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THE HEAT TRACE FOR THE DRIFTING LAPLACIAN AND SCHRÖDINGER OPERATORS ON MANIFOLDS*

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Abstract. We study the heat trace for both Schrödinger operators as well as the drifting Laplacian on compact Riemannian manifolds. In the case of a finite regularity (bounded and measurable) potential or weight function, we prove the existence of a partial asymptotic expansion of the heat trace for small times as well as a suitable remainder estimate. This expansion is sharp in the following sense: further terms in the expansion exist if and only if the potential or weight function is of higher Sobolev regularity. In the case of a smooth weight function, we determine the full asymptotic expansion of the heat trace for the drifting Laplacian for small times. We then use the heat trace to study the asymptotics of the eigenvalue counting function. In both cases the Weyl law coincides with the Weyl law for the Riemannian manifold with the standard Laplace-Beltrami operator. We conclude by demonstrating isospectrality results for the drifting Laplacian on compact manifolds.

Key words. Heat trace, drifting Laplacian, weighted Laplacian, Schrödinger operator, Weyl law, Weyl asymptotic, eigenvalue asymptotics.

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1. Introduction. Heat equation methods are incredibly useful; see for example, *The Ubiquitous Heat Kernel* [5]. One of the most standard ways to prove the spectral theorem for the Laplace operator on a Riemannian manifold is via the associated heat semi-group. Moreover, the heat equation provides a connection between probability theory and analysis. As noted in Kac's famous paper, one can thereby demonstrate analytical results using probabilistic methods [6]. Van den Berg and Srisatkunarajah also used probabilistic methods to demonstrate results for the short time asymptotic behavior of the heat trace on polygonal domains [10].

The heat trace of a Schrödinger operator on a manifold reflects many of the geometric quantities of the manifold intertwined with the potential term. It is also used to determined the rate at which the eigenvalues of the operator tend to infinity. In this article we focus on short time asymptotic expansions of the heat trace for Schrödinger operators on manifolds with an irregular potential, as well as the short time asymptotic expansions of the heat trace for drifting Laplacians on weighted manifolds.

To be more precise, we consider (M, g), a smooth, compact, Riemannian manifold of dimension n. The Laplace operator, Δ, is determined by the Riemannian metric, with

$$\Delta := -\sum_{i,j=1}^{n} \frac{1}{\sqrt{\det(g)}} \partial_i g^{ij} \sqrt{\det(g)} \partial_j .$$

In this article, we will consider the Schrödinger operator,

$$\Delta_V := \Delta + V,$$

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where the potential satisfies $V \in L^\infty(M)$. This is a self-adjoint operator which maps $H^2(M,g) \to L^2(M,g)$. We also consider the drifting Laplacian,

$$\Delta_f := \Delta + \nabla f \cdot \nabla.$$

Above, we assume that f is a function on M which satisfies $\nabla f \in L^\infty(M)$ and $\Delta f \in L^\infty(M)$. The operator Δ_f is also known as the weighted Laplacian or Bakry-Émery Laplacian. The drifting Laplacian is a self-adjoint operator as well, but with respect to the weighted volume measure

$$d\mu_f = e^{-f} d\mu,$$

where $d\mu$ is the volume measure given by the Riemannian metric.

Our first result is the existence of a small time asymptotic expansion for the trace of the heat kernels associated to Δ_V. We determine the first few coefficients in this expansion as well as a remainder estimate of order $o(t^{2-n/2})$ whenever the potential V is in L^∞. Let $e^{-t\Delta_V}$ denote the heat semigroup for the operator $\Delta + V$, and $e^{-t\Delta}$ denote the heat semigroup for the operator Δ.

Proposition 1.1. *Let (M,g) be a smooth, compact, Riemannian manifold of dimension n. Let $V \in L^\infty(M)$. Then $e^{-t\Delta_V}$ is trace class and has a short time asymptotic expansion as $t \downarrow 0$ given by*

$$\begin{aligned}\operatorname{tr} e^{-t\Delta_V} =& \frac{\operatorname{Vol}(M)}{(4\pi t)^{n/2}} + \frac{t}{(4\pi t)^{n/2}} \int_M \left(\frac{1}{6} K(z) - V(z)\right) d\mu(z) \\ &+ \frac{t^2}{(4\pi t)^{n/2}} \left[a_{0,2} + \frac{1}{2}\|V\|_{L^2}^2 - \frac{1}{6}\int_M K(z)V(z)\, d\mu(z)\right] + o(t^{2-n/2}),\end{aligned}$$

where $K(z)$ denotes the scalar curvature of (M,g), and $a_{0,2}$ is the coefficient of $(4\pi t)^{-n/2}t^2$ in the short time asymptotic expansion of $e^{-t\Delta}$ given in (26).

We prove Proposition 1.1 in Section 2. In Corollary 3.2 we will see that that a similar expansion also holds for Δ_f whenever $\Delta f, \nabla f \in L^\infty$. We show in Section §3 that Proposition 1.1 is sharp, and that further terms cannot be deduced unless the potential is of higher Sobolev regularity. Moreover, the remainder estimate in the asymptotic expansion given in Proposition 1.1 is determined by, and conversely it also distinguishes the regularity of the potential function. The main result of our paper is the following

Theorem 1.2. *Assume $V \in L^\infty$. Then the heat trace has an asymptotic expansion as $t \downarrow 0$ of the form*

$$\begin{aligned}\operatorname{tr} e^{-t\Delta_V} =& \frac{\operatorname{Vol}(M)}{(4\pi t)^{n/2}} + \frac{t}{3(4\pi t)^{n/2}} \int_M \left(\frac{1}{6} K(z) - V(z)\right) d\mu(z) \\ &+ \frac{t^2}{(4\pi t)^{n/2}} \left[a_{0,2} + \frac{1}{2}\|V\|_{L^2}^2 - \frac{1}{6}\int_M K(z)V(z)\, d\mu(z)\right] + O(t^{3-n/2}),\end{aligned}$$

if and only if $V \in H^1$.

This regularity result was inspired by the work of H. Smith and M. Zworski for Schrödinger operators on $\mathbb{R}^n$ [9]. Theorem 1.2 implies the following isospectrality result.

Corollary 1.3. *Assume* $V, \tilde{V} \in L^\infty$ *and that the Schrödinger operators* $\Delta_V, \Delta_{\tilde{V}}$ *are isospectral. Then the potential,* V*, is in* H^1 *if and only if* $\tilde{V}$ *is in* H^1*.*

A further consequence of the existence of a short-time asymptotic expansion of the heat trace like the one proven in Proposition 1.1 is Weyl's law for the growth rate of the eigenvalues of the operators, Δ_V and Δ_f.

Corollary 1.4. *Let* (M, g) *be a compact, smooth,* n*-dimensional Riemannian manifold with Laplace operator,* Δ*. Let* $V \in L^\infty(M, g)$*. Let* $\{\lambda_k\}$ *denote the eigenvalues of the Schrödinger operator,* $\Delta_V = \Delta + V$ *with* $\lambda_1 \leq \lambda_2 \leq \ldots$*. For a function* f *on* M *with*

$$\Delta f \text{ and } \nabla f \in L^\infty(M, g),$$

let $\{\mu_k\}$ *denote the eigenvalues of the drifting Laplacian* $\Delta_f = \Delta + \nabla f \cdot \nabla$ *with* $\mu_1 \leq \mu_2 \leq \ldots$*. Let*

$$N_V(\Lambda) := \#\{\lambda_k \leq \Lambda\}, \text{ and } N_f(\Lambda) := \#\{\mu_k \leq \Lambda\}.$$

Then

$$\lim_{k\to\infty} \frac{N_V(\Lambda)(2\pi)^n}{\Lambda^{n/2}\omega_n} = \mathrm{Vol}(M) = \lim_{k\to\infty} \frac{N_f(\Lambda)(2\pi)^n}{\Lambda^{n/2}\omega_n},$$

where above $\mathrm{Vol}(M)$ *denotes the volume of* M *with respect to the Riemannian metric* g*, and* ω_n *denotes the volume of the unit ball in* $\mathbb{R}^n$*.*

As one readily sees from the above result, Weyl's law for the drifting Laplacian is in fact independent of the weight function f, whenever ∇f and Δf are in L^∞. In fact, Weyl's law only depends on the dimension of the manifold and its Riemannian volume. Although some experts provide (without proof) the correct Weyl law in the case $f \in C^2$ (see for example p. 640 of [2]) it also often appears in literature (also without proof) with the weighted volume of the manifold. We anticipate that our paper will not only clarify this inconsistency, but also provide a more general setting in which the result is true. We prove Corollary 1.4 in §2.1 using the the short time asymptotic behavior of the heat trace together with the Karamata Lemma.

The independence of Weyl's law from the weight function f for the drifting Laplacian comes in contrast to estimates for its eigenvalues, which *do* depend on the weight function. For example, Setti demonstrated bounds for the first eigenvalue which depend on the weight function [8]. More recently, A. Hassennezhad [4] demonstrated upper bounds for all eigenvalues of Δ_V and hence Δ_f, under the assumption that V is continuous. J.-Y. Wu and P. Wu demonstrated lower bounds for the eigenvalues of Δ_f under the assumption that the f-Ricci curvature is bounded below [12]. Their estimates depend on lower bounds for the Bakry-Émery Ricci tensor, hence they also rely on f being a C^2 function. As a result, one would also not anticipate that a Weyl's law would hold for such general f as in Corollary 1.4, nor that it would be independent of f. Our work complements the upper and lower eigenvalue estimates obtained by other authors, and it may be useful to combine it with those results to obtain sharper eigenvalue estimates.

In the case that f is smooth, we compute in Section 4 the full small time asymptotic expansion expansion of the heat trace associated to Δ_f, $H_f(t, x, x)$.

THEOREM 1.5. *For a smooth weight function f, $H_f(t,x,x)$ has the short time asymptotic expansion*

$$H_f(t,x,x) \sim (4\pi t)^{-n/2} e^{f(x)} \sum_{i=0}^{\infty} u_i(x,x)\, t^i \tag{1}$$

where the u_i are defined by (22).

If $\{\lambda_i\}$ is the spectrum of the drifting Laplacian Δ_f, then

$$\sum_k e^{\lambda_k t} \sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} a_i\, t^i \tag{2}$$

where $a_i = \int_M u_i(x,x)\, d\mu(x)$.

We emphasize that the integrals defining the heat trace invariants, a_i, are with respect to the Riemannian volume form, $d\mu$. Finally, we obtain isospectrality results for the drifting Laplacian on compact manifolds.

THEOREM 1.6. *Consider two weighted manifold $(M^n, g_M, d\mu_{f_1})$ and $(N^m, g_N, d\mu_{f_2})$, with f_i as in Corollary 3.2 on the respective manifolds. Suppose that the drifting Laplacian Δ_{f_1} on M and the drifting Laplacian Δ_{f_2} on N are isospectral. Then the two manifolds must have the same dimension and the same Riemannian volume. If in addition M and N are orientable surfaces, and the two weight functions have equal Dirichlet norms, $\int_M |\nabla f_1|^2\, d\mu_M = \int_N |\nabla f_2|^2\, d\mu_N$, then they are diffeomorphic. If (M^n, g_M) is of dimension $n \leq 3$, then the set of isospectral drifting Laplacians, Δ_f, with smooth f, is compact in $C^\infty(M)$.*

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2. Heat Trace Expansions. Recall that (M,g) is a smooth, compact, Riemannian manifold of dimension n, and $\Delta_V = \Delta + V$ is a Schrödinger operator with $V \in L^\infty(M)$. We begin by proving Proposition 1.1, which gives the first few terms in the short time asymptotic expansion of the heat trace associated to such a Schrödinger operator. We shall see that the existence of further terms in such an expansion depends upon whether the potential, $V \in H^1$, or not.

Proof of Proposition 1.1. The estimates of [9] §3 p. 465, show that the semigroup $e^{-t\Delta_V}$ is trace class, but since their results are over $\mathbb{R}^n$, we recall the required calculations and estimates in the setting of a compact Riemannian manifold. Using Duhamel's principle, one can generate an expression for $e^{-t\Delta_V} - e^{-t\Delta}$ which is given by

$$e^{-t\Delta_V} - e^{-t\Delta} = \sum_{k \geq 1} W_k(t),$$

with

$$W_k(t) = (-1)^k \int_{0<s_1<\ldots<s_k<t} e^{-(t-s_k)\Delta} V e^{-(s_k-s_{k-1})\Delta} V \ldots V e^{-s_1\Delta} ds_1 \ldots ds_k. \tag{3}$$

We first observe that we have the L^2 operator norm bound

$$\|W_k(t)\|_{L^2 \to L^2} \leq \|V\|_\infty^k t^k / k!,$$

since the integrand is L^2 bounded by $\|V\|_\infty^k$, and the volume of integration is $t^k/k!$. Then, we also have a bound on the trace class norm,

$$\|W_k(t)\|_{L^1} \leq C^k k^{n/2} t^{k-n/2}/k!. \tag{4}$$

This follows immediately by the same arguments given on p. 365 of [9]. In fact, the proof can be simplified in our case: since we are working on a compact manifold, there is no need to use a cut-off function χ, and we can obtain the estimate directly by applying the operator $e^{-s\Delta}$.

Consequently, the operator $e^{-t\Delta_V} - e^{-t\Delta}$ is trace class, and the trace may be exchanged with summation to operator

$$\operatorname{tr}(e^{-t\Delta_V} - e^{-t\Delta}) = \sum_{k\geq 1} \operatorname{tr}(W_k(t)).$$

We compute the trace of the first term, W_1. Let $H_0(t,x,y)$ denote the heat kernel of the semigroup $e^{-t\Delta}$. The Schwartz kernel of W_1 is given by

$$W_1(t,x,y) = -\int_0^t \int_M H_0(s,x,z)H_0(t-s,z,y)V(z)\, dzds,$$

where to simplify notation, we have used dz to indicate $d\mu(z)$. To compute its trace, we set $y = x$ and integrate,

$$\operatorname{tr} W_1(t) = -\int_0^t \int_M \int_M H_0(s,x,z)H_0(t-s,z,x)V(z)\, dzdxds \tag{5}$$

Since we are on a compact manifold, the Laplacian has an L^2 orthonormal basis of eigenfunctions $\{\phi_k\}$ with corresponding eigenvalues λ_k. Its heat kernel then has a simple expression with respect to this basis

$$H_0(s,x,z) = \sum_{k\geq 1} e^{-\lambda_k t}\phi_k(x)\phi_k(z). \tag{6}$$

Expressing the heat kernels in (5) in terms of this orthonormal basis and using the orthonormality of the eigenfunctions, we compute the integral with respect to x to arrive at

$$\operatorname{tr} W_1(t) = -\int_0^t \int_M H_0(t,z,z)V(z)dzds = -t\int_M H_0(t,z,z)V(z)\, dz. \tag{7}$$

Next, we use the fact that the heat kernel has a local expansion along the diagonal as $t \downarrow 0$

$$H_0(t,z,z) = \frac{1}{(4\pi t)^{n/2}} + \frac{tK(z)}{6(4\pi t)^{n/2}} + r(t) \quad \text{where} \quad r(t) = O(t^{2-n/2}),$$

and $K(z)$ is the scalar curvature of (M,g) (see for example [7, Chapter 3]). Since $V \in L^\infty$ and M is compact, it then follows that as $t \downarrow 0$

$$\operatorname{tr} W_1(t) = -t(4\pi t)^{-n/2}\left[\int_M V(z)dz + \frac{t}{6}\int_M K(z)V(z)dz\right] + O(t^{3-n/2}). \tag{8}$$

In fact, we can also compute the full asymptotic expansion of the term $\operatorname{tr} W_1(t)$ as $t \downarrow 0$. Recalling the identity (7) for W_1 and applying the local heat trace expansion of H_0 together with the assumption that $V \in L^\infty$, we obtain

$$\operatorname{tr} W_1(t) \sim -t(4\pi t)^{-n/2} \sum_{j \geq 0} t^j \int_M u_{0,j}(z,z)V(z)dz.$$

The functions $u_{0,j}(z,z)$ are the local heat invariants of the heat kernel H_0 on the Riemannian manifold (M,g). Consequently, the $u_{0,j}$ are *independent of* V. It is in general very difficult to express these functions in terms of geometric and topological invariants of the manifold. We refer the interested reader to Rosenberg's results in [7], and perhaps more conveniently to equation (25) for an integral expression of the $u_{0,j}$. In the case $j = 1$, as we have mentioned above, $u_{0,1}(z,z) = K(z)/6$ where $K(z)$ is the scalar curvature of the manifold [7]. We would also like to remark that in the expansion of the trace of W_1 only V appears and none of its higher order derivatives.

To complete the proof, we continue to analyze the next term in the expansion. By definition (3), the trace of W_2 is

$$\operatorname{tr} W_2(t) = \int_{M^3} \int_{0<r<s<t} H_0(t-s,x,y)H_0(s-r,y,z)H_0(r,z,x)V(y)V(z)\, drdsdxdydz.$$

We again use the orthonormal basis expansion (6) to express $H_0(t-s,x,y)$ and $H_0(r,z,x)$, and integrate with respect to x to obtain

$$\operatorname{tr} W_2(t) = \int_{M^2} \int_{0<r<s<t} H_0(t-s+r,y,z)H_0(s-r,z,y)V(y)V(z)\, drdsdydz.$$

We can now apply a clever time-substitution technique to further simplify this expression (see for example [9, pp. 467–468]). Letting $u = t - s$ and suppressing the integrals over space we obtain the expression,

$$\int_{r+u<t,\ 0<r,u} H_0(u+r,y,z)H_0(t-u-r,y,z)V(y)V(z)\, drdu.$$

Setting $r = tv - u$, and observing that $drdu = tdvdu$, we have

$$\begin{aligned} &\int_{v=0}^{1} \int_{0<u<tv} H_0(tv,y,z)H_0(t(1-v),y,z)V(y)V(z)t\, dudv \\ &= t^2 \int_{v=0}^{1} H_0(tv,y,z)H_0(t(1-v),y,z)V(y)V(z)v\, dv, \end{aligned}$$

where for the last integration we have used that the integrand is independent of u. Moreover, the symmetry of the last integral with respect to the map $v \mapsto 1-v$, further simplifies it to

$$\frac{t^2}{2} \int_0^1 H_0(tv,y,z)H_0(t(1-v),y,z)V(y)V(z)\, dv.$$

We recall the local asymptotic expansion of the heat kernel of the Laplacian over a compact manifold. Let $\delta > 0$ be a uniform constant on the manifold such that the exponential map at p, $\exp_p$ is a diffeomorphism on the ball of radius δ in the tangent space at p. Then

$$H_0(t,y,z) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4t}} \sum_{j=0}^{k} t^j u_j(y,z) + r_k(t,y,z) \tag{9}$$

where the $u_j(y,z)$ are symmetric smooth functions on M^2 supported on a neighborhood of the diagonal, $\mathcal{N}_\delta = \{(y,z) \,|\, d(y,z) \le \delta\}$. Moreover, $u_0(y,y) = 1$, and the remainder term is of order

$$|r_k(t,y,z)| \le C_k t^{k+1-\frac{n}{2}} e^{-\frac{d(y,z)^2}{8t}}$$

on M^2 for $t \in (0,1]$. In Section 4, we use geometric methods to prove such an expansion for the heat kernel of the drifting Laplacian, but this is a classical result.

Hence,

$$H_0(tv,y,z) = (4\pi tv)^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4tv}} u_0(y,z) + r_0(tv,y,z),$$

and

$$H_0(t(1-v),y,z) = (4\pi t(1-v))^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4t(1-v)}} u_0(y,z) + r_0(t(1-v),y,z).$$

As a result, the product has a similar expansion which we split into the leading term and a remainder,

$$H_0(tv,y,z)H_0(t(1-v),y,z) = (4\pi t)^{-\frac{n}{2}} \left[4\pi v(1-v)t)^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4tv(1-v)}} u_0^2(y,z) + r(t,v,y,z)\right]$$

where[1]

$$|r(t,v,y,z)| \le Ct \cdot (v(1-v)t)^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{8tv(1-v)}}.$$

The heat kernel, $H_0(tv(1-v),y,z)$ has an analogous expansion, which we also split into the leading term and a remainder

$$H_0(tv(1-v),y,z) = (4\pi tv(1-v))^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4tv(1-v)}} u_0(y,z) + r_0(tv(1-v),y,z).$$

Comparing the two expressions we get,

$$\begin{aligned} &H_0(tv,y,z)H_0(t(1-v),y,z) - (4\pi t)^{-\frac{n}{2}} H_0(tv(1-v),y,z) \\ &= (4\pi t)^{-\frac{n}{2}} \Big[(4\pi tv(1-v))^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4tv(1-v)}} \left(u_0^2(y,z) - u_0(y,z)\right) \\ &\qquad + r(t,v,y,z) - r_0(tv(1-v),y,z)\Big]. \end{aligned}$$

We therefore have

$$\operatorname{tr} W_2(t) = \frac{t^2}{2} (4\pi t)^{-\frac{n}{2}} \int_{M^2} \int_{v=0}^1 H_0(tv(1-v),y,z)V(y)V(z)dvdydz + I + II,$$

where

$$I = \frac{t^2}{2} (4\pi t)^{-\frac{n}{2}} \int_{M^2} \int_{v=0}^1 (4\pi tv(1-v))^{-\frac{n}{2}} e^{-\frac{d(y,z)^2}{4tv(1-v)}} \left(u_0^2(y,z) - u_0(y,z)\right) V(y)V(z)dvdydz,$$

and

$$II = \frac{t^2}{2}(4\pi t)^{-\frac{n}{2}} \int_{M^2} \int_{v=0}^1 \left[r(t,v,y,z) - r_0(tv(1-v),y,z)\right] V(y)V(z)dvdydz.$$

[1]Throughout this proof, we may use C for any constants which do not depend on the variables in which we are demonstrating estimates.

From the asymptotic expression of the remainder terms, r and r_0, we have that

$$\begin{aligned}&\int_{M^2}\int_{v=0}^{1}\left[r(t,v,y,z)-r_0(tv(1-v),y,z)\right]V(y)V(z)dvdydz\\ &\leq C\,t\int_{M^2}\int_{v=0}^{1}(v(1-v)t)^{-\frac{n}{2}}e^{-\frac{d(y,z)^2}{8tv(1-v)}}\,|V(y)|\,|V(z)|\,dvdydz.\end{aligned}$$

However,

$$\begin{aligned}&\lim_{t\to 0}\int_{M^2}(v(1-v)t)^{-\frac{n}{2}}e^{-\frac{d(y,z)^2}{4tv(1-v)}}|V(y)|\,|V(z)|\,dydz\\ =\;&\lim_{s\to 0}C\int_{M^2}s^{-\frac{n}{2}}e^{-\frac{d(y,z)^2}{4s}}|V(y)|\,|V(z)|\,dydz\leq C\|V\|_\infty^2\leq C.\end{aligned}$$

To see this last inequality, we can split the integral into two parts, on the neighborhood $\mathcal{N}_\delta$ of the diagonal, and its complement. At each y, we can use the exponential map to express z as a point in the tangent space and $d(y,z)=|z|$. Since $s^{-\frac{n}{2}}e^{-\frac{|z|^2}{4s}}$ is the heat kernel on T_yM, the integral on $\mathcal{N}_\delta$ is uniformly bounded and independent of δ. For the complementary set, we use the fact that $d(y,z)\geq\delta$ and the rapid decay of the Gaussian on this set.

From the above we conclude that

$$II=\frac{t^2}{2}(4\pi t)^{-\frac{n}{2}}\;O(t)$$

For I, we recall that $u_0(z,z)=1$ and $u_0(z,y)=u_0(y,z)$, so we see that

$$\frac{\partial u_0(y,z)}{\partial y}\Big|_{y=z}=\frac{\partial u_0(y,z)}{\partial z}\Big|_{y=z}=0.$$

Since u_0 is smooth, and M is compact, this implies that there exist uniform constants C, $\varepsilon>0$ such that for $d(y,z)<\varepsilon$

$$|u_0(y,z)-u_0(z,z)|=|u_0(y,z)-1|\leq C\,d(y,z)^2.$$

Then

$$|u_0(y,z)^2-u_0(y,z)|=|u_0(y,z)||u_0(y,z)-1|<\left(1+C\,d(y,z)^2\right)C\,d(y,z)^2\;<C\,d(y,z)^2.$$

We let $\mathcal{N}_\varepsilon$ be a small neighborhood of the diagonal with $\varepsilon<\delta$. Then,

$$\begin{aligned}I&=\frac{t^2}{2}(4\pi t)^{-\frac{n}{2}}\int_{M^2}\int_{v=0}^{1}(4\pi tv(1-v))^{-\frac{n}{2}}e^{-\frac{d(y,z)^2}{4tv(1-v)}}\left(u_0^2(y,z)-u_0(y,z)\right)V(y)V(z)dvdydz\\ &=\frac{t^2}{2}(4\pi t)^{-\frac{n}{2}}\Bigg[\int_{\mathcal{N}_\varepsilon}\int_{v=0}^{1}(4\pi tv(1-v))^{-\frac{n}{2}}e^{-\frac{d(y,z)^2}{4tv(1-v)}}\left(u_0^2(y,z)-u_0(y,z)\right)V(y)V(z)dvdydz\\ &\qquad\qquad+O_\varepsilon(t^\infty)\Bigg]\\ &\leq\frac{t^2}{2}(4\pi t)^{-\frac{n}{2}}C\left[\int_{\mathcal{N}_\varepsilon}\int_{v=0}^{1}(4\pi tv(1-v))^{-\frac{n}{2}}d(y,z)^2e^{-\frac{d(y,z)^2}{4tv(1-v)}}dvdydz+O_\varepsilon(t^\infty)\right].\end{aligned}$$

The error term $O_\varepsilon(t^\infty)$ corresponds to the integral over $\mathcal{N}_\delta\setminus\mathcal{N}_\varepsilon$ and vanishes rapidly away from the diagonal.

Again using coordinates via the exponential map, we may write

$$\begin{aligned}&\int_{\mathcal{N}_\varepsilon} (4\pi tv(1-v))^{-\frac{n}{2}} d(y,z)^2 e^{-\frac{d(y,z)^2}{4tv(1-v)}}\, dydz \\ \leq\ & \int_M \int_{\mathbb{R}^n} C\, s^{-\frac{n}{2}} |z|^2 e^{-\frac{|z|^2}{4s}} \eta(y,z)\, dzdy, \quad s = tv(1-v).\end{aligned}$$

Above, η is a smooth bounded cut-off function which vanishes for z outside $B_\varepsilon(0) \subset T_yM = \mathbb{R}^n$. Given that the integral

$$\int_{\mathbb{R}^n} s^{-\frac{n}{2}} |z|^2 e^{-\frac{|z|^2}{4s}}\, dz = O(s),$$

and $0 \leq s = tv(1-v) \leq t$, we therefore have

$$I = \frac{t^2}{2}\,(4\pi t)^{-\frac{n}{2}}\, O(t).$$

We note that this estimate is independent of both ε and δ.

Hence, we have computed that

$$\operatorname{tr} W_2(t) = \frac{t^2}{2}\,(4\pi t)^{-\frac{n}{2}} \left[\int_{M^2} \int_{v=0}^1 H_0(tv(1-v), y, z) V(y) V(z) dvdydz + O(t)\right]. \quad (10)$$

Note that

$$\int_{y\in M} H_0(tv(1-v), y, z) V(y)\, dy = U_V(tv(1-v), z),$$

is the solution to the heat equation with initial data V at time $tv(1-v)$. By (10) we have

$$\left| 2(4\pi t)^{n/2} t^{-2} \operatorname{tr} W_2(t) - \int_0^1 \int_M U_V(tv(1-v), z) V(z)\, dzdv \right| = O(t). \quad (11)$$

We recall that the solution to the heat equation with initial data V at time $tv(1-v)$ converges to V as $t \downarrow 0$, uniformly in $L^2(M)$ for all $v \in (0,1)$ since $0 \leq tv(1-v) \leq t$. In other words,

$$\|U_V(tv(1-v), z) - V(z)\|_{L^2} \to 0 \text{ uniformly as } t \downarrow 0 \text{ for all } v \in (0,1).$$

As a result,

$$\begin{aligned}&\left| \int_0^1 \int_M U_V(tv(1-v), z) V(z) dvdz - \int_0^1 \int_M |V(z)|^2\, dz\, vdv \right| \\ \leq\ & \left(\|V\|_{L^2(M)} \sup_{v\in[0,1]} \|U_V(tv(1-v)) - V\|_{L^2(M)} \right) \to 0 \text{ uniformly as } t \downarrow 0.\end{aligned}$$

However, we do not know a priori the *rate* at which the right side converges as $t \downarrow 0$. Combining this estimate with (11) we therefore obtain

$$\operatorname{tr} W_2(t) = \frac{t^2}{2}(4\pi t)^{-n/2} \left[\|V\|_{L^2}^2 + o(1) \right]. \quad (12)$$

To complete the proof, we recall the bound in (4), which shows that

$$\operatorname{tr} \sum_{k \geq 3} W_k(t) \leq C t^{3-n/2}, \quad 0 < t \leq 1. \tag{13}$$

As a result, under the very general assumption that $V \in L^\infty$, we have a precise expression for the heat trace up to order $t^{2-n/2}$ and a remainder estimate,

$$\begin{aligned}\operatorname{tr} e^{-t\Delta_V} =& \frac{\mathrm{Vol}(M)}{(4\pi t)^{n/2}} + \frac{t}{(4\pi t)^{n/2}} \int_M \left(\frac{1}{3} K(z) - V(z) \right) \, d\mu(z) \\ &+ \frac{t^2}{(4\pi t)^{n/2}} \left[a_{0,2} + \frac{1}{2} \|V\|_{L^2}^2 - \frac{1}{6} \int_M K(z) V(z) \, d\mu(z) \right] + o(t^{2-n/2}),\end{aligned}$$

as $t \downarrow 0$. □

We note that the first, second, and fourth terms come from the corresponding terms in the short time heat trace expansion of $e^{-t\Delta}$ on M. Moreover, the potential V does not appear in the leading order asymptotics; it first makes an appearance in the coefficient of $t^{1-n/2}$. This is the crux of the matter and the reason why V does not affect Weyl's law for a Schrödinger operator. However, based on the work of Hassanezhad [4] and Wu and Wu [12], one *does* expect that more refined estimates of the eigenvalues of a Schrödinger operator may indeed depend on the potential.

2.1. Weyl's Law via the Heat Trace and Karamata's Lemma. In general, there are two main techniques which can be used to prove Weyl's law. The first technique is known as *Dirichlet-Neumann bracketing,* and this is the one used in Weyl's classical proof [11]. The second technique uses an element of the functional calculus of the Laplacian, such as the resolvent, wave group, or heat semi-group, together with a suitable Tauberian theorem. Here, we use the short time asymptotic behavior of the heat trace together with Karamata's Tauberian Lemma to prove Corollary 1.4. The proof is significantly simplified by using a generalization of Karamata's Lemma; originally, the function g below was required to be continuous.

LEMMA 2.1. *Let g be a bounded, non-negative, piecewise continuous function on* $[0, 1]$. *Assume that ν is a non-negative measure, $\alpha \geq 1$, and*

$$\int_0^\infty e^{-t\lambda} d\nu(\lambda) < \infty \, \forall t > 0, \quad \lim_{t \downarrow 0} t^\alpha \int_0^\infty e^{-t\lambda} d\nu(\lambda) = c \in (0, \infty).$$

Then, we have

$$\lim_{t \downarrow 0} t^\alpha \int_0^\infty g(e^{-t\lambda}) e^{-t\lambda} d\nu(\lambda) = \frac{c}{\Gamma(\alpha)} \int_0^\infty g(e^{-t}) e^{-t} t^{\alpha - 1} dt.$$

Proof. We begin by introducing the notations

$$F_t(g) = t^\alpha \int_0^\infty g\left(e^{-t\lambda}\right) e^{-t\lambda} \mathrm{d}\nu(\lambda), \quad G(g) = \frac{c}{\Gamma(\alpha)} \int_0^\infty g\left(e^{-t}\right) t^{\alpha-1} e^{-t} \mathrm{d}t. \tag{14}$$

Let $\{g_n^L\}$, $\{g_n^U\}$ be two sequences of continuous functions such that

$$0 \leq g_1^L(x) \leq \ldots \leq g_n^L(x) \leq \ldots \leq g(x) \leq \ldots \leq g_n^U(x) \ldots \leq g_1^U(x) \leq M, \text{ for a.e. } x \in [0, 1],$$

and

$$\lim_{n\to\infty} g_n^L(x) = \lim_{n\to\infty} g_n^U(x) = g(x) \quad \text{for a.e. } x \in [0,1].$$

Due to the assumption that g is non-negative and bounded, we have the estimate

$$0 \le G(g) \le \frac{c}{\Gamma(\alpha)} \int_0^\infty M t^{\alpha-1} e^{-t} dt = cM.$$

This will allow us to apply dominated convergence arguments.

Since all of the g_n^L and g_n^U are continuous, by Karamata's original Lemma,

$$\lim_{t\downarrow 0} F_t(g_n^L) = G(g_n^L) \text{ and } \lim_{t\downarrow 0} F_t(g_n^U) = G(g_n^U)$$

for each n. Therefore,

$$\limsup_{t\downarrow 0} F_t(g) \le \limsup_{t\downarrow 0} F_t(g_n^U) = G(g_n^U)$$

and

$$\liminf_{t\downarrow 0} F_t(g) \ge \liminf_{t\downarrow 0} F_t(g_n^L) = G(g_n^L).$$

Together these estimates give

$$G(g_n^L) \le \liminf_{t\downarrow 0} F_t(g) \le \limsup_{t\downarrow 0} F_t(g) \le G(g_n^U)$$

for all n.

By dominated convergence we have

$$\lim_{n\to\infty} G(g_n^L) = G(g) = \lim_{n\to\infty} G(g_n^U)$$

and as a result

$$\lim_{t\downarrow 0} F_t(g) = G(g).$$

□

Proof of Corollary 1.4. We assume that (M,g) is a compact Riemannian manifold, and $V \in L^\infty(M,g)$. Let $\{\lambda_k\}_{k\ge 1}$ be the spectrum of the Schrödinger operator, $\Delta + V$, on M. We shall apply the generalized Karamata Lemma to the measure

$$d\nu := \sum_{k\ge 1} \delta_{\lambda_k}.$$

Then we note that

$$\int_0^x d\nu(\lambda) = N(x) = \#\{\lambda_k \le x\} - n_-.$$

Above, n_- is the number of negative eigenvalues of the Schrödinger operator. It is well known that n_- is finite whenever the potential V is bounded below. We recall the brief argument for this for the sake of completeness. If ϕ_k is the unitary eigenfunction corresponding to λ_k, then

$$\lambda_k = \int_M |\nabla \phi_k|^2 + \int_M V\phi_k^2 \geq -\|V\|_\infty.$$

The spectral theorem in this setting implies that the eigenvalues may only accumulate at $\pm\infty$. By the above estimate, there can therefore exist at most finitely many negative eigenvalues.

We shall apply Lemma 2.1 to the function,

$$g(x) = \begin{cases} 0; & x \in \left[0, e^{-1}\right] \cup [1, \infty) \\ \frac{1}{x}; & x \in \left(e^{-1}, 1\right) \end{cases} . \tag{15}$$

By Proposition 1.1, and due to the fact that there are at most finitely many negative eigenvalues.

$$\lim_{t\downarrow 0} t^{n/2} \int_0^\infty e^{-t\lambda} d\nu(\lambda) = \lim_{t\downarrow 0} t^{n/2} \left(\operatorname{tr} e^{-t\Delta_V} - \sum_{\lambda_k<0} e^{-\lambda_k t} \right) = \frac{\mathrm{Vol}(M)}{(4\pi)^{n/2}}.$$

Let

$$c := \frac{\mathrm{Vol}(M)}{(4\pi)^{n/2}}, \quad \alpha = \frac{n}{2}$$

Applying Lemma 2.1 with the aforementioned g, c, and α we obtain

$$\lim_{t\downarrow 0} t^\alpha \int_0^\infty g\left(e^{-t\lambda}\right) e^{-t\lambda} \mathrm{d}\nu(\lambda) = \frac{c}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty g\left(e^{-t}\right) t^{\frac{n}{2}-1} e^{-t} \mathrm{d}t.$$

Substituting for g in this equation we have

$$\lim_{t\downarrow 0} t^\alpha \int_0^{1/t} e^{t\lambda} e^{-t\lambda} \mathrm{d}\nu(\lambda) = \lim_{t\downarrow 0} t^\alpha N\left(\frac{1}{t}\right) = \frac{c}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 e^t t^{n/2-1} e^{-t} \mathrm{d}t = \frac{c}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}.$$

Clearly then,

$$\lim_{t\to 0} t^{n/2} N\left(\frac{1}{t}\right) = \frac{2c}{n\Gamma(n/2)} \iff N(\lambda) \sim \frac{2\lambda^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} \frac{\mathrm{Vol}(M)}{(4\pi)^{n/2}}, \quad \lambda \to \infty.$$

To prove Weyl's law for the drifting Laplacian on a weighted manifold, we use the fact that it is unitarily equivalent to a Schrödinger operator. We consider a function f on the manifold, M, such that Δf and ∇f are both in $L^\infty(M, g)$. Let L^2_f denote the set of L^2 integrable functions on M with respect to the weighted measure $d\mu_f$, $L^2_f(M) = \{u \mid \int_M u^2 e^{-f}\, d\mu < \infty\}$. Using the transformation $T : L^2 \to L^2_f$ given by $T(u) = e^{\frac{1}{2}f} u$ we have that the drifting Laplacian is unitarily equivalent to the Schrödinger operator,

$$\Delta + V, \quad \text{where} \quad V = \frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2.$$

Under these assumptions the potential function $V = \frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2 \in L^\infty(M, g)$. Consequently, for the above V and f, the operators Δ_V and Δ_f are isospectral. We therefore apply the preceding proof of Weyl's law for the operator $\Delta + V$ and conclude that the operator Δ_f obeys the same Weyl law as $\Delta + V$. ∎

3. The Heat Trace and the Regularity of the Potential. In this section we demonstrate that the sharpness of the remainder estimate in the short time asymptotic expansion of the heat trace is equivalent to the regularity of the potential. It turns out that the regularity result of Theorem 1.2 is determined entirely by the term $\operatorname{tr} W_2$ we saw in the previous section.

PROPOSITION 3.1. *Assume that $V \in L^\infty$. If $\operatorname{tr} W_2(t)$ has an expansion as $t \downarrow 0$ of the form*

$$\operatorname{tr} W_2(t) = \frac{t^2}{2(4\pi t)^{n/2}} \left(c + O(t)\right).$$

then $V \in H^1$, and

$$c = \|V\|_{L^2}^2.$$

Conversely, if $V \in H^1$, then $\operatorname{tr} W_2(t)$ has such an expansion and

$$\operatorname{tr} W_2(t) = \frac{t^2}{2(4\pi t)^{n/2}} \left(\|V\|_{L^2}^2 + O(t)\right).$$

Proof. To prove the proposition, we begin by assuming that

$$\operatorname{tr} W_2(t) = \frac{1}{2} t^2 (4\pi t)^{-n/2} \left[c + O(t)\right].$$

Since $V \in L^\infty$, (12) gives $c = \|V\|_{L^2}^2$, and we now have

$$\operatorname{tr} W_2(t) = \frac{1}{2} t^2 (4\pi t)^{-n/2} \left[\|V\|_{L^2}^2 + O(t)\right]. \tag{16}$$

Comparing (10) to (16) we therefore obtain

$$\frac{1}{t}\left|\|V\|_{L^2}^2 - \int_0^1 \int_{M\times M} H_0(tv(1-v), y, z)\, V(y)V(z) dy dz dv\right| = \frac{O(t)}{t} \leq C$$

for t small enough.

By definition,

$$\int_{M\times M} H_0(tv(1-v), y, z)\, V(y)V(z) dy dz = (e^{-tv(1-v)\Delta_0} V, V),$$

therefore the above estimate also shows that

$$\frac{1}{t}\left|\|V\|_{L^2}^2 - \int_0^1 (e^{-tv(1-v)\Delta_0} V, V)\, dv\right| \leq C \text{ as } t \to 0^+. \tag{17}$$

We make the following observation. For any $\lambda_k \geq 0$

$$\begin{aligned}\lim_{t\to 0^+} \frac{1}{t}\left[1 - \int_0^1 e^{-tv(1-v)\lambda_k}\, dv\right] &= \int_0^1 v(1-v)\lambda_k\, dv = \frac{1}{6}\lambda_k \\ &= \frac{1}{6} \lim_{t\to 0^+} \frac{1}{t}\left[1 - \int_0^1 e^{-t\lambda_k}\, dv\right].\end{aligned}$$

Using the orthonormal basis expansion (6) for the heat kernel, we obtain

$$
\begin{aligned}
(e^{-tv(1-v)\Delta_0}V,V) &= \sum_{k\geq 1} e^{-\lambda_k tv(1-v)} \int_{M\times M} \phi_k(z')V(z')\phi_k(z)V(z)dzdz' \\
&= \sum_{k\geq 1} e^{-\lambda_k tv(1-v)} |\widehat{V_k}|^2,
\end{aligned}
$$

where

$$
\widehat{V_k} = \int_M V(z)\phi_k(z)dz,
$$

is the k^{th} Fourier coefficient of V with respect to the basis. We also note that

$$
\|V\|^2_{L^2(M)} = \sum_{k\geq 1} |\widehat{V_k}|^2.
$$

Consequently,

$$
\int_0^1 (e^{-tv(1-v)\Delta_0}V,V)dv - \|V\|^2_{L^2} = \int_0^1 \sum_{k\geq 1}(e^{-\lambda_k tv(1-v)} - 1)|\widehat{V_k}|^2 dv.
$$

Since $V \in L^2$, the above expression converges absolutely and uniformly. We may therefore compute

$$
\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\left[\int_0^1 (e^{-tv(1-v)\Delta_0}V,V)\,dv - \|V\|^2_{L^2}\right] \\
=\ & \lim_{t\to 0^+} \frac{1}{t}\int_0^1 \sum_{k\geq 1}(e^{-\lambda_k tv(1-v)} - 1)|\widehat{V_k}|^2 dv \\
=\ & \sum_{k\geq 1} \lim_{t\to 0^+} \frac{1}{t}\int_0^1 (e^{-\lambda_k tv(1-v)} - 1)|\widehat{V_k}|^2 dv \\
=\ & \frac{1}{6}\sum_{k\geq 1} \lim_{t\to 0^+} \frac{1}{t}\int_0^1 (e^{-\lambda_k t} - 1)|\widehat{V_k}|^2 dv.
\end{aligned}
\tag{18}
$$

On the other hand, by the definition of the heat operator,

$$
\sum_{k\geq 1} e^{-\lambda_k t}|\widehat{V_k}|^2 = (e^{-t\Delta_0}V,V).
$$

Substituting this in the right side of (18), we obtain

$$
\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\left[\int_0^1 (e^{-tv(1-v)\Delta_0}V,V)\,dv - \|V\|^2_{L^2}\right] \\
=\ & \frac{1}{6}\lim_{t\to 0^+} \frac{1}{t}\left[\int_0^1 (e^{-t\Delta_0}V,V)\,dv - \|V\|^2_{L^2}\right].
\end{aligned}
\tag{19}
$$

The definition of the heat operator [3] also gives that

$$
\begin{aligned}
\lim_{t\to 0^+} \frac{1}{t}\left[\int_0^1 (e^{-t\Delta_0}V,V)\,dv - \|V\|^2_{L^2}\right] &= -\lim_{t\to 0^+} \frac{d}{dt}(e^{-t\Delta_0}V,V) \\
&= \lim_{t\to 0^+} (\Delta_0 e^{-t\Delta_0}V,V) = \|\nabla V\|^2_{L^2}.
\end{aligned}
\tag{20}
$$

The estimate in (17) together with the identities (19) and (20) demonstrate that in fact $\|\nabla V\|_{L^2}^2 < \infty$ and we can conclude that V is in H_1. We have therefore proven that whenever $\operatorname{tr} W_2(t)$ has an expansion as above, this implies that $V \in H_1$.

Conversely, if we assume that $V \in H_1$, then by (20), we see that we obtain the improvement in the remainder estimate in $\operatorname{tr} W_2(t)$, so that

$$\operatorname{tr} W_2(t) = \frac{t^2}{2(4\pi t)^{n/2}} \left(\|V\|_{L^2}^2 + O(t) \right).$$

Hence, $\operatorname{tr} W_2(t)$ has an asymptotic expansion of this type if and only if $V \in H_1$. □

Finally, we use the proposition to prove Theorem 1.2.

Proof of Theorem 1.2. The à priori estimates on $\operatorname{tr} W_k(t)$ for $k \geq 3$ given in (4) show that

$$\operatorname{tr} \sum_{k \geq 3} W_k(t) \leq C t^{3-n/2}, \quad 0 < t \leq 1$$

and in consequence

$$\operatorname{tr} e^{-t\Delta_V} - \operatorname{tr} e^{-t\Delta} = \operatorname{tr} W_1(t) + \operatorname{tr} W_2(t) + O(t^{3-n/2}).$$

The asymptotic expansion for $\operatorname{tr} W_1(t)$ from (8) together with the asymptotic expansion for the trace of the semigroup $e^{-t\Delta}$ give

$$\operatorname{tr} e^{-t\Delta_V} - (4\pi t)^{-n/2} \left[\operatorname{Vol}(M) + t \int_M \left(\frac{1}{6} K(z) - V(z) \right) d\mu(z) \right.$$
$$\left. + t^2 \left(a_{0,2} - \frac{1}{6} \int_M K(z) V(z)\, d\mu(z) \right) \right] = \operatorname{tr} W_2(t) + O(t^{3-n/2}).$$

The theorem therefore follows from Proposition 3.1. □

We proved in Proposition 1.1 that for $V \in L^\infty$ we have an expansion for the heat trace of $\Delta + V$ with an error term of order $o(t^{2-n/2})$ for t small. Theorem 1.2 implies that if the error term is slightly better, then the potential must in fact belong to H^1. The converse is also true. This regularity result was inspired by the work of H. Smith and M. Zworski for Schrödinger operators on $\mathbb{R}^n$ [9]. There the authors were able to use the Fourier transform on Euclidean space to express the trace of the heat kernel and relate it to the H^m norms of the potential, but also to obtain estimates for the terms, W_k.

Here, we provide a generalization of some of their results to the case of compact manifolds. In our case, we are not able to use the nice properties of the Fourier transform, but instead rely on the definition of the H^1 norm with respect to the heat operator, as well the expansion of the heat kernel with respect to its orthonormal basis of L^2 eigenfunctions. We believe that similar results should also hold on non-compact manifolds where the Schrödinger operator or drifting Laplacian has a discrete spectrum and its heat operator can be expressed using such an orthonormal basis of eigenfunctions. We were recently informed by H. Smith that he has also independently worked out a generalization of his results with M. Zworski for complete manifolds.

Corollary 3.2. *Let (M, g) be a smooth, compact Riemannian manifold with Laplace operator Δ, and let f be a real-valued function on M, such that*

$$\Delta f \text{ and } \nabla f \in L^\infty(M, g).$$

Then, the heat kernel for drifting Laplacian $\Delta_f = \Delta + \nabla f \cdot \nabla$ has a small-time asymptotic expansion as $t \downarrow 0$ of the form

$$\begin{aligned} &\operatorname{tr} e^{-t\Delta_V} \\ = \ &(4\pi t)^{-n/2} \left[\mathrm{Vol}(M) + t \int_M \left(\frac{1}{6} K(z) - \frac{1}{4} |\nabla f(z)|^2 \right) d\mu(z) \right. \\ &\left. + t^2 \left(a_{0,2} - \int_M \frac{1}{6} K(z) \left(\frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2 \right) d\mu(z) + \frac{1}{2} \| \frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2 \|_{L^2}^2 \right) \right] \\ &+ R(t) \end{aligned}$$

where the remainder $R(t) = o(t^{2-n/2})$ as $t \downarrow 0$. Moreover, the remainder $R(t) = O(t^{3-n/2})$ if and only if $\frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2 \in H^1$.

Proof. The corollary follows immediately from the observation that the Schrödinger operator $\Delta + V$ and the drifting Laplacian $\Delta + \nabla f \cdot \nabla$ are unitarily equivalent, and that $\int_M V(z)\, d\mu(z) = 1/4 \int_M |\nabla f(z)|^2\, d\mu(z)$. □

Proof of Corollary 1.3. If the Schrödinger operators $\Delta + V$ and $\Delta + \tilde{V}$ are isospectral, then they have the same heat trace. By the preceding corollary, the remainders are either both $O(t^{3-n/2})$ or $o(t^{2-n/2})$, and therefore either both V and $\tilde{V}$ are in H^1, or they are both not in H^1. □

4. The Heat Trace for the Drifting Laplacian with a Smooth Weight Function. In this section we will give the classical method for obtaining the heat trace of the drifting Laplacian in the smooth case. This is done via the parametrix method, as in the case of the Laplacian, and works particularly well for a smooth weight function, f. Since we are mainly interested in the coefficients for the short time asymptotic expansion of the heat trace we will omit some of the simple technical elements of the arguments as they are identical to the unweighted case (we will refer the interested reader to [7] for the details).

The manifold is compact, so there exists a uniform constant $\varepsilon > 0$ such that for any $x \in M$ the exponential map at x is a diffeomorphism from the ball of radius ε in the tangent space onto $B_x(\varepsilon)$. Fix a point $x \in M$. For any $y \in B_x(\varepsilon)$ the Riemannian distance, $d(x, y)$, from x to y satisfies $d(x, y) < \varepsilon$. We let

$$U_\varepsilon = \{ (x, y) \in M \times M \mid y \in B_x(\varepsilon) \}.$$

Let

$$G(t, x, y) = (4\pi t)^{-n/2} e^{-\frac{d^2(x,y)}{4t}}$$

be the direct analogue of the Euclidean heat kernel on M which belongs to $C^\infty(\mathbb{R}^+ \times U_\varepsilon)$. Set

$$u(t, x, y) = u_0(x, y) + \ldots + u_k(x, y)\, t^k$$

where the functions $u_i(x, y)$ are to be determined. Define

$$S_k(t, x, y) = G(t, x, y)\, e^{\frac{1}{2}(f(x)+f(y))}\, u(t, x, y).$$

Denote the inner product on M by $\langle \cdot, \cdot \rangle$. Recalling that

$$\Delta(h \cdot g) = (\Delta h)g + h\Delta g - 2\langle \nabla h, \nabla g \rangle,$$

for $(x, y) \in U_\varepsilon$ we compute

$$\left(\frac{\partial}{\partial t} + \Delta_{f,y}\right) S_k = \left(\frac{\partial}{\partial t} + \Delta_y\right)(Gu) \cdot e^{\frac{1}{2}(f(x)+f(y))} + Gu \cdot \Delta_y(e^{\frac{1}{2}(f(x)+f(y))})$$
$$- 2\langle \nabla_y(Gu), \nabla_y(e^{\frac{1}{2}(f(x)+f(y))})\rangle + \langle \nabla_y f, \nabla_y(Gu\, e^{\frac{1}{2}(f(x)+f(y))})\rangle.$$

Above, the drifting Laplacian, Laplacian and gradient are taken with respect to the y variable. Using (3.8) of [7], we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \Delta_{f,y}\right) S_k \\ = \; & G\, e^{\frac{1}{2}(f(x)+f(y))} \Bigg[u_1 + \ldots + k t^{k-1} u_k + \frac{r}{2t}\frac{D'}{D}(u_0 + \ldots + t^k u_k) \\ & + \frac{r}{t}\left(\frac{\partial u_0}{\partial r} + \ldots + t^k \frac{\partial u_k}{\partial r}\right) + \Delta_y u_0 + \ldots + t^k \Delta_y u_k \Bigg] \\ & + G\,(u_0 + \ldots + t^k u_k)\; e^{\frac{1}{2}(f(x)+f(y))} \cdot \left[\frac{1}{2}\Delta_y f(y) + \frac{1}{4}|\nabla_y f(y)|^2\right] \end{aligned} \tag{21}$$

where

$$D = \det(d\exp_x)$$

is the determinant of the Riemannian volume form centered at x, and $D' = \partial D/\partial r$ is its derivative with respect to the radial function $r(y) = d(x, y)$.

To obtain the parametrix we will choose u_i inductively such that the coefficient of t^i vanishes for $-1 \le i \le k-1$. The coefficient of t^{-1} will vanish if we set

$$\frac{r}{2}\frac{D'}{D} u_0 + r\frac{\partial u_0}{\partial r} = 0.$$

This first order differential equation has a smooth solution for $r < \varepsilon$ given by

$$u_0(x, y) = D^{-1/2}(y)$$

which is a smooth function on U_ε independently of f and satisfies

$$u_0(x, x) = 1.$$

For t^{i-1} we obtain the equation

$$i\, u_i + \frac{r}{2}\frac{D'}{D} u_i + r\frac{\partial u_i}{\partial r} + \Delta_y u_{i-1} + \left[\frac{1}{2}\Delta_y f(y) + \frac{1}{4}|\nabla_y f(y)|^2\right] \cdot u_{i-1} = 0.$$

Letting $x(s)$ be the unit speed geodesic from x to y for $s \in [0, r]$ with $x(0) = x$ and $x(r) = y$, we may obtain a solution to the above equation from the integral equation

$$\begin{aligned} & u_i(x, y) \\ = \; & -\, r^{-i}(x, y) D^{-1/2}(y) \left[\int_0^r D^{1/2}(x(s)) \cdot (\Delta_{x(s)} u_{i-1})(x, x(s)) \cdot s^{i-1} ds \right. \\ & \left. + \int_0^r D^{1/2}(x(s)) \left(\frac{1}{2}\Delta f(x(s)) + \frac{1}{4}|\nabla f(x(s))|^2\right) u_{i-1}(x, x(s)) \cdot s^{i-1} ds \right]. \end{aligned} \tag{22}$$

We take this opportunity to correct a small misprint in [7] (3.12); in the above x stays fixed, and y varies along the geodesic from x. Observe that the functions u_i are smooth on $M \times M$.

As a result,

$$\left(\frac{\partial}{\partial t} + \Delta_{f,y}\right) S_k = G(t,x,y)\, t^k\, e^{\frac{1}{2}(f(x)+f(y))} \cdot \left[\Delta_y u_k(x,y) + \left(\frac{1}{2}\,\Delta f(y) + \frac{1}{4}\,|\nabla f(y)|^2\right) \cdot u_k(x,y)\right]. \tag{23}$$

Define

$$\eta(x,y) = \begin{cases} 1 & \text{on } U_{\varepsilon/2} \\ 0 & \text{on } M \times M \setminus U_\varepsilon \end{cases}$$

to be a smooth function with bounded first and second order derivatives. Then we extend S_k to $M \times M$ setting

$$h_k(t,x,y) = \eta(x,y) \cdot S_k(t,x,y) = \eta(x,y)\, G(t,x,y)\, e^{\frac{1}{2}(f(x)+f(y))} \sum_{i=1}^{k} u_i(x,y)\, t^i. \tag{24}$$

Each $h_k(t,x,y)$ is a smooth function on $(0,\infty) \times M \times M$. Moreover, for $k > n/2$, $h_k(t,x,y)$ is a local parametrix of the operator $\frac{\partial}{\partial t} + \Delta_f$ by virtue of the two properties it satisfies given in

Lemma 4.1. *The following two properties are satisfied by h_k,*

(*i*) $\frac{\partial}{\partial t} h_k + \Delta_{f,y} h_k \in C^0([0,\infty) \times M \times M)$;

(*ii*) $\lim_{t\to 0} \int_M h_k(t,x,y) g(y)\, d\mu_f(y) = g(x)$ *for any* $g \in L^2_f(M)$.

Proof. For (*i*) we need to show that $\frac{\partial}{\partial t} h_k + \Delta_{f,y} h_k$ extends to $t = 0$. This is true for $M \times M \setminus U_\varepsilon$ since $h_k \equiv 0$. On $U_{\varepsilon/2}$ (23) holds, and the right set tends to 0 as $t \to 0$ for $k > n/2$ and f smooth. On $U_\varepsilon \setminus U_{\varepsilon/2}$

$$\begin{aligned}(\partial/\partial t + \Delta_{f,y})\, h_k &= \eta\; (\partial/\partial t + \Delta_{f,y})\, S_k - 2\langle d\eta, dS_k\rangle + (\Delta_{f,y}\eta)\, S_k \\ &= (4\pi t)^{-n/2} e^{-\frac{d^2(x,y)}{4t}}\, \phi(t,x,y)\end{aligned}$$

where ϕ is a smooth function on $(0,\infty) \times M \times M$ and has a pole of order at most t^{-1} as $t \downarrow 0$. Since $d \geq \varepsilon/2$ on this set, we can also extend $\frac{\partial}{\partial t} h_k + \Delta_{f,y} h_k$ by zero to $t = 0$.

Note that for $k > l + n/2$, $(\partial/\partial t + \Delta_{f,y})\, h_k \in C^l([0,\infty) \times M \times M)$.

For (*ii*) we can show that for any $w \in L^2(M)$ (which is equivalent to $g = e^{\frac{1}{2}f} w \in L^2_f$)

$$\begin{aligned}&\lim_{t\to 0} \int_M h_k(t,x,y) e^{\frac{1}{2}f(y)} w(y)\, d\mu_f(y) \\ &= e^{\frac{1}{2}f(x)} \lim_{t\to 0} \sum_{i=0}^{k} t^i \cdot \int_M G(t,x,y)\, \eta(x,y)\, u_i(x,y)\, w(y)\; dy \\ &= e^{\frac{1}{2}f(x)} \lim_{t\to 0} \sum_{i=0}^{k} t^i\, u_i(x,x)\, w(x) \\ &= e^{\frac{1}{2}f(x)} w(x)\end{aligned}$$

since $G(t,x,y)$ is the heat kernel in $\mathbb{R}^n$ and $\eta(x,x) = u_0(x,x) = 1$.

Therefore, for any $g \in L^2_f(M)$ the result follows. □

LEMMA 4.2. *Let*

$$H_f(t,x,y) = h_k(t,x,y) - Q_k * h_k(t,x,y)$$

and $Q_k = \sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}(((\partial/\partial t + \Delta_{f,y})(h_k))^{*\lambda}$. *Then* $H_f(t,x,y) \in C^\infty((0,\infty)\times M \times M)$*, it is independent of* k *for* $k > 2 + n/2$ *and it is the heat kernel of the semigroup* $e^{-t\Delta_f}$.

The Lemma follows in exactly the same way as [7, Theorem 3.22], and simply relies on estimates of convolution operators which give that $|Q_k * h_k| \le C \cdot t^{k+1-n/2}$.

Proof of Theorem 1.5. Recall the definition that $A(t) \sim \sum_{i=0}^{\infty} a_i t^i$ as $t \downarrow 0$, if for all $k \ge K_o$

$$\lim_{t\to 0} \frac{1}{t^k}\left[A(t) - \sum_{i=K_o}^{k} a_i t^i\right] = 0.$$

The estimate (1) is now a direct consequence of the parametrix method and is due to the fact that $h_k(t,x,x) = \sum_{i=0}^{k} u_i(x,x)\, t^i$ and the term $Q_k * h_k(t,x,x)$ is at most of order $t^{k+1-n/2}$ for k sufficiently large (see [7, Proposition 3.23] for all the details).

For the second part of Theorem 1.5 we first note that under these hypotheses, by Corollary 1.4 and the fact that the eigenfunctions are an orthonormal basis of $L^2(M, e^{-f}d\mu)$, the heat kernel is trace class. By the heat trace formula and the asymptotic expansion (1),

$$\begin{aligned}
\operatorname{tr} e^{-t\Delta_f} &= \int_M H_f(t,x,x)\, d\mu_f(x) = \sum_k e^{\lambda_k t} \\
&\sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_M u_i(x,x) e^{f(x)}\, d\mu_f(x) \\
&= (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_M u_i(x,x)\, d\mu(x).
\end{aligned}$$

□

Note that for $f = 0$ the coefficient functions u_i corresponding to the heat kernel of the Laplacian over the manifold are inductively given by

$$u_{0,i}(x,y) = -r^{-i}(x,y) D^{-1/2}(y) \int_0^r D^{1/2}(x(s)) \cdot (\Delta_{x(s)} u_{i-1})(x, x(s)) \cdot s^{i-1} ds. \quad (25)$$

Denote

$$a_{0,i} = \int_M u_{0,i}(x,x)\, d\mu(x). \quad (26)$$

In this case, we also have $u_{0,0}(x,x) = 1$. As we have previously mentioned $u_{0,1}(x,x) = \frac{1}{6} K(x)$, where $K(x)$ is the scalar curvature of the manifold at the point x.

For the weighted case, the definition of the u_i in (22) gives

$$\begin{aligned} a_0 &= \mathrm{Vol}(M) \\ a_1 &= \int_M u_{0,1}(x,x)\, d\mu(x) - \int_M \left(\frac{1}{2}\, \Delta f(x) + \frac{1}{4}\, |\nabla f(x)|^2 \right) d\mu(x) \\ &= \int_M \frac{1}{6}\, K(x)\, d\mu(x) - \int_M \frac{1}{4}\, |\nabla f(x)|^2\, d\mu(x). \end{aligned}$$

These coincide with the coefficients of $(4\pi t)^{-n/2}$ and $(4\pi t)^{-n/2} t$ that we saw in Corollary 3.2 for (not necessarily) smooth f.

Using these estimates we can now prove the isospectrality results in Theorem 1.6

Proof of Theorem 1.6. By our assumption,

$$\operatorname{tr} e^{-t\Delta_{f_1}} = \operatorname{tr} e^{-t\Delta_{f_2}} = \sum_k e^{\lambda_k t}.$$

As a result, the leading term in (2) corresponding to $i = 0$ must have the same exponent in t, giving us $n = m$. Moreover, the coefficients a_0 must coincide, thereby requiring $\mathrm{Vol}(M) = \mathrm{Vol}(N)$.

Next we assume that M and N are orientable surfaces, and that the weight functions have equal Dirichlet norms. By isospectrality, the coefficients a_1 must also coincide, so that

$$\int_M K_M(x)\, d\mu_M(x) = \int_N K_N(x)\, d\mu_N(x).$$

By the Gauss-Bonnet theorem, M and N have identical Euler characteristic,

$$\chi(M) = \chi(N).$$

The result follows, since two compact oriented surfaces with the same Euler characteristic are diffeomorphic.

The compactness of the set of isospectral drifting Laplacians with smooth weight function follows immediately from the corresponding result for Schrödinger operators proven by J. Brüning, Theorem 3 of [1]. □

We conclude our paper with the following two remarks.

REMARK 4.3. *The unitary equivalence of the drifting Laplacian, Δ_f, and the Schrödinger operator, $\Delta + V$, with $V = \frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2$ for $V \in L^\infty$ implies that the heat kernel of $\Delta + V$, $H_V(t,x,y)$, is related to the heat kernel of the drifting Laplacian, $H_f(t,x,y)$ by the following formula*

$$H_f(t,x,y) = H_V(t,x,y) e^{\frac{1}{2}(f(x)+f(y))}.$$

Note, that the corresponding parametrix for $\partial/\partial t + \Delta + V$ would now be $h_k(t,x,y) = \eta(x,y)\, G(t,x,y)\, u(t,x,y)$ with the same u_i as in (22)

REMARK 4.4. *As we have seen in the proof of Corollary 1.4, a_0 determines Weyl's law for the eigenvalues of the operator. In the case of the drifting Laplacian this law is independent of the weight function f. This is also illustrated in the simple case of a constant function f. In this setting the eigenvalues of the drifting Laplacian* *coincide with the eigenvalues of the Laplacian on a compact manifold even though the weighted volume of manifold is different from its Riemannian volume. As a result* Δ_f *and* Δ *have the same heat trace, independently of* f*. What is fairly surprising is the fact that Weyl's asymptotic formula is independent of the function* f *for any* f *satisfying* $\frac{1}{2}\Delta f + \frac{1}{4}|\nabla f|^2 \in L^\infty$.

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