A}_{ij}^\beta$ we obtain by Leibniz' rule for connections

$$
\begin{aligned}
\nabla_l \widehat{r}_{kij} &= \widehat{\nabla}_\gamma \omega_{\alpha\beta} F_l^\gamma F_k^\alpha \widehat{A}_{ij}^\beta + \omega_{\alpha\beta} \widehat{A}_{lk}^\alpha \widehat{A}_{ij}^\beta + \omega_{\alpha\beta} F_k^\alpha \widehat{\nabla}_l^E \widehat{A}_{ij}^\beta \\
&\overset{(\widehat{\nabla}\omega=0)}{=} \omega_{\alpha\beta} \widehat{A}_{lk}^\alpha \widehat{A}_{ij}^\beta + \omega_{\alpha\beta} F_k^\alpha \widehat{\nabla}_l^E \widehat{A}_{ij}^\beta \,.
\end{aligned}
$$

Interchanging l and i and subtracting yields

$$\nabla_l \widehat{r}_{ki}{}^k - \nabla_i \widehat{r}_{kl}{}^k$$
$$= g^{kj} \omega_{\alpha\beta} (\widehat{A}_{lk}^\alpha \widehat{A}_{ij}^\beta - \widehat{A}_{ik}^\alpha \widehat{A}_{lj}^\beta) + g^{kj} \omega_{\alpha\beta} F_k^\alpha (\widehat{\nabla}_l^E \widehat{A}_{ij}^\beta - \widehat{\nabla}_i^E \widehat{A}_{lj}^\beta)$$
$$\stackrel{(7)}{=} 2 g^{kj} \omega_{\alpha\beta} \widehat{A}_{lk}^\alpha \widehat{A}_{ij}^\beta + g^{kj} \omega_{\alpha\beta} F_k^\alpha (-R_{jli}^m F_m^\beta + \widehat{R}_{\gamma\delta\epsilon}^\beta F_j^\gamma F_l^\delta F_i^\epsilon),$$

which can be rewritten as

$$(10) \qquad \nabla_l \widehat{r}_{ki}{}^k - \nabla_i \widehat{r}_{kl}{}^k = 2 g^{kj} \omega(\widehat{A}_{lk}, \widehat{A}_{ij}) + \omega_m{}^j R_{jli}^m + g^{kj} \omega(F_k, \widehat{R}(F_l, F_i) F_j).$$

Let us treat the first term. From

$$\widehat{A}_{lk} = \langle \widehat{A}_{lk}, F^m \rangle F_m + \langle \widehat{A}_{lk}, \phi F_m \rangle \eta^{mn} \phi F_n$$
$$= \widehat{s}^m{}_{lk} F_m + \eta^{mn} (\widehat{r}_{mlk} - \omega_m{}^p \widehat{s}_{plk}) \phi F_n$$

and

$$(11) \qquad \omega(F_m, \phi F_n) = \eta_{mn}, \quad \omega(\phi F_p, \phi F_q) = -\omega_{pq} + \omega_p{}^k \omega_q{}^l \omega_{kl}$$

we obtain

$$2 g^{kj} \omega(\widehat{A}_{lk}, \widehat{A}_{ij}) = 2 \omega(\widehat{s}^m{}_{lk} F_m + \eta^{mn}(\widehat{r}_{mlk} - \omega_m{}^p \widehat{s}_{plk}) \phi F_n,$$
$$\widehat{s}^u{}_i{}^k F_u + \eta^{uv}(\widehat{r}_{ui}{}^k - \omega_u{}^q \widehat{s}_{qi}{}^k) \phi F_v)$$
$$= 2 \widehat{s}^m{}_{lk} \eta^{uv} \widehat{r}_{ui}{}^k \eta_{mv} - 2 \eta^{mn} \widehat{r}_{mlk} \widehat{s}^u{}_i{}^k \eta_{nu} + (C_1 \lrcorner F^* \omega)_{li}$$
$$= 2 \widehat{s}_{mlk} \widehat{r}^m{}_i{}^k - 2 \widehat{s}_{mik} \widehat{r}^m{}_l{}^k + (C_1 \lrcorner F^* \omega)_{li},$$

for some tensor C_1, where here and in the following $C \lrcorner F^* \omega$ denotes any tensor that is formed by contracting an arbitrary tensor C with $F^* \omega$. Since $\widehat{\nabla}$ is metric, the skew symmetry

$$(12) \qquad \widehat{s}_{kij} = -\widehat{s}_{jik}$$

holds. Moreover, since $\widehat{\nabla}$ is also symplectic applying Leibniz' rule with the Levi-Civita connection ∇ acting on ω, we obtain

$$(13) \qquad \nabla_k \omega_{ij} = \omega(\widehat{A}_{ki}, F_j) + \omega(F_i, \widehat{A}_{kj}) = \widehat{r}_{ikj} - \widehat{r}_{jki}.$$

This implies

$$2 \widehat{s}_{mlk} \widehat{r}^m{}_i{}^k = \widehat{s}_{mlk} (\widehat{r}^m{}_i{}^k - \widehat{r}^k{}_i{}^m)$$
$$= \widehat{s}_{mlk} \nabla_i \omega^{mk}$$
$$= \nabla_i (\widehat{s}_{mlk} \omega^{mk}) - \nabla_i \widehat{s}_{mlk} \omega^{mk}.$$

Hence we conclude

$$(14) \qquad 2 g^{kj} \omega(\widehat{A}_{lk}, \widehat{A}_{ij}) = \nabla_i (\widehat{s}_{mlk} \omega^{mk}) - \nabla_l (\widehat{s}_{mik} \omega^{mk}) + (C_2 \lrcorner F^* \omega)_{li}$$

where $C_2 \lrcorner F^* \omega$ contains the term $\nabla_l \widehat{s}_{mik} \omega^{mk} - \nabla_i \widehat{s}_{mlk} \omega^{mk}$.

With these preparations, we are ready to prove the main proposition of this section:

PROPOSITION 1. *Let (N, ω, g, J) be an almost Kähler manifold and $\widehat{\nabla}$ a metric and complex connection on TN. Then the generalized mean curvature form $\widehat{H}$ for an almost Lagrangian smooth immersion $F : M \to N$ satisfies*

$$(15) \qquad\qquad d\widehat{H} = (C \lrcorner F^* \omega) - F^* \widehat{\rho}$$

for some smooth tensor field C. In particular, if F is Lagrangian and $\widehat{\nabla}$ Einstein, then $\widehat{H}$ is closed.

Proof. Plug equation (14) into equation (10) and recall equation (9), we derive

$$\nabla_l \widehat{H}_i - \nabla_i \widehat{H}_l = \nabla_l \widehat{r}_{ki}{}^k - \nabla_i \widehat{r}_{kl}{}^k + \nabla_l (\widehat{s}_{mik} \omega^{mk}) - \nabla_i (\widehat{s}_{mlk} \omega^{mk})$$
$$(16) \qquad\qquad = (C_3 \lrcorner F^* \omega)_{li} + g^{kj} \omega(F_k, \widehat{R}(F_l, F_i) F_j) \,.$$

Since $\widehat{\nabla}$ is metric and complex, we know that

$$\omega(\widehat{R}(V, W) X, Y) = \omega(\widehat{R}(V, W) Y, X) \text{ for any } V, W, X, Y \in TN \,.$$

The Ricci form $\widehat{\rho}$ is given by

$$\widehat{\rho}(V, W) := \frac{1}{2} \sum_{\alpha=1}^{2n} \omega(\widehat{R}(V, W) e_\alpha, e_\alpha) \,,$$

where e_α is an arbitrary orthonormal basis of TN. In terms of F_i and ϕF_i, we can rewrite the trace in the form

$$\widehat{\rho}(V, W) = \frac{1}{2} g^{ij} \omega(\widehat{R}(V, W) F_i, F_j) + \frac{1}{2} \eta^{ij} \omega(\widehat{R}(V, W) \phi F_i, \phi F_j) \,.$$

On the other hand,

$$\omega(\widehat{R}(V, W) \phi F_i, \phi F_j) = \omega(\widehat{R}(V, W)(J F_i - \omega_i{}^l F_l), J F_j - \omega_j{}^k F_k)$$
$$= \omega(J \widehat{R}(V, W) F_i, J F_j) + (C_4 \lrcorner F^* \omega)(V, W)$$
$$= \omega(\widehat{R}(V, W) F_i, F_j) + (C_4 \lrcorner F^* \omega)(V, W)$$

so that with (8) we obtain

$$(17) \qquad\qquad \widehat{\rho}(V, W) = g^{ij} \omega(\widehat{R}(V, W) F_i, F_j) + (C_5 \lrcorner F^* \omega)(V, W) \,.$$

Combining (16) with (17) and taking into account that the Levi-Civita connection ∇ is torsion free (so that $(d\widehat{H})_{li} = \nabla_l \widehat{H}_i - \nabla_i \widehat{H}_l$), the proposition is proved. $\square$

4. The generalized mean curvature flow in almost Kähler manifolds.

4.1. The generalized mean curvature vector. Suppose $F : M \to N$ is an almost Lagrangian submanifold of an almost Kähler manifold (N, ω, g, J). We recall that, with respect to the variation of volume defined by g, there is the classical mean curvature vector field $\overrightarrow{H}$ defined on M:

$$\overrightarrow{H} = \eta^{kl} (A_{kl})^\perp = \eta^{ij} g^{kl} \langle A_{kl}, \phi F_i \rangle \phi F_j.$$

DEFINITION 3. *Let $\widehat{\nabla}$ be a metric and complex connection. The generalized mean curvature vector of M with respect to $\widehat{\nabla}$ is defined to be*

$$\overrightarrow{\widehat{H}} := \overrightarrow{H} + \eta^{ij} g^{kl} \left(\langle \widehat{T}(\phi F_i, F_k), F_l \rangle + \langle \widehat{T}(F_i, F_k), \phi F_l \rangle \right) \phi F_j \,.$$

In the following, we derive a relation between the generalized mean curvature vector and the generalized mean curvature form $\widehat{H} = \widehat{H}_i dx^i$.

First we express the difference between $\widehat{\nabla}$ and the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$ in terms of the torsion $\widehat{T}$ in the next Lemma.

LEMMA 2. *The two connections $\widehat{\nabla}$ and ∇ on TN are related by*

$$2\langle \widehat{\nabla}_X Y - \nabla_X Y, Z \rangle = \langle \widehat{T}(X, Y), Z \rangle + \langle \widehat{T}(Z, X), Y \rangle + \langle \widehat{T}(Z, Y), X \rangle.$$

Proof. Using the fact that ∇ is torsion free and compatible with the metric, we have

$$X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle$$
$$= \langle 2\nabla_X Y - [X, Y], Z \rangle + \langle Y, [X, Z] \rangle + \langle X, [Y, Z] \rangle.$$

On the other hand, $\widehat{\nabla}$ is a metric connection, therefore

$$X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle = \langle 2\widehat{\nabla}_X Y - [X, Y] - \widehat{T}(X, Y), Z \rangle$$
$$+ \langle Y, [X, Z] + \widehat{T}(X, Z) \rangle + \langle X, [Y, Z] + \widehat{T}(Y, Z) \rangle \,.$$

Subtracting the two identities yields the desired formula. $\square$

LEMMA 3. *The generalized mean curvature vector $\overrightarrow{\widehat{H}}$ and the generalized mean curvature form $\widehat{H}$ with respect to the connection $\widehat{\nabla}$ are related by*

$$(18) \qquad \overrightarrow{\widehat{H}} = \eta^{ij} \left(\widehat{H}_i - \nabla^k \omega_{ki} - \omega_i{}^m \widehat{s}_{mk}{}^k - \omega^{mk} \widehat{s}_{mki} \right) \phi F_j \,.$$

In particular, if M is Lagrangian, then

$$(19) \qquad \overrightarrow{\widehat{H}} = g^{ij} \widehat{H}_i J F_j \,.$$

Proof. From (9), (12) and (13) we conclude

$$(20) \qquad \widehat{h}_{lki} = \widehat{h}_{ikl} + \nabla_k \omega_{li} - \omega_l{}^m \widehat{s}_{mki} + \omega_i{}^m \widehat{s}_{mkl} \,.$$

Moreover, since $\widehat{\nabla}$ has torsion we get

$$(21) \qquad \widehat{h}_{lik} = \widehat{h}_{lki} + \langle \widehat{T}(F_i, F_k), \phi F_l \rangle \,.$$

From these equations we then deduce

$$(22) \qquad \widehat{h}_{ik}{}^k + g^{kl} \langle \widehat{T}(F_i, F_k), \phi F_l \rangle = \widehat{H}_i - \nabla^k \omega_{ki} - \omega_i{}^m \widehat{s}_{mk}{}^k - \omega^{mk} \widehat{s}_{mki} \,.$$

In addition, Lemma 2 implies

$$\widehat{h}_{ik}{}^k = \langle \widehat{A}_k{}^k, \phi F_i \rangle$$
$$= \langle \overrightarrow{H}, \phi F_i \rangle + g^{kl} \langle \widehat{T}(\phi F_i, F_k), F_l \rangle \,.$$

Combining this with (22) we finally get

$$\langle \overrightarrow{\widehat{H}}, \phi F_i \rangle = \langle \overrightarrow{H}, \phi F_i \rangle + g^{kl} \langle \widehat{T}(\phi F_i, F_k), F_l \rangle + g^{kl} \langle \widehat{T}(F_i, F_k), \phi F_l \rangle$$
$$= \widehat{H}_i - \nabla^k \omega_{ki} - \omega_i{}^m \widehat{s}_{mk}{}^k - \omega^{mk} \widehat{s}_{mki} \,.$$

This proves the lemma. □

4.2. Proof of main theorems. We recall Theorem 1:

THEOREM 1. *Suppose (N, ω, g, J) is an almost Kähler manifold and $\widehat{\nabla}$ is a complex and metric connection on TN. For any initial smooth compact almost Lagrangian submanifold M_0, there exists a maximal time $T \in (0, \infty]$ so that the generalized mean curvature flow (1) exists smoothly on $[0, T)$ in the class of almost Lagrangian submanifolds.*

Proof. It suffices to prove that the operator $E[F] := \overrightarrow{\widehat{H}}[F]$ has no non-trivial degeneracies, i.e. we have to show that it is elliptic in the normal directions. By Definition 3, the generalized mean curvature vector differs from the classical mean curvature vector only by terms of lower order so that the symbol of our operator is the same as for the mean curvature flow and short-time existence follows. □

Next we recall Theorem 2:

THEOREM 2. *Suppose (N, ω, g, J) is an almost Kähler manifold and suppose $\widehat{\nabla}$ is a complex and metric connection that satisfies the Einstein condition. Then the generalized mean curvature flow (1) w.r.t. $\widehat{\nabla}$ preserves the Lagrangian condition.*

Proof. From Cartan's formula we know

$$\frac{\partial}{\partial t} F^* \omega = d \left(F^* \left(\frac{\partial F}{\partial t} \lrcorner \omega \right) \right) = d(F^*(\overrightarrow{\widehat{H}} \lrcorner \omega)) \,.$$

We fix some time interval $[0, t_0]$, $0 < t_0 < T$. From (15), the Einstein property of $\widehat{\nabla}$, and (18) we obtain

$$\frac{\partial}{\partial t} F^* \omega = -d\widehat{H} + dd^\dagger(F^* \omega) + C_1 \lrcorner F^* \omega + C_2 \lrcorner \nabla(F^* \omega)$$
$$= dd^\dagger(F^* \omega) + C_3 \lrcorner F^* \omega + C_2 \lrcorner \nabla(F^* \omega)$$

for smooth tensor fields C_1, C_2, C_3, where $d^\dagger(F^* \omega)$ is the 1-form $\nabla^k \omega_{ki} dx^i$. Since ω (and $F^* \omega$) is closed we have

$$\Delta(F^* \omega) = dd^\dagger(F^* \omega) + C_4 \lrcorner F^* \omega \,,$$

where C_4 depends on the Riemannian curvature of M. Combining the last identities we deduce

$$\frac{\partial}{\partial t} F^* \omega = \Delta(F^* \omega) + C_5 \lrcorner F^* \omega + C_2 \lrcorner \nabla(F^* \omega)$$

for smooth tensor fields C_2, C_5.

Together with Cauchy-Schwarz' inequality, we thus obtain an estimate of the form

$$\frac{\partial}{\partial t} |F^* \omega|^2 \le \Delta |F^* \omega|^2 + c|F^* \omega|^2$$

for all $t \in [0, t_0]$ and some constant c depending on t_0.

Thus the growth rate of $|F^*\omega|^2$ on $[0, t_0]$ is at most exponential, i.e.

$$\sup_{p \in M} |F^*\omega|^2(p, t) \leq \sup_{p \in M} |F^*\omega|^2(p, 0) e^{ct}, \forall t \in [0, t_0].$$

However, $|F^*\omega|^2(p, 0)$ is zero for all $p \in M$ as M_0 is Lagrangian. Since t_0 is arbitrary, the theorem follows. $\square$

The definition of generalized Lagrangian mean curvature flows only requires the existence of an "almost Einstein connection" on N in the sense that $\widehat{\rho} - f\omega$ is dd^c exact for some smooth function f. In fact, the latter condition implies the existence of an actual "Einstein connection" on N, see Example 2 in the next section.

5. Examples of almost Kähler manifolds with Einstein connections.

EXAMPLE 1. *If (N, ω, g, J) is Kähler-Einstein, then we can choose $\widehat{\nabla}$ to be the Levi-Civita connection ∇ of g and $f = K/2n$, where K is the scalar curvature of N and n is the complex dimension of N. In this case we recover the classical Lagrangian mean curvature flow.*

EXAMPLE 2. *Let (N, ω, g, J) be one of the Kähler manifolds that are almost Einstein considered in [B]. In this case the Levi-Civita connection ∇ is metric and complex and its Ricci form ρ satisfies*

$$\rho = \lambda\omega + ndd^c\psi$$

for some constant λ and some smooth function ψ. According to Lemma 1 the connections

$$\widehat{\nabla} := \nabla + \sigma \otimes J$$

for any $\sigma \in \Omega^1(N)$ are also complex and metric and the curvature tensors $R, \widehat{R}$ of ∇ resp. $\widehat{\nabla}$ are related by $\widehat{R} = R + d\sigma \otimes J$. For the Ricci forms we get

$$\widehat{\rho} = \rho - nd\sigma.$$

Therefore, if we choose $\sigma := d^c\psi$, then the Ricci form $\widehat{\rho}$ is conformal to ω (with $f = \lambda$). Since the torsion of $\widehat{\nabla}$ is given by $\widehat{T} = \sigma \wedge J$, Proposition 3 shows that the mean curvature vector $\overrightarrow{\widehat{H}}$ w.r.t. the connection $\widehat{\nabla}$ coincides with the mean curvature vector considered in [B] and we obtain the same flow. Since we do not need the integrability of J, we note that the same trick works for almost Kähler manifolds that are almost Einstein, so that the Kähler condition in [B] is actually not needed.

EXAMPLE 3. *Let $N := T^*M$ be the cotangent bundle of a Riemannian manifold (M, g). The cotangent bundle carries a natural almost Kähler structure and a complex and metric connection $\widehat{\nabla}$ completely determined by the Levi-Civita connection ∇ of (M, g) (see [VA] and [YI], §IV.6). In this case the Ricci form of the connection $\widehat{\nabla}$ even vanishes (i.e. $f = 0$) and our theorem applies. We will treat this example in great detail in a forthcoming paper.*

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