ASIAN J. MATH.
Vol. 22, No. 2, pp. 0269–0284, April 2018

004

ON HIGGS BUNDLES OVER SHIMURA VARIETIES OF BALL QUOTIENT TYPE*

KE CHEN†, XIN LU‡, SHENG-LI TAN§, AND KANG ZUO¶

Dedicated to Prof. Ngaiming Mok on the occasion of his sixtieth birthday

Abstract. We prove the generic exclusion of certain Shimura varieties of unitary and orthogonal types from the Torelli locus. The proof relies on a slope inequality on surface fibration due to G. Xiao, and the main result implies that certain Shimura varieties only meet the Torelli locus in dimension zero.

Key words. Coleman-Oort conjecture, Torelli locus, Shimura varieties, slope inequality.

Mathematics Subject Classification. Primary 11G15, 14G35, 14H40; Secondary 14D07, 14K22.

1. Introduction. The Coleman-Oort conjecture, cf. [8], predicts that the open Torelli locus $\mathcal{T}_g^\circ$ in the Siegel modular variety $\mathcal{A}_g$ contains at most finitely many CM points when g is sufficiently large. The behavior of CM points under Zariski closure is affirmed by the André-Oort conjecture: such closures have to be finite unions of Shimura subvarieties. For CM points in $\mathcal{A}_g$ this is recently proved in [11], and this leads to the following equivalent form of the Coleman-Oort conjecture which is of main concern in the present paper: when g is large enough, $\mathcal{T}_g^\circ$ does not contain any Zariski open subvariety of any Shimura subvariety in $\mathcal{A}_g$ of dimension > 0.

In [3] Hain has established the conjecture for a large class of Shimura subvarieties in $\mathcal{A}_g$ that do not contain locally symmetric divisors. In particular, it holds for Shimura subvarieties uniformized by Hermitian symmetric domains of rank at least 2. His proof makes use of rigidity property of mapping class groups, namely for an arithmetic group Γ coming from a simple $\mathbb{Q}$-group of $\mathbb{R}$-rank at least 2, any homomorphism from Γ to the mapping class group $\Gamma_{g,r}^n$ is of finite image.

Note that the phenomenon of rigidity in [3] is also related to the metric rigidity property studied by Ngaiming Mok in [7]:

EXAMPLE 1.1. The idea of the metric rigidity approach can be illustrated through the case of Hilbert modular varieties, which we refer to [4] for necessary backgrounds. Assume that $g \geq 5$ and let $M \subset \mathcal{A}_g$ be the Hilbert modular subvariety parametrizing abelian varieties of dimension g with real multiplication by O_F; here F is a totally real number field of degree g and O_F is an order in F. Consider an extremal situation where M is not only contained in $\mathcal{T}_g^\circ$, but actually lifts to $i : M \hookrightarrow \mathcal{M}_g$, giving rise to

*Received October 21, 2016; accepted for publication June 2, 2017. This work is supported by SFB/Transregio 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG (Deutsche Forschungsgemeinschaft), and by National Key Basic Research Program of China (Grant No. 2013CB834202) and National Natural Science Foundation of China (Grant No. 1171203, No. 11231003, No. 11301495).

†Department of Mathematics, Nanjing University, Hankou Road 22, Nanjing 210093, P. R. of China (kechen@ustc.edu.cn).

‡School of Mathematics, East China Normal University, Shanghai Key Laboratory of PMMP, Donchuan Road 500, Shanghai 200241, P. R. of China (x.lu@uni-mainz.de)

§Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Dongchuan Road 500, Shanghai 200241, P. R. of China (sltan@math.ecnu.edu.cn).

¶Institut für Mathematik, Universität Mainz, Mainz, 55099, Germany (zuok@uni-mainz.de).

a surjective pull-back $i^*\Omega^1_{\mathcal{M}_g} \to \Omega^1_M$. Note that Ω^1_M is merely semi-positive, dual to the proper semi-negativity of Tan_M studied in [7], and it admits non-trivial quotients which are NOT big, which is absurd because $\Omega^1_{\mathcal{M}_g}$, and thus $i^*\Omega^1_{\mathcal{M}_g}$ as well, is already big.

On the other hand, for Shimura varieties uniformized by simple Hermtian symmetric domains of rank 1, such as those associated to $\mathbf{SU}(n,1)$ and $\mathbf{Spin}(N,2)$, Hodge-theoretic techniques provide complements when the rigidity property fails. In our previous work [1], we have proved the Coleman-Oort conjecture for a class of Shimura varieties of these types. The starting point is a numerical property of semi-stable fibration of surfaces over curves, which is translated, via the Simpson correspondence, into constraints on the fundamental group representations for the Shimura varieties of interest, and one concludes by Satake's classification of rational symplectic representations. More precisely, the following inequality due to G. Xiao will play a crucial role in our study:

Theorem 1.2 (Xiao's inequality). *Let $f : S \to B$ be a non-isotrovial fibration of a smooth projective algebraic surface S over a smooth projective algebraic curve B. Assume that f is generically smooth and its fibers are semi-stable curves of genus $g \geq 2$. Then holds the inequality*

$$12 \deg f_*\omega_{S/B} \geq (2g - 2 + \operatorname{rank} A_{\max})\mu_{\max}$$

where $\mu_{\max}$ is the maximum of slopes of vector subbundles of $\omega_{S/B}$, and $A_{\max} \subset f_*\omega_{S/B}$ is the maximal subbundle of $f_*\omega_{S/B}$ of slope $\mu_{\max}$.

Remark 1.3. (1) The inequality above is slightly different from the original version of Xiao [12], where he obtains $12 \deg f_*\omega_{S/B} \geq (2g-2)\mu_{\max}$ and thus

$$\frac{\operatorname{rank} F^{1,0}}{g} \leq \frac{5}{6} + \frac{1}{6g}.$$

The proof of the version in Theorem 1.2 is exactly the same as in [12], which is not reproduced in this paper.

(2) In [1] we have made use of the following refined form

$$12 \deg f_*\omega_{S/B} \geq \big(4g - 4 - 2\operatorname{rank} F^{1,0}\big)\mu_{\max}, \quad \text{and} \quad \frac{\operatorname{rank} F^{1,0}}{g} \leq \frac{4}{5} + \frac{2}{5g},$$

where a decomposition $f_*\omega_{S/B} = A^{1,0} \oplus F^{1,0}$ is assumed, with $A^{1,0}$ semi-stable and ample while $F^{1,0}$ is the maximal flat part. Such a decomposition holds when B is constructed out of a Shimura curve contained generically in $\mathcal{T}_g$ due to properties of Higgs bundles on Shimura curves: only one single non-zero slope appears in the Harder-Narasimhan filtration of $f_*\omega_{S/B}$. The Shimura varieties studied in [1] do contain Shimura curves, while in the more general case of the present paper we can only resort to the inequality as in Theorem 1.2 because several different slopes might be involved.

Theorem 1.4 (Shimura varieties of $\mathbf{SU}(n,1)$-type). *Let $M \subset \mathcal{A}_V$ be a Shimura subvariety of $\mathbf{SU}(n,1)$-type defined over a totally real field F of degree d, such that the corresponding symplectic representation is primary of type Λ_m for some integer*

$m \in [1, n]$ of multiplicity r. Then M is NOT contained generically in $\mathcal{T}_g$ as long as the following inequality holds:

$$\frac{n+m-1}{n}\left(\frac{n+1}{m}\cdot d+\frac{1}{2}-\frac{1}{r\binom{n}{m-1}}\right)>12.$$

Remark 1.5. The theorem above actually implies that the Shimura variety M under consideration only meets the open Torelli locus $\mathcal{T}_g^\circ$ at finitely many points: otherwise M contains a curve C which is generically contained in $\mathcal{T}_g$, and the arguments in the proof (cf. Section 3) produces an inequality contradicting Xiao's estimation. Our naive bound on the flat part also treats the similar phenomenon for more general Shimura varieties, cf. Proposition 2.4 and Corollary 2.5, with a slight different inequality involving less parameters.

We also apply the theorem above to the Coleman-Oort conjecture for some Shimura varieties of orthogonal type containing Shimura varieties of unitary type. In fact, if $h : W \times W \to \mathbb{C}$ is an Hermitian form over $\mathbb{C}$ of signature $(n, 1)$, then its real part is a quadratic form of signature $(2n, 2)$, and one obtains a natural equivariant embedding of the corresponding Hermitian symmetric domains. Adding suitable arithmetic constraints we expand this example into the following theorem:

Theorem 1.6 (Shimura varieties of orthogonal types). *Let $M' \subset \mathcal{A}_V$ be a Shimura subvariety of $\mathbf{Spin}(N, 2)$-type given by some Shimura datum $(\mathbf{G}', X'; X'^+)$, namely associated to some quadratic space (W, q) over a totally real field F of degree d, of signature*

- *$(N, 2)$ along one fixed real embedding $\sigma = \sigma_1 : F \hookrightarrow \mathbb{R}$;*
- *definite along the other embeddings $\sigma_2, \cdots, \sigma_d : F \hookrightarrow \mathbb{R}$.*

Assume that M' contains a Shimura subvariety of $\mathbf{SU}(n, 1)$-type associated to some Hermitian space (H, h) over some CM quadratic extension E of F, such that the signature of h is

- *$(n, 1)$ along σ;*
- *definite along the other embeddings $\sigma_2, \cdots, \sigma_d$;*

which fits into an orthogonal direct sum decomposition $W = U \oplus \mathrm{Res}_{E/F} H$ with U some F-subspace of signature $(N - 2n, 0)$ along σ. If $N > 2n$ and the inclusion $M' \hookrightarrow \mathcal{A}_V$ is defined by a symplectic representation of primary type, then M' is NOT contained generically in $\mathcal{T}_g$ as long as $d > 3 + \frac{1}{m\cdot 2^{\lfloor (N+1)/2\rfloor}} - \frac{1}{2^{n+2}}$, where m is the multiplicity of the spinor representation in $W \otimes_{F,\sigma} \mathbb{R}$ for the group $\mathbf{Spin}(N, 2)$, the only non-compact factor in $\mathbf{G}'^{\mathrm{der}}(\mathbb{R})$.

In particular the inequality holds whenever $d \geq 4$; if $N > 2n + 4$, then it holds for $d \geq 3$.

This theorem deals with the case $N > 2n$, while the case $N = 2n$ is treated in detail taking care of the parity of n and the primary type of symplectic representations involved, cf. Proposition 4.4.

Similar to [1], the criteria obtained involve certain representation-theoretic parameters describing the symplectic representations defining Shimura subvarieties, and they are NOT pure bounds on the genus g.

The paper is organized as follows. In Section 2 we collect preliminaries on Shimura subvarieties in $\mathcal{A}_g$ and prove a naive bound on the compact factors in the Mumford-Tate groups for Shimura subvarieties contained generically in the Torelli locus. Section

3 computes the slopes of certain Higgs bundles on Shimura varieties of $\mathbf{SU}(n,1)$-type and proves Theorem 1.4. Finally in Section 4 we apply the results in Section 3 to a class of Shimura varieties of orthogonal types.

2. A naive bound for flat Higgs subbundles. In this paper Shimura varieties and Shimura subvarieties in $\mathcal{A}_g$ are connected algebraic varieties over $\mathbb{C}$, following the definitions given in [1, 5]:

Definition 2.1 (Shimura varieties). A (connected) Shimura datum is of the form $(\mathbf{G}, X; X^+)$ consisting of

- $(\mathbf{G}, X)$ a (pure) Shimura datum in the sense of [2];
- X^+ is a connected component of X.

Note that X is a homogeneous space under $\mathbf{G}(\mathbb{R})$ of homomorphisms $\mathbb{S} \to \mathbf{G}_\mathbb{R}$ subject to certain Hodge-theoretic constraints, with $\mathbb{S}$ the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_\mathrm{m}$; X^+ is homogeneous under $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$, and is an Hermitian symmetric domain.

Take $\Gamma \subset \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ a congruence subgroup, we have the (connected) Shimura variety $M = \Gamma\backslash X^+$, where Γ acts on X^+ through its image in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. The theorem of Baily-Borel compactification affirms that M is a normal quasi-projective algebraic variety over $\mathbb{C}$, and we always assume that Γ is torsion-free, so that M is smooth and its fundamental group is identified with Γ. Note that the general theory of Shimura varieties in the adelic setting affirms that a connected component of a Shimura variety at finite level is of the form $\Gamma'\backslash X^+$ where Γ' is a congruence subgroup in $\mathbf{G}(\mathbb{R})_+$, whose underlying complex algebraic variety is the same as $\Gamma\backslash X^+$ with $\Gamma = \Gamma' \cap \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$, hence our simplified condition using $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$.

The canonical projection $\wp = \wp_\Gamma : X^+ \to \Gamma\backslash X^+, x \mapsto \Gamma x$ is called the uniformization map. A Shimura subvariety in M is given as $M' = \wp(X'^+)$ for some connected Shimura subdatum $(\mathbf{G}', X'; X'^+)$, namely $(\mathbf{G}', X')$ is a Shimura subdatum of $(\mathbf{G}, X)$ in the sense of [2] and X'^+ is a connected component of X' contained in X^+. It is known that M' is a closed subvariety in M, and its fundamental group is isomorphic to $\Gamma \cap \mathbf{G}'^{\mathrm{der}}(\mathbb{R})^+$.

Example 2.2 (Siegel modular variety). Fix V a rational symplectic space of dimension $2g$, we have the Shimura datum $(\mathbf{GSp}_V, \mathcal{H}_V; \mathcal{H}_V^+)$ where $\mathcal{H}_V^+$ is the Siegel upper half space of genus g. Usually we assume that V comes from the standard symplectic $\mathbb{Z}$-module $V_\mathbb{Z} \simeq \mathbb{Z}^{2g}$ of discriminant 1, and thus a suitable choice of a congruence subgroup $\Gamma \subset \mathbf{GSp}_V(\mathbb{Q})$ defines Siegel modular varieties $\mathcal{A}_g = \mathcal{A}_V := \Gamma\backslash\mathcal{H}_V^+$ parameterizing principally polarized abelian variaties of dimension g with level-Γ structures.

We mainly consider Shimura subvarieties in $\mathcal{A}_V$. As is explained in [1], a Shimura subdatum $(\mathbf{G}, X; X^+)$ defining a Shimura subvariety $M \subset \mathcal{A}_V$ gives rise to a rational symplectic representation $\mathbf{G} \to \mathbf{GSp}_V$ satisfying Satake's condition (H2) in the sense of [10]. We always assume that the level structure Γ is suitably chosen so that the inclusion $M \hookrightarrow \mathcal{A}_g$ extends to their smooth toroidal compactifications $\overline{M} \hookrightarrow \overline{\mathcal{A}}_V$, which joins to M resp. to $\mathcal{A}_V$ finitely many boundary divisors.

We write $\mathcal{T}_g^\circ$ for the schematic image of

$$j : \mathcal{M}_g \to \mathcal{A}_g = \mathcal{A}_V,\ [C] \mapsto [\mathrm{Jac}(C)]$$

called the open Torelli locus, and $\mathcal{T}_g$ for its closure, called the Torelli locus. The slope inequality of Xiao is mainly applied to the following situation:

Proposition 2.3 (Higgs bundles for surface fibration). *Let C be a closed curve* *in $\overline{\mathcal{A}}_g$ contained generically in the Torelli locus, namely $C^\circ := C \cap \mathcal{T}_g^\circ$ is open in C. Let B° be the normalization of the preimage of C° in $\mathcal{M}_g$, giving rise to a family of curve $f^\circ : S^\circ \to B^\circ$ which is compactified into a surface fibration $f : S \to B$ with semi-stable fibers of genus g. Write $i : B \to C$ for the induced morphism from B into C, and $\mathcal{V}_C^{1,0}$ for the $(1,0)$-part of the logarithmic Higgs bundle $\mathcal{V}_C$ on C deduced from the variation of Hodge structure on $\mathcal{A}_g$ defined by the moduli problem. Then holds the isomorphism*

$$f_*\omega_{S/B} \simeq i^*\mathcal{V}_C^{1,0}$$

where $\omega_{S/B}$ is the relative dualizing sheaf for f. In particular we have the decomposition $f_*\omega_{S/B} = \mathcal{F}_B \oplus \mathcal{A}_B$, where $\mathcal{F}_B$ is the semi-stable subbundle of slope 0 corresponding to the Higgs subbundle in $\mathcal{V}_C$ given by the maximal unitary subrepresentation in of the $\mathbb{C}$-linear representation of $\pi_1(C)$ associated to $\mathcal{V}_C$ using Simpson's correspondence.

In the rest of this section we derive a naive bound on the flat part in the canonical Higgs bundle associated to a Shimura variety of dimension > 0 generically contained in $\mathcal{T}_g$ using Xiao's inequality.

Proposition 2.4 (naive bound on the flat part). *Let $M \subset \mathcal{A}_g$ be a Shimura subvariety defined by a subdatum $(\mathbf{G}, X; X^+)$, such that the derived part of the $\mathbb{Q}$-group $\mathbf{G}$ admits an isomorphism $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{H}$ for some totally real field F and some semi-simple F-group $\mathbf{H}$, and the representation $\mathbf{G}^{\mathrm{der}} \hookrightarrow \mathbf{Sp}_V$ decomposes into*

$$V = V_0 \oplus \mathrm{Res}_{F/\mathbb{Q}}W$$

with V_0 a trivial subrepresentation and $\mathbf{H}$ acting on W preserving some symplectic F-form. Assume further that F is of degree d over $\mathbb{Q}$, such that:

- *along $r(> 0)$ real embeddings $\sigma_1, \cdots, \sigma_r : F \hookrightarrow \mathbb{R}$, the Lie group $\mathbf{H}(\mathbb{R}, \sigma_i)$ is compact;*
- *along the other $d - r(> 0)$ real embeddings $\sigma_{r+1}, \cdots, \sigma_d : F \hookrightarrow \mathbb{R}$, the Lie group $\mathbf{H}(\mathbb{R}, \sigma_j)$ is non-compact.*

Here $\mathbf{H}(\mathbb{R}, \sigma_i)$ is the evaluation of $\mathbf{H}$ at $\sigma_i : F \hookrightarrow \mathbb{R}$.

If M is contained generically in the Torelli locus, then holds the inequality

$$\frac{r}{d} \leq \frac{5}{6} + \frac{1}{6g}.$$

Proof. The $\mathbb{Q}$-linear representation $V = V_0 \oplus \mathrm{Res}_{F/\mathbb{Q}}W$ decomposes into

$$V \otimes_{\mathbb{Q}} \mathbb{R} = V_0 \otimes_{\mathbb{Q}} \mathbb{R} \oplus \Big(\bigoplus_{i=1}^{d} W \otimes_{F,\sigma_i} \mathbb{R}\Big)$$

after the base change $\mathbb{Q} \hookrightarrow \mathbb{R}$, with $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ acting on $V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ trivially, and $\mathbf{H}(\mathbb{R}, \sigma_i)$ acting on $V \otimes_{\mathbb{Q}} \mathbb{R}$ through the summand $W \otimes_{F,\sigma_i} \mathbb{R}$. Let

$$\mathcal{V} = \mathcal{V}_M = \mathcal{V}_0 \oplus \bigoplus_{i=1}^{d} \mathcal{W}_i$$

be the Higgs bundle decomposition on $\overline{M}$ according to the decomposition of

$$V \otimes_{\mathbb{Q}} \mathbb{C} = (V \otimes_{\mathbb{Q}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C},$$

where $\mathcal{V}_0$ corresponds to V_0 and $\mathcal{W}_i$ corresponds to $W \otimes_{F,\sigma} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ respectively. Then the Higgs subbundles $\mathcal{V}_0$ and $\mathcal{W}_i$ for $i = 1, \cdots, r$ are flat as the $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$-action factors through compact Lie groups. In particular, the portion of of flat part in $\mathcal{V}$ is at least

$$\frac{\operatorname{rank} \mathcal{F}^{1,0}}{\operatorname{rank} \mathcal{V}^{1,0}} = \frac{\operatorname{rank} \mathcal{F}}{\operatorname{rank} \mathcal{V}} = \frac{\operatorname{rank} \mathcal{V}_0 + \sum_{i=1}^{r} \operatorname{rank} \mathcal{W}_i}{\operatorname{rank} \mathcal{V}} \geq \frac{\sum_{i=1}^{r} \operatorname{rank} \mathcal{W}_i}{\sum_{i=1}^{d} \operatorname{rank} \mathcal{W}_i} = \frac{r}{d}$$

since $\operatorname{rank} \mathcal{W}_i = 2 \dim_F W$ for all i.

Assume that M is contained generically in $\mathcal{T}_g$. Then the linear system of the ample line bundle of top degree automorphic forms on M produces a curve C in $\overline{M}$ such that $C \cap M$ is open and dense in C, and up to Hecke translation we may assume further that C is contained generically in $\mathcal{T}_g$. Restricting $\mathcal{V}_M$ to C gives us the Higgs bundle $\mathcal{V}_C$, which flat part $\mathcal{F}_C$ contains the pull-back of $\mathcal{F}$ to C, hence

$$\frac{\operatorname{rank} \mathcal{F}_C}{\operatorname{rank} \mathcal{V}_C} \geq \frac{\operatorname{rank} \mathcal{F}}{\operatorname{rank} \mathcal{V}} = \frac{r}{d}$$

It remains to notice that the map $i : B \to C$ is finite, and we have

$$\operatorname{rank} \mathcal{F}_B = \operatorname{rank} \mathcal{F}_C$$

for the flat part $\mathcal{F}_B$ in $f_* \omega_{S/B}$, hence Xiao's inequality Theorem 1.2 would fail for B as long as $\frac{r}{d} > \frac{5}{6} + \frac{1}{6g}$, which is absurd. □

Corollary 2.5. *Let $M \subset \mathcal{A}_V = \mathcal{A}_g$ be a Shimura subvariety defined by $(\mathbf{G}, X; X^+)$ with $\mathbf{G}^{\mathrm{der}} \simeq \operatorname{Res}_{F/\mathbb{Q}} \mathbf{H}$ for some semi-simple group $\mathbf{H}$ over a totally real field F, which is compact along r real embeddings of F and non-compact along the remaining $d - r$ embeddings, d being the degree $[F : \mathbb{Q}]$. If $\frac{r}{d} > \frac{5}{6} + \frac{1}{6g}$, then M only meets $\mathcal{T}_g^\circ$ at finitely many points.*

The proof is similar and immediate: otherwise one finds a curve in $\overline{M}$ contradicting Xiao's inequality.

3. Shimura subvarieties of $\mathbf{SU}(n,1)$-type. We first briefly recall the set-up for Shimura subvarieties of $\mathbf{SU}(n,1)$-type, which is slightly more general than the one used in [1], as we no longer require the existence of an Hermitian form over a CM field.

Definition 3.1 (Shimura subvarieties of $\mathbf{SU}(n,1)$-type). Let $\mathcal{A}_V = \mathcal{A}_g$ be the Siegel modular variety given as in Example 2.2, assoociated to a symplectic $\mathbb{Q}$-vector space V of dimension $2g$. A Shimura subvariety of $\mathcal{A}_V$ is said to be of $\mathbf{SU}(n,1)$-type if it is defined by a Shimura subdatum $(\mathbf{G}, X; X^+) \subset (\mathbf{GSp}_V, \mathcal{H}_V; \mathcal{H}_V^+)$ such that $\mathbf{G}^{\mathrm{der}} \simeq \operatorname{Res}_{F/\mathbb{Q}} \mathbf{H}$ where

- F is a totally real number field of degree d and $\mathbf{H}$ is a simple F-group;
- among the real embeddings $\sigma_1, \cdots, \sigma_d$, we have $\mathbf{H}(\mathbb{R}, \sigma_1) \simeq \mathbf{SU}(n,1)$, and $\mathbf{H}(\mathbb{R}, \sigma_i) \simeq \mathbf{SU}(n+1)$ for $i = 2, \cdots, d$.

We call F the definition field of the datum (which is in general different from the reflex field, a notion not involved in the present paper).

Note that when $n \geq 2$, $X = X^+$ is connected. The inclusion

$$(\mathbf{G}, X; X^+) \hookrightarrow (\mathbf{GSp}_V, \mathcal{H}_V; \mathcal{H}_V^+)$$

gives rise to the representation of $\mathbf{G}$ on V, and from [10] we know that the restriction of this representation to $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{H}$ is decomposed into $V = V_0 \oplus V'$ where

- $\mathbf{G}^{\mathrm{der}}$ acts on V_0 trivially;
- the representation $\mathbf{G}^{\mathrm{der}} \to \mathbf{GL}_{V'}$ is the scalar restriction of an F-linear representation $\mathbf{H} \to \mathbf{GL}_{W,F}$ where W is an F-vector space carrying a symplectic F-form preserved by $\mathbf{H}$.

Let M be the Shimura subvariety in $\mathcal{A}_V$ of $\mathbf{SU}(n,1)$-type in the sense above, defined by $(\mathbf{G}, X; X)$, isomorphic to $\Gamma\backslash X$ for a torsion-free congruence subgroup $\Gamma \subset \mathbf{G}^{\mathrm{der}}(\mathbb{Q})^+$, and we write $\mathcal{V}$ for the Higgs bundle of the $\mathbb{C}$-PVHS on M given by the moduli problem. Similar to the situation in [1] which we have also used in Section 2 for the naive bound, there exists a decomposition of Higgs bundles

$$\mathcal{V} = \mathcal{V}_0 \oplus \sum_{i=1}^{d} \mathcal{W}_i$$

with $\mathcal{V}_0$ a trivial Higgs bundle corresponding to V_0, and $\mathcal{W}_i$ the Higgs bundle associated to $W \otimes_{F,\sigma_i} \mathbb{R}$, in which the only non-flat part is given by $\mathcal{W}_1$.

According to the Satake classification over $\mathbb{R}$ cf.[9], along $\sigma_1 : F \hookrightarrow \mathbb{R}$ the base change admits a decomposition

$$\sigma_1^* W = W \otimes_{F,\sigma_1} \mathbb{R} \simeq \bigoplus_{m=1}^{n-1} \Lambda_m^{\oplus r_m}$$

where $\Lambda_m = \wedge^m_{\mathbb{C}}\mathrm{Std}$ is the m-th exterior power of the standard representation of $\mathbf{SU}(n,1)$ on $\mathbb{C}^{n+1}$. Note that the action of $\mathbf{SU}(n,1)$ on Λ_m preserves an Hermitian form of signature $\left(\binom{n}{m}, \binom{n}{m-1}\right)$, and $\dim_{\mathbb{R}} \Lambda_m = 2\binom{n+1}{m}$.

In the sequel we assume for simplicity that V is primary of type Λ_m in the sense of [1, Subsection 5.2], namely $V_0 = 0$ and $\sigma_1^* W \simeq (\wedge^m \mathrm{Std})^{\oplus r}$ for some multiplicity $r \geq 1$.

We need the following fact to compute the Harder-Narasimhan filtration on certain curves, cf. [6, 7.12]:

Lemma 3.2 (decomposition of the canonical Higgs bundle). *Let $\mathcal{E}$ be the Higgs bundle on M associated to the $\mathbb{C}$-representation*

$$\Gamma \hookrightarrow \mathbf{G}^{\mathrm{der}}(\mathbb{R}) \to \mathbf{H}(\mathbb{R}, \sigma_1) \stackrel{\mathrm{Std}}{\to} \mathbf{GL}_{\mathbb{C}}(\mathbb{C}^{n+1} \otimes_{\mathbb{R}} \mathbb{C})$$

(through the unique non-compact factor of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$). Then the Hodge decomposition of $\mathcal{E}$ is of the form $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ with

- $\mathcal{E}^{1,0} \simeq \Omega^1_M \otimes \mathcal{L}^\vee \oplus \mathcal{L}$;
- $\mathcal{E}^{0,1} \simeq (\mathcal{E}^{1,0})^\vee = \mathrm{Tan}_M \otimes \mathcal{L} \oplus \mathcal{L}^\vee$.

Here $\mathcal{L}$ is a line bundle on M such that $\mathcal{L}^{\otimes(n+1)} = \omega_M (= \Omega^n_M)$, and the summands in $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$ are stable subbundles.

In particular, writing $\mathcal{S} = \Omega^1_M \otimes \mathcal{L}^\vee$ which gives $\mathcal{E}^{1,0} = \mathcal{S} \oplus \mathcal{L}$, we have

$$c_1(\mathcal{L}) = \frac{1}{n+1} c_1(\omega_M) = c_1\big(\Omega^1_M \otimes \mathcal{L}^\vee\big), \qquad c_1(\mathcal{E}^{1,0}) = 2c_1(\mathcal{L}) = \frac{2}{n+1} c_1(\omega_M).$$

For the m-th exterior power

$$\wedge^m \mathcal{E}^{1,0} = \wedge^m \mathcal{S} \oplus \wedge^{m-1}\mathcal{S} \otimes \mathcal{L},$$

we have

$$\begin{aligned} c_1(\wedge^m \mathcal{S}) &= \binom{n-1}{m-1} \cdot \frac{1}{n+1} c_1(\omega_M), \\ c_1(\wedge^{m-1}\mathcal{S} \otimes \mathcal{L}) &= \left(2\binom{n}{m-1} - \binom{n-1}{m-1}\right) \cdot \frac{1}{n+1} c_1(\omega_M), \\ c_1(\wedge^m \mathcal{E}^{1,0}) &= \binom{n}{m-1} \cdot \frac{2}{n+1} c_1(\omega_M). \end{aligned}$$

Similar properties hold for the canonical logarithmic extensions of these structures to $\overline{M}$ (the smooth toroidal compactification of M).

Proof. The decomposition for $\mathcal{E}$ and the first Chern classes for $\mathcal{S}$ and $\mathcal{L}$ are given in [6]. The claim for higher exterior powers follows from the splitting principle of Chern classes (using the facts that $c_1(\wedge^m \mathcal{M}) = \binom{\mathrm{rank}\,(\mathcal{M})-1}{m-1} \cdot c_1(\mathcal{M})$ and $c_1(\mathcal{M}' \otimes \mathcal{M}'') = \mathrm{rank}\,(\mathcal{M}')c_1(\mathcal{M}'') + \mathrm{rank}\,(\mathcal{M}'')c_1(\mathcal{M}')$). □

THEOREM 3.3 (exclusion of Shimura subvarieties of $\mathbf{SU}(n,1)$-type). *Let $M \subset \mathcal{A}_V = \mathcal{A}_g$ be a Shimura subvariety of $\mathbf{SU}(n,1)$-type defined by a Shimura subdatum $(\mathbf{G}, X; X)$ as in Definition 3.1, such that the rational symplectic representation $\mathbf{G}^{\mathrm{der}} \hookrightarrow \mathbf{Sp}_W$ is primary of type Λ_m in the sense of [1].Then M is NOT contained generically in the Torelli locus in $\mathcal{A}_W$ as long as the following inequality holds:*

$$\frac{n+m-1}{n}\left(\frac{n+1}{m} \cdot d - \frac{2}{r\binom{n}{m-1}}\right) > 12.$$

Proof. Assume on the contrary that M is contained generically in $\mathcal{T}_g$. Then the same construction used in Proposition 2.4 produces a curve C in the n-dimensional subvariety $\overline{\mathcal{M}}(\subset \overline{\mathcal{A}}_V)$ using successive $n-1$ hyperplane sections of a fixed very ample power ω_M^N of the automorphic line bundle on M, such that $C^\circ = C \cap \mathcal{T}_g^\circ$ is open and dense in C, which gives rise to a surface fibration $f: S \to B$ and a finite morphism $i: B \to C$.

The logarithmic Higgs bundle $\mathcal{V}^{1,0}$ on $\overline{M}$ defined by the moduli problem $M \hookrightarrow \mathcal{A}_V$ already admits the following filtration

$$0 = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq \mathcal{V}_3 = \mathcal{V}^{1,0}$$

where:

- $\mathcal{V}_1$ is the summand $(\wedge^{m-1}\mathcal{S} \otimes \mathcal{L})^{\oplus r}$ in $\mathcal{W}_1$;
- $\mathcal{V}_2 = \mathcal{W}_1 \simeq \mathcal{V}_1 \oplus (\wedge^m \mathcal{S})^{\oplus r}$;
- $\mathcal{V}_3 = \mathcal{V}^{1,0}$ only differs from $\mathcal{W}_1$ by a direct summand $\mathcal{F}$ flat of degree zero.

The graded quotients of this filtration are already semi-stable. Computing their degrees over C we find that the maximal slope in $\mathcal{V}_C^{1,0}$ is realized on $\mathcal{V}_1$

$$\mu(\mathcal{V}_1) = \mu(\wedge^{m-1}\mathcal{S} \otimes \mathcal{L}) = \frac{2\binom{n}{m-1} - \binom{n-1}{m-1}}{\binom{n}{m-1}} \cdot d_C$$

with the constant $d_C = \frac{N^{n-1}}{n+1} c_1(\omega_{\overline{M}})^n$ as we have used the fixed very ample line bundle ω_M^N on M which extends to $\omega_{\overline{M}}^N$ on $\overline{M}$. Note also that

$$\deg(\mathcal{V}_C^{1,0}) = r\deg(\wedge^m\mathcal{E}^{1,0}) = 2r\binom{n}{m-1}\cdot d_C,$$
$$g = \frac{1}{2}\dim_{\mathbb{Q}} V = rd\binom{n+1}{m},$$

and that the maximal slope part is $\mathcal{V}_1 \simeq (\wedge^{m-1}\mathcal{S}\otimes\mathcal{L})^{\oplus r}$, which is of rank $r\binom{n}{m-1}$.

Passing from C to B only changes d_C (resp. $\mu(\mathcal{V}_1)$) to d_B (resp. $\mu(i^*\mathcal{V}_1)$) by a positive multiple $\deg i$, and Xiao's inequality gives us

$$12\cdot 2r\binom{n}{m-1}\cdot d_B \geq \left(2rd\binom{n+1}{m} - 2 + r\binom{n}{m-1}\right)\cdot\left(2 - \frac{\binom{n-1}{m-1}}{\binom{n}{m-1}}\right)\cdot d_B,$$

which is

$$\frac{n+m-1}{n}\left(\frac{n+1}{m}d + \frac{1}{2} - \frac{1}{r\binom{n}{m-1}}\right) \leq 12$$

and hence the generic exclusion when the inequality above fails. □

For example, when $m = 1$, we obtain the generic exclusion for such Shimura subvarieties satisfying

$$(n+1)d > \frac{23}{2} + \frac{1}{r}$$

which is

- $(n+1)d \geq 13$ when $r = 1, 2$; this is, for example, the case when the symplectic representation is directly obtained as the imaginary part of an Hermitian form of signature $(n, 1)$ over some CM quadratic extension E/F, and $g = d(n+1)$;
- $(n+1)d \geq 12$ when $r \geq 3$; this is the same as the inequality $d(n+1) \geq 12$ given in [1] under more restrictive assumptions.

4. Shimura varieties of orthogonal type. In this section we consider a class of Shimura varieties of orthogonal type containing Shimura subvarieties of $\mathbf{SU}(n,1)$-type. Recall the following:

Definition 4.1 (Shimura subvarieties of orthogonal type). A Shimura subvariety of orthogonal type, or more precisely, of $\mathbf{Spin}(N,2)$-type, in $\mathcal{A}_V$ is defined by a subdatum $(\mathbf{G}, X; X^+)$ such that $\mathbf{G}^{\mathrm{der}}(\mathbb{R}) \simeq \mathbf{Spin}(N,2)\times\mathbf{Spin}(N+2)^{d-1}$ for some d. Here $\mathbf{Spin}(a,b)$ is the spin group of the standard quadratic space $\mathbb{R}^{a+b}$ of signature (a,b).

The Hermitian symmetric domain X above is the one associated to $\mathbf{Spin}(N,2)$, and we often make use of the following two equivalent descriptions of X:

(1) X is the open subset of two-dimensional negative definite $\mathbb{R}$-subspaces in $\mathbb{R}^{N+2}$ in the Grassmannian $\mathbf{Gr}(2,\mathbb{R}^{N+2})$;
(2) X is the open subset of negative definite isotropic $\mathbb{C}$-lines in $\mathbb{P}(\mathbb{C}^{N+2})$, namely those $\mathbb{C}v$ such that $b(v,\bar{v}) = 0$ and $b(v,\bar{v}) < 0$ for b the quadratic form of signature on $\mathbb{R}^{N+2}$ extended to $\mathbb{C}^{N+2}$; in particular X is an open subset of the quadric defined by b in $\mathbb{P}(\mathbb{C}^{N+2})$.

The equivalence between (1) and (2) is well-known: starting with $u, u' \in \mathbb{R}^{N+2}$ which are negative definite and orthogonal to each other, we can choose a suitable $J \in \mathbb{C}^\times$ purely imaginary so that $v = u + Ju'$ defines a line in (2); conversely, given a line $\mathbb{C}v$ in (2), one produces a pair of negative definite vectors (u, u') orthogonal to each other giving rise to a negative definite $\mathbb{R}$-plane in (1).

The two descriptions above also give us a natural equivariant embedding of Hermitian symmetric domains: if (V, q) is a quadratic space of signature $(N, 2)$ over $\mathbb{R}$, and $U \subset V$ is a positive definite subspace of dimension $N - 2n (\geq 0)$, such that the restriction of q to the orthogonal complement of U in V is the real part of some Hermitian space (W, h) of signature $(n, 1)$, then we have a natural inclusion of semi-simple Lie groups $\mathbf{SU}(W, h) \hookrightarrow \mathbf{SO}(V, q)$ inducing an equivariant holomorphic embedding of the corresponding Hermitian symmetric domains $X(W, h) \hookrightarrow X(V, q)$, sending a negative definite $\mathbb{C}$-line in $X(W, h)$ to the associated negative definite $\mathbb{R}$-plane in $X(V, q)$. Note that this inclusion actually factors through

$$\mathbf{SU}(W, h) \hookrightarrow \mathbf{SO}(W, b) \hookrightarrow \mathbf{SO}(V, q)$$

and

$$X(W, h) \hookrightarrow X(W, b) \hookrightarrow X(V, q),$$

where we write b for the real part of h, equal to the restriction of q to W as a real subspace of V. Also the inclusions $\mathbf{SU}(W, h)$ into special orthogonal groups lift into homomorphisms into the corresponding spin groups, because special unitary groups are simply connected.

In this section we are only interested in the Coleman-Oort problem for Shimura varieties of $\mathbf{Spin}(N, 2)$-type containing Shimura varieties of $\mathbf{SU}(n, 1)$-type. We start with the case $N = 2n$ before entering the general case where $N > 2n$.

Let (W, h) be an Hermitian space over a CM field E, and write F for the totally real part of E. Assume that the signature of h is:

- $(n, 1)$ along one real embedding $\sigma : F \hookrightarrow \mathbb{R}$;
- definite along the other embeddings $\sigma_2, \cdots, \sigma_d : F \hookrightarrow \mathbb{R}$

with $d = [F : \mathbb{Q}]$. We also have the real part of (W, h), namely (U, q) with $U = \mathrm{Res}_{E/F} W$ and $q = \mathrm{tr}_{E/F} h$. Write $\mathbf{H} = \mathbf{SU}(W, h)$ for the special unitary F-group associated to (W, h), contained in $\mathbf{H}' = \mathbf{Spin}(U, q)$ the spin F-group of (U, q). Let $M' \subset \mathcal{A}_V = \mathcal{A}_g$ be a Shimura subvariety of $\mathbf{Spin}(2n, 2)$-type, i.e. defined by a Shimura subdatum $(\mathbf{G}', X'; X'^+)$ with $\mathbf{G}'^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{H}'$, which contains a Shimura subvariety M of $\mathbf{SU}(n, 1)$-type defined by a Shimura subdatum $(\mathbf{G}, X; X^+)$ with $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{H}$.

We have used spinor representations in [1]:

- the spinor representations of the Lie group $\mathbf{Spin}(2n, 2)$ are P_+ and P_-: choose any splitting of $\mathbb{C}^{2n+2}$ into $T \oplus T^\vee$ such that the quadratic form is equivalent to the pairing

$$((u, u^\vee), (v, v^\vee)) \mapsto u^\vee(v) + v^\vee(u)$$

  we have $P_+ = \wedge^+ T := \wedge^{\mathrm{even}}_{\mathbb{C}} T$ and $P_- = \wedge^- T := \wedge^{\mathrm{odd}}_{\mathbb{C}} T \simeq P_+^\vee$, both of dimension 2^n over $\mathbb{C}$.

We need to find suitable splitting for (U, q), at least over $\mathbb{C}$, using (W, h). Consider the following lemma for the quadratic extension $\mathbb{C}/\mathbb{R}$, in which we temporarily use

(U, q) and (W, h) to denote the quadratic and Hermitian forms involved:

LEMMA 4.2. *Let $h : W \times W \to \mathbb{C}$ be an Hermitian form over $\mathbb{C}$, with real part $q : U \times U \to \mathbb{R}$, and $q_{\mathbb{C}} : U_{\mathbb{C}} \times U_{\mathbb{C}} \to \mathbb{C}$. Then $U_{\mathbb{C}}$ admits a natural splitting by $W \hookrightarrow U_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$, and $U_{\mathbb{C}} \simeq W \oplus \overline{W}$ with respect to the real structure on $U_{\mathbb{C}}$ given by U.*

Proof. We may diagonalize (W, h) into direct sums of one dimensional $\mathbb{C}$-spaces, and reduce to the case when $\dim_{\mathbb{C}} W = 1$: $W = \mathbb{C}w$ for some basis w, and $h(zw, z'w) = c\bar{z}z'$ for $z, z' \in \mathbb{C}$ and $c \in \mathbb{R}^{\times}$. Thus U admits a basis $(w, \mathbf{i}w)$ for a fixed choice $\mathbf{i} = \sqrt{-1}$, and $q = \mathrm{tr}_{\mathbb{C}/\mathbb{R}} h$ sends $(aw + b\mathbf{i}w, a'w + b'\mathbf{i}w)$ to $c(aa' + bb')$. It suffices to choose the splitting to be

$$W \hookrightarrow U_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C},\ w \mapsto w \otimes (1 + \mathbf{i})$$

in which case $\overline{W}$ is simply $W \otimes (1 - \mathbf{i})$. □

Return to the general setting over a CM field E/F. Evaluate the inclusion $\mathbf{H} = \mathbf{SU}(W, h) \hookrightarrow \mathbf{H}' = \mathbf{Spin}(U, q)$ at $\sigma = \sigma_1 : F \hookrightarrow \mathbb{R}$, we obtain $\mathbf{SU}(n, 1) \hookrightarrow \mathbf{Spin}(2n, 2)$. The $\mathbb{C}$-representations $P_{\pm} \simeq \wedge^{\pm}(W_{\mathbb{C}})$ of $\mathbf{H}'(\mathbb{R}, \sigma)$ restricts to sums of wedge product $\mathbb{C}$-representations of $\mathbf{H}(\mathbb{R}, \sigma)$:

LEMMA 4.3. *Let $M \hookrightarrow M'$ be the inclusion of a Shimura variety of $\mathbf{SU}(n, 1)$-type into a Shimura variety of $\mathbf{Spin}(2n, 2)$-type associated to $\mathbf{H} \hookrightarrow \mathbf{H}'$ as above. Consider the Higgs bundles $\mathcal{P}_{\pm}$ on M' associated to the spinor representation $P_{\pm}$ of $\mathbf{H}'(\mathbb{R}, \sigma)$. Then the restriction of $\mathcal{P}_{\pm}$ to M decomposes into*

$$\mathcal{P}_+ = \bigoplus_{m \text{ even}} \mathcal{E}_m,\ \mathcal{P}_- = \bigoplus_{m \text{ odd}} \mathcal{E}_m$$

with $\mathcal{E}_m$ the Higgs bundle associated to the m-th exterior power Λ_m of the the representation

$$\mathbf{G}^{\mathrm{der}}(\mathbb{R}) \to \mathbf{H}(\mathbb{R}, \sigma_1) \stackrel{\mathrm{Std}}{\longrightarrow} \mathbf{GL}_{n+1}(\mathbb{C}).$$

In particular, $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}^{1,0} = \mathcal{S} \oplus \mathcal{L}$, and $\mathcal{E}_m^{1,0}$ decomposes into the direct sum of $\wedge^m \mathcal{S}$ and $\wedge^{m-1}\mathcal{S} \otimes \mathcal{L}$, whose ranks and Chern classes are given as in Lemma 3.2. Take C a generic curve in $\overline{M}$ produced from the linear system of ω_M^N a fixed very ample power of ω_M, the slopes of these summands on C are as follows:

$$\mu(\wedge^m \mathcal{S}_C) = \frac{m}{n} d_C, \quad \mu(\wedge^{m-1}\mathcal{S}_C \otimes \mathcal{L}_C) = \frac{2n - m}{n} d_C$$

for $m = 1, \cdots, n$, with $d_C = \frac{N^{n-1}}{n+1} c_1(\omega_{\overline{M}})^n$. We also have $c_1(\mathcal{E}_0^{1,0}) = 0$ and $c_1(\mathcal{E}_{n+1}^{1,0}) = \frac{2}{n+1} c_1(\omega_M)$, and thus $\mu(\mathcal{E}_{0,C}^{1,0}) = 0$ and $\mu(\mathcal{E}_{n+1,C}^{1,0}) = 2d_C$.

The proof is immediate after Lemma 3.2.

The following proposition takes care of the inclusion $\mathbf{SU}(n, 1) \hookrightarrow \mathbf{Spin}(2n, 2)$ in a natural arithmetic setting. Note that the case (1) is singled out for later use in Theorem 4.5.

PROPOSITION 4.4 (restriction on the real part). *Let $M' \subset \mathcal{A}_V = \mathcal{A}_g$ be a Shimura subvariety of $\mathbf{Spin}(2n, 2)$-type, containing a Shimura subvariety $M \subset M'$ of $\mathbf{SU}(n, 1)$-type in the sense above. Assume that the symplectic representation $\mathbf{G}'^{\mathrm{der}} \hookrightarrow \mathbf{Sp}_V$*

defining $M' \hookrightarrow \mathcal{A}_V$ admits no trivial subrepresentations. If M' is contained generically in $\mathcal{T}_g$, then the following hold:

(1) *if P_+ and P_- appear with equal multiplicity r, then $d \leq 3 + 2^{-n-1}(\frac{1}{r} - \frac{1}{2})$;*

(2) *if $M' \hookrightarrow \mathcal{A}_V$ is primary of type P_- of multiplicity r, then along the parity of n we have:*

- $d \leq 3 + 2^{-n}(\frac{1}{r} - \frac{1}{2})$ *when n is even;*
- $d \leq \frac{6n}{2n-1} + 2^{-n}(\frac{1}{r} - \frac{1}{2})$ *when n is odd;*

(3) *if $M' \hookrightarrow \mathcal{A}_V$ is primary of type P_+ of multiplicity r, then along the parity of n we have:*

- $d \leq \frac{3n}{n-1} + 2^{-n}(\frac{1}{r} - \frac{n}{2})$ *when n is even;*
- $d \leq 3 + 2^{-n}(\frac{1}{r} - \frac{1}{2})$ *when n is odd.*

Proof. We have $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{H}$ and $\mathbf{G}'^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{H}'$, and the inclusion $\mathbf{G}' \hookrightarrow \mathbf{Sp}_V$ is restricted from $\mathbf{H}' \hookrightarrow \mathbf{Sp}_W$ where W is an F-symplectic space, so that $V \simeq \mathrm{Res}_{F/\mathbb{Q}}W$ and the canonical logarithmic Higgs bundle $\mathcal{V} = \mathcal{V}_{M'}$ admits a decomposition $\mathcal{V} = \oplus_{i=1,\cdots,d}\mathcal{W}_i$ with $\mathcal{W}_i$ corresponds to the action of $\mathbf{G}'^{\mathrm{der}}(\mathbb{R})$ on $W \otimes_{F,\sigma_i} \mathbb{R}$ through $\mathbf{H}'(\mathbb{R}, \sigma_i)$. In particular, the $(1,0)$-parts $\mathcal{W}_2^{1,0}, \cdots, \mathcal{W}_d^{1,0}$ are already flat.

Similar to Theorem 3.3, we may assume that a generic curve C is chosen in $\overline{M}'$ such that C is contained generically in $\mathcal{T}_g$ and lifts to a semi-stable surface fibration $f : S \to B$ together with a finite morphism $i : B \to S$ satisfying Proposition 2.3.

(1) In this case we have

$$W \otimes_{F,\sigma_1} \mathbb{R} \simeq (P_+ \oplus P_-)^{\oplus r} \simeq (\wedge^{\cdot}(\mathbb{C}^{n+1}))^{\oplus r}$$

for some multiplicity r, and thus $\mathcal{W}_1^{1,0} \simeq (\oplus_{m=0}^{n+1}\mathcal{E}_m^{1,0})^{\oplus r}$. Therefore the maximal slope in $\mathcal{V}_C^{1,0}$ the restriction to C is given by

$$\mu_{\max} = \mu((\mathcal{E}_{n+1,C}^{1,0})^{\oplus r}) = \mu(\mathcal{E}_{n+1,C}^{1,0}) = 2d_C$$

with $(\mathcal{E}_{n+1,C}^{1,0})^{\oplus r}$ of rank r, while $\deg \mathcal{V}_C^{1,0} = \deg \mathcal{W}_{1,C}^{1,0} = r\sum_m \deg(\mathcal{E}_{m,C}^{1,0}) = r{\cdot}2^{n+1}d_C$. and the flat part is

$$(\wedge^0\mathcal{E}_1^{1,0})^{\oplus r} \oplus \mathcal{W}_2^{1,0} \oplus \cdots \oplus \mathcal{W}_d^{1,0}.$$

Here we have used the same constant d_C as in Theorem 3.3. Passing to B using $i : B \to C$, the slopes and degrees only differ by a common multiple $\deg(i)$, namely one replaces the constant d_C by $d_B = \deg(i)d_C$.

When M' is contained generically in $\mathcal{T}_g$, we may choose C in $\overline{M} \subset \overline{M}'$ such that C is contained generically in $\mathcal{T}_g$, so that Xiao's inequality for $f : S \to B$ gives

$$12 \cdot 2^{n+1}r \geq (2 \cdot 2^{n+1}rd - 2 + r)2$$

which is $d \leq 3 + 2^{-n-1}(\frac{1}{r} - \frac{1}{2})$.

(2) In this case $\mathcal{V}_C^{1,0} = \mathcal{W}_{1,C}^{1,0} \oplus \cdots \mathcal{W}_{d,C}^{1,0}$ is of rank $rd \cdot 2^n$, with $\mathcal{W}_{2,C}^{1,0}, \cdots, \mathcal{W}_{d,C}^{1,0}$ flat, and $\mathcal{W}_C^{1,0} \simeq (\mathcal{P}_-^{1,0})^{\oplus r}$ contains no flat part, and it already contains $(\mathcal{E}_{1,C}^{1,0})^{\oplus r}$ of slope $(2 - \frac{1}{n})d_C$, which is:

(2-1) maximal when $\mathcal{E}_{n+1,C}$ does not contribute, namely n is odd, and in this case the maximal slope is realized on $\mathcal{L}^{\oplus r}$ contained in $(\mathcal{E}_{1,C}^{1,0})^{\oplus r}$, of rank r;

(2-2) strictly smaller than the maximal slope $2d_C = \mu(\mathcal{E}_{n+1,C}^{1,0})$ when n is even, and the maximal slope is realized on $(\mathcal{E}_{n+1,C}^{1,0})^{\oplus r}$ of rank r.

Note that $\mathcal{W}_C^{1,0} \simeq (\mathcal{P}_-^{1,0})^{\oplus r}$ is of rank $2^n r$ and degree $2^n r d_C$, hence Xiao's inequality implies:

(2-1) when n is odd: $12 \cdot 2^n r \geq (2^{n+1} rd - 2 + r)(2 - \frac{1}{n})$, which is $d \leq \frac{6n}{2n-1} + 2^{-n}(\frac{1}{r} - \frac{1}{2})$;

(2-2) when n is even: $12 \cdot 2^n r \geq (2^{n+1} rd - 2 + r) \cdot 2$, which is $d \leq 3 + 2^{-n}(\frac{1}{r} - \frac{1}{2})$.

(3) In this case $\mathcal{W}_C^{1,0} \simeq (\mathcal{P}_+^{1,0})^{\oplus r}$ is of rank $2^n r$. Along the parity of n we have:

(3-1) when n is even: the maximal slope comes from the summand $\mathcal{S}_C \otimes \mathcal{L}_C$ in $\mathcal{E}_{2,C}^{1,0}$, which is $(2 - \frac{2}{n})d_C$ of rank rn, and Xiao's inequality becomes

$$12 \cdot 2^n r \geq (2^{n+1} rd - 2 + rn)(1 - \frac{1}{n}) \cdot 2,$$

namely $d \leq \frac{3n}{n-1} + 2^{-n}(\frac{1}{r} - \frac{n}{2})$ (note that $n \geq 2$).

(3-2) when n is odd: the maximal slope comes from the summand $\mathcal{E}_{n+1,C}^{1,0}$, which is $2d_C$ of rank r, and similar to the case (2-2), Xiao's inequality is $12 \cdot 2^n r \geq (2^{n+1} rd - 2 + r) \cdot 2$, namely $d \leq 3 + 2^{-n}(\frac{1}{r} - \frac{1}{2})$.□

In particular, when $r \geq 3$, the generic exclusion of M holds as long as $d \geq \frac{3n}{n-1}$ for n even and $d \geq \frac{6n}{2n-1}$ for n odd, which are slightly finer that the bound $d \geq 6$ given in [1].

We proceed to the remaining case when $N > 2n$:

THEOREM 4.5 (generic exclusion of Shimura varieties of orthogonal types). *Let $M' \subset \mathcal{A}_V$ be a Shimura variety of $\mathbf{Spin}(N, 2)$-type which contains a Shimura subvariety M of $\mathbf{SU}(n, 1)$-type in the sense above, namely M' is associated to some quadratic space (W, q) over some CM field E/F subject to the constraints of signature used in this section, and let U be a positive definite subspace of signature $(N - 2n, 0)$, whose orthogonal complement is the real part of some Hermitian space (H, h) over E/F giving rise to the Shimura subvariety M mentioned above.*

Assume that the symplectic representation defining $M' \hookrightarrow \mathcal{A}_V$ is primary. Then M' is NOT contained generically in $\mathcal{T}_g$ as long as $[F : \mathbb{Q}] > 3 + \frac{1}{m \cdot 2^{\lfloor (N+1)/2 \rfloor}} - \frac{1}{2^{n+2}}$, where m is the multiplicity of the spinor representation in $W \otimes_{F,\sigma} \mathbb{R}$ for the group $\mathbf{Spin}(N, 2)$.

Proof. Recall the behavior of spinor representations to spin subgroups used in [1], where we consider the case over $\mathbb{C}$ for simplicity:

- the restriction of the spinor representation P of $\mathbf{Spin}(2k + 1)$ to $\mathbf{Spin}(2k)$, using an orthogonal decomposition of the form $\mathbb{C}^{2k+1} = L \oplus \mathbb{C}^{2k}$ for some line L, is the direct sum of P_+ and P_-, the two half-spin representations of $\mathbf{Spin}(2k)$;
- similarly, the restrictions of P_+ and P_- of $\mathbf{Spin}(2k + 2)$ to $\mathbf{Spin}(2k + 1)$ are both isomorphic to the spinor representation of $\mathbf{Spin}(2k + 1)$ as long as the restriction comes from an orthogonal decomposition $\mathbb{C}^{2k+2} = L \oplus \mathbb{C}^{2k+1}$ by some line L.

In our setting, the condition $N - 2n > 0$ refines the inclusion $W \supset \mathrm{Res}_{E/F} H$ into $W \supset W' \supsetneq \mathrm{Res}_{E/F} H$ with W' of dimension $3 + 2n$ over F, such that

- $W = U' \oplus W'$ is an orthogonal direct sum decomposition with U' of signature $(N - 2n - 1, 0)$ along σ;
- $W' = L \oplus \mathrm{Res}_{E/F} H$ is an orthogonal direct sum decomposition with L a positive definite line.

The inclusion of Shimura varieties $M \subset M'$ is thus refined into $M \subset M'' \subset M'$ with M'' the Shimura variety of $\mathbf{Spin}(2n+1,2)$-type associated to $(W', q|_{W'})$, and we write $(\mathbf{G}'', X''; X''^+)$ for the corresponding Shimura subdatum of $(\mathbf{GSp}_V, \mathcal{H}_V; \mathcal{H}_V^+)$. Since $M' \hookrightarrow \mathcal{A}_V$ is defined by some symplectic representation $\mathbf{G}'^{\mathrm{der}} \hookrightarrow \mathbf{Sp}_V$ primary of spinor type, its restriction to $\mathbf{G}''^{\mathrm{der}} \hookrightarrow \mathbf{Sp}_V$ is primary of spinor type, namely it is the scalar restriction from $\rho : \mathbf{H}'' \hookrightarrow \mathbf{Sp}_{V''}$ along F over $\mathbb{Q}$, where $\mathbf{H}''$ is the spin F-group of W'' and V'' is an F-symplectic space such that $V = \mathrm{Res}_{F/\mathbb{Q}} V''$ and the base change of ρ along $\sigma : F \hookrightarrow \mathbb{R}$ decomposes into a direct sum of copies of the unique spinor representation P for $\mathbf{Spin}(2n+1,2)$.

We have explained that the restriction of P to $\mathbf{Spin}(2n,2)$ is the direct sum of the two half-spin representations P_- and P_+ of equal multiplicity, hence it suffices to apply Proposition 4.4 (1) to the further inclusion $\mathbf{SU}(n,1) \subset \mathbf{Spin}(2n,2) \subset \mathbf{Spin}(2n+1,2)$ to deduce the bound $d = [F : \mathbb{Q}] > 3 + 2^{-n-1}\left(\frac{1}{r} - \frac{1}{2}\right)$ from Xiao's inequality.

It remains to make precise the multiplicity r along the parity of N. Assume that the inclusion $M' \hookrightarrow \mathcal{A}_V$ is primary of multiplicity m: in other words, the rational symplectic representation defining $M' \hookrightarrow \mathcal{A}_V$ is the scalar restriction of some F-linear symplectic representation $\mathbf{H} \to \mathbf{Sp}_W$, and $W \otimes_{F,\sigma} \mathbb{R}$ is isomorphic to m-copies of the spinor representation of $\mathbf{Spin}(N,2)$.

- If $N = 1 + 2N' > 2n$ is odd, with $N' \geq n$, then there exists only one spinor representation P' for $\mathbf{H}'(\mathbb{R},\sigma) \simeq \mathbf{Spin}(1+2N',2)$, and its restriction to $\mathbf{H}''(\mathbb{R},\sigma) \simeq \mathbf{Spin}(2n,2)$ is isomorphic to $2^{N'-n}$ copies of $P_+ \oplus P_-$. Since P' is of multiplicity m in $W \otimes_{F,\sigma} \mathbb{R}$, this gives $r = m \cdot 2^{N'-n}$ for the multiplicity of $P_+ \oplus P_-$ in the representation of $\mathbf{H}''(\mathbb{R},\sigma)$.
- If $N = 2N' > 2n$ is even, with $N' > n$, then either of the two half-spin representations P'_+ and P'_- of $\mathbf{Spin}(2N',2)$ restricts to the unique spinor representation of $\mathbf{Spin}(2N'-1,2)$ and further to $2^{N'-n-1}$ copies of $P_+ \oplus P_-$ of $\mathbf{Spin}(2n,2)$, and one obtains $r = m \cdot 2^{N'-n-1}$ for the multiplicity of $P_+ \oplus P_-$ in the representation of $\mathbf{H}''(\mathbb{R},\sigma)$.

Hence the multiplicity is always $r = m \cdot 2^{\lfloor (N-1)/2 \rfloor - n} = m \cdot 2^{\lfloor (N+1)/2 \rfloor - n - 1}$, with m the multiplicity of W, and the proof is completed. ∎

Acknowledgement. It is our pleasure and honor to dedicate this work to Prof. Ngaiming Mok at his sixtieth anniversary. Prof. Mok has been well-known for his contribution to complex differential geometry and algebraic geometry, and his works have seen growing influences on young geometers. We congratulate him sincerely on this special occasion, and wish him a fruitful career yet to come.

REFERENCES

[1] K. Chen, X. Lu, and K. Zuo, *On the Oort conjecture for Shimura varieties of unitary and orthogonal types*, Compositio Mathematica, 152 (2016), pp. 889–917.

[2] P. Deligne, *Variétés de Shimura, interpretation modulaire et construction de modèles canoniques, Automorphic Forms, Representations, and L-functions, Part 2*, Proceedings of Symposia in Pure Mathematics, vol. 33 (AMS Providence RI, 1979), pp. 247–289.

[3] R. Hain, *Locally symmetric families of curves and Jacobians*, Aspects of Mathematics, vol. E33 (Vieweg, Braunschwig, 1999), pp. 91–108.

[4] J. de Jong and S. Zhang, *Generic Abelian varieties with real multiplication are not Jacobians*, Diophantine Geometry, CRM Series, vol. 4 (Edizione della Normale, Pisa, 2007), pp. 165–172.

[5] X. Lu and K. Zuo, *The Oort conjecture for Shimura curves in the Torelli locus of curves*, Journal de Mathématiques Pures et Appliqueées (9) 108(2017), no.4, 532-552

[6] M. Möller, E. Viehweg, and K. Zuo, *Stability of Hodge bundles and characterization of Shimura curves*, Journal of Differential Geometry, 92 (2012), pp. 71–151.

[7] N. Mok, *Metric rigidity theorems on Hermitian locally symmetric manifolds*, Series in pure mathematics, vol. 6, World Scientific.
[8] B. Moonen and F. Oort, *The Torelli locus and special subvarieties*, Handbook of Moduli, Vol. II, pp. 549–594.
[9] I. Satake, *Holomorphic embeddings of symplectic domains into a Siegel space*, Americal Journal of Mathematics, 87 (1965), pp. 425–461.
[10] I. Satake, *Symplectic representations of algebraic groups satisfying a certain analyticity condition*, Acta Mathematica, 117 (1965), pp. 215–279.
[11] J. Tsimerman, *A proof of the André-Oort conjecture*, Annals of Mathematics (2) 187(2018), no.2, 379-390
[12] G. Xiao, *Fibred algebraic surfaces with low slope*, Mathematische Annalen, 276:3 (1987), pp. 449–466.