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EXPLOITING LOG-CAPACITY IN CONVEX GEOMETRY*

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Abstract. This article is devoted to an exploitation of the log-capacity for convex bodies - especially - its connections to volume-radius, mean-width, Hadamard-type variational formula, Minkowski-type problem, and Yau-type problem.

Key words. Log-capacity, volume-radius, mean-width, Hadamard-type variation, Minkowski-type problem, Yau-type problem.

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1. Introduction.

1.1. Background. Thanks to its role in two-dimensional potential theory that is the study of planar harmonic functions in mathematics and mathematical physics, the logarithmic capacity in the Euclidean plane $\mathbb{R}^2$ has been studied systemically; see [28, 2, 24, 33, 44, 45, 43, 54] for some relatively recent publications on this topic. However, the higher dimensional extension (i.e., to the Euclidean space $\mathbb{R}^n$, $n \geq 3$) of the planar logarithmic capacity has received relatively little attention due to a nonlinear nature; see [6, 11, 18, 19, 3, 4, 32] (see also [1, 23, 39] for some function-space-based capacities) only because of the author's limited knowledge of other references.

Fortunately, in their paper [11] Colosanti-Cuoghi were able to utilize an equilibrium potential to introduce a kind of the logarithmic capacity (in short, log-capacity) for $2 \leq n$-dimensional convex bodies. To be more precise, let $\mathscr{C}^n$ be the class of all non-empty convex compact subsets of $\mathbb{R}^n$, and denote by $\mathscr{K}^n$ the class of all $K \in \mathscr{C}^n$ with non-empty interior K°. For $K \in \mathscr{K}^n$ let $u = u_K$ be its log-equilibrium potential, i.e., the unique weak solution to the following boundary value problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{n-2}\nabla u) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus K; \\ u = 0 \quad \text{on} \quad \partial K; \\ u(x) \sim \log|x| \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{1.1}$$

where $\log(\cdot)$ is the base-e (i.e., natural) logarithm, $\sim$ means that there exists a constant $c > 0$ such that

$$c^{-1} \leq \frac{u(x)}{\log|x|} \leq c \quad \text{as} \quad |x| \to \infty.$$

In accordance with Kichenassamy-Veron's [31, Theorem 1.1 and Remarks 1.4-1.5], $u(x) - \log|x|$ tends to a constant depending on K as $|x| \to \infty$, and so the following

$$\mathrm{ncap}(K) = \exp\Big(-\lim_{|x|\to\infty}\big(u(x) - \log|x|\big)\Big) \tag{1.2}$$

was employed by Colosanti-Cuoghi in [11] to define the log-capacity of K since the case $n = 2$ of (1.2) is just the logarithmic capacity on $\mathbb{R}^2$.

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According to [11, Remarks 2.2&2.3], the log-capacity ncap$(\cdot)$ enjoys the following basic properties:

- $\mathrm{ncap}\big(\overline{\mathbb{B}^n}\big)=1$ provided $\overline{\mathbb{B}^n}=\{x\in\mathbb{R}^n:|x|\le 1\}$;
- $\mathrm{ncap}(x_0+\rho K)=\rho\,\mathrm{ncap}(K)$ provided $x_0+\rho K=\{x_0+\rho x: x\in K\}$ and $(x_0,\rho,K)\in\mathbb{R}^n\times(0,\infty)\times\mathscr{K}^n$;
- $\mathrm{ncap}(K_1)\le\mathrm{ncap}(K_2)$ provided $K_1,K_2\in\mathscr{K}^n$ with $K_1\subseteq K_2$.

Naturally, the log-capacity of an arbitrary $K\in\mathscr{C}^n$ is defined as:

$$\mathrm{ncap}(K)=\inf_{K\subseteq L\in\mathscr{K}^n}\mathrm{ncap}(L).$$

Such a definition induces not only the last two properties for $\mathscr{C}^n$ but also the following downward-monotone-convergence

- $\mathrm{ncap}(\cap_{j=1}^\infty K_j)=\lim_{j\to\infty}\mathrm{ncap}(K_j)$ provided $K_j\in\mathscr{C}^n$ with $K_j\supseteq K_{j+1}$.

1.2. Overview. In this paper we study five problems which are naturally associated with the above-defined log-capacity. First of all, we discover the optimal relationship among the volume-radius, the log-capacity and the mean-width (cf. Theorem 2.1). Secondly, we find an integral identity and a lower bound estimate for the non-tangential limit of the gradient of the log-equilibrium potential on the boundary of a $\mathscr{K}^n$-member (cf. Theorems 3.1 & 3.2). Thirdly, we establish Hadamard's variational formula for (1.2) (cf. Theorem 4.4). Fourthly, we handle the existence and uniqueness of Minkowski's problem for the log-capacity (cf. Theorem 5.1). Last of all, we settle the log-capacity analogue of Yau's [56, Problem 59] (the prescribed mean curvature problem) in a weak sense (cf. Theorem 6.1). Here it is perhaps appropriate to point out that since our log-capacity generalization is from the linear case $n=2$ (where the classical $2=n$-harmonic functions are often taken into account) to the nonlinear case $n\ge 3$ (where only the nonlinear $3\le n$-harmonic functions can be used), in all situations we have to seek an unified way, which turns out to be highly non-trivial, to deal with these issues.

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2. Volume-radius and mean-width via log-capacity.

2.1. Log-capacity breaking iso-mean-width inequality. Given $K\in\mathscr{C}^n$. Following [48, (1.7)], we say that

$$h_K(x)=\sup_{y\in K}x\cdot y\quad\forall\quad x\in\mathbb{R}^n$$

is the support function of K, and

$$\mathrm{b}(K)=\frac{2}{\sigma_{n-1}}\int_{\mathbb{S}^{n-1}}h_K\,d\theta$$

is the mean-width of K whose case $n=2$ gives $\pi\,\mathrm{b}(K)=S(K)$, the perimeter of K (cf. [48, p. 318]) – here and henceforth $d\theta$ is the uniform surface area measure on $\mathbb{S}^{n-1}$, i.e., the $n-1$ dimensional spherical Lebesgue measure, where $\sigma_{n-1}=n\omega_n$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. The sharp iso-mean-width (or Uryasohn's) inequality

$$\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}\le\frac{\mathrm{b}(K)}{2}\tag{2.1}$$

is well known for any $K \in \mathscr{C}^n$ (cf. [48, (6.25)]), where the left-hand quantity of (2.1) is called the volume-radius of K and the right-hand quantity of (2.1) is dominated by a half of the diameter diam(K) of K. Surprisingly, the following result indicates that (2.1) can be split by ncap(K), just like the well-known planar case (cf. [44, Theorem 5.3.5] and [7, Example 7.4]).

THEOREM 2.1. *Let $K \in \mathscr{C}^n$. Then*

$$\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}} \le ncap(K) \le \frac{b(K)}{2}. \tag{2.2}$$

And, (2.2) *is optimal in the sense that if K is either a ball or a singleton then two equalities of* (2.2) *hold.*

Proof. A straightforward computation shows that each equality of (2.2) occurs whenever K is either a ball or or a singleton.

In order to proceed further, let us recall the definition of the conformal capacity ncap(O, K) for a given open set $O \subset \mathbb{R}^n$ containing a compact set K (cf. [23, p.287]):

$$\text{ncap}(O, K) = \inf_{f \in \mathsf{W}(O,K)} \int_{O \setminus K} |\nabla f|^n \, dV,$$

where dV is the Lebesgue volume element and $\mathsf{W}(O, K)$ comprises all $f \in C_0^\infty(O)$ (infinitely differentiable functions with compact support in O) enjoying $f \ge 1$ on K - according to [23, p.27], without affecting ncap(O, K) the class $\mathsf{W}(O, K)$ can be replaced by

$$\mathsf{W}_0(O, K) = \{f \in W_0^{1,n}(O) \cap C(O) : \ f \ge 1 \ \text{ on } \ K\},$$

where $C(O)$ consists of all continuous functions in O and $W_0^{1,n}(O)$ is the closure of $C_0^\infty(O)$ in the Sobolev $(1, n)$-space $W^{1,n}(O)$ equipped with the norm

$$\|f\|_{W^{1,n}(O)} = \left(\int_O |f|^n \, dV\right)^{\frac{1}{n}} + \left(\int_O |\nabla f|^n \, dV\right)^{\frac{1}{n}}.$$

For an arbitrary subset E of O, the above definition is extended by

$$\text{ncap}(O, E) = \inf_{E \subseteq \text{open } U \subseteq O} \ \sup_{\text{compact } K \subseteq U} \text{ncap}(O, K).$$

Below are the known facts (cf. [23, Theorem 2.2] and [17, (2.10)]):

(i) ncap$(O, K_1) \le$ ncap(O, K_2) as $K_1 \subseteq K_2$ are compact;

(ii) ncap$(O_1, K) \ge$ ncap(O_2, K) as $O_1 \subseteq O_2$ are open and K is compact;

(iii) ncap$(O, \cap_{j=1}^\infty K_j) = \lim_{j\to\infty}$ ncap(O, K_j) as $K_j \supseteq K_{j+1}$ are compact;

(iv) $\left(\frac{\text{ncap}(O,K)}{\sigma_{n-1}}\right)^{\frac{1}{1-n}} \le \log\left(\frac{V(O)}{V(K)}\right)^{\frac{1}{n}}$.

Due to the definition of ncap(K) for $K \in \mathscr{C}^n$, it is enough to verify (2.2) under the assumption $K \in \mathscr{K}^n$ in what follows.

On the one hand, we check the left-hand inequality of (2.2). To do so, let $r \in (0,\infty)$ be large enough such that $K \subseteq r\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < r\}$ and set u_r be the unique solution to

$$\begin{cases} -\mathrm{div}(|\nabla u_r|^{n-2}\nabla u_r) = 0 \quad \text{in} \quad r\mathbb{B}^n \setminus K; \\ \\ u_r = 0 \quad \text{on} \quad \partial K \quad \& \quad u_r(x) = \log r \quad \text{as} \quad |x| = r, \end{cases}$$

According to the argument for [11, Theorem 2.2], $\{u_r\}$ has a subsequence, still denoted by $\{u_r\}$, convergent to u which is the unique weak solution of (1.1) and makes that

$$\alpha = \lim_{|x|\to\infty} \big(u(x) - \log|x|\big)$$

is finite. According to [31], we have that if $|x| \to \infty$ then

$$u(x) = \log|x| + \alpha + o(1) \quad \& \quad |\nabla u(x)| = |x|^{-1} + o(|x|^{-1}).$$

Consequently, by the maximum principle we get

$$0 \le u(x) \le \max_{|y|=r} u(y) \quad \forall \quad x \in \overline{r\mathbb{B}^n} = \{y \in \mathbb{R}^n : |y| \le r\}.$$

If

$$v_r(x) = \frac{u(x)}{\max_{|y|=r} u(y)} \quad \forall \quad x \in \overline{r\mathbb{B}^n},$$

then for $r \to \infty$ and $0 < t \to 1$ we have

$$\begin{aligned} v_r(x) = t &\Leftrightarrow \log|x| + \alpha + o(1) = t\big(\log r + \alpha + o(1)\big) \\ &\Leftrightarrow |x| = r^t \exp\Big((t-1)\big(\alpha + o(1)\big)\Big) \equiv r_*. \end{aligned}$$

Note that

$$\begin{cases} -\mathrm{div}(|\nabla v_r|^{n-2}\nabla v_r) = 0 \quad \text{in} \quad r\mathbb{B}^n \setminus K; \\ v_r = 0 \quad \text{on} \quad \partial K; \\ 0 \le v_r(x) \le 1 \quad \text{as} \quad x \in \overline{r\mathbb{B}^n}. \end{cases}$$

So, using (ii), the definition of $\mathrm{ncap}(\cdot, K)$, the test function $v_{r,t} := 1 - t^{-1}v_r$ which belongs to the class $\mathsf{W}_0\big(\{\{v_r(x) < t\}, K\big)$ via setting $v_r(x) = 0$ as $x \in K$, the divergence theorem and an integration-by-part, we get

$$\begin{aligned} \mathrm{ncap}(r\mathbb{B}^n, K) &\le \mathrm{ncap}\big(\{v_r(x) < t\}, K\big) \\ &\le \int_{\{v_r(x)<t\}} |\nabla v_{r,t}|^n \, dV \\ &\le \int_{\{v_r(x)=t\}} |\nabla v_{r,t}|^{n-1} \, dS \\ &= t^{1-n} \int_{\{v_r(x)=t\}} \left(\frac{|\nabla u|}{\max_{|y|=r} u(y)}\right)^{n-1} dS \\ &= t^{1-n} \int_{|x|=r_*} \left(\frac{1+o(1)}{r_*(\log r + \alpha + o(1))}\right)^{n-1} dS \\ &= t^{1-n}\sigma_{n-1} \left(\frac{1+o(1)}{\log r + \alpha + o(1)}\right)^{n-1}. \end{aligned}$$

That is to say,

$$\left(\frac{\mathrm{ncap}(r\mathbb{B}^n, K)}{t^{1-n}\sigma_{n-1}}\right)^{\frac{1}{n-1}} \le \frac{1+o(1)}{\log r + \alpha + o(1)}.$$

Now, an application of (iv) derives

$$\frac{\log r + \alpha + o(1)}{t^{-1}\big(1+o(1)\big)} \le \log \frac{r}{\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}} = \log r - \log\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}$$

thereby finding (thanks to: $t \to 1$; $r \to \infty$; $o(1) \to 0$)

$$\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}} \le e^{-\alpha} = \mathrm{ncap}(K).$$

On the other hand, we demonstrate the right-hand inequality of (2.2). For $x \in \mathbb{R}^n$, we have

$$\frac{|x|\mathrm{b}(K)}{2} = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} h_K(|x|\theta)\, d\theta. \tag{2.3}$$

The right side of (2.3) can be approximated by $\sum_{k=1}^m h_K(|x|\theta_k)\lambda_k$ – the support function of $\sum_{k=1}^m \lambda_k T_k K$, where $\lambda_k \in (0,1)$, $\sum_{k=1}^m \lambda_k = 1$, and $T_k K$ is a rotation of K generated by θ_k. Meanwhile, according to Colesanti-Cuoghi's [11, Theorem 3.1], we have

$$\mathrm{ncap}\left(\sum_{k=1}^m \lambda_k T_k K\right) \ge \sum_{k=1}^m \lambda_k \mathrm{ncap}(T_k K) = \mathrm{ncap}(K) \tag{2.4}$$

due to the easily-checked rotation-invariance of $\mathrm{ncap}(\cdot)$. Note also that the left side of (2.3) is the support function of a ball of radius $2^{-1}\mathrm{b}(K)$. So, the above approximation, the correspondence between a support function and an element of $\mathscr{C}^n$, and (2.4) yield the desired inequality. □

2.2. Another look at volume-radius and log-capacity. Here, we can say more about volume-radius and log-capacity through the solution u_K of (1.1).

REMARK 2.2. For $K \in \mathscr{K}^n$, let $(\nabla u_K)|_{\partial K}$ be the non-tangential limit of ∇u_K at ∂K (cf. [35, Theorem 3] and [34, Theorem 4.3]). If $|\nabla u_K|$ equals a positive constant c on ∂K, then $c^{-1} = \big(V(K)/\omega_n\big)^{\frac{1}{n}}$ is the volume-radius of K and hence $|\nabla u_K|$ exists as a kind of weak mean curvature on the level surfaces of $u = u_K$. In fact,

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = 0 \ \text{ in } \ \mathbb{R}^n \setminus K \ \ \& \ \ |\nabla u|_{\partial K} = c$$

implies that if

$$X = n(x \cdot \nabla u)|\nabla u|^{n-2}\nabla u - |\nabla u|^n x,$$

ν stands for the outer unit normal vector, and $r \to \infty$, then

$$\begin{aligned}(n-1)nc^n V(K) &= (1-n)\int_{\partial K} x \cdot \nabla u |\nabla u|^{n-1}\, dS \\ &= \left(\frac{n-1}{n}\right)\int_{\partial(r\mathbb{B}^n)} X \cdot \nu\, dS \\ &= (n-1)\sigma_{n-1} + o(1),\end{aligned}$$

Thus, $c = \Big(\omega_n/V(K)\Big)^{\frac{1}{n}}$, as desired.

Moreover, if U is n-harmonic, i.e., $\mathrm{div}(|\nabla U|^{n-2}\nabla U) = 0$, in $\mathbb{R}^n \setminus K$, U is continuous on ∂K, and $U(x)$ has a finite limit $U(\infty)$ as $x \to \infty$, then the divergence theorem is utilized to produce

$$U(\infty) = \frac{1}{\sigma_{n-1}}\left(\int_{\partial K} U|\nabla u_K|^{n-1}\,dS + \int_{\mathbb{R}^n\setminus K}\frac{\nabla u_K\cdot\nabla U}{\big(|\nabla u_K|^{n-2} - |\nabla U|^{n-2}\big)^{-1}}\,dV\right).$$

In particular, if $n = 2$ then this formula reduces to [28, (6.3)], and consequently, if $U(x) = u_K(x) - \log|x|$ (which is $2 = n$-harmonic in $\mathbb{R}^n \setminus K$) then

$$\mathrm{ncap}(K) = \exp\left(\sigma_{n-1}^{-1}\int_{\partial K}(\log|x|)|\nabla u_K(x)|^{n-1}\,dS(x)\right) \quad \text{for} \quad n = 2.$$

It is our conjecture that this last formula is still valid for $n \geq 3$.

3. Boundary estimation of log-equilibrium.

3.1. An identity for the unit sphere area through log-equilibrium. In the above and below, by a convex body in $\mathbb{R}^n$ we mean an element of $\mathscr{K}^n$. For $K \in \mathscr{K}^n$, the Gauss map $g : \partial K \to \mathbb{S}^{n-1}$ is defined almost everywhere with respect to the surface measure dS and determined by $g(x) = \nu$, the outer unit normal at $x \in \partial K$. In the process of finding a representation of the log-capacity $\mathrm{ncap}(K)$ in terms of the integral of $|\nabla u_K|^n$ of the log-equilibrium potential u_K on ∂K, we get the following result whose case $n = 2$ is essentially known; see also [28].

THEOREM 3.1. *If $K \in \mathscr{K}^n$, then*

$$\int_{\partial K} h_K(g)|\nabla u_K|^n\,dS = \sigma_{n-1}. \tag{3.1}$$

In other words, if $g_*(|\nabla u_K|^n\,dS)$ is defined by

$$\int_{g^{-1}(E)} |\nabla u_K|^n\,dS \quad \forall \quad \text{Borel set} \;\; E \subset \mathbb{S}^{n-1},$$

then

$$\int_{\mathbb{S}^{n-1}} h_K g_*(|\nabla u_K|^n\,dS) = \sigma_{n-1}.$$

Consequently,

$$\int_{\mathbb{S}^{n-1}} \xi g_*(|\nabla u_K|^n\,dS)(\xi) = 0. \tag{3.2}$$

Proof. For $K \in \mathscr{K}^n$, let $u = u_K$. Suppose that ν is the outer unit normal. Two cases are in order.

Case 1. K is of class $C^{2,+}$ - namely - ∂K is of class C^2 and its Gauss curvature $G(K, x)$ is positive at any $x \in \partial K$. Then

$$|\nabla u| = -\frac{\partial u}{\partial \nu} \quad \text{on} \quad \partial K; \tag{3.3}$$

see also [46].

Observe that if

$$X = n(x \cdot \nabla u)|\nabla u|^{n-2}\nabla u - |\nabla u|^n x$$

then $\mathrm{div} X = 0$ in $\mathbb{R}^n \setminus K$ and hence by an integration-by-part,

$$\int_{\partial K} X \cdot \nu \, dS = \int_{\partial(r\mathbb{B}^n)} X \cdot \nu \, dS \quad \text{as} \quad r \to \infty.$$

However, the right side of the last formula tends to σ_{n-1} as $r \to \infty$ thanks to the expansion of u at infinity. So, from (3.3) it follows that

$$(n-1)\int_{\partial K} (x \cdot \nabla u)\Big(-\frac{\partial u}{\partial \nu}\Big)^{n-1} dS = \left(\frac{1-n}{n}\right)\int_{\partial K} X \cdot \nu \, dS = (n-1)\sigma_{n-1}.$$

Consequently, (3.1) follows from

$$\int_{\partial K} h_K(g)|\nabla u|^n \, dS = \int_{\partial K} (x \cdot \nabla u)\left(-\frac{\partial u}{\partial \nu}\right)^{n-1} dS = \sigma_{n-1}.$$

To reach (3.2), note that σ_{n-1} is a dimensional constant and the support function of $L = K + x_0$ is

$$h_L(\xi) = h_K(\xi) + x_0 \cdot \xi \quad \text{for} \quad \xi \in \mathbb{S}^{n-1},$$

where $x_0 \in \mathbb{R}^n$ is arbitrarily given. So, an application of (3.1) to L yields

$$\int_{\partial K} x_0 \cdot g(x)|\nabla u_K(x)|^n \, dS(x) = 0$$

and consequently, the following vector equation

$$\int_{\partial K} g(x)|\nabla u_K(x)|^n \, dS(x) = 0$$

holds. This gives (3.2).

Case 2. K just belongs to $\mathscr{K}^n$. To prove (3.1) under this general situation, recall first that the Hausdorff metric d_H on $\mathscr{C}^n$ is determined by

$$d_H(K_1, K_2) = \sup_{x \in K_1} d(x, K_1) + \sup_{x \in K_2} d(x, K_2) \quad \forall \quad K_1, K_2 \in \mathscr{C}^n,$$

where $d(x, E)$ stands for the distance from the point x to the set E.

Of course, the interior of the above K is a Lipschitz domain. According to Lewis-Nyström's [35, Theorem 3] (cf. [15] and [29] for harmonic functions), we see that ∇u_K has non-tangential limit, still denoted by ∇u_K, almost everywhere on ∂K with respect to dS. Moreover, $|\nabla u_K|$ is n-integrable on ∂K under dS, i.e.,

$$\int_{\partial K} |\nabla u_K|^n \, dS < \infty. \tag{3.4}$$

For $0 < t < 1$ let

$$L_t = \{x \in \mathbb{R}^n \setminus K : \ u_K(x) > t\} \quad \& \quad K_t = \mathbb{R}^n \setminus L_t.$$

Then $K_t \in \mathscr{K}^n$ is of class $C^{2,+}$ (cf. [11, Theorem 2.2]). Note that $u_K - t$ is equal to the log-equilibrium potential u_{K_t} of K_t, and note that continuity of u_K on ∂K yields $\lim_{t\to 0} d_H(K_t, K) = 0$. So,

$$\sigma_{n-1} = \int_{\partial K_t} (x \cdot \nabla u_K(x))|\nabla u_K(x)|^{n-1}\, dS(x).$$

This, plus (3.4) and the dominated convergence theorem, derives

$$\begin{aligned}\sigma_{n-1} &= \lim_{t\to 0} \int_{\partial K_t} (x \cdot \nabla u_K(x))|\nabla u_K(x)|^{n-1}\, dS(x)\\ &= \int_{\partial K} (x \cdot \nabla u_K(x))|\nabla u_K(x)|^{n-1}\, dS(x),\end{aligned}$$

whence yielding (3.1) and its consequence (3.2). □

3.2. A lower bound for the gradient of log-equilibrium. Being motivated by [13, Lemma 2.18] we find the following lower bound estimate for the gradient of the equilibrium potential of (1.1) on the boundary of a convex body.

THEOREM 3.2. *For $K \in \mathscr{K}^n$ let u_K be its equilibrium potential. If $K \subset r\mathbb{B}^n$, then there exists a constant $c > 0$ depending only on r and n such that $|\nabla u_K| \geq c$ almost everywhere on ∂K with respect to dS.*

Proof. Suppose that $u = u_K$ and $t_0 \in (0,1)$ obey

$$K_t = \{x \in \mathbb{R}^n \setminus K : \ u(x) \leq t\} \subset r\mathbb{B}^n \quad \forall \quad t \in (0, t_0).$$

Note that K_t is of class $C^{2,+}$ and the existence of t_0 is ensured by the continuity of u in $\mathbb{R}^n \setminus K$ (cf. [11, Theorem 2.2]). Now, for $t \in (0, t_0)$ let

$$\check{u}_t(x) = u(x) - t \quad \forall \quad x \in \mathbb{R}^n \setminus K_t.$$

Then $\check{u}_t$ is the solution of (1.1) for K_t, and in $C^2(\mathbb{R}^n \setminus K_t)$. For $\tau \in [0, 1)$ let

$$\check{K}_\tau = \{x \in \mathbb{R}^n \setminus K_t : \ \check{u}_t(x) \leq \tau\}$$

and $h(\cdot, \tau)$ be its support function $h_{\check{K}_\tau}$. Since $\check{K}_0 = K_t \subset r\mathbb{B}^n$, $\check{u}_t$ is controlled, via the maximum principle, by the log-equilibrium potential of $r\mathbb{B}^n$. Consequently, there is a constant $c_0 > 0$ depending on n and r such that

$$\operatorname{diam}(\check{K}_{2^{-1}}) = \operatorname{diam}(\{x \in \mathbb{R}^n : 2^{-1} < u(x) \leq 1\}) \leq c_0.$$

Moreover, we have

$$0 \leq \inf_{x \in \mathbb{S}^{n-1}} h(x, 2^{-1}) \leq \sup_{x \in \mathbb{S}^{n-1}} h(x, 2^{-1}) \leq c_0,$$

whence deriving

$$h(x, 0) = h(x, 2^{-1}) - \int_0^{2^{-1}} \frac{\partial h}{\partial \tau}(x, \tau)\, d\tau \quad \forall \quad x \in \mathbb{S}^{n-1}.$$

From [11, Theorem A.2] it follows that $s \mapsto \frac{\partial h}{\partial \tau}(x, \tau)$ is a non-decreasing function on $[0, 1)$. This monotonicity and the mean-value theorem for derivatives yield

$$\frac{\partial h}{\partial \tau}(x, \tau)\Big|_{\tau=0} \leq 2\big(h(x, 2^{-1}) - h(x, 0)\big) \leq 2h(x, 2^{-1}) \leq 2c_0 \ \ \forall \ \ x \in \mathbb{S}^{n-1}.$$

Meanwhile, an application of [11, Theorem A.1] gives

$$\frac{\partial h}{\partial \tau}(x,\tau)|_{\tau=0} = |\nabla \check{u}_t(x)|^{-1},$$

where $x \in \partial K_t$ satisfies

$$x = (\nabla \check{u}_t(x))|\nabla \check{u}_t(x)|^{-1} \quad \& \quad \check{u}_t(x) = 0.$$

As a result, we get

$$\inf_{x\in\partial K_t} |\nabla u(x)| = \inf_{x\in\partial K_t} |\nabla \check{u}_t(x)| \geq (2c_0)^{-1}.$$

The desired assertion follows by letting $t \to 0$ and using the existence of the non-tangential maximal function of $|\nabla u|$ on ∂K. □

4. Hadamard's variation for log-capacity.

4.1. Hadamard's variation: smooth case. For $K_1, K_2 \in \mathscr{K}^n$ and $0 \leq t_1, t_2$ define

$$t_1K_1 + t_2K_2 = \{x = t_1x_1 + t_2x_2 : \ x_j \in K_j\}.$$

In accordance with Colesant-Cuoghi's [11, Theorem 3.1] (cf. Borell [7] for $n = 2$), we have the following Brunn-Minkowski inequality for $t \in [0,1]$ and $K_1, K_2 \in \mathscr{K}^n$:

$$\text{ncap}(tK_1 + (1-t)K_2) \geq t\text{ncap}(K_1) + (1-t)\text{ncap}(K_2) \tag{4.1}$$

with equality if and only if K_1 is a translate and a dilate of K_2.

Notice that (4.1) implies that

$$\frac{d^2}{dt^2}\text{ncap}(tK_1 + (1-t)K_2)\big|_{t=0} \leq 0.$$

So, we get the following assertion extending the smooth two-dimensional Hadamard's variation formula (cf. [47]).

THEOREM 4.1. *If $K_0, K_1 \in \mathscr{K}^n$ are of class $C^{2,+}$, then*

$$\frac{d}{dt}\log ncap(K_0 + tK_1)\big|_{t=0} = \sigma_{n-1}{}^{-1}\int_{\partial K_0} h_{K_1}(g)|\nabla u_{K_0}|^n \, dS, \tag{4.2}$$

equivalently,

$$\frac{d}{dt}\log ncap((1-t)K_0 + tK_1)\big|_{t=0} = \sigma_{n-1}{}^{-1}\int_{\partial K_0} \frac{|\nabla u_{K_0}|^n}{\big(h_{K_1}(g) - h_{K_0}(g)\big)^{-1}} \, dS. \tag{4.3}$$

Consequently,

$$\frac{\sigma_{n-1}}{ncap(K_0)} \leq \int_{\partial K_0} |\nabla u_{K_0}|^n \, dS \tag{4.4}$$

with equality if K_0 is a ball.

Proof. To derive (4.2), note again that

$$u(x) = u_K(x) = \log|x| - \log \mathrm{ncap}(K) + o(1) \quad \forall \ \ x \in \mathbb{R}^n \setminus K.$$

Proving (4.2) is equivalent to establishing the first variation of u. To do so, for an arbitrary small number $\epsilon > 0$ let K_ϵ be such a convex body that its boundary ∂K_ϵ is obtained by shifting ∂K an infinitesimal distance $\delta\nu = \epsilon\rho(s)$ along its outer unit normal ν, where ρ is a smooth function on ∂K:

$$\partial K_\epsilon = \{x + \epsilon\rho(x)\nu(x) : \ x \in \partial K\},$$

and denote by $u_\epsilon = u_{K_\epsilon}$.

For convenience, set

$$K^c = \mathbb{R}^n \setminus K \ \ \& \ \ K^c_\epsilon = \mathbb{R}^n \setminus K_\epsilon,$$

and define

$$u(x) = 0 \ \forall \ x \in K \ \ \& \ \ u_\epsilon(x) = 0 \ \forall \ x \in K_\epsilon.$$

Consider the following difference

$$\mathrm{Dif}(\epsilon) = \int_{K^c} |\nabla u|^{n-2}\nabla u \cdot \nabla u_\epsilon \, dV - \int_{K^c_\epsilon} |\nabla u_\epsilon|^{n-2}\nabla u_\epsilon \cdot \nabla u \, dV. \tag{4.5}$$

On the one hand,

$$\begin{aligned} \mathrm{Dif}(\epsilon) &= \int_{K^c\setminus K^c_\epsilon} |\nabla u|^{n-2}\nabla u \cdot \nabla u_\epsilon \, dV + \int_{K^c_\epsilon} (|\nabla u|^{n-2} - |\nabla u_\epsilon|^{n-2})\nabla u_\epsilon \cdot \nabla u \, dV \\ &= \epsilon \int_{\partial K^c} |\nabla u|^{n-1}\Big(\frac{\partial u_\epsilon}{\partial \nu}\Big)\rho \, dS + \int_{K^c_\epsilon} (|\nabla u|^{n-2} - |\nabla u_\epsilon|^{n-2})\nabla u_\epsilon \cdot \nabla u \, dV. \end{aligned}$$

This yields

$$\lim_{\epsilon\to 0} \frac{\mathrm{Dif}(\epsilon)}{\epsilon} = -\int_{\partial K} |\nabla u|^{n-1}\Big(\frac{\partial u}{\partial \nu}\Big)\rho \, dS.$$

On the other hand, note that

$$\begin{cases} -\mathrm{div}(|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon = 0 & \text{in} \quad K^c_\epsilon; \\ -\mathrm{div}(|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon = 0 & \text{in} \quad K^c, \end{cases}$$

and

$$\begin{cases} \mathrm{div}(u|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon) = u\mathrm{div}\big(|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon\big) + |\nabla u_\epsilon|^{n-2}\nabla u_\epsilon \cdot \nabla u; \\ \mathrm{div}(u_\epsilon|\nabla u|^{n-2}\nabla u) = u_\epsilon\mathrm{div}\big(|\nabla u|^{n-2}\nabla u\big) + |\nabla u|^{n-2}\nabla u \cdot \nabla u_\epsilon. \end{cases}$$

So, an application of the divergence theorem gives

$$\begin{aligned}&\int_{K^c}|\nabla u|^{n-2}\nabla u\cdot\nabla u_\epsilon\,dV\\&=\int_{K^c}\operatorname{div}(u_\epsilon|\nabla u|^{n-2}\nabla u)\,dV\\&=\lim_{r\to\infty}\int_{K^c\setminus(r\mathbb{B}^n)^c}\operatorname{div}(u_\epsilon|\nabla u|^{n-2}\nabla u)\,dV\\&=\int_{\partial K^c}u_\epsilon|\nabla u|^{n-2}\nabla u\cdot\nu\,dS-\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}u_\epsilon|\nabla u|^{n-2}\nabla u\cdot\nu\,dS\\&=-\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}u_\epsilon|\nabla u|^{n-2}\nabla u\cdot\nu\,dS.\end{aligned}$$

Similarly, we have

$$\int_{K_\epsilon^c}|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon\cdot\nabla u\,dV=-\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}u|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon\cdot\nu\,dS.$$

Consequently,

$$\begin{aligned}&\operatorname{Dif}(\epsilon)\\&=-\lim_{r\to\infty}\left(\int_{\partial(r\mathbb{B}^n)^c}u_\epsilon|\nabla u|^{n-2}\nabla u\cdot\nu\,dS-\int_{\partial(r\mathbb{B}^n)^c}u|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon\cdot\nu\,dS\right)\\&=\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}\nu\cdot\left(\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|^{2-n}}-\frac{\nabla u}{|\nabla u|^{2-n}}\right)u\,dS-\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}\frac{(u_\epsilon-u)\nabla u}{|\nabla u|^{2-n}}\cdot\nu\,dS.\end{aligned}$$

This derives via (3.1)

$$\begin{aligned}\lim_{\epsilon\to0}\frac{\operatorname{Dif}(\epsilon)}{\epsilon}&=\lim_{\epsilon\to0}\left(\frac{\log\operatorname{ncap}(K_\epsilon)-\log\operatorname{ncap}(K)}{\epsilon}\right)\lim_{r\to\infty}\int_{\partial(r\mathbb{B}^n)^c}\frac{\nabla u\cdot\nu}{|\nabla u|^{2-n}}\,dS\\&=-\sigma_{n-1}\lim_{\epsilon\to0}\frac{\log\operatorname{ncap}(K_\epsilon)-\log\operatorname{ncap}(K)}{\epsilon}.\end{aligned}$$

The above two formulas for $\lim_{\epsilon\to0}\epsilon^{-1}\operatorname{Dif}(\epsilon)$ derive

$$\lim_{\epsilon\to0}\frac{\log\operatorname{ncap}(K_\epsilon)-\log\operatorname{ncap}(K)}{\epsilon}=\int_{\partial K}|\nabla u|^n\rho\,\frac{dS}{\sigma_{n-1}},$$

and thereby verifying (4.2) through letting $K=K_0$ and $\rho=h_{K_1}\circ g$.

Through the chain rule and the homogeneous property of the support function, (4.2) immediately derives (4.3) and vice visa. Now, because $t\mapsto\operatorname{ncap}\big((1-t)K_0+tK_1\big)$ is concave on $[0,1]$; see also [11], if $K_1=\overline{r\mathbb{B}^n}$ and $r=\operatorname{ncap}(K_0)$ then an application of (4.3) gives

$$\begin{aligned}0&\le\frac{d}{dt}\log\operatorname{ncap}\big((1-t)K_0+tK_1\big)\big|_{t=0}\\&=\left(\frac{1}{\operatorname{ncap}(K_0)}\right)\frac{d}{dt}\operatorname{ncap}\big((1-t)K_0+tK_1\big)\big|_{t=0}\\&=\int_{\partial K_0}\big(h_{K_1}(g)-h_{K_0}(g)\big)|\nabla u_{K_0}|^n\,\frac{dS}{\sigma_{n-1}}\\&=\int_{\partial K_0}\big(r-h_{K_0}(g)\big)|\nabla u_{K_0}|^n\,\frac{dS}{\sigma_{n-1}},\end{aligned}$$

whence reaching (4.4) via (3.1). □

4.2. Hadamard's variation: non-smooth case. To generalize Theorem 4.1, without loss of generality we may assume that the origin is an interior point of $K, K_j \in \mathscr{K}^n$, write $\varrho_K : \mathbb{S}^{n-1} \to \partial K$ and $\varrho_{K_j} : \mathbb{S}^{n-1} \to \partial K_j$ for the radial projections

$$\mathbb{S}^{n-1} \ni \theta \mapsto \varrho_K(\theta) = r_K(\theta)\theta \in \partial K$$

and

$$\mathbb{S}^{n-1} \ni \theta \mapsto \varrho_{K_j}(\theta) = r_{K_j}(\theta)\theta \in \partial K_j$$

respectively, where $r_K(\theta)$ and r_{K_j} are the unique positive numbers ensuring $r_K(\theta)\theta \in \partial K$ and $r_{K_j}(\theta)\theta \in \partial K_j$ respectively, and set

$$D(\theta) = |\nabla u_K(\varrho_K(\theta))| r_K(\theta) (h_K\big(g(\varrho_K(\theta))\big))^{-\frac{1}{n}}$$

and

$$D_j(\theta) = |\nabla u_{K_j}(\varrho_{K_j}(\theta))| r_{K_j}(\theta) (h_{K_j}\big(g(\varrho_{K_j}(\theta))\big))^{-\frac{1}{n}}$$

respectively.

In the sequel, we will use the fact that $dS(x) = |x|^n (x \cdot g(x))^{-1} d\theta$ holds for $\theta = x/|x|$.

THEOREM 4.2. *For $\{K, K_1, K_2, ...\} \subseteq \mathscr{K}^n$, $\epsilon > 0$ and $\alpha > 0$, there exist $s_0 > 0$, $\eta > 0$ and a family of balls $\mathcal{B}$ on $\mathbb{S}^{n-1}$ such that:*

(i) *every member in $\mathcal{B}$ has radius s_0;*

(ii) *there is a constant $N > 0$ depending only on the inner and outer radii of K, such that any point of $\mathbb{S}^{n-1}$ belongs to at most N balls of $\mathcal{B}$;*

(iii) *$S(\mathbb{S}^{n-1} \setminus F) < \epsilon$ where $F = \cup_{B \in \mathcal{B}} B$;*

(iv) *if $d_H(K_j, K) < \eta$, then for any $B \in \mathcal{B}$ we have*

$$s_0^{1-n} \left(\int_B \left| \left(\frac{D_j(\theta)}{D(\theta)} \right)^{n-1} - 1 \right|^\alpha d\theta + \int_B \left| \left(\frac{D(\theta)}{D_j(\theta)} \right)^{n-1} - 1 \right|^\alpha d\theta \right) < \epsilon\,;$$

(v)

$$\lim_{j \to \infty} \int_{\mathbb{S}^{n-1}} \left| D_i^n(\theta) - D^n(\theta) \right| d\theta = 0\,.$$

Proof. The following argument comes from an appropriate modification of the argument for Lemmas 4.4-4.5-4.6 in [13]. According to Jerison's [27, Lemma 3.3], we have that for any $\epsilon > 0$ there exists $\eta > 0$ and a finite disjoint collection of open balls $B_{r_k}(z_k)$ (centered at z_k with radius r_k) such that $z_k \in \partial K$ and for any convex body $L \in \mathscr{K}^n$ for which $d_H(L, K) < \eta$:

(a) $S\big(\partial L \setminus \cup_k B_{r_k}(z_k)\big) < \epsilon$;

(b) after a suitable rotation and translation depending on k, we have that ∂K and ∂L are given on $B_{r_k}(z_k)$ by the graphs of functions ϕ and ψ respectively, enjoying

$$\sup\big\{|\nabla\phi(x)| + |\nabla\psi(x)| \,:\, |x| < \epsilon^{-1} r_k,\ \phi \,\&\, \psi \text{ differentiable at } x\big\} \le \epsilon\,.$$

Now, given $\epsilon > 0$. Following the beginning part of the proof of Jerison's [27, Lemma 3.7] we choose a sufficiently small number $s_0 < \min\{r_k\}$ such that the Jacobians of the change of variables ϱ_{K_k} and ϱ_K vary by at most ϵ as θ varies by the distance $s > 0$ and $\varrho_K(\theta)$ is contained in $\cup_k B_{r_k}(z_k)$. As a consequence, we can select $\mathcal{B}$ obeying (i)-(ii)-(iii) described as above.

Meanwhile, from Lewis-Nyström's [36, Theorem 2] it follows that for each $s \in (0, s_0)$ and each ball B of radius s in the concentric ϵ^{-1} multiple of any element in $\mathcal{B}$, there is a constant c_B such that

$$s^{1-n} \int_B |\log D(\theta) - c_B|\, d\theta < \epsilon. \tag{4.6}$$

Furthermore, using the previously-stated (a)-(b) we can take $\delta > 0$ small enough to obtain

$$s^{1-n} \int_B |\log D_j(\theta) - c_B|\, d\theta < \epsilon \quad \forall \quad s \in (0, \delta). \tag{4.7}$$

A combination of (4.6) and (4.7) gives

$$s^{1-n} \int_B \Big| \log \frac{D_j(\theta)}{D(\theta)} \Big|\, d\theta < 2\epsilon.$$

Applying John-Nirenberg's exponential inequality (cf. [30]) for a BMO-function to (4.6), we obtain that given $\alpha > 0$ and for arbitrarily small $\epsilon' > 0$ one can take $\eta' > 0$ and s_0 so small that for each $B \in \mathcal{B}$ there is a constant c'_B ensuring

$$s_0^{1-n} \int_B \Big| c'_B \Big(\frac{D_j(\theta)}{D(\theta)} \Big)^{n-1} - 1 \Big|^{\alpha} d\theta < \epsilon'. \tag{4.8}$$

Note that η' and s_0 can be chosen small enough to ensure that for each $B \in \mathcal{B}$ we have

$$\frac{\int_B D_j^{n-1}(\theta)\, d\theta}{\int_B D^{n-1}(\theta)\, d\theta} = \Big(1 + \mathcal{O}(\epsilon')\Big) \frac{\int_{\varrho_{\Omega_j}(B)} |\nabla u_{K_j}|^{n-1}\, dS}{\int_{\varrho_\Omega(B)} |\nabla u_K|^{n-1}\, dS}, \tag{4.9}$$

where $\mathcal{O}(\epsilon')$ is a positive big-oh function of ϵ'.

Next, we are about to show that c'_B in (4.8) is equal to 1. To this end, let us fix s_0 and allow η to rely on s_0. Note that the quotient on the right side of (4.9) is the ratio of the n-harmonic measures (cf. [38]) of the sets $\varrho_j(B)$ and $\varrho(B)$. So, employing the maximum principle to compare n-harmonic functions in $\mathbb{R}^n \setminus K_j$ to n-harmonic functions in $\mathbb{R}^n \setminus \rho K$ (where ρK means a ρ-dilation of K) , we can take $\eta > 0$ smaller still, relying on s_0 such that

$$\left| \frac{\int_B D_j^{n-1}(\theta)\, d\theta}{\int_B D^{n-1}(\theta)\, d\theta} - 1 \right| \lesssim \epsilon' \tag{4.10}$$

holds for any $B \in \mathcal{B}$. In the above and below, $U \lesssim V$ stands for $U \le c_n V$ for a dimensional constant $c_n > 0$.

Using the $q > n$-harmonic setting of Lewis-Nyström's [35, Theorem 3] and the Hölder inequality we find that

$$\left(\frac{1}{S(\varrho_\Omega(B))} \int_{\varrho_\Omega(B)} |\nabla u_K|^n\, dS \right)^{\frac{n-1}{n}} \lesssim \frac{1}{S(\varrho_K(B))} \int_{\varrho_K(B)} |\nabla u_K|^{n-1}\, dS \tag{4.11}$$

is valid for any ball centered at ∂K. Clearly, a similar estimate is valid for each ∂K_j. Thus,

$$\left(s_0^{1-n}\int_B D^n(\theta)\,d\theta\right)^{\frac{n-1}{n}} \lesssim s_0^{1-n}\int_B D^{n-1}(\theta)\,d\theta \tag{4.12}$$

and similarly for D_j. Now, using Hölder's inequality plus (4.12), (4.8) and (4.11), we get that for each $B\in\mathcal{B}$,

$$\begin{aligned}
\frac{\int_B c'_B D_j^{n-1}(\theta)\,d\theta}{\int_B D^{n-1}(\theta)\,d\theta} - 1 &= \frac{\int_B c'_B\Big(\big(\frac{D_j(\theta)}{D(\theta)}\big)^{n-1} - 1\Big)D^{n-1}(\theta)\,d\theta}{\int_B D^{n-1}(\theta)\,d\theta}\\
&\lesssim \left(\int_B \Big(c'_B\big(\frac{D_j(\theta)}{D(\theta)}\big)^{n-1} - 1\Big)^n d\theta\right)^{\frac{1}{n}} \left(\frac{\Big(\int_B D^n(\theta)\,d\theta\Big)^{\frac{n-1}{n}}}{\int_B D^{n-1}(\theta)\,d\theta}\right)\\
&\lesssim \left(s_0^{1-n}\int_B \Big(c'_B\big(\frac{D_j(\theta)}{D(\theta)}\big)^{n-1} - 1\Big)^n d\theta\right)^{\frac{1}{n}}\\
&\lesssim \epsilon'.
\end{aligned}$$

In a similar manner, we replace $c'_B D_j/D$ by $(D/c'_B)D_j$ in the above estimates to obtain

$$\frac{\int_B D^{n-1}(\theta)\,d\theta}{\int_B c'_B D_j^{n-1}(\theta)\,d\theta} - 1 \lesssim \epsilon'.$$

Since (4.10) yields

$$\left|\frac{\int_B D_j^{n-1}(\theta)\,d\theta}{\int_B D^{n-1}(\theta)\,d\theta} - 1\right| \lesssim \epsilon',$$

we must have $|c'_B - 1| \lesssim \epsilon'$, whence getting $c'_B = 1$. As a consequence of this and (4.8), we find

$$s_0^{1-n}\int_B \left|\Big(\frac{D_j(\theta)}{D(\theta)}\Big)^{n-1} - 1\right|^\alpha d\theta \lesssim \epsilon' \;\;\&\;\; s_0^{1-n}\int_B \left|\Big(\frac{D(\theta)}{D_j(\theta)}\Big)^{n-1} - 1\right|^\alpha d\theta \lesssim \epsilon',$$

whence completing the proof of (iv).

Although the idea of verifying (v) is motivated by the argument for [27, Proposition 4.3], we still need more effort to adapt it to our nontrivial situation. Because of $q>n$ in [35, Theorem 3], it is possible to find $\beta\in(1,\infty)$ such that $n\beta/(\beta-1)=q$. Given $\epsilon>0$, take $\eta>0$ and F in accordance with (i)-(iv). Using the inequality

$$|a^n - b^n| \le \frac{(a+b)|a^{n-1}-b^{n-1}|}{n^{-1}(n-1)} \quad \forall\, a,b\ge 0\,,$$

the Hölder inequality and (3.1), we achieve

$$\begin{aligned}&\int_F |D_j^n(\theta)-D^n(\theta)|\,d\theta\\&\le\Big(\frac{n}{n-1}\Big)\int_F\big|D_j^{n-1}(\theta)-D^{n-1}(\theta)\big|\big(D_j(\theta)+D(\theta)\big)\,d\theta\\&\lesssim\bigg(\int_F\big|D_j^{n-1}(\theta)-D^{n-1}(\theta)\big|^{\frac{n}{n-1}}\,d\theta\bigg)^{\frac{n-1}{n}}\bigg(\int_F\big(D_j(\theta)+D(\theta)\big)^n\,d\theta\bigg)^{\frac1n}\\&\lesssim\big(2\sigma_{n-1}\big)^{\frac1n}\bigg(\int_F\bigg|\Big(\frac{D_j(\theta)}{D(\theta)}\Big)^{n-1}-1\bigg|^{\frac{n}{n-1}}D^n(\theta)\,dS(\theta)\bigg)^{\frac{n-1}{n}}\\&\lesssim\bigg(\int_F\bigg|\Big(\frac{D_j(\theta)}{D(\theta)}\Big)^{n-1}-1\bigg|^{\frac{n\beta}{n-1}}dS(\theta)\bigg)^{\frac{n-1}{n\beta}}\bigg(\int_F D^q(\theta)\,d\theta\bigg)^{\frac{n-1}{q}},\end{aligned}$$

thereby deducing

$$\int_F\big|D_j^n(\theta)-D^n(\theta)\big|\,d\theta\lesssim\epsilon\quad\text{as}\quad j\to\infty,\tag{4.13}$$

via (iv) with $\alpha=q$ as well as [35, Theorem 3] insuring $\int_{\mathbb{S}^{n-1}}D^q(\theta)\,d\theta<\infty$.

On the other hand, by the Hölder inequality with $q>n$ we derive

$$\begin{aligned}\int_{\mathbb{S}^{n-1}\setminus F}|D_j^n(\theta)-D^n(\theta)|\,d\theta&\le\int_{\mathbb{S}^{n-1}\setminus F}\big(D_j^n(\theta)+D^n(\theta)\big)\,d\theta\\&\lesssim\big(S(\mathbb{S}^{n-1}\setminus F)\big)^{\frac{q}{q-n}}\bigg(\int_{\mathbb{S}^{n-1}\setminus F}\big(D_j^q(\theta)+D^q(\theta)\big)\,d\theta\bigg)^{\frac nq},\end{aligned}$$

whence getting (v) through (iii), (4.13) and [35, Theorem 3] which especially guarantees

$$\sup_j\int_{\mathbb{S}^{n-1}\setminus F}\big(D_j^q(\theta)+D^q(\theta)\big)\,d\theta<\infty.$$

□

With the help of Theorem 4.2, we can establish the following weak convergence result for the measure induced by Theorem 3.1.

THEOREM 4.3. *Let* $K,K_j\in\mathscr{K}^n$ *and* $\lim_{j\to\infty}d_H(K_j,K)=0$. *If* u,u_j *are the log-equilibrium potentials of* K,K_j *respectively, then* $d\mu_j=(g_j)_*(|\nabla u_j|^n\,dS)$ *converges weakly to* $d\mu=g_*(|\nabla u|^n\,dS)$, *i.e.*,

$$\lim_{j\to\infty}\int_{\mathbb{S}^{n-1}}f\,d\mu_j=\int_{\mathbb{S}^{n-1}}f\,d\mu\quad\forall\quad f\in C(\mathbb{S}^{n-1}).$$

Proof. The following argument is analogous to [9, Section 5] (cf. [27, the proof of Theorem 3.1] and [13, the proof of Lemma 4.3]). Recall that the push-forward measures $d\mu$ & $d\mu_j$ on $\mathbb{S}^{n-1}$ are determined respectively by

$$\mu(E)=\int_{g^{-1}(E)}|\nabla u|^n\,dS\;\;\&\;\;\mu_j(E)=\int_{g_j^{-1}(E)}|\nabla u_j|^n\,dS\;\forall\;\text{ Borel set }\;E\subset\mathbb{S}^{n-1},$$

where g and g_j are the Gauss maps attached to K and K_j respectively. It remains to verify that μ is the weak limit of μ_j as $j \to \infty$.

An application of Theorem 4.2(v) yields

$$\lim_{j\to\infty}\Big(\mu(\mathbb{S}^{n-1})-\mu_j(\mathbb{S}^{n-1})\Big)=\lim_{j\to\infty}\int_{\mathbb{S}^{n-1}}\big(D^n(\theta)-D^n_j(\theta)\big)\,d\theta=0. \tag{4.14}$$

Note that $g^{-1}(E)\subseteq\partial K$ and $g_j^{-1}(E)\subseteq\partial K_j$ are closed (cf. [9] and [26, 27]) for any Borel set $E\subseteq\mathbb{S}^{n-1}$, and that if $\xi_j\in g_j(x_j)$ approaches ξ and if $x_j\to x$ then $\xi\in g(x)$ and $x\in\partial K$. So, for any open neighborhood U in ∂K of the closed set $g^{-1}(E)$ we have that $\varrho^{-1}_{K_j}\big(g_j^{-1}(E)\big)\subseteq\varrho_K^{-1}(U)$ as $j\to\infty$, whence finding

$$\limsup_{j\to\infty}\mu_j(E)\le\lim_{j\to\infty}\int_{\varrho_K^{-1}(U)}D^n_j(\theta)\,d\theta\le\int_{\varrho_K^{-1}(U)}D^n(\theta)\,d\theta.$$

When the infimum is over all $U\supseteq g^{-1}(E)$, we get $\limsup_{j\to\infty}\mu_j(E)\le\mu(E)$. This last inequality and (4.14) imply that for any open subset O of $\mathbb{S}^{n-1}$,

$$\begin{aligned}\liminf_{j\to\infty}\mu_j(O)&=\liminf_{j\to\infty}\big(\mu_j(O)-\mu_j(\mathbb{S}^{n-1}\setminus O)\big)\\&\ge\liminf_{j\to\infty}\mu_j(\mathbb{S}^{n-1})-\mu(\mathbb{S}^{n-1}\setminus O)\\&=\mu(\mathbb{S}^{n-1})-\mu(\mathbb{S}^{n-1}\setminus O)=\mu(O).\end{aligned}$$

If $\tilde\mu$ is any weak limit of a subsequence of μ_j, then the above inequalities on $\limsup_{j\to\infty}$ and $\liminf_{j\to\infty}$ deduce that $\tilde\mu(C)\le\mu(C)$ and $\mu(O)\le\tilde\mu(O)$ hold for any closed $C\subseteq\mathbb{S}^{n-1}$ and any open $O\subseteq\mathbb{S}^{n-1}$. Consequently, for any closed $C\subseteq\mathbb{S}^{n-1}$ we have

$$\mu(C)\ge\tilde\mu(C)=\inf\{\tilde\mu(O):\text{open }O\supseteq C\}\ge\inf\{\mu(O):\ \text{open }O\supseteq C\}=\mu(C),$$

and hence $\tilde\mu=\mu$. □

The following is the general variational result.

THEOREM 4.4. *(4.2)-(4.3)-(4.4) are valid for $K_0,K_1\in\mathscr{K}^n$.*

Proof. Given $K_0,K_1\in\mathscr{K}^n$. There are two $C^{2,+}$-sequences $\{K_{0,j}\},\{K_{1,j}\}$ in $\mathscr{K}^n$ such that

$$\lim_{j\to\infty}d_H(K_{0,j},K_0)=0=\lim_{j\to\infty}d_H(K_{1,j},K_1).$$

Now, for $t\in(0,1)$ and $j=1,2,...$ set

$$\begin{cases}K_t=(1-t)K_0+tK_1,\quad K_{t,j}=(1-t)K_{0,j}+tK_{1,j};\\ \Phi(t)=\mathrm{ncap}(K_0+tK_1),\quad \Phi_j(t)=\mathrm{ncap}(K_{0,j}+tK_{1,j});\\ \Psi(t)=\mathrm{ncap}(K_t),\quad \Psi_j(t)=\mathrm{ncap}(K_{t,j}).\end{cases}$$

Note that

$$t\mapsto\Psi_j(t)=(1-t)\Phi_j\Big(\frac{t}{1-t}\Big)$$

is a concave function on $(0,1)$. So,

$$\Psi_j'(t)\le\frac{\Psi_j(t)-\Psi_j(0)}{t}\le\Psi_j'(0)\quad\forall\quad t\in(0,1). \tag{4.15}$$

A simple computation gives

$$\Psi_j'(t) = -\Phi_j\Big(\frac{t}{1-t}\Big) + (1-t)^{-1}\Phi_j'\Big(\frac{t}{1-t}\Big)$$

and

$$\begin{aligned}\Psi_j'(0) &= -\Phi_j(0) + \Phi_j'(0)\\ &= \frac{\mathrm{ncap}(K_{0,j})}{\sigma_{n-1}}\left(-\sigma_{n-1} + \int_{\partial K_{0,j}} h_{K_{1,j}}(g)|\nabla u_{K_{0,j}}|^n\, dS\right)\\ &= \frac{\mathrm{ncap}(K_{0,j})}{\sigma_{n-1}}\int_{\partial K_{0,j}}\Big(h_{K_{1,j}}(g) - h_{K_{0,j}}(g)\Big)|\nabla u_{K_{0,j}}|^n\, dS,\end{aligned}$$

owing to (3.1) and (4.3). Upon letting $j \to \infty$ and $t \to 0$ in (4.15), we use Theorem 4.3 to obtain

$$\Psi'(0) = \frac{\mathrm{ncap}(K_0)}{\sigma_{n-1}}\int_{\partial K_0}\Big(h_{K_1}(g) - h_{K_0}(g)\Big)|\nabla u_{K_0}|^n\, dS,$$

whence establishing (4.3), equivalently, (4.2), and thus (4.4). □

5. Minkowski's problem for log-capacity.

5.1. Prescribing volume variation. Given $K \in \mathscr{K}^n$. From the Gauss map $g : \partial K \to \mathbb{S}^{n-1}$ one can introduce the area set function $\mathcal{H}^{n-1}_{\partial K}$ of ∂K via setting

$$\mathcal{H}^{n-1}_{\partial K}(E) = S\big(\{x \in \partial K :\ g(x) \cap E \neq \emptyset\}\big) \quad \forall \quad \text{Borel subset } E \subset \mathbb{S}^{n-1}.$$

This measure $d\mathcal{H}^{n-1}_{\partial K}$ is treated as the push-forward measure $g_*(dS)$ on $\mathbb{S}^{n-1}$ of the $n-1$ dimensional surface measure dS on ∂K through the inverse map g^{-1} of g. Obviously, $\mathcal{H}^{n-1}_{\partial K}(\mathbb{S}^{n-1}) = S(K)$, i.e., the surface area of K. Two more special facts on this measure are worth recalling. The first is that if ∂K is polyhedron then $d\mathcal{H}^{n-1}_{\partial K} = \sum_k c_k\delta_{\nu_k}$, where δ_{ν_k} is the unit point mass at ν_k and c_k is the $(n-1)$ dimensional measure of the face of ∂K with outward unit normal being ν_k. The second is that if $K \in \mathscr{K}^n$ is of class $C^{2,+}$ then $d\mathcal{H}^{n-1}_{\partial K}$ is absolutely continuous with respect to $d\theta$ and so decided by the reciprocal of the Gauss curvature $G(K,\cdot)$ of ∂K.

The classical Minkowski problem is to ask under what conditions on a given nonnegative Borel measure on $\mathbb{S}^{n-1}$ one can get a convex body $K \in \mathscr{K}^n$ such that $d\mathcal{H}^{n-1}_{\partial K} = d\mu$. As is well-known in convex geometry, this problem is solvable if and only if the support of μ is not contained in any equator (the intersection of $\mathbb{S}^{n-1}$ with any hype-plane through the origin) and μ has centroid at the origin. Moreover, the above K is unique up to translation – this follows from the equality case of the well-known Brunn-Minkowski inequality for $V(\cdot)$:

$$V(K_0 + tK_1)^{\frac{1}{n}} \geq V(K_0)^{\frac{1}{n}} + tV(K_1)^{\frac{1}{n}} \quad \forall \quad K_0, K_1 \in \mathscr{K}^n \quad \& \quad t \in [0,1].$$

The foregoing inequality and the following Hadamard's variation formula:

$$\frac{d}{dt}V(K_0 + tK_1)\big|_{t=0} = \int_{\partial K_0} h_{K_1}(g)\, dS = \int_{\mathbb{S}^{n-1}} h_{K_1}\, d\mathcal{H}^{n-1}_{\partial K_0} \quad \forall \quad K_0, K_1 \in \mathscr{K}^n$$

give

$$\int_{\mathbb{S}^{n-1}} h_{K_1}\, d\mathcal{H}^{n-1}_{\partial K_0} \geq nV(K_0)^{1-\frac{1}{n}}V(K_1)^{\frac{1}{n}},$$

whence ensuring that if K_0 is fixed and K_1 varies with $V(K_1) \geq 1$ then $\int_{\mathbb{S}^{n-1}} h_{K_1}\, d\mathcal{H}^{n-1}_{\partial K_0}$ reaches its minimum whenever $K_1 = V(K_0)^{-\frac{1}{n}}K_0$. So, the just-described Minkowski problem is equivalent to the problem prescribing the first variation of volume, i.e., the following minimum problem

$$\inf\left\{\int_{\mathbb{S}^{n-1}} h_K\, d\mu :\ K \in \mathscr{K}^n \ \&\ V(K) \geq 1\right\}$$

for a given nonnegative Borel measure μ on $\mathbb{S}^{n-1}$; see e.g.[10, 12, 42, 41].

5.2. Prescribing log-capacity variation. As $V(\cdot)$ is replaced by ncap$(\cdot)$, we empoy Theorem 4.1 and (4.1) to obtain that

$$\int_{\partial K_0} h_{K_1}(g)|\nabla u_{K_0}|^n\, dS = \left(\frac{\sigma_{n-1}}{\mathrm{ncap}(K_0)}\right)\frac{d}{dt}\mathrm{ncap}(K_0 + tK_1)\Big|_{t=0} \geq \frac{\sigma_{n-1}\mathrm{ncap}(K_1)}{\mathrm{ncap}(K_0)}$$

holds for all $K_0, K_1 \in \mathscr{K}^n$. Clearly, if $K_0 \in \mathscr{K}^n$ is fixed and $K_1 \in \mathscr{K}^n$ changes under ncap$(K_1) \geq 1$, then

$$\int_{\mathbb{S}^{n-1}} h_{K_1}\, g_*(|\nabla u_{K_0}|^n\, dS) = \int_{\partial K_0} h_{K_1}(g)|\nabla u_{K_0}|^n\, dS \geq \frac{\sigma_{n-1}}{\mathrm{ncap}(K_0)}$$

with equality (i.e., the most right quantity exists as the infimum of the most left integral) if $K_1 = K_0/\mathrm{ncap}(K_0)$. This implication plus the review about the problem of prescribing the first variation of volume as well as [28, Corollaries 2.7 & 6.6] leads to a consideration of the Minkowski-type problem for the first variation of the log-capacity. Below is our result.

THEOREM 5.1. *Let μ be a nonnegative Borel measure on $\mathbb{S}^{n-1}$.*

(i) *If*

$$\inf_{\zeta\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\zeta \cdot \eta|\, d\mu(\eta) > 0 = \int_{\mathbb{S}^{n-1}} \theta \cdot \eta\, d\mu(\eta) \quad \forall \quad \theta \in \mathbb{S}^{n-1}, \tag{5.1}$$

then

$$\mathcal{M}_{ncap} = \inf\left\{\int_{\mathbb{S}^{n-1}} h_K d\mu :\ K \in \mathscr{C}^n \ \&\ ncap(K) \geq 1\right\} > 0,$$

and hence there is a $K \in \mathscr{C}^n$ with $ncap(K) \geq 1$ such that

$$\mathcal{M}_{ncap} = \int_{\mathbb{S}^{n-1}} h_K\, d\mu.$$

(ii) *Conversely, if $K \in \mathscr{K}^n$ with $ncap(K) = 1$ is a minimizer for $\mathcal{M}_{ncap}$, then it satisfies*

$$g_*(|\nabla u_K|^n\, dS) = \sigma_{n-1}\, d\mu, \tag{5.2}$$

and hence (5.1) *holds.*

(iii) *The minimizer in (ii) is unique up to translation.*

Proof. (i) For convenience, let

$$\mathbb{S}^{n-1} \ni \xi \mapsto \mathcal{P}_\mu(\xi) = \int_{\mathbb{S}^{n-1}} \max\{0, \xi\cdot\eta\}\, d\mu(\eta)$$

be the projection body function. Then (5.1) amounts to

$$0<\min_{\mathbb{S}^{n-1}}\mathcal{P}_\mu\le\max_{\mathbb{S}^{n-1}}\mathcal{P}_\mu<\infty\ \ \&\ \ P_\mu(\xi)=P_\mu(-\xi)\ \ \forall\ \ \xi\in\mathbb{S}^{n-1}.$$

In order to prove $\mathcal{M}_{\text{ncap}}>0$, observe that the equation in (5.1) ensures that $\int_{\mathbb{S}^{n-1}}h_K\,d\mu$ is translation invariant. So, we may assume that the origin is at the midpoint of a diameter of $K\in\mathscr{C}^n$ with $\text{ncap}(K)\ge1$. Let $2R=\text{diam}(K)$. According to Theorem 2.1, we have:

$$\text{ncap}(K)\ge1\Rightarrow 2R\ge \text{b}(K)\ge 2\text{ncap}(K)\ge2.$$

If $\mathbf{e}$ is a unit vector with $\pm R\mathbf{e}\in\partial K$, then $h_K(\xi)\ge R|\mathbf{e}\cdot\xi|$ holds for all $\xi\in\mathbb{S}^{n-1}$, and hence

$$0<2\min_{\mathbb{S}^{n-1}}\mathcal{P}_\mu\le2R\mathcal{P}_\mu(\mathbf{e})\le\int_{\mathbb{S}^{n-1}}R|\mathbf{e}\cdot\xi|\,d\mu(\xi)\le\int_{\mathbb{S}^{n-1}}h_K\,d\mu.$$

This, along with the definition of $\mathcal{M}_{\text{ncap}}$, yields $\mathcal{M}_{\text{ncap}}>0$. Furthermore, when $K\in\mathscr{C}^n$ satisfies

$$\text{ncap}(K)\ge1\quad\&\quad\int_{\mathbb{S}^{n-1}}h_K\,d\mu\le2\mathcal{M}_{\text{ncap}},$$

we have

$$0<\text{diam}(K)\min_{\mathbb{S}^{n-1}}\mathcal{P}_\mu=2R\min_{\mathbb{S}^{n-1}}\mathcal{P}_\mu\le2\mathcal{M}_{\text{ncap}}.$$

Now, suppose that $\{K_j\}_{j=1}^\infty$ is a sequence in $\mathscr{C}^n$ which satisfies

$$\mathcal{M}_{\text{ncap}}=\lim_{j\to\infty}\int_{\mathbb{S}^n}h_{K_j}\,d\mu\ \ \&\ \ \text{ncap}(K_j)\ge1.$$

Then

$$2\le2\text{ncap}(K_j)\le\text{diam}(K_j)\le\frac{2\mathcal{M}_{\text{ncap}}}{\min_{\mathbb{S}^{n-1}}\mathcal{P}_\mu}\ \ \text{as}\ \ j\to\infty.$$

In accordance with the Blaschke selection principle (see e.g. [48, Theorem 1.8.6]), $\{K_j\}_{j=1}^\infty$ has a subsequence, still denoted by $\{K_j\}_{j=1}^\infty$, that converges to a $K\in\mathscr{C}^n$ with respect to the Hausdorff distance $d_H(\cdot,\cdot)$. Consequently, $h_{K_j}\to h_K$. Now, if $\text{ncap}(K)<1$, then from the definition of $\text{ncap}(K)$ it follows that there is an $L\in\mathscr{K}^n$ enjoying

$$K\subset L\ \ \&\ \ \text{ncap}(L)<1.$$

But, as j is sufficiently large we have $K_j\subset L$, and consequently by the monotonicity of $\text{ncap}(\cdot)$,

$$1\le\text{ncap}(K_j)\le\text{ncap}(L)<1,$$

a contradiction. Therefore, one must have $\text{ncap}(K)\ge1$.

(ii) Suppose that $K\in\mathcal{K}^n$ with $\text{ncap}(K)=1$ is a minimizer for $\mathcal{M}_{\text{ncap}}$. For $(t,L)\in(0,1)\times\mathcal{K}^n$ one has $K+tL\in\mathcal{K}^n$ and $h_{K+tL}=h_K+th_L$. Consequently, K is a critical point of the functional

$$\mathcal{D}(K+tL)=\int_{\mathbb{S}^{n-1}}h_{K+tL}\,d\mu-\text{ncap}(K+tL).$$

This, along with (4.2) in Theorem 4.4 and $\mathrm{ncap}(K) = 1$, gives

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{D}(K+tL)\Big|_{t=0} \\ &= \int_{\mathbb{S}^{n-1}} h_L\,d\mu - \sigma_{n-1}^{-1}\int_{\partial K} h_L(g)|\nabla u_K|^n\,dS \\ &= \int_{\mathbb{S}^{n-1}} h_L\,d\mu - \sigma_{n-1}^{-1}\int_{\mathbb{S}^{n-1}} h_L g_*\big(|\nabla u_K|^n\,dS\big). \end{aligned}$$

An application of [48, Lemmas 1.7.9 & 1.8.10] implies

$$\sigma_{n-1}\int_{\mathbb{S}^{n-1}}\phi\,d\mu = \int_{\mathbb{S}^{n-1}}\phi\,g_*\big(|\nabla u_K|^n\,dS\big) \quad \forall \quad \phi\in C(\mathbb{S}^{n-1})$$

thereby producing (5.2). Accordingly, a combination of both (3.2) and (5.2) derives

$$0 = \int_{\mathbb{S}^{n-1}}\theta\cdot\xi\,g_*(|\nabla u_K|^n\,dS)(\xi) = \sigma_{n-1}\int_{\mathbb{S}^{n-1}}\theta\cdot\xi\,d\mu(\xi) \quad \forall \quad \theta\in\mathbb{S}^{n-1}.$$

Therefore, the equality in (5.1) holds. Meanwhile, an application of (5.2) (for $K\in\mathscr{K}^n$ with $\mathrm{ncap}(K)=1$) and Theorem 3.2 (with a positive constant c) deduces

$$\begin{aligned} \inf_{\theta\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}|\theta\cdot\eta|\,d\mu(\eta) &= \sigma_{n-1}^{-1}\inf_{\theta\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}|\theta\cdot\eta|\,g_*(|\nabla u_K|^n\,dS)(\eta) \\ &= \sigma_{n-1}^{-1}\inf_{\theta\in\mathbb{S}^{n-1}}\int_{\partial K}|\theta\cdot g(x)||\nabla u_K(x)|^n\,dS(x) \\ &\ge c^n\sigma_{n-1}^{-1}\inf_{\theta\in\mathbb{S}^{n-1}}\int_{\partial K}|\theta\cdot g(x)|\,dS(x) \\ &= c^n\sigma_{n-1}^{-1}\inf_{\theta\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}|\theta\cdot\eta|\,d\mathcal{H}^{n-1}_{\partial K}(\eta) \\ &> 0. \end{aligned}$$

Thus, the inequality in (5.1) is true.

(iii) Our argument for the uniqueness is inspirited by [8, Section 5]. Now, assume that $K_0, K_1\in\mathscr{K}^n$ are two minimizers of $\mathcal{M}_{\mathrm{ncap}}$ in (ii). Then

$$\begin{cases} g_*(|\nabla u_{K_0}|^n\,dS) = g_*(|\nabla u_{K_1}|^n\,dS); \\ \mathrm{ncap}(K_0) = 1 = \mathrm{ncap}(K_1). \end{cases}$$

If

$$\psi(t) = \mathrm{ncap}\big((1-t)K_0 + tK_1\big),$$

then Theorems 4.4 & 3.1 yield

$$\begin{aligned}\psi'(0) &= \frac{\mathrm{ncap}(K_0)}{\sigma_{n-1}}\int_{\partial K_0}\big(h_{K_1}(g)-h_{K_0}(g)\big)|\nabla u_{K_0}|^n\,dS\\ &= \sigma_{n-1}^{-1}\Big(\int_{\partial K_0} h_{K_1}(g)|\nabla u_{K_0}|^n\,dS-\sigma_{n-1}\Big)\\ &= {\sigma_{n-1}}^{-1}\Big(\int_{\mathbb{S}^{n-1}} h_{K_1}\,g_*(|\nabla u_{K_0}|^n\,dS)-\sigma_{n-1}\Big)\\ &= {\sigma_{n-1}}^{-1}\Big(\int_{\mathbb{S}^{n-1}} h_{K_1}\,g_*(|\nabla u_{K_1}|^n\,dS)-\sigma_{n-1}\Big)\\ &= {\sigma_{n-1}}^{-1}\Big(\sigma_{n-1}-\sigma_{n-1}\Big)\\ &= 0.\end{aligned}$$

Note that $t\mapsto\psi(t)$ is concave on $[0,1]$. So this function is constant, in particular, we have

$$\mathrm{ncap}(K_1)=\psi(1)=\psi(t)=\psi(0)=\mathrm{ncap}(K_0). \tag{5.3}$$

Since the equality of (4.1) holds, K_1 is a translate and a dilate of K_0. But (5.3) is valid, so K_1 is only a translate of K_0 thanks to the uniqueness of the Brunn-Minkowski inequality for $\mathrm{ncap}(\cdot)$ over $\mathscr{K}^n$ proved in [11]. □

6. Yau's problem for log-capacity.

6.1. Prescribing mean curvature. On [56, p. 683], Yau posed the following problem:

"Let h be a real-valued function on $\mathbb{R}^3$. Find (reasonable) conditions on h to insure that one can find a closed surface with prescribed genus in $\mathbb{R}^3$ whose mean curvature (or curvature) is given by h. F. Almgren made the following comments: For "suitable" h one can obtain a compact smooth submanifold ∂A in $\mathbb{R}^3$ having mean curvature h by maximizing over bounded open sets $A\subset\mathbb{R}^3$ the quantity

$$F(A)=\int_A h\,d\mathcal{L}^3-Area(\partial A).$$

A function h would be suitable, for example, in case it were continuous, bounded, and $\mathcal{L}^3$ summable, and $\sup F>0$. However, the relation between h and the genus of the resulting extreme ∂A is not clear."

Although not yet completely solved, this problem for mean curvature or Gaussian curvature has a solution at least for the closed surface of genus zero, see [50, 5, 25] or [51, 52]. The following, essentially contained in [55, Corollary 1.2], may be regarded as a resolution of Yau's problem in a special form - if $I\in L^1(\mathbb{R}^n)$ is positive and continuous, k is nonnegative integer, $\alpha\in(0,1)$, and

$$\mathcal{I}(K)=S(K)-\int_K I\,dV,$$

then:

- There is $K_0\in\mathscr{C}^n$ such that $\mathcal{I}(K_0)=\inf_{K\in\mathscr{C}^n}\mathcal{I}(K)\le 0$ if and only if there is $L_0\in\mathscr{C}^n$ such that $\mathcal{I}(L_0)\le 0$.

- Suppose that $K \in \mathscr{K}^n$ is a minimizer for $\mathcal{I}(\cdot)$. Then there exists a measure μ_K on $\mathbb{S}^{n-1}$ such that the weak mean curvature equation $d\mu_K = g_*(I|_{\partial K}\, dS)$. Moreover, if K is of class $C^{2,+}$ then the mean curvature $H(K,x)$ (i.e., the arithmetic mean of $n-1$ principal curvatures at $x \in \partial K$) equals $(n-1)^{-1}I(x)$.
- If I is of $C^{k,\alpha}(\mathbb{R}^n)$ and $K \in \mathscr{K}^n$, being of class $C^{2,+}$, is a minimizer for $\mathcal{I}(\cdot)$, then K is of $C^{k+2,\alpha}$.

6.2. Prescribing log-capacitary curvature. Thanks to the relationship between the mean-width and the log-capacity explored in Section 2, as well as the discussion on the Minkowski-type problem above, it seems interesting to consider the log-capacity analogue of Yau's problem. More precisely, using the log-capacity in place of the surface area we study the functional

$$\mathcal{J}(K) = \mathrm{ncap}(K) - \int_K J\, dV,$$

thereby obtaining the following result.

THEOREM 6.1. *Let J be positive and continuous function on $\mathbb{R}^n$ with*

$$\|J\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} J\, dV < \infty,$$

$(k+1,\alpha,\beta) \in \mathbb{N}\times(0,1)\times(0,\infty)$, and $\mathscr{K}^n_\beta$ comprise all $K \in \mathscr{K}^n$ whose inradii $r_{in}(K)$ are not less than β.

(i) *There exists $K_0 \in \mathscr{K}^n_\beta$ such that $\mathcal{J}(K_0) = \inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K)$. Moreover, $\inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K) \le 0$ if and only if there exists $L_0 \in \mathscr{K}^n_\beta$ such that $\mathcal{J}(L_0) \le 0$.*

(ii) *Suppose that $K \in \mathscr{K}^n_\beta$ is a minimizer for $\mathcal{J}(\cdot)$. Then such a K satisfies the weak log-capacitary curvature equation*

$$ncap(K)\sigma_{n-1}^{-1} g_*\big(|\nabla u_K|^n\, dS\big) = g_*(J|_{\partial K}\, dS). \tag{6.1}$$

Moreover, if K is of class $C^{2,+}$, then we have the log-capacitary curvature equation

$$ncap(K)\sigma_{n-1}^{-1}|\nabla u_K(x)|^n = J(x) \quad \forall \quad x \in \partial K. \tag{6.2}$$

(iii) *If J is of $C^{k,\alpha}(\mathbb{R}^n)$ and $K \in \mathscr{K}^n_\beta$, being of class $C^{2,+}$, is a minimizer for $\mathcal{J}(\cdot)$, then K is of $C^{k+1,\alpha}$.*

Proof. (i) Since

$$\mathcal{J}(K) \ge \mathrm{ncap}(K) - \|J\|_{L^1(\mathbb{R}^n)} \ge -\|J\|_{L^1(\mathbb{R}^n)} \quad \forall \quad K \in \mathscr{K}^n_\beta,$$

it follows that $\inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K)$ is finite. Consequently, there is a sequence $\{K_j\}$ from $\mathscr{K}^n_\beta$ such that

$$\lim_{j\to\infty} \mathcal{J}(K_j) = \inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K).$$

Using the linear structure of $\mathrm{ncap}(\cdot)$, Theorem 2.1 and $\mathrm{r}_{K_j} \ge \beta > 0$, we get

$$0 < 2\beta \le 2\mathrm{r}_{K_j} = 2\mathrm{ncap}(\mathrm{r}_{K_j}\mathbb{B}^n) \le 2\mathrm{ncap}(K_j) \le \mathrm{b}(K_j) \le \mathrm{diam}(K_j). \tag{6.3}$$

An application of (2.2) implies

$$\mathcal{J}(K_j) \geq \mathrm{ncap}(K_j) - \|J\|_{L^1(\mathbb{R}^n)} \geq \left(\frac{V(K_j)}{\omega_n}\right)^{\frac{1}{n}} - \|J\|_{L^1(\mathbb{R}^n)}.$$

So, if $\{\mathrm{diam}(K_j)\}$ is unbounded, then (6.3) is used to ensure that $\{V(K_j)\}$ is unbounded, and hence $\{\mathcal{J}(K_j)\}$ has a subsequence $\{\mathcal{J}(K_{j_k})\}$ which tends to ∞ as $k \to \infty$. But, $\lim_{k\to\infty} \mathcal{J}(K_{j_k})$ exists as a finite value. Therefore, $\{\mathrm{diam}(K_j)\}$ has a uniform upper bound. Now, taking into account of the above-mentioned Blaschke selection principle, we can get a subsequence of $\{K_j\}$ which is convergent to an element $K_0 \in \mathscr{K}^n_\beta$ due to $K_j \in \mathscr{K}^n_\beta$. Note that $\mathcal{J}(\cdot)$ is continuous. Thus, K_0 is a minimizer of $\mathcal{J}(\cdot)$ over $\mathscr{K}^n_\beta$, i.e., $\mathcal{J}(K_0) = \inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K)$, as desired.

Furthermore, if $\inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K) \leq 0$, then the previously-found minimizer $K_0 \in \mathscr{K}^n_\beta$ satisfies $\mathcal{J}(K_0) \leq 0$. Conversely, if there is $L_0 \in \mathscr{K}^n_\beta$ such that $\mathcal{J}(L_0) \leq 0$, then $\inf_{K\in\mathscr{K}^n_\beta} \mathcal{J}(K) \leq \mathcal{J}(L_0) \leq 0$.

(ii) For $K \in \mathscr{K}^n$, $t > 0$ and $\phi \in C(\mathbb{S}^{n-1})$ let

$$K_t = \left\{x \in \mathbb{R}^n :\ x\cdot\theta \leq h_K(\theta) + t\phi(\theta) \quad \forall \quad \theta \in \mathbb{S}^{n-1}\right\}.$$

Then $K_t \in \mathscr{K}^n$ and $h_{K_t} = h_K + t\phi$. Using Theorem 4.4 (plus the ideas presented in [28, Sections 3-4]) as well as Tso's variation formula [52, (4)], we produce

$$\frac{d}{dt}\mathcal{J}(K_t)\Big|_{t=0} = \left(\frac{\mathrm{ncap}(K)}{\sigma_{n-1}}\right)\int_{\partial K}\phi(g)|\nabla u_K|^n\,dS - \int_{\partial K}\phi(g)J\,dS. \tag{6.4}$$

Obviously, if K is a minimizer of $\mathcal{J}(\cdot)$, then it is a critical point of $\mathcal{J}(K_t)$ and hence $\frac{d}{dt}\mathcal{J}(K_t)\Big|_{t=0} = 0$. This last equation, along with (6.4), gives

$$\begin{aligned}\left(\frac{\mathrm{ncap}(K)}{\sigma_{n-1}}\right)\int_{\mathbb{S}^{n-1}}\phi g_*(|\nabla u_K|^n\,dS) &= \left(\frac{\mathrm{ncap}(K)}{\sigma_{n-1}}\right)\int_{\partial K}\phi(g)|\nabla u_K|^n\,dS\\ &= \int_{\partial K}\phi(g)J\,dS\\ &= \int_{\mathbb{S}^{n-1}}\phi g_*(J\,dS).\end{aligned}$$

Owing to the fact that $\phi \in C(\mathbb{S}^{n-1})$ is arbitrary, we arrive at (6.1). Furthermore, if K is of class $C^{2,+}$, then $g : \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism (cf. [14, 22]), and hence

$$\left(\frac{\mathrm{ncap}(K)}{\sigma_{n-1}}\right)|\nabla u_K(x)|^n = J(x) \quad \forall \quad x \in \partial K$$

validates (6.2).

(iii) Suppose $J \in C^{k,\alpha}(\mathbb{R}^n)$ with k being a nonnegative integer. Since K is of class $C^{2,+}$, an application of [37, Theorem 1] and [40, Theorem 4.1] (cf. [20, 16, 49, 53, 21]) yields that $u_K \in C^{1,\hat{\alpha}}(\partial K)$ holds for some $\hat{\alpha} \in (0,1)$, and more importantly, the Gauss map from ∂K to $\mathbb{S}^{n-1}$ is a diffeomorphism. Therefore, (6.2) is true. Using (6.2) and $J \in C^{k,\alpha}(\mathbb{R}^n)$ with $\alpha \in (0,1)$, we obtain that $|\nabla u_K|$ belongs to $C^{k,\alpha}(\partial K)$. Note again that K is of class $C^{2,+}$. So, it follows that K is of $C^{k+1,\alpha}$ from the fact that $|\nabla u_K|\big|_{\partial K}$ is bounded above and below by two positive constants (cf. (6.2) and Theorem 3.2). □

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