

JET VANISHING ORDERS AND EFFECTIVITY OF KOHN'S ALGORITHM IN DIMENSION 3*

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Dedicated to Professor Ngaiming Mok on the occasion of his 60th birthday

Abstract. We propose a new class of geometric invariants called *jet vanishing orders*, and use them to establish a new selection algorithm in the Kohn's construction of subelliptic multipliers for special domains in dimension 3, inspired by the work of Y.-T. Siu [S10]. In particular, we obtain effective termination of our selection algorithm with explicit bounds both for the steps of the algorithm and the order of subellipticity in the corresponding subelliptic estimates. Our procedure possesses additional features of certain stability under high order perturbations, due to deferring the step of taking radicals to the very end, see Remark 1.2 for more details.

We further illustrate by examples the sharpness in our technical results (in Section 3) and demonstrate the complete procedure for arbitrary high order perturbations of the Catlin-D'Angelo example [CD10] in Section 5.

Our techniques here may be of broader interest for more general PDE systems, in the light of the recent program initiated by the breakthrough paper of Y.-T. Siu [S17].

Key words. Finite type, Multiplier ideals, Jacobian, subelliptic estimates, jets, germs of holomorphic functions, order of contact, $\bar{\partial}$ -Neumann problem.

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1. Introduction. In his seminal paper [K79], J.J. Kohn invented a purely algebraic construction of ideals of subelliptic multipliers for the $\bar{\partial}$ -Neumann problem. The goal of this note is to propose a new class of geometric invariants, called *jet vanishing orders*, that permits us to obtain a fine-grained control of the effectiveness in the Kohn's construction procedure of subelliptic multipliers for the so-called *special domains* of finite D'Angelo type [D79, D82] in \mathbb{C}^3 .

Since this paper is dedicated to Professor Ngaiming Mok, we would like to mention a striking parallel between the Kohn's *subelliptic multiplier ideals* and the *varieties of minimal rational tangents* (VMRT) pioneered by N. Mok and J.-M. Hwang [HM98]. Both theories connect *global* PDE or algebraic-geometric structures with *local* differential-geometric objects that can be treated by local geometric and analytic methods, subsequently leading to important consequences of global nature. For more details on the VMRT, see the articles and lecture notes by Hwang and Mok [Mo08, H14]. Another important parallel is historical. The VMRT theory was developed to study deformation rigidity problems originated from the celebrated Kodaira-Spencer's work on deformation of complex structures, with an important ingredient coming from D. Spencer's program to generalize the Hodge theory of harmonic integrals. And it was the same program that led Spencer to formulate the $\bar{\partial}$ -Neumann problem, for which Kohn invented his multiplier ideals approach to tackle the problem of local regularity (see [K-etal04] for a more detailed account).

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Note that Kohn's original procedure in [K79] gives no effective bound on the order of subellipticity ε in subelliptic estimates, as illustrated by examples of G. Heier [He08] and D.W. Catlin and J.P. D'Angelo [CD10], see §1.2 and §5 below. On the other hand, Y.-T. Siu [S10, S17] obtained a new effective procedure for special domains and outlined an extension of the special domain approach to general real-analytic and smooth cases. Also note that for the special domains of the so-called *triangular form* in \mathbb{C}^n , a different effective procedure in Kohn's algorithm was given by Catlin-D'Angelo [CD10, Section 5]. Triangular systems were first introduced by D'Angelo [D95] under the term *regular coordinate domains*, where also an effective procedure for obtaining subelliptic estimates was provided. See also D.W. Catlin and J.-S. Cho [CC08] and T.V. Khanh and G. Zampieri [KhZa14] for subelliptic estimates for triangular systems by a different method, and A. Basyrov, A.C. Nicoara and the second author [BNZ17] for an approximation of general smooth pseudoconvex boundaries by triangular systems of sums of squares matching the Catlin multitype. We next mention that A.C. Nicoara [N14] proposed a construction for the termination of the Kohn algorithm in the real-analytic case with an indication of the ingredients needed for the effectivity. We further refer to [K79, K84, D95, DK99, S01, S02, K04, S05, S09, CD10, S10, S17] for more profound and extensive discussion of subelliptic multipliers (see also [Ch06] for an algebraic approach), as well as surveys [Si91, BSt99, FS01, M03, St06, MV15] and books [Mr66, FK72, Tr80, Ho90, D93, CS01, O02, Za08, St10, Ta11, Ha14, O15] for the $\bar{\partial}$ -Neumann problem in broader context.

In relation to Kohn's foundational work on the subelliptic multipliers, we would like to mention a remarkable new development in the field due to Siu [S17] providing new techniques of generating multipliers for general systems of partial differential equations, including, as a special case, a new procedure even for the case of the $\bar{\partial}$ -Neumann problem.

Finally we mention the related important Nadel's multiplier ideal sheaves [Na90] that were originally motivated by the ones defined by Kohn [K79] and are in some sense dual to them. See [S01, §4], [S09, 1.4.6] for more detailed discussions of the relation between both types of multipliers, and the expository articles and books [S01, S02, De01, L04, S05, S09] for further information.

1.1. Kohn's algorithm for special domains and subelliptic estimates.

We shall consider so-called *special domains* as introduced by Kohn in [K79], defined locally near the origin by holomorphic functions in the first n variables:

$$\Omega := \left\{ \operatorname{Re}(z_{n+1}) + \sum_{j=1}^m |F_j(z_1, \dots, z_n)|^2 < 0 \right\} \subset \mathbb{C}^{n+1}, \quad (1.1)$$

where F_1, \dots, F_m is any collection of holomorphic functions vanishing at the origin in \mathbb{C}^n (and defined in its neighborhood).

Understanding Kohn's multipliers for the special domains is important, on the one hand, due to their link with local analytic and algebraic geometry, and on the other hand, through their connection with more general cases via D'Angelo's construction of associated families of holomorphic ideals [D93, Chapter 3] and Siu's program relating special domains approach with general real-analytic and smooth cases [S10, II.3, II.4].

Next recall the *Kohn's algorithm* or, more precisely, its holomorphic variant for special domains (1.1) (corresponding to $q = 1$ in the notation of [K79] and to algebraic geometric formulation in [S17, 2.9.4]). For any set S of germs of holomorphic functions

at 0 in \mathbb{C}^n , consider the ideal $J(S)$ generated by all function germs g satisfying

$$df_1 \wedge \dots \wedge df_n = g dz_1 \wedge \dots \wedge dz_n, \quad f_1, \dots, f_n \in S,$$

i.e. by all *Jacobian determinants*

$$g = \frac{\partial(f_1, \dots, f_n)}{\partial(z_1, \dots, z_n)} = \det \left(\frac{\partial f_i}{\partial z_j} \right). \quad (1.2)$$

Then the Kohn's 0th *multiplier ideal* for (1.1) is defined as the radical

$$I_0 := \sqrt{J(S)}, \quad S := \{F_1, \dots, F_m\}, \quad (1.3)$$

and for any $k > 0$, the Kohn's k th *multiplier ideal* is defined inductively by

$$I_k := \sqrt{J(S \cup I_{k-1})}. \quad (1.4)$$

The obtained increasing sequence of ideals $I_0 \subset I_1 \subset \dots$ is said to *terminate at k* if I_k contains the unit 1.

Kohn proved in [K79] that the necessary and sufficient condition for the ideal termination for special domains (in fact, for his more general algorithm applied to domains with *real-analytic* boundaries) is the finiteness of the D'Angelo type of $\partial\Omega$ at 0. Note that Kohn's algorithm is actually defined for domains with arbitrary smooth boundaries, in which generality, however, the corresponding termination remains a major open problem. For a special domain (1.1), the type equals $2\tau(S)$, where S is given by (1.3) and

$$\tau(S) := \sup_{\gamma} \inf_{f \in S} \frac{\nu(f \circ \gamma)}{\nu(\gamma)} \quad (1.5)$$

is the type of the set S of holomorphic function germs, where the supremum is taken over all nonconstant germs of holomorphic curves $\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$, and $\nu(g)$ denotes the vanishing order of a smooth vector function g at 0, given by the lowest order of its nonvanishing partial derivative at 0. We refer to [D82, D93] for more detailed discussions of finite type and [M92b, BSt92, FIK96, D17, MM17, Z17] for equivalent characterizations in various particular cases, see also [FLZ14].

It follows from Kohn's work [K79] that ideal termination at a boundary point $p \in \partial\Omega$ leads to a *subelliptic estimate of some order of subellipticity* $\varepsilon > 0$ at that point, namely

$$\|u\|_{\varepsilon}^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2), \quad (1.6)$$

where $\|\cdot\|_{\varepsilon}$ and $\|\cdot\|$ are respectively the tangential Sobolev norm of the (fractional) order ε and the standard L^2 norm on Ω , u is any $(0, 1)$ form in the domain of the adjoint operator $\bar{\partial}^*$ (with respect to the standard L^2 product on Ω), smooth up to the boundary and with compact support in a fixed neighborhood U of p in the closure $\bar{\Omega}$, such that both U and the real constant $C > 0$ are independent of u , see [K79, Definition 1.11] for more details. (Note that Kohn obtained his results for (p, q) forms with arbitrary p, q .) We mention that, even though the estimate (1.6) depends on the choice of local coordinates (or the metric used in the L^2 product), the subellipticity property (i.e. the existence of an estimate (1.6) for given ε) is invariant [Sw72, Ce08, CeSt09].

In the same paper, Kohn further proved that his algorithm does terminate for pseudoconvex domains with real-analytic boundaries of finite type, leading to subelliptic estimates, based on a result of K. Diederich and J.E. Fornæss [DiF78] (see also E. Bedford and J.E. Fornæss [BF81], Siu [S10, Part IV] and Kohn [K10]). For pseudoconvex domains with smooth boundaries of D'Angelo finite type, subelliptic estimates are due to D.W. Catlin [C87] by a different method, whereas the termination of Kohn's algorithm in that generality remains a major open problem (see e.g. the first problem in [DK99, Section 14]). Note that the finite type condition is also necessary for subelliptic estimates due to the work of P. Greiner [Gr74] in \mathbb{C}^2 and D.W. Catlin [C83] in \mathbb{C}^n .

It is crucial to point out the particularly remarkable feature of Kohn's procedure that allows us to study functional-analytic estimates such as (1.6) with purely geometric methods applied to ideals of holomorphic functions. Consequently, our approach here is accessible to any geometer without familiarity with the functional-analytic aspect of the problem.

1.2. Multipliers and effectivity. Recall [K79] that a germ at $p \in \partial\Omega$ of a smooth function on the closure $\bar{\Omega}$ is a *subelliptic multiplier with order of subellipticity* ε if, for some representative f of the germ, the estimate similar to (1.6) with $\|u\|_\varepsilon^2$ replaced by $\|fu\|_\varepsilon^2$ holds under the same assumptions, namely

$$\|fu\|_\varepsilon^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2), \quad (1.7)$$

where both U and C may depend on f . It follows from Kohn's work [K79, Section 4 and 7] that the ideals I_k as defined in (1.4) consist of subelliptic multipliers, i.e. satisfy (1.7) *with some order of subellipticity* ε that may depend on the actual multipliers. More precisely, Kohn showed that taking a Jacobian determinant in (1.2) reduces the order of subellipticity by $1/2$, whereas taking a root of order s in the radical reduces it by $1/s$. *It is the last step where an effective control of ε may get lost*, as it is a priori not clear what root order is required in order to obtain the full radical. In fact, examples of Heier [He08, Section 1.1] and Catlin-D'Angelo [CD10, Proposition 4.4] illustrate precisely that, i.e. the lack of control of ε as the parameter K in the example goes to infinity, whereas the type remains bounded (see Section 5 below). See also Siu [S17, 4.1] for a more elaborate and detailed explanation of this important phenomenon. A perturbation of Catlin-D'Angelo's example is treated in Section 5 below.

In order to regain the effectiveness, one needs to restrict the the number of steps and the root orders allowed in the radicals in terms of only the type and dimension, leading to an *effective Kohn algorithm* in the terminology of Siu [S17, 2.6]). In addition, it is desired to have algorithmic selection rules for constructing sequences of the actual multipliers ending with 1. Our goal here is to obtain a fine effectiveness control in terms of our new invariants (that are in turn controlled by the type) and a new selection procedure for the multipliers leading to an effective termination. In particular, we are going to the extreme to avoid taking radicals until the very last step, see Remark 1.2 below for details.

1.3. Applications and further directions. Among notable applications of subelliptic estimates (1.6), J.J. Kohn and L. Nirenberg [KN65] proved that the latter imply *local boundary regularity* of the Kohn's solution of the $\bar{\partial}$ -Neumann problem $\bar{\partial}u = f$, i.e. the solution u is smooth at those boundary points where f is. See [K72, DK99] and the references therein for more detailed discussions of this and many other applications.

Another important application of subelliptic estimates, specifically demonstrating the *importance of the effectivity*, is a lower bound on the Bergman metric directly related to the order of subellipticity, due to J.D. McNeal [M92a]. The effective control in subelliptic estimates also plays important role in the construction of peak functions by J.E. Fornæss and J.D. McNeal [FM94] and in other results.

It should be noted that applications of Kohn's algorithm and its effectiveness are not limited to subelliptic estimates. In their remarkable paper [DiF79], K. Diederich and J.E. Fornæss discovered a direct way of using Kohn's multipliers to construct so-called plurisubharmonic "bumping" functions for arbitrary domains with real-analytic boundaries of finite type. The bumping functions are subsequently applied in the same paper for Kobayashi metric estimates and Hölder regularity of proper holomorphic maps. However, the exponents in the crucial estimates could not be effectively controlled (in terms of the type and dimension) due to some steps involving Lojasiewicz inequality. Here effective procedures for generating multipliers would allow for more explicit quantitative conclusions.

In a more recent work, Kohn [K00, K02, K04] has demonstrated some new use of subelliptic multipliers by relating them to certain new microlocal subelliptic multipliers on real hypersurfaces, and establishing hypoellipticity of the \square_b and the $\bar{\partial}$ -Neumann operator. In [K05] Kohn introduced an analogue of his theory of subelliptic multipliers for new classes of differential operators. See also L. Baracco [Ba15] and L. Baracco, S. Pinton and G. Zampieri [BaPZa15] for recent related results.

Inspired by Kohn's original subelliptic multipliers, analogous notion of multiplier ideals for the compactness estimate were studied by M. Çelik [Ce08], E.J. Straube [St08], in their joint work [CeSt09] and by M. Çelik and Y.E. Zeytuncu [CeZ17]. Finally, we mention results by D. Chakrabarti and M.-C. Shaw [ChS11] connecting properties of the Kohn's solution for individual domains with corresponding properties for to their products, with some of their results recently used by X.-X. Chen and S.K. Donaldson [ChD13] to study rigidity properties of complex structures.

1.4. Main results. Our main new tool is the following invariant number associated to a set S of germs of holomorphic functions at 0 in \mathbb{C}^n , a germ of analytic subvariety $(V, 0)$ in \mathbb{C}^n and an integer k :

$$\tau_V^k(S) := \sup_{\gamma} \inf_{f \in S} \frac{\nu(j^k f \circ \gamma)}{\nu(\gamma)}, \quad k \geq 0, \quad (1.8)$$

where $j^k f = (\partial^\alpha f)_{|\alpha| \leq k}$ is the vector of all partial derivatives up to order k , and the supremum is taken over the set of all nonconstant germs of holomorphic maps $\gamma: (\mathbb{C}, 0) \rightarrow (V, 0)$, and the vanishing order ν is as above. We call τ_V^k the *k-jet type of S along V* . In particular, for $k = 0$, we obtain the D'Angelo type (1.5). It is easy to see that the k -jet types form a non-increasing sequence for $k = 0, 1, \dots$, and become equal to 0 whenever k is greater or equal the minimum vanishing order of a germ in S . Note that, even though the space of curves in (1.8) is infinite-dimensional, the computation of the k -jet type in \mathbb{C}^2 can be reduced to certain finite number of curves by a result of J.D. McNeal and A. Némethi [MN05] (see also [He08, LT08] for further results in this direction).

Since the effective termination is well-known if at least one function F_j has order 1 (see e.g. [CD10]), we shall assume that all functions F_j vanish of order ≥ 2 at 0. The following is a simplified version (with rougher bounds) of the main result of the paper:

THEOREM 1.1. *In the context of Kohn’s algorithm for special domains (1.1) in \mathbb{C}^3 of D’Angelo finite type $\leq 2T$ at 0, there is an effective algorithmic construction of a sequence of multipliers*

$$f_1, \dots, f_l, \quad l \leq T(T - 1) + 4, \tag{1.9}$$

with $f_l = 1$, based on the invariants (1.8). Every f_j is obtained by taking a Jacobian determinant of a linear combination of elements in the set

$$\{F_1, \dots, F_m, f_1, \dots, f_{j-1}\}, \tag{1.10}$$

except the two multipliers $f_{l-2} = z_1$, $f_{l-1} = z_2$ that are in the radical of the ideal $I := (f_1, \dots, f_{l-3})$ with a root order $s \leq T^2(T - 1)^3$, i.e. $z_1^s, z_2^s \in I$.

REMARK 1.2. We would like to point out the following particular features in our multiplier construction in comparison to other known procedures. From the three different procedures in Kohn’s construction, namely (1) taking Jacobian determinants, (2) forming ideals and (3) taking the radicals, only the first one is used in our process to obtain a pair of multipliers generating an ideal of finite (effectively controlled) multiplicity. Only then the ideal is formed and the radical is taken to obtain the linear multipliers, leading to the termination.

In particular, our procedure possesses certain stability under high order perturbations, as illustrated in §5. Such stability is generally not available in procedures based on taking radicals. For instance, in the ring of germs of holomorphic functions at 0 in $\mathbb{C}_{z,w}^2$, $f = w$ is in the radical of the ideal generated by $f^2 = w^2$, but any perturbation $g = f^2 + z^m = w^2 + z^m$ with m odd generates an ideal I_m having no other germs in its radical, i.e. $\sqrt{I_m} = I_m$ (since the zero variety of I_m is locally irreducible at 0). This problem does not occur in our construction because we only take radicals from ideals of finite codimension, where any germ vanishing at 0 is in the radical.

COROLLARY 1.3. *For special domains (1.1) of finite type $\leq 2T$ at 0 in \mathbb{C}^3 , a subelliptic estimate (1.6) at 0 holds with the order of subellipticity*

$$\varepsilon \geq \frac{1}{2^{l-1}s} \geq \frac{1}{2^{T(T-1)+3} T^2(T - 1)^3},$$

where l is the number of multipliers in (1.9) and s is the radical root order in Theorem 1.1.

Proof. It follows from Kohn’s work [K79, Sections 4 and 7], that in the context of Theorem 1.1, a Jacobian determinant of the F_j has the order of subellipticity $\geq 1/4$ (see [K79, (4.29, 4.64)] and [S17, 2.9.2]), and taking any further Jacobian determinant reduces the order of subellipticity by $1/2$ (see [K79, (4.42, 4.64)]), whereas taking a root of order s reduces it by $1/s$ (see [K79, 4.36]). Applying these calculations to the construction in Theorem 1.1, we obtain the desired conclusion. \square

Note that we obtained more refined bounds than those given in Theorem 1.1 and Corollary 1.3 in terms of our new invariants (1.8). Also we illustrate in Section 5 how our procedure can be applied to the Catlin-D’Angelo’s example as well as its higher order perturbations.

1.5. Overview of our procedure. Following [S10, S17], we call the functions F_j in (1.1) *pre-multipliers*, to distinguish them from the *multipliers* obtained via the algorithm. It is important to emphasize that the pre-multipliers are only used inside the Jacobian determinants (1.2) but are never added to the ideals directly.

On a large scale, there are 3 *major steps*, each reducing the dimension of the variety defined by the multipliers (resembling Kohn’s original algorithm for real-analytic hypersurfaces [K79]).

The *first step* consists of constructing the first multiplier f_1 as Jacobian determinant from the pre-multipliers with effectively bounded vanishing order at 0. Our method gives a bound of at most $\leq T(T - 1)$. In fact, a finer bound is given in terms of the k -jet types (as defined by (1.8)) of the pre-multipliers along their zero curves, see (4.1). Note that other bounds are known from the work of D’Angelo [D82, D93], Siu [S10, S17] and Nicoara [N12]. Any known effective bound d can be used at this step to proceed with our construction.

The *second major step* aims to reduce the dimension of the variety

$$V := \{f_1 = 0\}$$

from 1 to 0. Our method here is based on a sequence of *minor steps* constructing new multipliers that gradually reduce the jet order $k \geq 0$ for which the k -jet type along V can be effectively bounded. The construction starts with $k = d \leq T(T - 1)$ (the vanishing order of f_1), and at every minor step, the order k is reduced from k_j to $k_{j+1} < k_j$, where the new k_{j+1} -jet type along V gets an effective bound equal to the previous bound for the k_j -jet type plus at most $(k_j - k_{j+1})(T - 1)$. In other words, lowering the jet order by a number N increases the type bound by $N(T - 1)$. In fact, the method gives a finer bound with T replaced by the vanishing order of pre-multipliers only along curves in V . Based on the configuration of the k -jet types, multiple possible choices for k_j are available (see Section 3 for details), where fewer Jacobian determinant iterations can be traded for possibly higher vanishing order bound and vice versa. At the end of this major step, we obtain a multiplier f_{l-3} whose type (i.e. the (0-jet) type as defined for $k = 0$ in (1.8)) along V is effectively bounded by an estimate no worse than $d(T - 1)$ (where $d \leq T(T - 1)$ is the vanishing order of f_1). The proof is based on the core technical results in Section 3 with examples given illustrating the sharpness of the assumptions.

Finally, the *third major step* consists of taking the coordinate functions $f_{l-2} := z_1$ and $f_{l-1} := z_2$ in the radical of the finite type ideal $I(f_1, f_{l-3})$, and subsequently their Jacobian determinant $f_l = 1$. The corresponding radical root order can be effectively bounded by

$$td \leq d^2(T - 1) \leq T^2(T - 1)^3$$

in terms of the vanishing order d of f_1 and the type $t \leq d(T - 1)$ of f_{l-3} along V . See Lemma 4.3 for the bound td .

2. Vanishing orders, contact orders and jets.

2.1. Normalised vanishing order. We write $f: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$ for a germ at 0 of a holomorphic function in \mathbb{C}^n (without specifying the value $f(0)$), and

$$\nu(f) := \min\{|\alpha| : \partial_z^\alpha f(0) \neq 0\} \in \mathbb{N} \cup \{\infty\}, \quad \mathbb{N} = \{0, 1, \dots\}$$

for the *vanishing order* of f , also called *multiplicity* in the literature, (the minimum is ∞ if the set is empty), where

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| := \alpha_1 + \dots + \alpha_n,$$

is a multiindex and

$$\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n}$$

is the corresponding partial derivative with respect to chosen coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$. Clearly $\nu(f)$ does not depend on the choice of local holomorphic coordinates.

More generally, for any set S of germs of holomorphic maps (on the same \mathbb{C}^n), define its vanishing order to be the minimum vanishing order for its elements:

$$\nu(S) := \min\{\nu(f) : f \in S\}.$$

It is clear that $\nu(S) = \nu(I(S))$, where I is the ideal generated by S . In particular, for any holomorphic map germ

$$f = (f_1, \dots, f_m) : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^m,$$

define its vanishing order to be the vanishing order of the set of its components, i.e.

$$\nu(f) := \min_{j=1, \dots, m} \nu(f_j).$$

It is again easy to see that $\nu(f)$ also does not depend on the choice of local holomorphic coordinates in \mathbb{C}^m in a neighborhood of $f(0)$.

Now following D'Angelo [D82, D93], for every germ of a holomorphic map

$$\gamma \neq 0 : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0), \quad f : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^\ell,$$

define its *normalized vanishing order of f along γ* by

$$\nu_\gamma(f) := \frac{\nu(f \circ \gamma)}{\nu(\gamma)}. \tag{2.1}$$

In case $m = 1$ (considered here), the normalized vanishing order is invariant under singular parameter changes $\gamma \mapsto \gamma \circ \varphi$, where $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ is any nonconstant germ of a holomorphic map. Similarly, for a set S of holomorphic map germs, define its normalized vanishing order along γ by

$$\nu_\gamma(S) := \min\{\nu_\gamma(f) : f \in S\},$$

and again $\nu_\gamma(S) = \nu_\gamma(I(S))$ for the ideal $I(S)$ generated by S .

Note that in general, the normalized vanishing order may not be an integer, and, in fact, can be any rational number $p/q > 1$ as the example

$$\gamma(t) := (t^p, t^q), \quad f(z, w) := z, \quad \nu_\gamma(f) = p/q,$$

shows.

It is easy to see that given f , one has $\nu_\gamma(f) = \nu(f)$ for a generic linear map γ . In general, $\nu_\gamma(f)$ can only become larger:

LEMMA 2.1. *The normalized vanishing order of f along any map γ is always greater or equal than the vanishing order:*

$$\nu_\gamma(f) \geq \nu(f). \tag{2.2}$$

Proof. Expanding into a power series $f = \sum f_\alpha z^\alpha$ with $|\alpha| \geq \nu(f)$, and substituting γ , we conclude $\nu(f \circ \gamma) \geq \nu(f) \nu(\gamma)$ as desired. \square

2.2. Contact order and finite type. Let S be a set of germs of holomorphic functions $f: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$. Then in view of Lemma 2.1 (and the remark before it), the vanishing order satisfies

$$\nu(S) = \min_{\gamma} \nu_{\gamma}(S).$$

At the opposite end, define (as in [D82] and [D93, 2.3.2, Definition 9]) the *contact order* or *the type* of S by

$$\tau(S) := \sup_{\gamma} \nu_{\gamma}(S) = \sup_{\gamma} \min_{f \in S} \nu_{\gamma}(f).$$

(Note that the minimum in the right-hand side is always achieved, since the denominator in (2.1) is fixed.) We say that the set S is of *finite type* $T = \tau(S)$ if the latter number is finite. Since $\nu_{\gamma}(S) = \nu_{\gamma}(I(S))$ for the ideal $I(S)$ generated by S , we also have

$$\nu(S) = \nu(I(S)), \quad \tau(S) = \tau(I(S)).$$

2.3. Contact order along subvarieties.. Let $f: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^m$ be a germ of a holomorphic map as before and $V \subset \mathbb{C}^n$ any complex-analytic subvariety passing through 0. Then the *contact order* of f along V is defined by

$$\tau_V(f) := \sup_{\gamma} \nu_{\gamma}(f) \tag{2.3}$$

where the supremum is taken over all non-constant germs of holomorphic maps $\gamma: (\mathbb{C}, 0) \rightarrow (V, 0)$.

In particular, if V is (a germ of) an irreducible curve at 0, the right-hand side in (2.3) is independent of γ , since in this case, any two nonzero germs $\gamma: (\mathbb{C}, 0) \rightarrow (V, 0)$ are related by a sequence of singular reparametrizations. In general, the contact order along V is the maximum contact order along the irreducible components of V at 0.

2.4. Jet vanishing orders. The main new tool in this paper is the following notion of jet vanishing order.

For every integer $k \geq 0$, and a germ of a holomorphic map $f: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^m$, consider its k -jet $j^k f$, which in local coordinates can be regarded as a germ of a holomorphic map

$$F = j^k f = (\partial_z^{\alpha} f)_{|\alpha| \leq k}: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^N,$$

(for suitable integer N dependent on n, m and k) given by all partial derivatives of the components of f up to order k . Then define the *k -jet vanishing order* of f by

$$\nu^k(f) := \nu(j^k f) = \min_{|\alpha| \leq k} \nu(\partial_z^{\alpha} f), \tag{2.4}$$

which is also equal to $\max(\nu(f) - k, 0)$.

Further, define the *k -jet normalized vanishing order* of f along a nonzero germ $\gamma: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ by

$$\nu_{\gamma}^k(f) := \nu_{\gamma}(j^k f) = \min_{|\alpha| \leq k} \nu_{\gamma}(\partial_z^{\alpha} f),$$

(which are the minimum vanishing orders of the partial derivatives up to order k along γ), and the k -jet contact order of f along a subvariety V by

$$\tau_V^k(f) := \tau_V(j^k f) = \sup_{\gamma} \nu_{\gamma}^k(f), \tag{2.5}$$

where as in (2.3), the supremum is taken over all non-constant germs of holomorphic maps $\gamma: (\mathbb{C}, 0) \rightarrow (V, 0)$. Note that again, this definition is independent of holomorphic local coordinates, and we have the monotonicity:

$$\nu^{k_1}(f) \geq \nu^{k_2}(f), \quad \nu_{\gamma}^{k_1}(f) \geq \nu_{\gamma}^{k_2}(f), \quad \tau_V^{k_1}(f) \geq \tau_V^{k_2}(f), \quad k_1 \leq k_2. \tag{2.6}$$

As consequence of (2.4) and the definitions, we have the stabilisation property

$$\nu^k(f) = \nu_{\gamma}^k(f) = \tau_V^k(f) = 0, \quad k \geq \nu(f), \tag{2.7}$$

for any γ and any V .

Similarly, for any set S of germs of holomorphic maps $f: (\mathbb{C}^n, 0) \rightarrow \mathbb{C}^m$, define the numbers

$$\nu^k(S) := \nu(j^k S), \quad \nu_{\gamma}^k(S) := \nu_{\gamma}(j^k S), \quad \tau^k(S) := \tau(j^k S), \quad \tau_V^k(S) := \tau_V(j^k S),$$

that, in fact, depend only on the ideal generated by S , where $j^k S$ denotes the set of all k -jets of elements in S .

3. Control of jet vanishing orders for Jacobian determinants. This section is the technical core of the paper. Our goal is to obtain fine control of how the jet vanishing orders along curves change under taking Jacobian determinants. The main idea is to have a control of certain “good” terms in the multiplier expansion and to avoid possible cancellations with other terms. This is achieved via certain technical conditions comparing jet vanishing orders for different jet orders. We also illustrate by examples that our technical conditions are sharp.

For a holomorphic curve germ $\gamma \neq 0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$, we can always change coordinates $(z, w) \in \mathbb{C}^2$ to achieve

$$\gamma = (\alpha, \beta), \quad \nu(\alpha) > \nu(\beta) \geq 1. \tag{3.1}$$

We first give a comparison condition between jet vanishing orders $\nu_{\gamma}^{k-1}(F)$ and $\nu_{\gamma}^k(F)$ of a function germ F along γ to guarantee that the minimum vanishing order along γ among partial derivatives of F is achieved for its transversal derivatives.

LEMMA 3.1. *Given γ satisfying (3.1) and a germ of a holomorphic map*

$$F: (\mathbb{C}^2, 0) \rightarrow \mathbb{C},$$

assume that the jet vanishing orders of F along γ satisfy

$$\nu_{\gamma}^{k-1}(F) > \nu_{\gamma}^k(F) + 1 \tag{3.2}$$

for some $k \geq 1$.

Then

$$\nu_{\gamma}^k(F) = \nu_{\gamma}(\partial_z^k F), \tag{3.3}$$

i.e. the minimum vanishing order among all partial derivatives of order k is achieved for the k th derivative transversal to (the image of) γ :

$$\nu_\gamma(\partial_z^k F) = \min_{a+b \leq k} \nu_\gamma(\partial_z^a \partial_w^b F).$$

Proof. Assume on the contrary that

$$\nu_\gamma(\partial_z^a \partial_w^b F) < \nu_\gamma(\partial_z^k F)$$

for some $a < k$ and $b \leq k - a$. We can choose a and b such that the minimum vanishing order is achieved, i.e.

$$\nu_\gamma^k(F) = \nu_\gamma(\partial_z^a \partial_w^b F) < \nu_\gamma(\partial_z^k F). \tag{3.4}$$

In view of (3.2), we must have the top order $a + b = k$ here, in particular, $b = k - a \geq 1$. Then

$$\nu(\partial_z^a \partial_w^b F(\gamma)) \leq \nu(\partial_z^{a+1} \partial_w^{b-1} F(\gamma)) < \nu\left(\partial_z^{a+1} \partial_w^{b-1} F(\gamma) \frac{\alpha'}{\beta'}\right), \tag{3.5}$$

as we have assumed $\nu(\alpha) > \nu(\beta)$. Differentiating in the parameter of γ , we obtain

$$(\partial_z^a \partial_w^{b-1} F(\gamma))' = \beta' \left(\partial_z^{a+1} \partial_w^{b-1} F(\gamma) \frac{\alpha'}{\beta'} + \partial_z^a \partial_w^b F(\gamma) \right),$$

from which, using (3.5), we conclude

$$\nu((\partial_z^a \partial_w^{b-1} F(\gamma))') = \nu(\partial_z^a \partial_w^b F(\gamma)) + \nu(\beta) - 1. \tag{3.6}$$

Since our definition of the k -jet vanishing order implies

$$\nu((\partial_z^a \partial_w^{b-1} F(\gamma))') \geq \nu(\beta) \nu_\gamma^{k-1}(F) - 1,$$

we have in view of (3.4) and (3.6),

$$\nu(\beta) \nu_\gamma^k(F) = \nu(\partial_z^a \partial_w^b F(\gamma)) = \nu((\partial_z^a \partial_w^{b-1} F(\gamma))') - \nu(\beta) + 1 \geq \nu(\beta) \nu_\gamma^{k-1}(F) - \nu(\beta). \tag{3.7}$$

On the other hand, by our assumption (3.2), we have

$$\nu(\beta) \nu_\gamma^{k-1}(F) - \nu(\beta) > \nu(\beta) \nu_\gamma^k(F),$$

which contradicts (3.7) completing the proof. \square

EXAMPLE 3.2. For

$$\gamma(t) = (0, t), \quad F(z, w) = w^2 + zw^2,$$

compute

$$\nu_\gamma^0(F) = \nu_\gamma(F) = 2, \quad \nu_\gamma(\partial_z F) = 2, \quad \nu_\gamma(\partial_w F) = 1, \quad \nu_\gamma^1(F) = 1.$$

Hence (3.2) is violated and the conclusion of Lemma 3.1 fails. This shows the sharpness of our assumption (3.2).

The following is our main technical result. Again, we need to assume a comparison condition between different jet vanishing orders to guarantee that the terms from lower order jets do not cancel with the terms providing the needed vanishing order control. Here we use different letters F and φ for the function germs to emphasize their different roles. The role of F is to provide the jet vanishing orders $\nu_\gamma^k(F)$, whereas for φ , only the usual vanishing order $\nu_\gamma(\varphi)$ is used.

LEMMA 3.3. *Given germs of holomorphic maps*

$$\gamma \neq 0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0), \quad F, \varphi: (\mathbb{C}^2, 0) \rightarrow \mathbb{C},$$

suppose that

$$\nu(\varphi) \geq 2, \tag{3.8}$$

and for some $k \geq 1$,

$$\nu_\gamma^{k-1}(F) > \nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1, \tag{3.9}$$

where ν_γ^k is the k -jet vanishing order along γ . Then the Jacobian determinant

$$G := \det \begin{pmatrix} \partial F \\ \partial \varphi \end{pmatrix}$$

satisfies

$$\nu_\gamma^{k-1}(G) = \nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1.$$

Proof. By definition, we have

$$G = \partial_z F \partial_w \varphi - \partial_w F \partial_z \varphi,$$

and hence

$$\partial_z^a \partial_w^b G = \partial_z^{a+1} \partial_w^b F \partial_w \varphi - \partial_z^a \partial_w^{b+1} F \partial_z \varphi + \text{error terms}, \quad a + b = k - 1, \tag{3.10}$$

where the error terms are bilinear expressions in $j^{k-1}F$ and $j^k\varphi$.

After a coordinate change if necessary, we may assume that the assumptions of Lemma 3.1 are satisfied, in particular

$$\nu(\gamma) = \nu(\beta)$$

and $\beta \neq 0$.

Next our assumption (3.9) implies that the (normalized) vanishing order of the error terms in (3.10) along γ is strictly bigger than

$$\nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1.$$

Therefore, to prove the lemma, it is enough to show that

$$\min_{a+b=k-1} (\nu_\gamma(\partial_z^{a+1} \partial_w^b F \partial_w \varphi - \partial_z^a \partial_w^{b+1} F \partial_z \varphi)) = \nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1. \tag{3.11}$$

Since

$$(\partial_z^a \partial_w^b F(\gamma))' = \partial_z^{a+1} \partial_w^b F(\gamma) \alpha' + \partial_z^a \partial_w^{b+1} F(\gamma) \beta'$$

and

$$(\varphi(\gamma))' = \partial_z \varphi(\gamma) \alpha' + \partial_w \varphi(\gamma) \beta',$$

substituting these into the first two terms in the right-hand side of (3.10), we obtain

$$\begin{aligned} & \partial_z^{a+1} \partial_w^b F(\gamma) \partial_w \varphi(\gamma) - \partial_z^a \partial_w^{b+1} F(\gamma) \partial_z \varphi(\gamma) \\ &= \partial_z^{a+1} \partial_w^b F(\gamma) \frac{(\varphi(\gamma))'}{\beta'} - \frac{(\partial_z^a \partial_w^b F(\gamma))'}{\beta'} \partial_z \varphi(\gamma). \end{aligned} \tag{3.12}$$

Since for $a + b = k - 1$,

$$\begin{aligned} \nu \left(\frac{(\partial_z^a \partial_w^b F(\gamma))'}{\beta'} \partial_z \varphi(\gamma) \right) &= \nu((\partial_z^a \partial_w^b F(\gamma))') - \nu(\beta') + \nu(\partial_z \varphi(\gamma)) \\ &= \nu((\partial_z^a \partial_w^b F(\gamma))) - 1 - \nu(\beta) + 1 + \nu(\partial_z \varphi(\gamma)) \\ &= \nu(\beta) (\nu_\gamma(\partial_z^a \partial_w^b F) - 1 + \nu_\gamma(\partial_z \varphi)) \\ &\geq \nu(\beta) (\nu_\gamma^{k-1}(F) - 1 + \nu_\gamma(\partial_z \varphi)) \end{aligned}$$

we obtain by (3.9), (3.8) and Lemma 2.1,

$$\nu \left(\frac{(\partial_z^a \partial_w^b F(\gamma))'}{\beta'} \partial_z \varphi(\gamma) \right) > \nu(\beta) (\nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1), \quad a + b \leq k - 1. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned} \nu \left(\partial_z^{a+1} \partial_w^b F(\gamma) \frac{(\varphi(\gamma))'}{\beta'} \right) &= \nu(\partial_z^{a+1} \partial_w^b F(\gamma)) + \nu((\varphi(\gamma))') - \nu(\beta') \\ &= \nu(\partial_z^{a+1} \partial_w^b F(\gamma)) + \nu(\varphi(\gamma)) - \nu(\beta) \\ &= \nu(\beta) (\nu_\gamma(\partial_z^{a+1} \partial_w^b F) + \nu_\gamma(\varphi) - 1), \end{aligned}$$

where in view of (3.3), the minimum of the vanishing order of the last expression for $a + b = k - 1$ is achieved for $(a, b) = (k - 1, 0)$ and equals to

$$\nu(\beta) (\nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1). \tag{3.14}$$

Together with (3.13), this implies that the minimum for $a + b = k - 1$ of the right-hand side in (3.12) equals (3.14). This gives the desired relation (3.11), completing the proof. \square

EXAMPLE 3.4. Let

$$F(z, w) = z^k + z^{k-1}w^s, \quad \varphi(z, w) = w^l + azw, \quad \gamma(t) = (0, t), \quad l \geq 2.$$

Then

$$\nu_\gamma^{k-1}(F) = s, \quad \nu_\gamma^k(F) = 0, \quad \nu_\gamma(\varphi) = l.$$

In particular, our comparison assumption (3.9) is violated if and only if $s \leq l - 1$. Then for the Jacobian determinant G as Lemma 3.3, we have

$$G = \det \begin{pmatrix} kz^{k-1} + (k-1)z^{k-2}w^s & sz^{k-1}w^{s-1} \\ aw & lw^{l-1} + a(l-s)zw \end{pmatrix}$$

and hence for $s = l - 1$ and $a = lk/s$, the terms with $z^{k-1}w^{l-1}$ cancel and we obtain

$$\nu_\gamma^{k-1}(G) = \min\left(\nu_\gamma^{k-1}(z^{k-2}w^{s+l-1}), \nu_\gamma^{k-1}(z^{k-1}w^{s+1})\right) = \min(s + l - 1, s + 1) = l,$$

failing the conclusion (3.9) of Lemma 3.3. Hence our assumption (3.9) is sharp.

As first direct consequence in combination with Lemma 2.1, we obtain an estimate for the (total) vanishing order of the Jacobian determinant:

COROLLARY 3.5. *Under the assumptions of Lemma 3.3,*

$$\nu(G) \leq \nu_\gamma^k(F) + \nu_\gamma(\varphi) + k - 1.$$

As next immediate consequence of Lemma 3.3, we obtain:

COROLLARY 3.6. *Given germs of holomorphic maps*

$$\gamma \neq 0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0), \quad F, \varphi: (\mathbb{C}^2, 0) \rightarrow \mathbb{C},$$

suppose that

$$\nu(\varphi) \geq 2. \tag{3.15}$$

Then

$$\nu_\gamma^{k-1}(G) \leq \nu_\gamma^k(F) + \nu_\gamma(\varphi) - 1,$$

where either $G = F$ or $G = \det \begin{pmatrix} \partial F \\ \partial \varphi \end{pmatrix}$.

Repeatedly applying Corollary 3.6, we obtain:

COROLLARY 3.7. *Given germs of holomorphic maps*

$$\gamma \neq 0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0), \quad f, \varphi: (\mathbb{C}^2, 0) \rightarrow \mathbb{C},$$

suppose that $\nu(\varphi) \geq 2$, and define f_{j+1} inductively by

$$f_1 := f, \quad f_{j+1} := \det \begin{pmatrix} \partial f_j \\ \partial \varphi \end{pmatrix}, \quad j = 1, 2, \dots$$

Then for every $k \in \{0, \dots, \nu(f)\}$, there exists $j \in \{1, \dots, \nu(f) - k + 1\}$ such that

$$\nu_\gamma^k(f_j) \leq (\nu(f) - k)(\nu_\gamma(\varphi) - 1).$$

In particular, for $k = 0$,

$$\nu_\gamma(f_j) \leq \nu(f)(\nu_\gamma(\varphi) - 1)$$

holds for some $j \in \{1, \dots, \nu(f) + 1\}$.

Proof. Setting $m := \nu(f)$, we obtain from the definition of the m -jet vanishing order that

$$\nu_\gamma^m(f_1) = \nu_\gamma^m(f) = 0.$$

Then Corollary 3.6 implies that there exist $j \in \{1, 2\}$ such that

$$\nu_\gamma^{m-1}(f_j) \leq \nu_\gamma(\varphi) - 1.$$

Next let $k = m - 1$ and repeat the argument for $F = f_j$, to conclude that there exist $j \in \{1, 2, 3\}$ such that

$$\nu_\gamma^{m-2}(f_j) \leq 2(\nu_\gamma(\varphi) - 1).$$

We repeat this process to show inductively that for any positive integer $\ell \leq m$, there exists $j \in \{1, \dots, \ell + 1\}$ such that

$$\nu_\gamma^\ell(f_j) \leq (m - \ell)(\nu_\gamma(\varphi) - 1)$$

as desired. \square

4. Selection algorithm and Proof of Theorem 1.1. We shall write \mathcal{C} for the set of nonzero germs of holomorphic maps $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$. Let S be a set of holomorphic function germs $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ with minimal vanishing order

$$m := \nu(S) \geq 2,$$

and the finite type

$$T := \tau(S).$$

4.1. Step 1: selecting a multiplier with minimal vanishing order. Choose any $f_0 = f \in S$ with minimal (total) vanishing order

$$\nu(f) = \nu(S) = m.$$

Then choose any $\gamma \in \mathcal{C}$ in the zero set of f (i.e. $\nu_\gamma(f) = \infty$). Finally choose any $\varphi \in S$ with

$$\nu_\gamma(\varphi) \leq T$$

and set

$$\mu := \min\{k - 1 + \nu_\gamma^k(f) : \nu_\gamma^{k-1}(f) > \nu_\gamma^k(f) + \nu_\gamma(\varphi) - 1, 1 \leq k \leq m\}.$$

Since $\nu_\gamma(f) = \infty$, it is easy to see that

$$\mu \leq k_0 - 1 + \nu_\gamma^{k_0}(f) \leq k_0 - 1 + (m - k_0)(\nu_\gamma(\varphi) - 1) \leq (m - 1)(\nu_\gamma(\varphi) - 1),$$

where

$$k_0 := \max\{k : \nu_\gamma^{k-1}(f) > \nu_\gamma^k(f) + \nu_\gamma(\varphi) - 1, 1 \leq k \leq m\}.$$

Then the multiplier

$$f_1 := \det \begin{pmatrix} \partial f \\ \partial \varphi \end{pmatrix}.$$

satisfies for some k ,

$$k + \nu_\gamma^k(f_1) \leq \mu + \nu_\gamma(\varphi) - 1$$

in view of Lemma 3.3. In particular, it follows that the vanishing order

$$\nu(f_1) \leq \mu + \nu_\gamma(\varphi) - 1 \leq m(\nu_\gamma(\varphi) - 1) \leq m(T - 1). \tag{4.1}$$

Thus we have found a multiplier $f_1 \in J(S)$ (as defined in §1.1) whose vanishing order is bounded by any of the above estimates.

REMARK 4.1. The main outcome of this step is to construct a multiplier f_1 with controlled vanishing order. Our method gives an explicit estimate for this order. Note that the initial function (pre-multiplier) $f_0 \in S$ is only used in this step. Once a multiplier f_1 is constructed, it will be used in the sequel for the curve selection and the computation of jet vanishing orders.

4.2. Step 2: selecting a multiplier with effectively bounded order along a zero curve of another multiplier. Now choose any multiplier f_1 with vanishing order

$$\nu(f_1) \leq d \leq m(T - 1).$$

Note that Step 1 yields such f_1 , however, any other f_1 with a sharper estimate d can be chosen. Let $\gamma_i \in \mathcal{C}$ be germs of curves parametrizing irreducible components of the zero set of f_1 . Choose $\varphi \in \text{span}(S)$ with

$$\nu_{\gamma_i}(\varphi) \leq T,$$

which can be any generic linear combination of elements in S . Then Corollary 3.7 implies that

$$\nu_{\gamma_i}^{m-1}(f_j) \leq T - 1,$$

where $j \in \{1, 2\}$ depending on i , and

$$f_2 := \det \begin{pmatrix} \partial f_1 \\ \partial \varphi \end{pmatrix}.$$

LEMMA 4.2. *Let Φ, S_0 be two sets of germs of holomorphic maps $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ and define inductively*

$$S_{j+1} := S_j \cup \left\{ \det \begin{pmatrix} \partial f \\ \partial \varphi \end{pmatrix} : f \in S_j \cup \Phi, \varphi \in \Phi \right\}, \quad j \geq 0.$$

Write

$$d := \nu(S_0)$$

for the minimum vanishing order of functions in S_0 and assume that the type

$$\tau(\Phi) := \sup_\gamma \nu_\gamma(\Phi) = \sup_\gamma \min_{\varphi \in \Phi} \nu_\gamma(\varphi)$$

is finite. Assume further that

$$\nu(\Phi) \geq 2.$$

Then for $k < d$, the k -jet type of S_j satisfies

$$\tau^k(S_{d-k}) \leq (d - k)(\tau(\Phi) - 1).$$

In particular, for $k = 0$, we obtain

$$\tau(S_d) \leq d(\tau(\Phi) - 1).$$

Proof. Choose $f \in S_0$ with

$$\nu(f) = d$$

and let γ parametrize a curve in \mathbb{C}^2 . By the definition of type, there exists $\varphi \in \Phi$ with

$$\nu_\gamma(\varphi) \leq \tau(\Phi).$$

Then Corollary 3.7 implies that there exists a sequence $F_k \in S_k$ such that

$$\nu_\gamma^{d-k}(F_k) \leq (d - k)(\nu_\gamma(\varphi) - 1),$$

which completes the proof. \square

4.3. Step 3: selecting a pair of multipliers with effectively bounded multiplicity and taking the radical. Let Φ be the set of pre-multipliers and let

$$S_0 := \left\{ \det \begin{pmatrix} \partial f \\ \partial \varphi \end{pmatrix} : f, \varphi \in \Phi \right\}.$$

Then Step 2 implies that there exists an integer $d \leq m(T - 1)$ such that

$$\tau(S_d) \leq d(T - 1).$$

Choose $F \in S_0$ such that

$$\nu(F) = d.$$

Then by the definition of $\tau(S_d)$, for each irreducible component C_j of $\{F = 0\}$, there exists $F_j \in S_d$ such that

$$\nu_{C_j}(F_j) \leq d(T - 1).$$

Choose a generic linear combination \tilde{F} of $\{F_j\}$. Then \tilde{F} is a multiplier satisfying

$$\nu_{C_j}(\tilde{F}) \leq d(T - 1) \text{ for all } j. \tag{4.2}$$

LEMMA 4.3. *Let $I = (F, \tilde{F})$ be the ideal generated by F and \tilde{F} . Let $d := \nu(F)$ be the vanishing order and $t := \tau_{\{F=0\}}(\tilde{F})$ the contact order of \tilde{F} along the zero curve of F (i.e. the maximal vanishing order along an irreducible component). Then the multiplicity*

$$D(I) := \dim(\mathbb{C}\{z, w\}/I) \tag{4.3}$$

satisfies $D(I) \leq dt$.

Proof. The proof can be obtained by following the arguments of D’Angelo’s proof of Theorem 2.7 in [D82], more precisely, by combining [D82, (2.12)] with [D82, Lemma 2.11]. \square

REMARK 4.4. A pair (F, \tilde{F}) satisfying the assumptions of Lemma 4.3 can be called an *effective regular sequence*. Recall that a *regular sequence* (see e.g. [D93, 2.2.3, Definition 6]) is any sequence (f_1, \dots, f_s) in a local ring R , if each f_{j+1} is a non-zero-divisor in the quotient $R/(f_1, \dots, f_j)$ for any $j = 1, \dots, s$ (for $j = 1$, the quotient needs to be interpreted as R itself). In our case, the additional *effectiveness* of the regular sequence condition comes from the estimates of the two orders – the vanishing order of F and the contact order of \tilde{F} along the zero curve C of F . Note that for the conclusion of Lemma 4.3 to hold, we need to estimate the *contact order* (the maximum order over irreducible components of C) for \tilde{F} rather than its vanishing order along C (the minimum order over the irreducible components).

COROLLARY 4.5. *Under the assumptions of Lemma 4.3, for $a := dt$, one has $h^a \in I$ for any germ of holomorphic function h with $h(0) = 0$, i.e. I contains the a -th power of the maximal ideal.*

Proof. The statement follows directly from the inequality $K(I) \leq D(I)$ in [D82, Theorem 2.7], see also [D93, 2.1.6, Page 57], where $K(I)$ is the minimal power of the maximal ideal contained in I and $D(I)$ is the multiplicity. \square

Proof of Theorem 1.1. By Corollary 4.5, we have z^a, w^a as multipliers, where

$$a = dt, \quad d \leq T(T - 1), \quad t \leq d(T - 1)$$

implying, in particular,

$$a \leq T^2(T - 1)^3.$$

Taking the a th roots z, w of z^a, w^a , and taking their Jacobian determinant, we obtain the last multiplier 1 leading to the algorithm termination. Furthermore, taking generic linear combinations at each step, we obtain a single sequence f_1, \dots, f_l of multipliers with $f_{l-3} = \tilde{F}$ satisfying (4.2), and hence satisfying the desired properties as in the statement of Theorem 1.1. This completes the proof. \square

5. Perturbations of Heier’s and Catlin-D’Angelo’s examples. Heier [He08] and Catlin and D’Angelo [CD10] gave examples of special domains in \mathbb{C}^3 , where the original Kohn’s procedure [K79] of taking full radicals at every step does not lead to an effective estimate (in terms of the type and the dimension) for the order of subellipticity in subelliptic estimate. The main reason is the lack of control of the root order in the radical. Heier’s example is

$$\operatorname{Re} z_3 + |z_1^3 + z_1 z_2^K|^2 + |z_2|^2, \quad K \geq 2,$$

where the set of pre-multipliers is $S = \{z_1^3 + z_1 z_2^K, z_2\}$ and a calculation of Kohn’s multiplier ideals yields, in the notation of Section 1.1:

$$I_0 = \sqrt{J(S)} = J(S) = (3z_1^2 + z_2^K), \quad J(S \cup I_0) = (z_1, z_2^K), \quad I_1 = \sqrt{J(S \cup I_0)} = (z_1, z_2),$$

where the last radical requires taking elements of the root order K that can be arbitrarily high in comparison with the D’Angelo type 6. As consequence, the corresponding order of subellipticity ε in the subelliptic estimate (1.6) obtained this way

is not effectively controlled by the type. Note that since $z_2 \in S$, $z_1 \in J(S \cup I_0)$, it is easy to regain the effectivity by taking another Jacobian determinant instead of the radical to obtain

$$1 \in J(S \cup J(S \cup I_0)),$$

leading to an effective subelliptic estimate with $\varepsilon = 1/8$. In particular, the step of taking radicals for this example can be avoided completely, due to the presence of the linear pre-multiplier $z_2 \in S$.

A similar lack of effectivity phenomenon is exhibited in the Catlin-D’Angelo’s example as explained in [CD10, Section 4] and [S17, Section 4] (where, however, it is not possible to avoid taking radicals when there is no pre-multiplier of vanishing order 1). Furthermore, the same lack of control with the same argument also applies to higher order perturbations of that example (that may not be of a triangular form).

More precisely, consider perturbations $\Omega \subset \mathbb{C}^3$ of the Catlin-D’Angelo’s example given by

$$\operatorname{Re} z_3 + |F_1(z_1, z_2)|^2 + |F_2(z_1, z_2)|^2 < 0,$$

$$F_1 = z_1^M + O(L), \quad F_2 = z_2^N + z_1^K z_2 + O(L), \quad K > M \geq 2, \quad N \geq 3,$$

where we use the notation

$$O(L) = O(|(z_1, z_2)|^L).$$

Then for L sufficiently large, the explicit calculations in [CD10, Proposition 4.4] and [S17, Section 4.1] show that the root order required for the radical I_1 of $J_1 := J(S \cup I_0)$ (in the notation of §1.1) is at least K , i.e. $(I_1)^{K-1} \not\subset J_1$. As consequence, the order of subellipticity obtained this way is not effectively controlled by the type.

We now illustrate how our selection procedure modifying the original Kohn’s algorithm applies to this case. For simplicity, we shall assume L sufficiently large but effectively bounded from below by the type $T = \max(M, N)$ (i.e. $L \geq \Phi(T)$ for suitable function Φ that can be directly computed), and K being sufficiently large (when the above non-effectivity occurs). The first multiplier generating $J_0 := J(S)$ is

$$\begin{aligned} f_1 := \det \begin{pmatrix} \partial F_1 \\ \partial F_2 \end{pmatrix} &= \det \begin{pmatrix} Mz_1^{M-1} & 0 \\ Kz_1^{K-1}z_2 & Nz_2^{N-1} + z_1^K \end{pmatrix} \\ &= MNz_1^{M-1}z_2^{N-1} + Mz_1^{K+M-1} + O(L-1), \end{aligned} \tag{5.1}$$

where the perturbation error estimate $L - 1$ is only rough for the sake of simplicity. Then a monodromy argument implies that the zero set of f_1 is the union of the (possibly reducible) curves

$$z_1^{M-1} = O(L - N), \quad Nz_2^{N-1} + z_1^K = O(L - M). \tag{5.2}$$

For any parametrisation γ of an irreducible component of the first curve, we take in our Step 2

$$\varphi = F_2, \quad \nu_\gamma(\varphi) = N,$$

and compute

$$f_2 = \det \begin{pmatrix} \partial f_1 \\ \partial \varphi \end{pmatrix} \sim z_1^{M-2}z_2^{2(N-1)} + O(L-2) \pmod{z_1^{K-1}},$$

where we write \sim for the equality up to a constant factor. The corresponding jet vanishing orders of f_1 are

$$\begin{aligned} \nu_\gamma^0(f_1), \dots, \nu_\gamma^{M-2}(f_1) &\geq \frac{L-N}{M-1}, \quad \nu_\gamma^{M-1}(f_1) = N-1, \\ \nu_\gamma^M(f_1) &= N-2, \quad \dots, \quad \nu_\gamma^{M+N-2}(f_1) = 0. \end{aligned}$$

Then our comparison condition (3.10) holds for $k = M - 1$, and hence Lemma 3.3 gives the desired control of the next lower jet vanishing order

$$\nu_\gamma^{M-2}(f_2) = 2(N-1).$$

Next, continuing as in Step 2 in the previous section, we obtain the sequence of multipliers

$$f_j = \det \left(\frac{\partial f_{j-1}}{\partial \varphi} \right) \sim z_1^{M-j} z_2^{j(N-1)} + O(L-j) \pmod{z_1^{K-j+1}}, \quad j = 1, 2, \dots, M,$$

where the last multiplier

$$f_M = z_2^{M(N-1)} + O(L-M) \pmod{z_1^{K-M+1}}$$

has the finite order $M(N-1)$ along γ .

Similarly, for any curve γ in the variety defined by the second equation (5.2), we take in our Step 1 the other pre-multiplier

$$\psi = F_1 = z_1^M + O(L), \quad \nu_\gamma(\psi) = M.$$

Then following our Step 2 procedure, we obtain the sequence of multipliers

$$g_1 = f_1, \quad g_j = \det \left(\frac{\partial g_{j-1}}{\partial \psi} \right) \sim z_1^{j(M-1)} z_2^{N-j} + O(L-j), \quad j = 2, \dots, N,$$

where the last multiplier

$$g_N = z_1^{N(M-1)} + O(L-N)$$

has the finite order $N(M-1)$ along γ .

Finally, in Step 3, we take

$$F = f_1 = MNz_1^{M-1}z_2^{N-1} + Mz_1^{K+M-1} + O(L-1),$$

$$\tilde{F} = f_M + g_N = z_2^{M(N-1)} + z_1^{N(M-1)} + O(L - \max(M, N)) \pmod{z_1^{K-M+1}}$$

such that

$$d = \nu(F) = M + N - 2, \quad t = \nu_{F=0}(\tilde{F}) = \max(N(M-1), M(N-1))$$

and the ideal generated by F and \tilde{F} has the multiplicity $\leq dt$ which bounds the root order in the radical, to obtain the multipliers z_1 and z_2 , and hence their Jacobian determinant, leading to the desired termination. Furthermore, the number of steps and the root order when taking the radical, and hence the order of subellipticity in Corollary 1.3 are explicitly controlled in terms of the type $2T = 2 \max(M, N)$.

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