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NUMERICAL BOUNDEDNESS ON RATIONAL EQUIVALENCES OF ZERO CYCLES ON ALGEBRAIC VARIETIES WITH TRIVIAL CH_0*

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Dedicated to the 60th birthday of Professor Ngaiming Mok

Abstract. The main purpose of this article is to show that there exists numerical bound with respect to rational equivalences in some sense. We also prove that finite dimensionality of the zero dimensional Chow groups are preserved by degeneration.

Key words. Algebraic cycles, rational equivalence.

Mathematics Subject Classification. 14C25.

1. Introduction. Let X be a smooth projective variety over an algebraically closed field of characteristic 0 and Z_0X denote the group of 0-dimensional algebraic cycles of X (see §1.3 Ch. I [1]). We call a finite set $\{(C_i, \phi_i)\}$ where C_i's are curves on X and ϕ_i's rational functions on C_i a rational equivalence datum. For a zero-cycle $\alpha \in Z_0X$ which is rationally equivalent to zero, a rational equivalence datum for α is defined to be a rational equivalence datum $\{(C_i, \phi_i)\}$ which satisfies

$$\alpha = \sum \mathrm{div}(\phi_i) = \sum \mathrm{zero}(\phi_i) - \sum \mathrm{pole}(\phi_i) \in Z_0X.$$

Therefore α and β are rationally equivalent if and only if there exists a rational equivalence datum $\{(C_i, \phi_i)\}$ for $\alpha - \beta$. To a rational equivalence datum, one associates several numerical data: let m be the number of curves of $\{C_i\}$, let p_a be the maximum of the arithmetic genera $p_a(C_i)$ and let d the maximum of the degrees $\deg(\phi_i)$. Certainly, the triple (m, p_a, d) depends on the choice of rational equivalence data for $\alpha - \beta$. Now we consider the case where the zero Chow group $\mathrm{CH}_0(X)$ is trivial, which means that any two closed points in X, namely two effective cycles of degree one, are rationally equivalent. Our basic question is whether there exists a uniform bound on the triple (m, p_a, d) for *all* $p - q$ where p, q go through all pairs of closed points of X. More precisely, we are asking whether there exists a triple $(m, p_a, d) \in \mathbb{N}^3$ such that there is always a rational equivalence datum for all $p - q$ whose numerical data does not exceed the given triple. One of our main results is to give the affirmative answer to this question in the case that k is uncountable.

THEOREM 1.1. *Let X be a smooth projective variety over an algebraically closed field of characteristic 0 which is uncountable. Assume $\mathrm{CH}_0(\mathrm{X}) = \mathbb{Z}$. Then there exists a uniform bound (m, p_a, d) of rational equivalences data for all $p - q \in Z_0X$ where p, q go through all pairs of closed points of X.*

See Theorem 2.4 for a more general formulation of the above theorem. It is a challenging problem to find an effective bound in the theorem. As an application, we obtain the following result:

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THEOREM 1.2. *Let C be a nonsingular curve over an algebraically closed field of characteristic 0 which is uncountable, $0 \in C$ a closed point, $f : \mathcal{X} \to C$ a flat projective morphism with integral fibres and $\mathcal{Y}$ a subvariety of $\mathcal{X}$. Let $C' = C - \{0\}$, $\mathcal{X}' = C' \times_C \mathcal{X}$ and $\mathcal{Y}' = C' \times_C \mathcal{Y}$. Assume that $\mathcal{Y}$ is of dimension at most 2 and, for any closed point $c \in C'$, the natural map $i_{c*} : \mathrm{CH}_0(\mathcal{Y}_c) \to \mathrm{CH}_0(\mathcal{X}_c)$ is surjective, that is, $\mathrm{CH}_0(\mathcal{X}_c)$ is representable. Then $\mathrm{CH}_0(\mathcal{X}_0)$ is representable.*

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2. Rational equivalence of special type. Let k be an algebraically closed field of characteristic 0 and let X be a projective variety over k. We start with definition:

DEFINITION 2.1. A *rational equivalence datum* is a finite set $\{(V_i, \phi_i)\}$ where the V_i's are $(n+1)$-dimensional subvarieties of X, and $\phi_i \in k(V_i)$. Let $\alpha \in Z_n X$ be a n-dimensional algebraic cycle which is rationally equivalent to zero. We call $\{(V_i, \phi_i)\}$ a *rational equivalence datum for* α if

$$\alpha = \sum \mathrm{div}(\phi_i) = \sum \mathrm{zero}(\phi_i) - \sum \mathrm{pole}(\phi_i) \in Z_n X.$$

In the case that $n = 0$, we associate a triple $(m, p_a, d) \in \mathbb{N}^3$ with a rational equivalence datum $\{(C_i, \phi_i)\}$: let m be the number of curves of $\{C_i\}$, let p_a be the maximum of the arithmetic genera $p_a(C_i)$ and let d the maximum of the degrees $\deg(\phi_i)$. We say a triple (m, p_a, d) is bounded by a triple (M, P, D) if $m \le M, p_a \le P, d \le D$. Let two 0-cycles α, β be rationally equivalent. We call (M, P, D) a bound of rational equivalence of α, β if there is a rational equivalence datum for $\alpha - \beta$ whose triple is bounded by (M, P, D).

Note that an effective zero-cycle $\alpha = \sum n_i p_i$ on X of degree $n = \sum n_i$ can be viewed as a closed point of the Chow variety $\mathrm{Chow_{0,n}(X)}$ parametrizing effective algebraic cycles of dimension 0 and degree n (see §3.21 Ch. III [3]).

DEFINITION 2.2. Two effective zero-cycles α, β of degree n are said to be *m-connected* if there exist $m+1$ effective 0-cycles $\alpha = \alpha_0, \alpha_1, \dots, \alpha_m = \beta$ in $S^n X$ such that for each $i = 0, 1 \dots, m-1$, there is a morphism $f_i : \mathbf{P}^1_k \to S^n X$ such that $f_i(0) = \alpha_i, f_i(\infty) = \alpha_{i+1}$. We call the morphism f_i a *one-connection datum* for (α_i, α_{i+1}).

It is known that two zero-cycles α, β are rationally equivalent if and only if there exists a certain effective zero-cycle γ such that $\alpha + \gamma$ and $\beta + \gamma$ are effective and one-connected (see Example 1.6.3, [1]). Now we fix a base point $p \in X$.

LEMMA 2.3. *Let α, β be two rationally equivalent effective zero-cycles of degree m. Then there exists a natural number n such that $\alpha + np$ is 3-connected to $\beta + np$.*

Proof. Take an arbitrary rational equivalence datum for $\beta - \alpha$:

$$\beta - \alpha = \sum \mathrm{zero}(\phi_i) - \sum \mathrm{pole}(\phi_i).$$

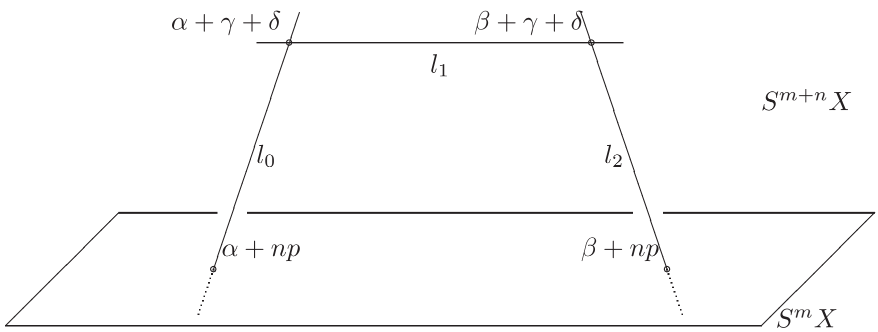
Set $\gamma = \sum \mathrm{zero}(\phi_i)$. Take an irreducible curve $C \subset X$ which passes through both p and $\mathrm{supp}(\gamma)$. By Serre's vanishing on C, we find a rational function ψ on C such that $\mathrm{zero}(\psi) = np$ and $\delta := \mathrm{pole}(\psi) - \gamma$ are effective divisors on C. Put

$$\begin{aligned}
\alpha_0 &= \alpha + np = \alpha + \mathrm{zero}(\psi)\\
\alpha_1 &= \alpha + \mathrm{pole}(\psi) = \beta + \sum \mathrm{pole}(\phi_i) + \delta\\
\alpha_2 &= \beta + \sum \mathrm{zero}(\phi_i) + \delta = \beta + \mathrm{pole}(\psi)\\
\alpha_3 &= \beta + \mathrm{zero}(\psi) = \beta + np
\end{aligned}$$

And define three morphisms $\mathbf{P}^1_k \to S^{m+n}X$ as follows:

$$l_0(t) := \alpha + \psi^{-1}(t), \quad l_1(t) = \beta + \sum \phi_i^{-1}(t^{-1}) + \delta, \quad l_2(t) = \beta + \psi^{-1}(t^{-1}).$$

It follows from the construction that l_i gives a one-connection datum between α_i and α_{i+1}. We draw a line configuration in $S^{m+n}X$ to illustrate the situation:



□

We will prove the following result.

THEOREM 2.4. *Let k be an algebraically closed filed of characteristic zero which is uncountable and $\mathcal{X}, S$ be two algebraic varieties over k. Let $f : \mathcal{X} \to S$ be a flat projective morphism with integral fibers and $\mathcal{Y} \subset \mathcal{X}$ be a closed subvariety.*

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{i} & \mathcal{X}\\
& \searrow & \downarrow{\scriptstyle f}\\
& & S
\end{array}$$

Assume that for any closed point $s \in S$, the natural map $i_{s*} : \mathrm{CH}_0(\mathcal{Y}_s) \to \mathrm{CH}_0(\mathcal{X}_s)$ is surjective. Then there exists a triple $(M, P, D) \in \mathbb{N}^3$ such that, for each closed point $p_s \in \mathcal{X}_s$, we can find a 0-cycle $Z_s \in Z_0\mathcal{Y}_s$ which is rationally equivalent to p_s in $Z_0\mathcal{X}_s$ with a rational equivalence bounded by (M, P, D).

For the proof of Theorem 2.4, we use Proposition 2.5 as follows.

PROPOSITION 2.5. *Let k be an algebraically closed filed of characteristic zero which is uncountable and $\mathcal{X}, S$ be two algebraic varieties over k. Let $f : \mathcal{X} \to S$ be a flat projective morphism with integral fibers and $\mathcal{Y} \subset \mathcal{X}$ be a closed subvariety.*

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{i} & \mathcal{X}\\
& \searrow & \downarrow{\scriptstyle f}\\
& & S
\end{array}$$

Let $p : S \to \mathcal{X}$ be a section and assume that $[p(s)] \in \operatorname{image}(\mathrm{CH}_0(\mathcal{Y}_s) \to \mathrm{CH}_0(\mathcal{X}_s))$ for all closed point $s \in S$. Then there exists a triple $(M, P, D) \in \mathbb{N}^3$ such that, for each closed point $s \in S$, we can find a 0-cycle $Z_s \in Z_0\mathcal{Y}_s$ which is rationally equivalent to $p(s)$ in $Z_0\mathcal{X}_s$ with a rational equivalence bounded by (M, P, D).

Proof of Proposition 2.5. We shall employ a Hilbert scheme argument. For each pair of natural numbers n, m, set

$$S_{n,m} := \{s \in S | np(s) + Z_s^- \overset{3-con}{\sim} (n-1)p(s) + Z_s^+, Z_s^+ \in E_{m+1}(\mathcal{Y}_s), Z_s^- \in E_m(\mathcal{Y}_s)\}$$

where $E_m(\mathcal{Y}_s)$ consists of all effective algebraic cycles of degree m in $Z_0\mathcal{Y}_s$. By assumption, for any $s \in S$ we can take two effective 0-cycles Z_s^+, Z_s^- of $\mathcal{Y}_s$ such that $Z_s^+ - Z_s^- = p(s) \in \mathrm{CH}_0(\mathcal{X}_s)$. Now Lemma 2.3 asserts that

$$\cup S_{n,m} = S.$$

We claim $S_{n,m} \subset S$ is a countable union of constructible sets. Let C_3 be the unique connected curve consisting of three copies of $\mathbb{P}^1$ whose dual graph is a tree. Let us label three copies as $C_\alpha, C_\beta, C_\gamma$, such that C_β is the unique copy which intersect with the remaining two copies. Let us also mark two closed points $\{0, \infty\} \subset C_3$, where 0 belongs to C_α while ∞ belongs to C_γ. We remark that each point of $\mathrm{Hom}_S(C_3 \times S, \mathrm{Chow}_{0,n+m}(\mathcal{X}/S))$ can be regarded as a morphism g_s from C_3 to $\mathrm{Chow}_{0,n+m}(\mathcal{X}/S)_s$ for a point $s \in S$. Consider the closed subset $\mathrm{Hom}^0_{n,m} \subset \mathrm{Hom}_S(C_3 \times S, \mathrm{Chow}_{0,n+m}(\mathcal{X}/S))$ consisting of those points whose corresponding morphisms g_s have the following property:

$$g_s(0) \in np(s) + \mathrm{Chow}_{0,m}(\mathcal{Y}/S)_s, \quad g_s(\infty) \in (n-1)p(s) + \mathrm{Chow}_{0,m+1}(\mathcal{Y}/S)_s.$$

We equip $\mathrm{Hom}^0_{n,m}$ with the reduced closed subscheme structure. Then, the image of the composite

$$\mathrm{Hom}^0_{n,m} \subset \mathrm{Hom}_S(C_3 \times S, \mathrm{Chow}_{0,n+m}(\mathcal{X}/S)) \to S$$

is nothing but $S_{n,m}$. By the theorem of Chevalley, $S_{n,m}$ is a countable union of constructible sets $S_{n,m,i}$.

Lemma 2.6 below allows us to conclude that $S = \cup S_{n,m,i}$ has a finite subcover.

LEMMA 2.6. *Let X be an algebraic variety over an algebraically closed field k which is uncountable and $\{X_n\}_{n\in\mathbb{N}}$ a countable set of constructible subsets of X such that every closed point of X lies on some X_n. Then there exists a natural number N such that $X = \cup_{n=1}^N X_n$.*

Proof. We proceed by induction on $\dim X$. If $\dim X = 0$, then X consists of finite closed points and we always have such N. Let $\dim X = m$, and assume that lemma holds for lower dimensional cases. As X is covered by finitely many affine open subschemes, we may assume X is an affine variety from the beginning.

We prove some X_n contains a nonempty Zariski open subset U. If not, all X_n are of dimension at most $m-1$. For each irreducible component $X_{n,i}$ of X_n, we take a closed point $x_{n,i} \in X_{n,i}$. We will find a nonempty hypersurface D which misses all $x_{n,i}$. The condition for D to contain $x_{n,i}$ can be written as a nonzero linear equation. Since k is uncountable and the number of linear equations is countable, we have a D which doesn't satisfy any equations and thus is the one we want. For this D, let D_i be any irreducible component. We have $\dim(X_n \cap D_i) \le m-2$, for all n. By inductive hypothesis, finitely many $X_n \cap D_i$ should cover D_i, contradiction by the dimension reason.

To complete the proof, we again apply the inductive hypothesis to each component of the complement $X - U$. □

We use the following lemma to conclude Proposition 2.5.

LEMMA 2.7. *Notations as above. Let $Y \subset \mathrm{Hom}^0_{n,m}$ be an irreducible component. Then the triples associated with the rational equivalences data corresponding to Y are uniformly bounded.*

Proof. Let Σ be the corresponding incidence variety:

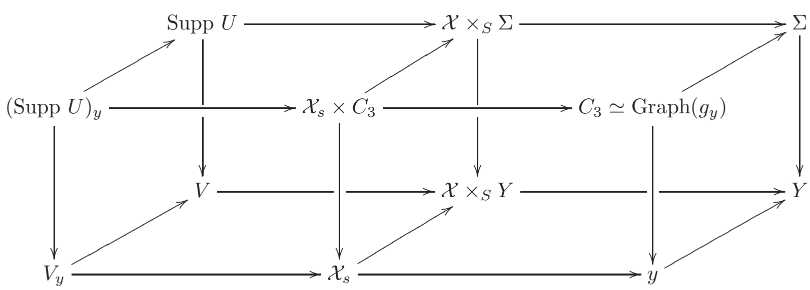
$$\begin{array}{ccc} \Sigma & \hookrightarrow & Y \times_S (C_3 \times S) \times_S \mathrm{Chow}_{0,n+m}(\mathcal{X}/S) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \\ Y. & & \end{array}$$

Then, by applying the Chow functor (see [3, Ch. I, §3.21]), the third projection $\mathrm{pr}_3 : \Sigma \to \mathrm{Chow}_{0,n+m}(\mathcal{X}/S)$ gives a family of nonnegative proper, algebraic cycles $(U \to \Sigma)$ of $\mathcal{X} \times_S \Sigma/\Sigma$.

For the closed subscheme Supp $U \hookrightarrow \mathcal{X} \times_S \Sigma$, let V be its scheme theoretic image with its reduced scheme structure under the following projective morphism

$$1_{\mathcal{X}} \times \mathrm{pr}_1 : \mathcal{X} \times_S \Sigma \to \mathcal{X} \times_S Y.$$

We write $\pi : V \subset \mathcal{X} \times_S Y \to Y$ for the composition, where the latter morphism is the second projection. Note that for any closed point $y \in \pi(V) = Y$ which is mapped to $s \in S$, the corresponding morphism $g_y : C_3 \to \mathrm{Chow}_{0,n+m}(\mathcal{X}_s)$ gives rise to a 3-connection in $\mathcal{X}_s$. Moreover, we have the following Cartesian diagram:



According to [1, Ch. I, Prop. 1.6], the one-dimensional components of V_y are those curves occurred in the 3-connection given by g_y. We show that all the curves have uniformly bounded arithmetic genera.

Consider the morphism $\pi : V \to Y$ introduced as above. There is a flattening stratification $Y_1, Y_2, \ldots, Y_k$ which are locally closed subschemes such that $V \times_Y Y_i \to Y_i$ is flat (cf. [4, Section 8]). So we may assume π is flat. Let C be a one dimensional component of V_y equipped with its reduced structure. We claim the arithmetic genus $p_a(C) = h^1(C, \mathcal{O}_C) \le h^1(V_y, \mathcal{O}_{V_y})$. Let $\mathcal{I}_C$ be the ideal sheaf of C. Then the short exact sequence $0 \to \mathcal{I}_C \to \mathcal{O}_{V_y} \to \mathcal{O}_C \to 0$ induces a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(V_y, \mathcal{I}_C) \longrightarrow H^0(V_y, \mathcal{O}_{V_y}) \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\alpha^0} \\ \quad\; H^1(V_y, \mathcal{I}_C) \longrightarrow H^1(V_y, \mathcal{O}_{V_y}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow 0 \end{array}$$

So we have $h^1(C, \mathcal{O}_C) \leq h^1(V_y, \mathcal{O}_{V_y})$.

Since $h^1(V_y, \mathcal{O}_{V_y})$ is an upper semi-continuous function on Y, $h^1(V_y, \mathcal{O}_{V_y})$ is uniformly bounded. Hence there is a uniform bound of arithmetic genera.

It remains to show the number of curves and the maximum of the degrees associated with each of these rational equivalences data are bounded by constants. For any rational equivalence datum given by a morphism $C_3 \to \mathrm{Chow}_{0,n+m}(\mathcal{X}_s)$, the sum of the degrees associated with it is bounded by $3(n+m)$. Therefore, the two numerical numbers are both uniformly bounded by $3(n+m)$. □

Applying the lemma to our situation, we can find a triple (M, P, D) which is a universal bound and then complete the proof of Proposition 2.5. □

Proof of Theorem 2.4. We just take the base change and apply Proposition 2.5:

$$\begin{array}{ccc} \mathcal{X} \times_S \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & S \end{array}$$

□

In the case that S is a point and the Chow group is $\mathbb{Z}$, we have the following special version:

Corollary 2.8. *Let X be a smooth projective variety over $\mathbb{C}$ with $\mathrm{CH}_0(X) = \mathbb{Z}$. Then there exists a natural number N such that for all $p, q \in X$, Np is 3-connected to $(N-1)p + q$.*

3. Applications. As in Section 2, let k be an algebraically closed field of characteristic zero which is uncountable. We have seen that rational equivalence of two effective cycles can be realized by $\mathbb{P}^1$ connections of corresponding points in the moduli of effective cycles, namely the Chow variety. It is well-known that the degeneration of rational curves is still rational (see Proposition 3.2). By this observation, we get the following theorem:

Theorem 3.1. *Let C be a nonsingular curve over k, $0 \in C$ a point, $f : \mathcal{X} \to C$ a flat projective morphism with integral fibres and $\mathcal{Y}$ a subvariety of $\mathcal{X}$. Let $C' = C - \{0\}$, $\mathcal{X}' = C' \times_C \mathcal{X}$ and $\mathcal{Y}' = C' \times_C \mathcal{Y}$.*

$$\begin{array}{ccccc} \mathcal{X}' & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_0 \\ {\scriptstyle f'}\downarrow & \square & {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f_0} \\ C' & \longrightarrow & C & \longleftarrow & \{0\} \end{array}$$

Assume that, for any closed point $c \in C'$, the natural map $i_{c*} : \mathrm{CH}_0(\mathcal{Y}_c) \to \mathrm{CH}_0(\mathcal{X}_c)$ is surjective. Then the morphism $i_{0*} : \mathrm{CH}_0(\mathcal{Y}_0) \to \mathrm{CH}_0(\mathcal{X}_0)$ is surjective. In particular, if $\mathcal{Y}$ is of dimension at most 2, $\mathrm{CH}_0(\mathcal{X}_0)$ is representable (see [5, Definition 10.6]).

Proof. We take the base change $\mathcal{X} \times_C \mathcal{X} \to \mathcal{X}$ with the diagonal as a section. For simplicity, we write $S = \mathcal{X}$, $X = \mathcal{X} \times_C \mathcal{X}$ and $\mathcal{Y}_S = \mathcal{X} \times_C \mathcal{Y}$. Let n, m be natural numbers and $\mathrm{Hom}^0_{n,m}$ as in the proof of Proposition 2.5, and Y an irreducible component of $\mathrm{Hom}^0_{n,m}$. $\mathrm{Hom}_S(C_3 \times S, \mathrm{Chow}_{0,n+m}(X/S))$ is an open subscheme of

$\mathrm{Hilb}((C_3 \times S) \times_S \mathrm{Chow}_{0,n+m}(X/S)/S)$. Let $\overline{Y}$ be the closure of Y in $\mathrm{Hilb}((C_3 \times S) \times_S \mathrm{Chow}_{0,n+m}(X/S)/S)$, and $\overline{\Sigma}$ be the incidence variety corresponding to $\overline{Y}$:

$$\begin{array}{ccc} \Sigma & \longrightarrow & \overline{\Sigma} \\ {\scriptstyle u}\downarrow & \square & \downarrow{\scriptstyle \overline{u}} \\ Y & \longrightarrow & \overline{Y} \end{array}$$

Now we use the following general fact:

PROPOSITION 3.2. *Let C be a nonsingular curve over k, $0 \in C$ a point and $f : \mathcal{M} \to C$ a flat projective morphism such that we have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{M} - f^{-1}(0) & \xrightarrow{\mathrm{res} f} & C - \{0\} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C_3 \times (C - \{0\}) & \xrightarrow{\mathrm{pr}_2} & C - \{0\} \end{array}$$

where φ is an isomorphism of schemes. Then the central fibre $f^{-1}(0)$ is connected and each component is a rational curve.

Proof. The connectedness follows from Stein factorization. Since $\mathcal{M} - f^{-1}(0)$ has a trivial C_3 fibration and f is flat, $\mathcal{M}$ has three components $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. And for each i we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_i - f^{-1}_{|\mathcal{M}_i}(0) & \xrightarrow{\mathrm{res} f_{|\mathcal{M}_i}} & C - \{0\} \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{P}^1 \times (C - \{0\}) & \xrightarrow{\mathrm{pr}_2} & C - \{0\} \end{array}$$

where φ_i is an isomorphism. Then by semi-stable reduction (see [2, CH. II]), there is a finite morphism $\pi : C' \to C$ with C' nonsingular and $\pi^{-1}(0)$ consists of one point, say $0'$, and a projective morphism p as follows:

$$\begin{array}{cccc} \mathcal{M}'_i & & & \\ & \searrow^{p} & & \\ & & \mathcal{M}_i \times C' & \longrightarrow & \mathcal{M}_i \\ & {\scriptstyle f'}\searrow & \downarrow & & \downarrow{\scriptstyle f} \\ & & C' & \xrightarrow{\pi} & C \end{array}$$

where p is an isomorphism over $C' - \{0'\}$, $\mathcal{M}'_i$ is nonsingular, and the fibre $f'^{-1}(0')$ is reduced with nonsingular components crossing normally. As f' is still flat, the Hilbert polynomial of the fibre $f'^{-1}(0')$ is 0. Then the structure sheaf of every smooth component has vanishing first cohomology. It follows that every component is isomorphic to $\mathbb{P}^1$. This completes the proof. $\square$

By Proposition 3.2, $\overline{u}^{-1}(\overline{y})$ is connected for any $\overline{y} \in \overline{Y}$ and each component is a rational curve. This means that each point of $\overline{Y}$ gives a rational equivalence. Let S_Y be the image of the composition

$$\overline{Y} \subset \mathrm{Hilb}((C_3 \times S) \times_S \mathrm{Chow}_{0,n+m}(X/S)/S) \to S.$$

We see that S_Y is a closed subset of S satisfying $[p(s)] \in \mathrm{image}(\mathrm{CH}_0(\mathcal{Y}_s) \to \mathrm{CH}_0(\mathcal{X}_s)), \forall s \in S_Y$. Now by Theorem 2.4, there are finitely many Y_i's, with possibly different (n, m)'s, such that $\cup S_{Y_i} \supset (S - f^{-1}(0))$. As $\cup S_{Y_i}$ is closed, we have $\cup S_{Y_i} = S$. This completes the proof. ☐

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