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A RESULT ON RICCI CURVATURE AND THE SECOND BETTI NUMBER*

JIANMING WAN†

Abstract. We prove that the second Betti number of a compact Riemannian manifold vanishes under certain Ricci curved restriction. As consequences we obtain an interesting curved restriction for compact Kähler-Einstein manifolds and a homology sphere theorem in dim $= 4, 5$.

Key words. Ricci curvature, Betti number.

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1. Introduction. The study of relation between curvature and topology is the central topic in Riemannian geometry. One of the strong tool is Bochner technique. It plays a very important role in understanding relation between curvature and Betti numbers. The first result in this field is Bochner's classical result (c.f. [6])

THEOREM 1.1. *(Bochner 1946) Let M be a compact Riemannian manifold with Ricci curvature $Ric_M > 0$. Then the first Betti number $b_1(M) = 0$.*

Berger investigated that in what case the second Betti number vanishes. He proved the following (c.f. [1], also see [2] theorem 2.8)

THEOREM 1.2. *(Berger) Let M be a compact Riemannian manifold of dimension $n \geq 5$. Suppose that n is odd and the sectional curvature satisfies that $\frac{n-3}{4n-9} \leq K_M < 1$. Then the second Betti number $b_2(M) = 0$.*

Consider a different curvature condition, Micallef and Wang proved (c.f. [4], also see [2] theorem 2.7)

THEOREM 1.3. *(Micallef-Wang) Let M be a compact Riemannian manifold of dimension $n \geq 4$. Suppose that n is even and M has positive isotropic curvature. Then the second Betti number $b_2(M) = 0$.*

Here positive isotropic curvature means, for any four othonormal vectors $e_1, e_2, e_3, e_4 \in T_pM$, the curvature tensor satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2|R_{1234}|.$$

Recall that the Rauch-Berger-Klingenberg's sphere theorem (c.f. [1]) states that a simple connected compact Riemannian manifold is homeomorphic to a sphere if the sectional curvatures lie in $(\frac{1}{4}, 1]$. A generalization of sphere theorem (dues to Micallef-Moore c.f. [5]) says that a compact simply connected Riemannian manifold with positive isotropic curvature is a homotopy sphere. Hence with the help of Poincare conjecture it is homeomorphic to a sphere. From the two theorems we know that theorems 1.2 and 1.3 can not cover too many examples.

In this note we shall use Ricci curvature to give a relaxedly sufficient condition for the second Betti number vanishing. Our main result is

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†Department of Mathematics, Northwest University, Xi'an 710127, China (wanj_m@aliyun.com; wan@nwu.edu.cn). The research is supported by the National Natural Science Foundation of China N0.11301416.

THEOREM 1.4. *Let M be a compact Riemannian manifold. The dimension $\dim(M) = 2m$ or $2m+1$. Let $\bar{k}$ (resp. $\underline{k}$) be the maximal (resp. minimal) sectional curvature of M. If the Ricci curvature of M satisfies that*

$$Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}), \tag{1.1}$$

then the second Betti number $b_2(M) = 0$.

Particularly, if M is a compact Riemannian manifold with nonnegative sectional curvature, then the second Betti number vanishes provided

$$Ric_M > \frac{2m+1}{3}\bar{k}. \tag{1.2}$$

Note that there is no dimensional restriction in theorem 1.4.

Any compact Kähler manifold does not satisfy (1.1) since it has $b_2 \geq 1$.

The condition 1.1 is a Ricci pinching condition. We mention that several other Ricci pinching type theorems obtained by Gu and Xu (c.f. [3] [7],).

As an immediate consequence, we obtain a curvature restriction for special Einstein manifolds.

COROLLARY 1.5. *Let M be a compact Einstein manifold with nonzero second Betti number. Then the Ricci curvature satisfies*

$$Ric \leq \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}). \tag{1.3}$$

In addition, if the sectional curvature is nonnegative, one must have

$$Ric \leq \frac{2m+1}{3}\bar{k}. \tag{1.4}$$

Particularly (1.3) holds for any compact Kähler-Einstein manifold.

REMARK 1.6. 1) The condition (1.1) implies that the maximal sectional curvature $\bar{k} > 0$: If $\bar{k} \leq 0$, then

$$\bar{k} \geq Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}).$$

We get $\bar{k} < \underline{k}$. This is a contradiction.

2) Since $\bar{k} > 0$, of course (1.1) implies $Ric_M > 0$.

3) If the minimal sectional curvature $\underline{k} < 0$. Since $\bar{k} > 0$. If $\dim(M) = 2m+1$, from

$$2m\bar{k} \geq Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}),$$

one has

$$\bar{k} > \frac{2m-2}{4m-1}|\underline{k}|.$$

Similarly

$$\bar{k} > \frac{1}{2}|\underline{k}|$$

provided $\dim(M) = 2m$.

We use theorem 1.4 to test some simple examples.

EXAMPLE 1.7. 1) The space form S^n, $\bar{k} = \underline{k} = 1$, $Ric = n-1 = \bar{k}$ for $n = 2$ and $Ric = n - 1 > \bar{k}$ for $n \neq 2$, $b_2(S^2) = 1$ and $b_2(S^n) = 0$ for $n \neq 2$.

2) $S^2 \times S^2$ with product metric, $\bar{k} = 1, \underline{k} = 0$, $Ric = 1 < \bar{k} + \frac{2n-2}{3}(\bar{k} - \underline{k})$, $b_2(S^2 \times S^2) = 2$.

3) $S^m \times S^m, m > 4$ with product metric, $\bar{k} = 1, \underline{k} = 0$, $Ric = m - 1 > \frac{2m+1}{3}\bar{k}$, $b_2 = 0$.

4) $\mathbb{CP}^n$ with Fubini-Study metric, $\bar{k} = 4, \underline{k} = 1$, $Ric = 2n + 2 = \bar{k} + \frac{2n-2}{3}(\bar{k} - \underline{k})$, $b_2(\mathbb{CP}^n) = 1$.

From the examples we know that the inequality (1.1) is sharp.

The proof of theorem 1.4 is also based on Bochner technique. But comparing with Berger and Micallef-Wang's results, we consider a different side. This allows us get a uniform result (without dimensional restriction).

2. Proof of the theorem.

2.1. Bochner formula. Let M be a compact Riemannian manifold. Let

$$\Delta = d\delta + \delta d$$

be the Hodge-Laplacian, where d is the exterior differentiation and δ is the adjoint to d.

Let $\varphi \in \Omega^k(M)$ be a smooth k-form. Then we have the well-known Weitzenböck formula (c.f. [6])

$$\Delta\varphi = \sum_i \nabla^2_{v_i v_i}\varphi - \sum_{i,j} \omega^i \wedge i(v_j) R_{v_i v_j}\varphi, \tag{2.1}$$

here $\nabla^2_{XY} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$ and $R_{XY} = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$. The $\{v_i, 1 \leq i \leq n\}$ are the local orthonormal vector fields and $\{\omega_i, 1 \leq i \leq n\}$ are the duality.

A k-form φ is called harmonic if $\Delta\varphi = 0$.

The famous *Hodge theorem* states that the de Rham cohomology $H^k_{dR}(M)$ is isomorphic to the space spanned by k-harmonic forms.

Let $\varphi = \sum_{i,j} \varphi_{ij}\omega^i \wedge \omega^j$ be a harmonic 2-form. By (2.1), under the normal frame we can get (c.f. [2] or [1])

$$\Delta\varphi_{ij} = \sum_k (Ric_{ik}\varphi_{kj} + Ric_{jk}\varphi_{ik}) - 2\sum_{k,l} R_{ikjl}\varphi_{kl}, \tag{2.2}$$

where $R_{ijkl} = \langle R(v_i, v_j)v_k, v_l\rangle$ is the curvature tensor and $Ric_{ij} = \sum_k \langle R(v_k, v_i)v_k, v_j\rangle$ is the Ricci tensor.

So we have

$$\begin{aligned}
\Delta|\varphi|^2 &= 2\sum_{i,j}\varphi_{ij}\Delta\varphi_{ij} + 2\sum_{i,j}\sum_k (v_k\varphi_{ij})^2 \\
&\geq 2\sum_{i,j}\varphi_{ij}\Delta\varphi_{ij} \\
&\triangleq 2F(\varphi).
\end{aligned}$$

Note that by (2.1) one has the global form of above formula

$$0 = -\langle \Delta\varphi, \varphi\rangle = \sum_i |\nabla_{v_i}\varphi|^2 + \langle \sum_{i,j} \omega^i \wedge i(v_j) R_{v_i v_j}\varphi, \varphi\rangle - \frac{1}{2}\Delta|\varphi|^2.$$

The $F(\varphi)$ is just the curvature term $\langle \sum_{i,j} \omega^i \wedge i(v_j) R_{v_i v_j}\varphi, \varphi\rangle$.

2.2. Proof of Theorem 1.4. By Hodge theorem, we only need to show that every harmonic 2-form vanishes.

Case 1: Assume $\dim(M) = 2m$. For any $p \in M$, we can choose an orthonormal basis $\{v_1, w_1, ..., v_m, w_m\}$ of T_pM such that $\varphi(p) = \sum_\alpha \lambda_\alpha v^*_\alpha \wedge w^*_\alpha$ (for instance c.f. [1] or [2]). Here $\{v^*_\alpha, w^*_\alpha\}$ is the dual basis. Then

$$F(\varphi) = \sum_{\alpha=1}^m \lambda_\alpha^2 [Ric(v_\alpha, v_\alpha) + Ric(w_\alpha, w_\alpha)] - 2\sum_{\alpha,\beta=1}^m \lambda_\alpha\lambda_\beta R(v_\alpha, w_\alpha, v_\beta, w_\beta) \tag{2.3}$$

The term

$$\begin{aligned}
&-2\sum_{\alpha,\beta=1}^m \lambda_\alpha\lambda_\beta R(v_\alpha, w_\alpha, v_\beta, w_\beta)\\
&= -2\sum_{\alpha\neq\beta} \lambda_\alpha \cdot \lambda_\beta \cdot R(v_\alpha, w_\alpha, v_\beta, w_\beta) - 2\sum_{\alpha=1}^m \lambda_\alpha^2 R(v_\alpha, w_\alpha, v_\alpha, w_\alpha)\\
&\geq -\frac{4}{3}(\bar{k} - \underline{k}) \sum_{\alpha\neq\beta} |\lambda_\alpha| \cdot |\lambda_\beta| - 2\bar{k}\sum_{\alpha=1}^m \lambda_\alpha^2\\
&\geq -\frac{2}{3}(\bar{k} - \underline{k}) \sum_{\alpha\neq\beta} (\lambda_\alpha^2 + \lambda_\beta^2) - 2\bar{k}|\varphi|^2\\
&= -\frac{2}{3}(\bar{k} - \underline{k})(2m-2)|\varphi|^2 - 2\bar{k}|\varphi|^2\\
&= -2[\bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k})]|\varphi|^2.
\end{aligned}$$

The first " $\geq$ " follows from Berger's inequality (c.f. [1]): For any orthonormal 4-frames $\{e_1, e_2, e_3, e_4\}$, one has

$$|R(e_1, e_2, e_3, e_4)| \leq \frac{2}{3}(\bar{k} - \underline{k}).$$

On the other hand, by the condition (1.1) we have

$$\sum_{\alpha=1}^m \lambda_\alpha^2 [Ric(v_\alpha, v_\alpha) + Ric(w_\alpha, w_\alpha)] \geq 2[\bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k})]|\varphi|^2,$$

the equality holds if and only if $\varphi(p) = 0$.

This leads to

$$F(\varphi) \geq 0$$

with equality if and only if $\varphi(p) = 0$. Since

$$\int_M F(\varphi) \leq \frac{1}{4}\int_M \Delta|\varphi|^2 = 0,$$

we get

$$F(\varphi) \equiv 0.$$

Thus the harmonic 2-form $\varphi \equiv 0$.

Case 2: If $\dim(M) = 2m+1$. For any $p \in M$, we also can choose an orthonormal basis $\{u, v_1, w_1, ..., v_m, w_m\}$ of T_pM such that $\varphi(p) = \sum_\alpha \lambda_\alpha v_\alpha^* \wedge w_\alpha^*$ (c.f. [1] or [2]). We also have

$$F(\varphi) = \sum_{\alpha=1}^{m} \lambda_\alpha^2 [Ric(v_\alpha, v_\alpha) + Ric(w_\alpha, w_\alpha)] - 2 \sum_{\alpha,\beta=1}^{m} \lambda_\alpha \lambda_\beta R(v_\alpha, w_\alpha, v_\beta, w_\beta).$$

Thus the argument is same to the even dimensional case.

This completes the proof of the theorem.

3. Sphere theorem in $\dim 4$ and 5.

THEOREM 3.1. *Let M be a compact Riemannian manifold.* $\dim M = 4$ *or 5. If*

$$Ric_M > \frac{5\bar{k} - 2\underline{k}}{3},$$

then M is a real homology sphere, i.e. $b_i(M) = 0$ for $1 \leq i \leq \dim M - 1$.

Proof. Since $Ric_M > 0$, from theorem 1.1 we know that $b_1(M) = 0$. Theorem 1.4 implies that $b_2(M) = 0$. With the help of Poincare duality, we obtain the theorem. □

Finally we metion a differential sphere theorem for Ricci curvature obtained by Gu and Xu (c.f. [3] theorem D).

THEOREM 3.2. *Let M be a simple conncted compact Riemannian n-manifold. If*

$$Ric_M > (n - \frac{11}{5})\bar{k},$$

then M is diffeomorphic to S^n.

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