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QUASI-UNIPOTENT MOTIVES AND MOTIVIC NEARBY SHEAVES*

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Abstract. Let k be an algebraically closed field of characteristic zero. We consider a relative version over a general k-scheme of the category of quasi-unipotent motives introduced by J. Ayoub over k. We introduce a monodromic version of the nearby motivic sheaf functor associated with a function $f : X \to \mathbf{A}^1_k$ on a separated k-scheme of finite type and show that the motives obtained by applying it are quasi-unipotent. Using this construction, we prove a comparison between this monodromic version of the motivic nearby sheaf of J. Ayoub and the theory of virtual nearby cycles of J. Denef and F. Loeser that takes into account the monodromy action.

Key words. Motivic sheaves, Nearby motivic sheaves, quasi-unipotent motives, Motivic Milnor fiber.

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1. Introduction. Let k be a field of characteristic zero and X be smooth k-variety. To every morphism $f\colon X \to \mathbf{A}^1_k$, one classically attaches the following degeneration diagram:

$$\begin{array}{ccccc} X_\eta & \longrightarrow & X & \longleftarrow & X_\sigma \\ \big\downarrow{\scriptstyle f_\eta} & \square & \big\downarrow{\scriptstyle f} & \square & \big\downarrow{\scriptstyle f_\sigma} \\ \eta := \mathbf{G}_{m,k} & \xrightarrow{j} & \mathbf{A}^1_k & \xleftarrow{i} & \sigma := \mathsf{Spec}(k), \end{array} \tag{1.1}$$

where i is the zero section of the structural morphism of the affine line and j its complement. Inspired by the theory of nearby cycles, in [9], J. Denef and F. Loeser introduced the so-called *motivic nearby cycles* ψ_f, which can be defined by

$$\psi_f = \sum_{\emptyset \neq J \subset I} [\tilde{D}^\circ_J, \hat{\boldsymbol{\mu}}](\mathbf{1} - \mathbf{L})^{|J|-1} \in \mathrm{K}_0^{\hat{\boldsymbol{\mu}}}(\mathbf{Var}_{X_\sigma}) \tag{1.2}$$

as an element of a monodromic version of the usual Grothendieck ring of varieties. This formula can be computed, *via* motivic integration, on *every* log-resolution of the singularities of the pair (X, X_σ). (See remark 5.3.3 for complements.) In this context, $D = \sum_{i\in I} m_i D_i$ is the exceptional divisor of the chosen log-resolution, and $\tilde{D}^\circ_J$ is an étale covering of D°_J for every nonempty subset $J \subseteq I$ (see the end of the introduction for the notation). Independently, in [2, §3.5], J. Ayoub developed a functorial theory of motivic nearby cycles that provides a *nearby motivic sheaf* $\Psi_f(\mathbb{1}_{X_\eta}) \in \mathbf{SH}_{\mathfrak{M}}(X_\sigma)$, which can be interpreted as an incarnation of the classical nearby cycles in the world of motives.

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With J. Ayoub, the authors in [7] have shown that the *nearby motivic sheaf* $\Psi_f(\mathbb{1}_{X_\eta})$ can be compared to Denef-Loeser's construction (see also [20, Theorem 5.1] for an earlier version in the category of étale motives). Precisely, one has the following statement:

THEOREM (See [7, Corollary 8.7]). *Keep the notation of diagram (1.1). We have the following formula*

$$[\Psi_f(\mathbb{1}_{X_\eta})] = \chi_{X_\sigma,\mathrm{c}}(\psi_f) \in \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(X_\sigma)). \tag{1.3}$$

In this statement, the ring morphism $\chi_{X_\sigma,\mathrm{c}}\colon \mathrm{K}_0(\mathbf{Var}_{X_\sigma}) \to \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(X_\sigma))$ is the motivic Euler characteristic defined by motives with compact support of X_σ-varieties. Let us stress that formula (1.3) does not take into account the monodromy action which is yet crucial in the theory of nearby cycles. One explanation of this weakness is the lack, in the theory of motives of Ayoub-Voevodsky, of a reasonnable notion of quasi-unipotent motives in the relative settings and a description of the nearby motivic sheaf as a quasi-unipotent motives over the special fiber.

The goal of the present article is to define the category of quasi-unipotent motives over a scheme, provide such a description of $\Psi_f(\mathbb{1}_{X_\eta})$ and extend formula (1.3) to the monodromic context. After a presentation, in section 2, of results on group actions useful for establishing our main results, we introduce and study, in section 3, the category $\mathbf{QUSH}_{\mathfrak{M}}(S)$ of quasi-unipotent motives on a scheme S.

The main results of this section are used to prove theorem 4.1.1 and theorem 4.2.1 which, in particular, lift the nearby motivic sheaf $\Psi_f(\mathbb{1}_{X_\eta})$ to a quasi-unipotent motive $\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$. More precisely, our first main theorem is the following.

THEOREM. *Keep the notation of diagram (1.1). Let* $f^{\mathbf{G}_m} : \mathsf{Spec}(\mathscr{O}_X[T,T^{-1}]) \to \mathbf{A}^1_k$ *be the morphism obtained from* f *by multplication with the parameter* T. *Then,*

$$\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) := \Psi_{f^{\mathbf{G}_m}}(\mathbb{1}_{\mathbf{G}_{m,X_\eta}})$$

is in $\mathbf{QUSH}_{\mathfrak{M}}(X_\sigma)$, *and there is a canonical isomorphism*

$$1^*_{X_\sigma}\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \xrightarrow{\sim} \Psi_f(\mathbb{1}_{X_\eta}).$$

Thanks to the results of section 3, we construct, in section 5, a monodromic motivic Euler characteristic with values in the Grothendieck ring of quasi-unipotent motives. This ring morphism allows us to formulate and prove our second main result which can be interpreted as a monodromic version of formula (1.3) (see theorem 5.3.1):

THEOREM. *For every k-variety S, the motivic Euler characteristic $\chi_{S\times\mathbf{G}_m,\mathrm{c}}$ induces a morphism of rings*

$$\chi^{\hat\mu}_{S,\mathrm{c}}\colon \mathrm{K}_0^{\hat\mu}(\mathbf{Var}_S) \to \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S))$$

which verifies the following formula:

$$\chi^{\hat\mu}_{X_\sigma,\mathrm{c}}(\psi_f)) = [\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})] \in \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(X_\sigma)).$$

Conventions, notation. In this article, we denote by k an algebraically closed field of characteristic zero and by S a separated noetherian k-scheme of finite Krull dimension. For every k-scheme Y, a *Y-variety* is a separated Y-scheme of finite type. For every k-scheme Y, we denote by $\mathsf{Spec}(\mathcal{O}_Y[T,T^{-1}])$ the k-scheme $Y \times_k \mathbf{G}_{m,k}$ when we have to fix coordinates on $\mathbf{G}_{m,k}$. For every k-scheme Y, we denote by $1_Y : Y \to \mathbf{G}_{m,Y}$ the unit section morphism. For every k-scheme Y, we denote by $\mathrm{K}_0(\mathbf{Var}_Y)$ the usual Grothendieck ring of Y-varieties defined, e.g., in [10]. For every morphism of k-schemes $f : Y' \to Y$, we denote by $\check{f} : \mathbf{G}_{m,Y'} \to \mathbf{G}_{m,Y}$ the morphism of k-schemes obtained by base-change over k. Let Y be a scheme and $(D_i)_{i\in I}$ be a set of irreductible components of Y. Given a nonempty subset $J \subseteq I$, we denote by D_J and $D(J)$ the reduced closed subschemes of Y given by $D_J = \cap_{i\in J} D_i$, $D(J) = \cup_{i\in J} D_i$ and $D_J^\circ := D_J \setminus D(I \setminus J)$.

2. Preliminaries on group actions. If G be an algebraic k-group scheme, acting on a k-variety X, we say that the action is *good* if every G-orbit is contained in an affine open subscheme of X according to the terminology used in [9]. We set $\mathbf{B}_k$ for $\mathbf{A}^1_k$ or $\mathbf{G}_{m,k}$, and $\mathbf{C}_k$ for $\mathbf{G}_{m,k}$ or $\boldsymbol{\mu}_{n,k} := \mathsf{Spec}(k[T]/\langle T^n - 1\rangle)$. The k-scheme $\mathbf{B}_k$ is endowed with a good $\mathbf{C}_k$-action, called *multiplication*,

$$m_k : \mathbf{B}_k \times_k \mathbf{C}_k \to \mathbf{B}_k \tag{2.1}$$

which corresponds, at the level of the A-points, for a k-algebra A, to the map $m_k(A) : (\alpha, \beta) \mapsto \alpha\beta$. We denote by m_S the base-change morphism $\mathrm{Id}_S \times m_k$.

2.1. Diagonally monomial $\mathbf{G}_m$-actions. Let $n \in \mathbf{N}^\times$ be an integer. We denote by $e_n : \mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$ the morphism of k-schemes associated with the morphism of k-algebras $T \mapsto T^n$. Let $p : Y \to S \times_k \mathbf{G}_{m,k}$ be a $\mathbf{G}_{m,S}$-variety. A good $\mathbf{G}_{m,k}$-action $\sigma : Y \times_k \mathbf{G}_{m,k} \to Y$ on the k-scheme Y is said to be *diagonally monomial* (simply called *gdm* action) of weight n if the diagram

$$\begin{array}{ccc} Y \times_k \mathbf{G}_{m,k} & \xrightarrow{\ \sigma\ } & Y \\ \big\downarrow{\scriptstyle p\times e_n} & & \big\downarrow{\scriptstyle p} \\ S \times_k (\mathbf{G}_{m,k} \times_k \mathbf{G}_{m,k}) & \xrightarrow[m_S]{} & S \times_k \mathbf{G}_{m,k} \end{array} \tag{2.2}$$

is commutative. Let us stress that the action σ is *gdm* if and only if p becomes an equivariant morphism when $\mathbf{G}_{m,S}$ is endowed with the action $(S \times_k \mathbf{G}_{m,k}) \times_k \mathbf{G}_{m,k} \to S \times_k \mathbf{G}_{m,k}$ given by $(s,u,v) \to (s, uv^n)$. We denote by $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$ the category whose objects are the pairs $(p : Y \to \mathbf{G}_{m,S}, \sigma)$ where p is a $\mathbf{G}_{m,S}$-variety and σ is a *dgm* $\mathbf{G}_{m,k}$-action of weight n on Y. The morphisms in this category are the morphisms of $\mathbf{G}_{m,S}$-schemes which are equivariant with respect to the $\mathbf{G}_{m,k}$-action. Let us note that the structural morphism of $\mathbf{G}_{m,S}$ is $\mathrm{Id}_{\mathbf{G}_{m,S}}$ and the *gdm* $\mathbf{G}_{m,k}$-action is given by $\mathrm{Id}_S \times e_n$.

Since the field k is assumed to be of characteristic zero and algebraically closed, the k-group scheme $\boldsymbol{\mu}_{n,k}$ is identified with the finite group object in the category $\mathbf{Sch}_k$ associated with the finite group of the n-th roots of unity in k. Let us introduce the category $\mathbf{Var}^{\boldsymbol{\mu}_n}_S$ whose objects are the pairs $(p : Y \to S, \sigma)$ formed by a S-variety Y endowed with a good $\boldsymbol{\mu}_{n,k}$-action. The morphisms of this category are the morphisms of S-schemes which are equivariant with respect to the $\boldsymbol{\mu}_{n,k}$-action. We assume that S is endowed with the trivial action. By [17, Lemma 2.5] (see also [25]), we have the following link:

Proposition 2.1.1. *Let $n \in \mathbf{N}^\times$ be an integer. The categories $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$, $\mathbf{Var}^{\boldsymbol{\mu}_n}_S$ are equivalent.*

We refer to *loc. cit.* for the proof of proposition 2.1.1, but provide a quick description of the involved quasi-inverse functors which will be useful for our own purposes. With an object $(p\colon Y \to \mathbf{G}_{m,S}, \sigma)$ of the category $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$, the first functor associates the S-variety Y_1 defined by the following cartesian square:

$$\begin{array}{ccc} Y_1 & \dashrightarrow & Y \\ \vdots & \square & \downarrow \\ S & \xrightarrow[1_S]{} & \mathbf{G}_{m,S}. \end{array} \tag{2.3}$$

This S-variety Y_1 naturally carries on a good $\boldsymbol{\mu}_{n,k}$-action σ_1 since the $\mathbf{G}_{m,k}$-action is assumed to be good and monomially diagonal. The quasi-inverse functor is defined as follows. To every S-variety $p\colon Y \to S$, endowed with a good $\boldsymbol{\mu}_{n,k}$-action σ, one attaches the $\mathbf{G}_{m,S}$-variety $p \times e_n\colon Y \times_k \mathbf{G}_{m,k} \to \mathbf{G}_{m,S}$, with the $\boldsymbol{\mu}_{n,k}$-action given, for every k-algebra A, on the A-points by the map

$$Y(A) \times A^\times \times \boldsymbol{\mu}_{n,k}(A) \to Y(A) \times A^\times \tag{2.4}$$

defined by $(x, \lambda, \mu) \mapsto (x\mu, \lambda\mu^{-1})$. The required functor then associates with Y the $\mathbf{G}_{m,S}$-variety $Y \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k}$ defined to be the geometric quotient of $Y \times_k \mathbf{G}_{m,k}$ by the action (2.4). The action σ of $\mathbf{G}_{m,k}$ on $Y \times_k \mathbf{G}_{m,k}$ induces a *gdm* $\mathbf{G}_{m,k}$-action $\tilde{\sigma}$ on $Y \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k}$.

Remark 2.1.2. Let Y, X be schemes with a factorization $Y \to X \to Y$ of the identiy of Y. Then, if X is reduced, so is Y. Using this elementary remark and diagram (2.6), we observe that the scheme Y_1 contructed by diagram (2.3) is reduced when the $\mathbf{G}_{m,S}$-variety Y is assumed to be reduced. Conversely, [27, §2/(2)] implies that $Y \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k}$ is reduced if the S-scheme Y is reduced.

2.2. Examples. Given a S-variety Y, $g \in \mathscr{O}_Y(Y)^\times$ and $n \in \mathbf{N}^\times$, Ayoub defines the following $\mathbf{G}_{m,S}$-scheme $Q_n^{gm}(Y, g)$:

$$\mathsf{Spec}(\mathscr{O}_Y[T, T^{-1}, V]/\langle V^n - gT\rangle) \to \mathsf{Spec}(\mathscr{O}_S[T, T^{-1}]) = \mathbf{G}_{m,S}. \tag{2.5}$$

We can view this scheme as an object of the category $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$ when we endow it with the unique *gdm* $\mathbf{G}_{m,k}$-action σ of weight n defined, for every k-algebra A, on the A-points by the map

$$Q_n^{gm}(Y, g)(A) \times A^\times \to Q_n^{gm}(Y, g)(A)$$

which sends $((y, t, v), \alpha)$ to $(y, \alpha^n t, \alpha v)$ where $y \in Y(A)$, $\alpha \in A^\times$ and t, v are the images of T, V in $\mathscr{O}_Y[T, T^{-1}, V]/\langle V^n - gT\rangle$.

Let $p : Y \to \mathbf{G}_{m,S}$ be a morphism of k-schemes. Let us explain how to interpret the composed morphism (as in diagram (2.2))

$$Y \times_k \mathbf{G}_{m,k} \xrightarrow{p\times e_n} \mathbf{G}_{m,S} \times_k \mathbf{G}_{m,k} \xrightarrow{m_S} \mathbf{G}_{m,S}$$

in term of schemes of the form Q_n^{gm}. Let $g \in \mathscr{O}(Y)^\times$ be the unit on Y given by the inverse image of T^{-1} along the morphism

$$Y \to \mathbf{G}_{m,S} \to \mathbf{G}_{m,k} := \mathsf{Spec}(k[T,T^{-1}]).$$

Then, we observe that the morphism $m_S \circ (p \times e_n)$ coincides with the structural morphism

$$\begin{array}{c} Q_n^{gm}(Y,g) := \mathsf{Spec}(\mathscr{O}_Y[T,T^{-1},V]/\langle V^n - gT\rangle) \cong Y \times_k \mathbf{G}_{m,k} \\ \Big\downarrow{\scriptstyle \pi} \\ \mathsf{Spec}(\mathscr{O}_S[T,T^{-1}]) = \mathbf{G}_{m,S} \end{array}$$

of the $\mathbf{G}_{m,S}$-scheme $Q_n^{gm}(Y,g)$. In particular, if the $\mathbf{G}_{m,S}$-scheme Y is endowed with a *gdm* $\mathbf{G}_{m,k}$-action σ of weight n, then the morphism $p \circ \sigma (= m_S \circ (p \times e_n))$ equals the morphism π. Let us also stress that this morphism is smooth as soon as the morphism $\mathrm{pr}_1 \circ p\colon Y \to S$ is smooth. Furthermore, the action σ and the unit section induce a factorization of the identity of Y:

$$Y \xrightarrow{1_Y} Q_n^{gm}(Y,g) \xrightarrow{\sigma} Y. \tag{2.6}$$

We conclude this subsection by a technical result which will be crucial in the proof of theorem 5.3.1. Let $n \in \mathbf{N}^\times$ be an integer. Let $p : Y \to S$ be a S-variety, g be an element in $\mathscr{O}(Y)^\times$. We consider the pair $(Q_n^{\mu}(Y,g),\sigma)$ in $\mathbf{Var}_S^{\boldsymbol{\mu}_n}$ where the S-variety $Q_n^{\mu}(Y,g)$ equals $\mathsf{Spec}(\mathscr{O}_Y[U,U^{-1}]/\langle U^n - g\rangle)$ and σ is the good action of $\boldsymbol{\mu}_{n,k} = \mathsf{Spec}(k[T]/\langle T^n - 1\rangle)$ associated with the morphism of k-algebras defined by $U \mapsto TU$.

Lemma 2.2.1. *The object in* $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m,n}$ *associated with* $Q_n^{\mu}(Y,g)$ *is*

$$Q_n^{gm}(Y,g) : \mathsf{Spec}(\mathscr{O}_Y[T,T^{-1},V]/\langle V^n - gT\rangle) \to \mathsf{Spec}(\mathscr{O}_S[T,T^{-1}]).$$

Proof. This follows from the fact that $Q_n^{gm}(Y,g)$ is endowed with a canonical *gdm* $\mathbf{G}_{m,k}$-action for which the fiber over 1 is $(Q_n^{\mu}(Y,g),\sigma)$ and proposition 2.1.1. □

2.3. Equivariant compactifications. In the proof of theorem 3.4.1, we will use the following equivariant version of the Nagata compactification theorem.

Proposition 2.3.1. *Let* $n \in \mathbf{N}^\times$. *Let* $(p : Y \to \mathbf{G}_{m,S},\sigma)$ *be an object in* $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m,n}$. *Then, there exist an object* $(\overline{p} : \overline{Y} \to \mathbf{G}_{m,S},\overline{\sigma})$ *and an equivariant dense open immersion* $j : Y \hookrightarrow \overline{Y}$ *such that the morphism* $\overline{p}$ *is proper. Furthermore, if* S *is a* k*-variety and if the* k*-variety* Y *is smooth, we may assume that the* k*-variety* $\overline{Y}$ *also is smooth.*

Proof. Let G be finite (abstract) group and $\mathbf{G}$ its associated S-group scheme. Nagata's compactification theorem admits an $\mathbf{G}$-equivariant formulation as follows. Let us assume that $\mathbf{G}$ acts trivially on S. Let Y be a S-variety with a $\mathbf{G}$-action. By the Nagata compactification theorem, there exists a dense open immersion $j : Y \hookrightarrow \overline{X}$ into a proper S-variety $\overline{X}$. Consider the S-variety

$$P := \underbrace{\overline{X} \times_S \cdots \times_S \overline{X}}_{|G| \text{ terms}}$$

and the immersion $Y \hookrightarrow P$ given by the morphisms $j \circ g : Y \to \overline{X}$. Then, the closure $\overline{Y}$ of Y in P is a proper S-variety with an action of $\mathbf{G}$ for which the morphism $Y \hookrightarrow \overline{Y}$ is an equivariant dense open immersion.

By using this general remark, let us prove now the assertion of the proposition. Let Y_1 the fiber over 1 with its $\boldsymbol{\mu}_{n,k}$-action σ_1. By this remark, there exists an equivariant open immersion $j_1 : Y_1 \to \overline{Y}_1$ into a proper S-variety $\overline{p}_1 : \overline{Y}_1 \to S$ endowed with an action of $\boldsymbol{\mu}_{n,k}$. Let

$$\overline{Y} := \overline{Y}_1 \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k}$$

the associated $\mathbf{G}_{m,S}$-variety with a *gdm* $\mathbf{G}_{m,k}$-action. The morphism

$$j_1 \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k} : Y \to \overline{Y}$$

is a dense equivariant open immersion and $\overline{Y}$ is proper over $\mathbf{G}_{m,S}$. Indeed, we have the commutative diagram

$$\begin{array}{ccc} \overline{Y}_1 \times_k \mathbf{G}_{m,k} & \longrightarrow & \overline{Y} := \overline{Y}_1 \times_k^{\boldsymbol{\mu}_{n,k}} \mathbf{G}_{m,k} \\ & \searrow^{\overline{p}_1 \times_k e_n} & \downarrow \\ & & S \times_k \mathbf{G}_{m,k} \end{array}$$

and the result follows from [14, Corollaire (5.4.3)] since the quotient map is surjective. Now, if Y is smooth over k, using strong equivariant resolution of singularities as stated in [22, §3.3 and §3.4], we may also assume that $\overline{Y}$ is smooth over k. □

3. The categories of relative quasi-unipotent motives. In this section, we introduce the category $\mathbf{QUSH}_{\mathfrak{M}}(S)$ of quasi-unipotent motives over S. Our definition generalizes the one given in [5, Définition 1.3.25] when $S = \mathsf{Spec}(k)$. Our main statement in this section is theorem 3.4.1 which proves that the motive with compact support of a $\mathbf{G}_{m,S}$-variety endowed with a *gdm* $\mathbf{G}_m$-action of weight $n \in \mathbf{N}^\times$ is quasi-unipotent.

3.1. Quick review of the categories of motivic sheaves. In this article, we consider the category of motivic sheaves $\mathbf{SH}_{\mathfrak{M}}(S)$ that appears in [2, Définition 4.5.21] under the name $\mathbf{SH}^T_{\mathfrak{M}}(S)$, where T stands for a projective replacement of the presheaf

$$\frac{\mathbf{G}_{m,S} \otimes \mathbb{1}}{1_S \otimes \mathbb{1}}.$$

(The choice of T will not play any role in this article.) In the construction of $\mathbf{SH}_{\mathfrak{M}}(S)$, for which we refer to [2, §4.5], we choose as Grothendieck topology, either the Nisnevich topology or the étale topology. The stable (model) category $\mathfrak{M}$ will be either the category $\mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{\mathrm{op}}\mathcal{S}et_\bullet)$ of symmetric S^1-spectra or the category $\mathbf{Compl}(\Lambda)$ of complexes of Λ-modules over some ring Λ.

For $\mathfrak{M} = \mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{\mathrm{op}}\mathcal{S}et_\bullet)$ and the choice of the Nisnevich topology, the category obtained is the stable homotopy category of S-schemes of Morel–Voevodsky (see [21, 26, 29]). For $\mathfrak{M} = \mathbf{Compl}(\Lambda)$ and the choice of the étale topology, the category obtained is the category $\mathbf{DA}^{\text{ét}}(S, \Lambda)$ of étale motives with Λ-coefficients. The homological motive of a S-scheme of finite type $p : X \to S$ is the motive $p_! p^! \mathbb{1}_S$. If p is smooth, it is canonically isomorphic to $p_\sharp \mathbb{1}_X$.

The theory developed in [1, 2] provides the categories $\mathbf{SH}_{\mathfrak{M}}(-)$ with the six operations of Grothendieck and the formalism of nearby cycles. In particular, if $f\colon X \to \mathbf{A}^1_k$ is a morphism of k-varieties, with diagram (1.1), Ayoub associates a triangulated functor

$$\Psi_f : \mathbf{SH}_{\mathfrak{M}}(X_\eta) \to \mathbf{SH}_{\mathfrak{M}}(X_\sigma).$$

The object $\Psi_f(\mathbb{1}_{X_\eta}) \in \mathbf{SH}_{\mathfrak{M}}(X_\sigma)$ is called the *motivic nearby sheaf.*

Let us note that in $\mathbf{SH}_{\mathfrak{M}}(S)$, the category of compact objects coincides with the category of constructible motives introduced in [1, 2.2]. Moreover all the operations (the six operations and the nearby cycles functor) preserves constructible motives as shown in [1, Scholie 2.2.34, théorème 2.2.37] and [2, Théorème 3.5.14].

3.2. Quasi-unipotent motives. In this subsection, we define the categories of relative quasi-unipotent motives.

Let us recall some useful classical terminology. Let $\mathscr{T}$ be a triangulated category in which all (small) direct sums are representable. Let $\mathcal{E}$ be a set (or class of objects) in $\mathscr{T}$. We denote by $\langle\mathcal{E}\rangle$ (resp. $\langle\mathcal{E}\rangle^{\mathrm{ct}}$) the smallest full triangulated subcategory of $\mathscr{T}$ containing the objects in $\mathcal{E}$ (resp. and stable under direct factors). We denote by $\langle\langle\mathcal{E}\rangle\rangle$ the smallest full triangulated subcategory of $\mathscr{T}$ stable under all (small) direct sums and containing the objects in $\mathcal{E}$. Note that the category $\langle\langle\mathcal{E}\rangle\rangle$ is pseudo-abelian and thus stable by direct factors (see [1, Lemme 2.1.17]). We have the following inclusion

$$\langle\mathcal{E}\rangle \subseteq \langle\mathcal{E}\rangle^{\mathrm{ct}} \subseteq \langle\langle\mathcal{E}\rangle\rangle.$$

We denote by $\mathscr{T}_{\mathrm{comp}}$ the full subcategory of compact objects in $\mathscr{T}$. This is a triangulated subcategory of $\mathscr{T}$ stable by direct factors. Recall that $\mathscr{T}$ is said to be *compactly generated* if there exists a set $\mathcal{E}$ of compact objects in $\mathscr{T}$ such that $\mathscr{T} = \langle\langle\mathcal{E}\rangle\rangle$. In that case, we have $\mathscr{T}_{\mathrm{comp}} = \langle\mathcal{E}\rangle^{\mathrm{ct}}$ by [1, Proposition 2.1.24].

We denote $\mathcal{Q}_S$ the set of objects of $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$ of the form

$$\pi_\sharp \mathbb{1}_{Q_n^{gm}(Y,g)}(r) = \pi_! \pi^! \mathbb{1}_{\mathbf{G}_{m,S}}(r)$$

where Y is a smooth S-scheme of finite type, g is an element in $\mathscr{O}(Y)^\times$, $n \in \mathbf{N}^\times, r \in \mathbf{Z}$ are integers and $\pi : Q_n^{gm}(Y,g) \to \mathbf{G}_{m,S}$ is the structural morphism of the $\mathbf{G}_{m,S}$-scheme $Q_n^{gm}(Y,g)$. The category of quasi-unipotent motives over S is defined by

$$\mathbf{QUSH}_{\mathfrak{M}}(S) = \langle\langle\mathcal{Q}_S\rangle\rangle.$$

Hence, by definition, $\mathbf{QUSH}_{\mathfrak{M}}(S)$ is a full triangulated subcategory of $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$. Since the motives in $\mathcal{Q}_S$ are compact in $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$, the full subcategory of $\mathbf{QUSH}_{\mathfrak{M}}(S)$ formed by the compact objects is given by

$$\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S) = \mathbf{QUSH}_{\mathfrak{M}}(S) \cap \mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(\mathbf{G}_{m,S}).$$

When $S = \mathsf{Spec}(k)$, our definition coincides with the one given in [5, Définition 1.3.25] and used in [4, 6].

The following stability properties follows from the definition.

LEMMA 3.2.1. *Let S' be a separated noetherian schemes of finite Krull dimension and $f : S' \to S$ be a morphism of schemes.*

(1) If $B \in \mathbf{QUSH}_{\mathfrak{M}}(S)$, then $\breve{f}^*B$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S')$.

(2) If f is a smooth morphism and $A \in \mathbf{QUSH}_{\mathfrak{M}}(S')$, then $\breve{f}_\sharp A$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$.

Proof. (1) This is an easy consequence of base change (see [1, Définition 1.4.1]) and the fact that $\breve{f}^*$ commute with (small) direct sums (since it is a left adjoint).

(2) It follows from the definition. □

To prove that (2) in lemma 3.2.1 holds more generally, it will be useful to consider a bigger set of generators for the category $\mathbf{QUSH}_{\mathfrak{M}}(S)$. Consider the set $\mathcal{Q}'_S$ of objects of $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$ of the form

$$\breve{t}_!\pi_!\pi^!\mathbb{1}_{\mathbf{G}_{m,S'}}(r) = \breve{t}_!\pi_\sharp\mathbb{1}_{Q_n^{gm}(Y,g)}(r)$$

where S' is a subscheme of S and $t : S' \to S$ is the corresponding immersion, and $\pi : Q_n^{gm}(Y,g) \to \mathbf{G}_{m,S'}$ is the structural morphism of the $\mathbf{G}_{m,S'}$-scheme $Q_n^{gm}(Y,g)$ where Y is a smooth S'-scheme of finite type, g is an element in $\mathscr{O}(Y)^\times$ and $n \in \mathbf{N}^\times$, $r \in \mathbf{Z}$ are integers.

Lemma 3.2.2. *The set $\mathcal{Q}'_S$ generates the category $\mathbf{QUSH}_{\mathfrak{M}}(S)$ of quasi-unipotent motives, that is, $\langle\langle \mathcal{Q}_S \rangle\rangle = \langle\langle \mathcal{Q}'_S \rangle\rangle$.*

Proof. As $\mathcal{Q}_S \subseteq \mathcal{Q}'_S$, we only have to prove, with the above notation, that the motive $A := \breve{t}_!\pi_\sharp\mathbb{1}_{Q_n^{gm}(Y,g)}$ is quasi-unipotent. Since for an open immersion $\breve{t}_! = \breve{t}_\sharp$, we may assume without loss of generality that t is a closed immersion.

Let $(S_i)_{i\in I}$ be an affine open covering of S. Consider, for a nonempty subset $J \subseteq I$, the intersection $S_J := \cap_{i\in J} S_i$ and the corresponding open immersion $s_J : S_J \to S$. Then, $\breve{t}_!\pi_\sharp\mathbb{1}$ belongs to the triangulated category generated by the motives $(\breve{s_J})_\sharp(\breve{s_J})^*\breve{t}_!\pi_\sharp\mathbb{1}$ where $J \subseteq I$ is a nonempty subset (see [1, Lemme 2.2.13]). Consider the cartesian square

$$\begin{array}{ccc} S'_J & \xrightarrow{s'_J} & S' \\ \Big\downarrow{\scriptstyle t_J} & & \Big\downarrow{\scriptstyle t} \\ S_J & \xrightarrow{s_J} & S. \end{array} \qquad (3.1)$$

By applying the proper base change theorem [1, page 208], to the base change of (3.1) along the projection $\mathbf{G}_{m,k} \to \mathsf{Spec}(k)$, we get an isomorphism

$$(\breve{s_J})_\sharp(\breve{s_J})^*\breve{t}_!\pi_\sharp\mathbb{1} \simeq (\breve{s_J})_\sharp(\breve{t_J})_!(\breve{s'_J})^*\pi_\sharp\mathbb{1} \simeq (\breve{s_J})_\sharp(\breve{t_J})_!(\pi_J)_\sharp\mathbb{1}$$

where $Y_J := Y \times_{S'} S'_J$ and $\pi_J : Q_n^{gm}(Y_J, g) \to \mathbf{G}_{m,S_J}$ is the structural morphism. By lemma 3.2.1, it is enough to prove that $(\breve{t_J})_!(\pi_J)_\sharp\mathbb{1}$ is quasi-unipotent. Therefore, we may replace S by the affine open subset S_J and assume S affine.

Now consider an open covering $(Y_i)_{i\in I}$ of Y. Again consider, for a nonempty subset $J \subseteq I$, the intersection $Y_J := \cap_{i\in J} Y_i$ and the corresponding open immersion $u_J : Q_n^{gm}(Y_J, g) \to Q_n^{gm}(Y,g)$. Then, $\breve{t}_!\pi_\sharp\mathbb{1}$ belongs to the triangulated category generated by the motives

$$\breve{t}_!\pi_\sharp(u_J)_\sharp(u_J)^*\mathbb{1} \simeq \breve{t}(\pi_J)_\sharp\mathbb{1}$$

where $J \subseteq I$ is a nonempty subset and $\pi_J : Q_n^{gm}(Y_J, g) \to \mathbf{G}_{m,S'}$ is the structural morphism. Hence, we may replace Y by the affine open subset Y_J.

Therefore, we may assume that Y is affine, S is affine and that there exists an étale morphism $e : Y \to \mathbf{A}^m_{S'}$. Let $A' := \mathscr{O}(S')$. We may even assume that the morphism e is (isomorphic to) the one induced by the morphism of A-algebras

$$\Gamma(\mathbf{A}^m_{S'}, \mathscr{O}) := A'[T_1, \dots, T_m] \to (A'[T_1, \dots, T_n][T]/\langle P'\rangle)_{a'\cdot\partial_T P'\cdot Q'}$$

where $P', Q' \in A'[T_1, \dots, T_n, T]$, $\partial_T P'$ is the derivative of P' with respect to the variable T and a' is the leading coefficient of P'. Since $A' := \mathscr{O}(S')$ is a quotient of $A := \mathscr{O}(S)$, we can lift P', Q' to polynomials $P, Q \in A[T_1, \dots, T_n, T]$. Let a be the leading coefficient of P and consider the smooth S-scheme Z given by

$$\mathsf{Spec}(A[T_1, \dots, T_n][T]/\langle P\rangle)_{a\cdot\partial_T P\cdot Q}.$$

Then Z is a smooth S-scheme, such that $Z \times_S S' = Y$. The invertible function $g \in \mathscr{O}(Y)^\times$ can be lifted to a function $h \in \mathscr{O}(Z)$. By replacing Z with the open subset on which h is invertible, we may assume that $h \in \mathscr{O}(Z)^\times$.

Let U be the complement of S' in S and $Z|_U := Z \times_S U$. Let $h|_U$ be the restriction of h to the open subscheme $Z|_U$. Denote by $j : U \hookrightarrow S$ the open immersion and by $\pi_Z : Q^{gm}_n(Z, h) \to \mathbf{G}_{m,S}$ and $\pi_U : Q^{gm}_n(Z|_U, h|_U) \to \mathbf{G}_{m,S}$ the structural morphisms. By base change, the localization triangle

$$j_\sharp j^* \to \mathrm{Id} \to t_! t^* \xrightarrow{+1}$$

gives us an exact triangle

$$(\pi_U)_\sharp \mathbb{1}_{Q^{gm}_n(Z|_U, h|_U)} \to (\pi_Z)_\sharp \mathbb{1}_{Q^{gm}_n(Z,h)} \to t_! \pi_\sharp \mathbb{1}_{Q^{gm}_n(Y,g)} \xrightarrow{+1} .$$

This shows that $t_! \pi_\sharp \mathbb{1}_{Q^{gm}_n(Y,g)}$ is quasi-unipotent as claimed. □

Proposition 3.2.3. *Let S' be a separated noetherian schemes of finite Krull dimension and $f : S' \to S$ be a morphism of schemes. If $A \in \mathbf{QUSH}_{\mathfrak{M}}(S')$, then $\breve{f}_! A$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$.*

Proof. Note that $\breve{f}_!$ commute with (small) direct sums (since it is a left adjoint). By writing f as the composition of a closed immersion and a smooth morphism, we may consider separately the two cases. For a closed immersion, it is a consequence of lemma 3.2.2 since $\breve{t}_!$ maps $\mathcal{Q}_S$ to $\mathcal{Q}'_S$ by definition.

So we assume f smooth. In that case, there is a canonical isomorphism $\breve{f}_! A \simeq f_\sharp \mathsf{Th}^{-1}(\Omega_f) A$ where Ω_f is the locally free $\mathscr{O}$-module of relative differential. Let $(S'_i)_{i\in I}$ be an open covering of S'. Consider, for a nonempty subset $J \subseteq I$, the intersection $S'_J := \cap_{i\in J} S_i$ and the corresponding open immersion $s'_J : S'_J \to S'$. By [1, Lemme 2.2.13], $\breve{f}_! A$ belongs to the category generated by $\breve{f}_! (\breve{s}'_J)_! (\breve{s}'_J)^* A$ where $J \subseteq I$ is nonempty. Since $(\breve{s}'_J)^* A$ is quasi-unipotent by lemma 3.2.1, it is enough to prove the statement for $f \circ s'_J$. Therefore, we may assume that Ω_f is isomorphic to $\mathscr{O}^r_{S'}$ for some integer $r \geqslant 0$. In that case $\mathsf{Th}^{-1}(\Omega_f) A \simeq A(-r)[-2r]$ and we are reduce to show that $f_\sharp A$ is quasi-unipotent. Then, we may assume $A = \pi_\sharp \mathbb{1}_{Q^{gm}_n(Y,g)}$ where Y is a smooth S-scheme of finite type, $g \in \mathscr{O}(Y)^\times$ and $n \in \mathbf{N}^\times$. In that case, it follows from the definition. □

The above proposition has the following very useful corollary.

Corollary 3.2.4. *Let $z : Z \hookrightarrow S$ be a closed immersion and $u : U \hookrightarrow S$ be the open immersion of the complement. Then, an object $A \in \mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$ if and only if $\breve{u}^* A$ is in $\mathbf{QUSH}_{\mathfrak{M}}(U)$ and $\breve{z}^* A$ is in $\mathbf{QUSH}_{\mathfrak{M}}(Z)$.*

Proof. We have the localization triangle

$$\breve{u}_!\breve{u}^*A \to A \to \breve{z}_*\breve{z}^*A \xrightarrow{+1} .$$

Therefore, the result follows from lemma 3.2.3. □

3.3. Tensor product and weak tannakian formalism. As shown by the next proposition, quasi-unipotent motives are also stable under tensor product. This implies that $\mathbf{QUSH}_{\mathfrak{M}}(S)$ inherits also a symmetric monoidal structure from the one of $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,S})$ detailed in [1, 2].

PROPOSITION 3.3.1. *If A, B are in $\mathbf{QUSH}_{\mathfrak{M}}(S)$ then $A\otimes B$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$.*

Proof. Assume that Y is a smooth S-scheme of finite type, $g \in \mathscr{O}(Y)^\times$ is a unit and $n \in \mathbf{N}^\times$ is an integer. Let us first observe that, for any integer $d \geqslant 1$, the $\mathbf{G}_{m,S}$-scheme $Q_n^{gm}(Y,g)$ is a retract of $Q_{nd}^{gm}(Y[U,U^{-1}],U^n g)$. Indeed, the quasi-coherent $\mathscr{O}_Y[T,T^{-1}]$-algebra

$$\mathscr{A} = \frac{(\mathscr{O}_Y[U,U^{-1}])[T,T^{-1},R]}{\langle R^{nd} - U^n gT\rangle}$$

is isomorphic as a $\mathscr{O}_Y[T,T^{-1}]$-algebra to

$$\frac{\mathscr{O}_Y[U,U^{-1},T,T^{-1},R,V]}{\langle V^n - gT, R^d - UV\rangle}$$

and therefore to

$$\left(\frac{\mathscr{O}_Y[T,T^{-1},V]}{\langle V^n - gT\rangle}\right)[R,R^{-1}].$$

This shows our claim. This implies that the (homological) motive of $Q_n^{gm}(Y,g)$ is a direct factor of the (homological) motive of $Q_{nd}^{gm}(Y[U,U^{-1}],U^n g)$. Hence, to show the proposition we may first assume $A, B \in \mathcal{Q}_S$ and then, using the previous observation, we are reduce to showing that if Y, Z are smooth S-schemes of finite type, $n \in \mathbf{N}^\times$ and g, h are elements in $\mathscr{O}(Y)^\times$ and $\mathscr{O}(Z)^\times$ respectively, then the (homological) motive of the $\mathbf{G}_{m,S}$-scheme

$$Q_n^{gm}(Y,g) \times_{\mathbf{G}_{m,S}} Q_n^{gm}(Z,h) = \mathsf{Spec}\left(\frac{\mathscr{O}_{Y\times_S Z}[T,T^{-1},V,W]}{\langle V^n - gT, W^n - hT\rangle}\right)$$

is quasi-unipotent. For this, it suffices to remark that the $\mathscr{O}_Y[T,T^{-1}]$-algebras

$$\frac{\mathscr{O}_Y[T,T^{-1}][V,W]}{\langle V^n - gT, W^n - hT\rangle}, \quad \frac{\left(\frac{\mathscr{O}_Y[R]}{\langle gR^n - h\rangle}\right)[T,T^{-1},V]}{\langle V^n - gT\rangle}$$

are isomorphic. □

In [1, 2], Ayoub developed a weak tannakian formalism and applied it to construct motivic Galois groups or motivic fundamental groups. One of the key feature in this context of quasi-unipotent motives is [4, Proposition 2.10].

Let $(\mathscr{R}, \mathbf{\Delta} \times \mathbf{N}^\times)$ be the diagram of schemes over $\mathbf{G}_{m,k}$ indexed by the category $\mathbf{\Delta} \times \mathbf{N}^\times$ defined in [2, Definition 3.5.3]). Let us denote by

$$(\theta^{\mathscr{R}}, p_{\mathbf{\Delta}\times\mathbf{N}^\times}) : (\mathscr{R}, \mathbf{\Delta} \times \mathbf{N}^\times) \to \mathbf{G}_{m,k}$$

the structural morphism. Let $(\theta^{\mathscr{R}}_S, p_{\mathbf{\Delta}\times\mathbf{N}^\times}) : (\mathscr{R}_S, \mathbf{\Delta}\times\mathbf{N}^\times) \to \mathbf{G}_{m,S}$ be the diagram of schemes obtained by base change along the projection morphism $\mathrm{pr}_2 : \mathbf{G}_{m,S} = S \times_k \mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$. We set

$$\mathscr{U}_S := (p_{\mathbf{\Delta}\times\mathbf{N}^\times})_\sharp(\theta^{\mathscr{R}}_S)_*\mathbb{1}_{(\mathscr{R}_S,\mathbf{\Delta}\times\mathbf{N}^\times)}.$$

Note that by [1, Corollaire 2.4.21, Théorème 2.4.22], we have a canonical isomorphism $\mathrm{pr}_2^*\mathscr{U}_k \xrightarrow{\sim} \mathscr{U}_S$. Let $q_S : \mathbf{G}_{m,S} \to S$ be the projection.

Then, [4, Proposition 2.10] holds more generally with the same proof:

Proposition 3.3.2. *The functor*

$$\mathscr{U}_S \otimes q_S^* : \mathbf{SH}_{\mathfrak{M}}(S) \to \mathbf{QUSH}_{\mathfrak{M}}(S)$$

is a right adjoint to the functor $1_S^* : \mathbf{QUSH}_{\mathfrak{M}}(S) \to \mathbf{SH}_{\mathfrak{M}}(S)$.

3.4. Quasi-unipotence of some motives with compact support. Our main result in this section is

Theorem 3.4.1. *Let S be a k-variety. Let $(Y \xrightarrow{p} \mathbf{G}_{m,S}, \sigma)$ be an object in $\mathbf{Var}^{\mathbf{G}_{m},n}_{S\times\mathbf{G}_m}$. Then, the constructible motive $p_!\mathbb{1}_Y$ belongs to $\mathbf{QUSH}_{\mathfrak{M}}(S)$.*

To prove theorem 3.4.1, we will use a duality argument relying on the notion of strong dualizability in symmetric monoidal categories. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. Recall that an object $X \in \mathscr{C}$ is said to be *strongly dualizable* if there exist an object $X^\vee$ in $\mathscr{C}$ and morphisms

$$\mathbb{1} \xrightarrow{\mathrm{coev}} X^\vee \otimes X \qquad X \otimes X^\vee \xrightarrow{\mathrm{ev}} \mathbb{1} \tag{3.2}$$

such that the morphisms

$$X \xrightarrow{\mathrm{Id}\otimes\mathrm{coev}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{ev}\otimes\mathrm{Id}} X,\ X^\vee \xrightarrow{\mathrm{coev}\otimes\mathrm{Id}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{Id}\otimes\mathrm{ev}} X^\vee$$

are equal to the identity maps.

Remark 3.4.2. The property of being strongly dualizable is a property which holds at finite distance if and only if it holds in the limit. More precisely, let I be a small cofiltered category and $(\mathscr{C}_i)_{\in I}$ be an inductive system of symmetric monoidal categories $(\mathscr{C}_i, \otimes_i, \mathbb{1}_i)$ such that the transition functor $F_{j\to i} : \mathscr{C}_i \to \mathscr{C}_j$ is symmetric monoidal for every morphism $j \to i$ in I. Let $\mathscr{C}$ be the 2-colimit of the system with the induced symmetric monoidal structure. Given $i \in I$, let us denote by $F_{\infty\to i}$ the canonical functor $\mathscr{C}_i \to \mathscr{C}$ (which is symmetric monoidal). Let $i \in I$ and $A \in \mathscr{C}_i$. Then, $F_{\infty\to i}A$ is strongly dualizable in $\mathscr{C}$, if and only if there exists a morphism $j \to i$ such that $F_{j\to i}A$ is strongly dualizable in $\mathscr{C}_j$.

Let F be a field of characteristic zero. Recall the following lemma which is a direct consequence of [7, Lemma 4.10] and the equivalence of categories between $\mathbf{RigSH}_{\mathfrak{M}}(F(\!(t)\!))$ and $\mathbf{QUSH}_{\mathfrak{M}}(F)$ proved in [5, Scholie 1.3.26].

Lemma 3.4.3. *Let F be a field of characteristic zero. The objects in $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(F)$ are strongly dualizable in the symmetric monoidal category $\mathbf{QUSH}_{\mathfrak{M}}(F)$.*

In the relative setting, constructible quasi-unipotent motives will not be strongly dualizable in general. However, as the proposition below shows, they are still generically strongly dualizable.

PROPOSITION 3.4.4. *Let S be a integral separated k-scheme of finite and A be an object in $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)$. Then, there exists a dense open immersion $j : U \hookrightarrow S$ such that j^*A is strongly dualizable in $\mathbf{QUSH}_{\mathfrak{M}}(U)$. In particular, the motive $\underline{\mathsf{Hom}}(j^*A, \mathbb{1}_{\mathbf{G}_{m,U}})$ belongs to $\mathbf{QUSH}_{\mathfrak{M}}(U)$.*

The proof of the proposition relies on the following continuity property.

LEMMA 3.4.5. *Let I be a small cofiltered category. Let $(S_i)_{i\in I}$ be a pro-object in the category of schemes such that $f_{j\to i} : S_j \to S_i$ is an affine morphism for every morphism $j \to i$ in I. Let $S = \mathrm{Lim}_{i\in I} S_i$ be the projective limit in the category of schemes. We assume that S and S_i, for every $i \in I$ are noetherian separated of finite Krull dimension. Then, the category $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)$ is the 2-colimit of the categories $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S_i)$*

Proof. For $i \in I$, we denote by $f_{\infty\to i} : S \to S_i$ the canonical morphism. By [5, Proposition 1.A.1], it is enough to show that given a compact object A in $\mathbf{QUSH}_{\mathfrak{M}}(S)$, there exist $i \in I$ and a compact object $B \in \mathbf{QUSH}_{\mathfrak{M}}(S_i)$ such that $f^*_{\infty\to i}B$ is isomorphic to A. We may assume that A is in $\mathcal{Q}_S$. Let $t : T \to S$ be an immersion, Y be a smooth T-scheme of finite type, g be an element in $\mathscr{O}(Y)^\times$ and n be an element in $\mathbf{N}^\times$ such that $A = t_!p_\sharp\mathbb{1}$. By [15, Théorème (8.8.2), Proposition (8.10.5)] and [16, Proposition (17.7.8)], there exist $i \in I$, an immersion $T_i \to S_i$, a smooth T_i-scheme of finite type Y_i and a unit $g_i \in \mathscr{O}(Y_i)^\times$ such that $Y \to T \to S$ is obtained from $Y_i \to T_i \to S_i$ by pullback along the morphism $f_{\infty\to i} : S \to S_i$ and g is the image of g_i by the morphism $\mathscr{O}(Y_i)^\times \to \mathscr{O}(Y)^\times$. In particular, one has a cartesian square

$$\begin{array}{ccc} Q_n^{gm}(Y,g) & \longrightarrow & Q_n^{gm}(Y_i,g_i) \\ \downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle p_i} \\ \mathbf{G}_{m,T} & \xrightarrow[f_{\infty\to i}]{} & \mathbf{G}_{m,T_i}. \end{array}$$

Therefore $f^*_{\infty\to i}(t_i)_!(p_i)_\sharp\mathbb{1}$ is isomorphic to $A = t_!p_\sharp\mathbb{1}$. We may thus take $B = (t_i)_!(p_i)_\sharp\mathbb{1}$. □

Proof of proposition 3.4.4. Let I be the cofiltered category of nonempty affine open subschemes of S. Then $(U)_{U\in I}$ is a pro-object in the category schemes with affine transition morphisms and its projective limit is the spectrum of the function field F of S. Let $j_\infty : \mathsf{Spec}(F) \to S$ the canonical morphism. Since $j^*_\infty A$ is in $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(F)$, the result follows from lemma 3.4.3 and lemma 3.4.5 and the fact that the pullback along the transition morphisms are triangulated symmetric monoidal functors (see remark 3.4.2). □

The proof of theorem 3.4.1 relies on the following lemma.

LEMMA 3.4.6. *Let S be a connected smooth k-scheme of finite type. Let $(p : Y \to \mathbf{G}_{m,S}, \sigma)$ be an object in $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$ such that p is proper and Y is smooth over S. Then, there exists a dense open immersion $j : U \hookrightarrow S$ such that $j^*p_*\mathbb{1}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(U)$.*

Proof. In that case, the morphism $p_!\mathbb{1} \to p_*\mathbb{1}$ is an isomorphism. Since σ is a *gdm* $\mathbf{G}_m$-action of weight n, the morphism $p \circ \sigma$ coincides with the structural morphism $q : Q_n^{gm}(Y,g) \to \mathbf{G}_{m,S}$ where g is the image of $1/T$ by the morphism $k[T,T^{-1}] \to \mathscr{O}(Y)^\times$ induced by p (see subsection 2.1). Since 1_Y is a section of the morphism σ, the motive $p_*\mathbb{1}$ is a direct summand of the motive $q_*\mathbb{1}$ and it is sufficient to show that $q_*\mathbb{1}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$. By definition $q_\sharp\mathbb{1}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$. Since $q_\sharp\mathbb{1}$ is compact, by proposition 3.4.4, there exists a dense open immersion $j : U \hookrightarrow S$ such that $j^*q_\sharp\mathbb{1}$ is strongly dualizable in $\mathbf{QUSH}_{\mathfrak{M}}(U)$. In particular, $\underline{\mathsf{Hom}}(j^*q_\sharp\mathbb{1}, \mathbb{1})$ is in $\mathbf{QUSH}_{\mathfrak{M}}(U)$. Let $Y_U = Y \times_S U$ and denote by g_U the image of g by the morphism $\mathscr{O}(Y)^\times \to \mathscr{O}(Y_U)^\times$. We have a cartesian square

$$\begin{array}{ccc} Q_n^{gm}(Y_U, g_U) & \longrightarrow & Q_n^{gm}(Y, g) \\ \downarrow{\scriptstyle q_U} & \square & \downarrow{\scriptstyle q} \\ \mathbf{G}_{m,U} & \longrightarrow & \mathbf{G}_{m,S} \end{array}$$

where q_U is the structural morphism of the $\mathbf{G}_{m,U}$-scheme $Q_n^{gm}(Y_U, g_U)$. Hence $j^*q_\sharp\mathbb{1} \to (q_U)_\sharp\mathbb{1}$ is an isomorphism. Since S is smooth over k, U is regular and by [1, Proposition 2.3.52] and [1, Proposition 2.3.54] there is a canonical isomorphism

$$\underline{\mathsf{Hom}}((q_U)_\sharp\mathbb{1}, \mathbb{1}) \simeq (q_U)_*\mathbb{1}.$$

This implies that $j^*q_*\mathbb{1} \simeq (q_U)_*\mathbb{1}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(U)$ as desired. □

Proof of theorem 3.4.1. We may assume S reduced since the canonical functor

$$\mathbf{QUSH}_{\mathfrak{M}}(S_{\mathrm{red}}) \to \mathbf{QUSH}_{\mathfrak{M}}(S)$$

is an equivalence. By induction on the number of irreducible components of S, we may further assume S to be integral. Indeed assume that S is not irreducible and let $i : C \hookrightarrow S$ be the closed immersion of an irreductible component C of S. Denote by $j : U \hookrightarrow S$ the open immersion of the complement of C in S. Consider the cartesian squares

$$\begin{array}{ccccc} Y_U & \longrightarrow & Y & \longleftarrow & Y_C \\ \downarrow{\scriptstyle p_U} & \square & \downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle p_C} \\ U \times_k \mathbf{G}_{m,k} & \xrightarrow{\breve{j}} & S \times_k \mathbf{G}_{m,k} & \xleftarrow{\breve{i}} & C \times_k \mathbf{G}_{m,k}. \end{array}$$

By the proper base change theorem of [1], we have $\breve{i}^*p_!\mathbb{1}_Y \simeq (p_C)_!\mathbb{1}_{Y_C}$ and $\breve{j}^*p_!\mathbb{1}_Y \simeq (p_U)_!\mathbb{1}_{Y_U}$. Our claim follows from corollary 3.2.4.

It follows from lemma 3.4.6 that, if S has dimension zero over k, then $p_*\mathbb{1}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$ as soon as Y is smooth over S and p is proper. This is in particular the case if Y has dimension zero over k. We prove the theorem by induction on the dimension of S and Y over k.

Reduction A. Let us first start with the following observation: we may replace S by a dense open subscheme. In particular we may assume that S is smooth over k.

Let $u : U \hookrightarrow S$ be an open immersion and $z : Z \hookrightarrow S$ be the closed immersion of its complement. Consider the cartesian squares

$$\begin{array}{ccccc} Y_U & \longrightarrow & Y & \longleftarrow & Y_Z \\ \downarrow{\scriptstyle p_U} & \square & \downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle p_Z} \\ U \times_k \mathbf{G}_{m,k} & \xrightarrow{\breve{u}} & S \times_k \mathbf{G}_{m,k} & \xleftarrow{\breve{z}} & Z \times_k \mathbf{G}_{m,k}. \end{array}$$

By the proper base change theorem of [1], we have $\breve{z}^*p_!\mathbb{1}_Y \simeq (p_Z)_!\mathbb{1}_{Y_Z}$ and $\breve{u}^*p_!\mathbb{1}_Y \simeq (p_U)_!\mathbb{1}_{Y_U}$. If U is dense in S, then $\dim_k(Z) < \dim_k(S)$ and by induction $(p_Z)_!\mathbb{1}_{Y_Z}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(Z)$. Hence, by corollary 3.2.4, to show that $p_!\mathbb{1}_Y$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$, it is enough to show that $(p_U)_!\mathbb{1}_{Y_U}$ is in $\mathbf{QUSH}_{\mathfrak{M}}(U)$.

Reduction B. Let us now make a second observation: we may replace Y by a dense open subscheme stable under the action of σ. In particular we may assume that Y is smooth over k.

Let $u : U \hookrightarrow Y$ be the open immersion of a dense open subset U of Y stable under the action and smooth over k. Let Z be the closed complement of U in X and $z : Z \hookrightarrow Y$ the associated closed immersion. Then, Z is also stable under σ and both $(U \xrightarrow{p\circ u} \mathbf{G}_{m,k}, \sigma|_U)$ and $(Z \xrightarrow{p\circ z} \mathbf{G}_{m,k}, \sigma|_Z)$ belong to $\mathbf{Var}^{\mathbf{G}_m,n}_{S\times\mathbf{G}_m}$. The localization triangle of [1, Lemme 1.4.6] provides a distinguished triangle

$$(p\circ u)_!\mathbb{1}_U \to p_!\mathbb{1}_Y \to (p\circ z)_!\mathbb{1}_Z \xrightarrow{+1} .$$

Since $\dim(Z) < \dim(Y)$, by induction on the dimension of Y, $(p\circ z)_!\mathbb{1}_Z$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$. Hence, to show that $p_!\mathbb{1}_Y$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$ it is enough to show that $(p\circ u)_!\mathbb{1}_U$ is in $\mathbf{QUSH}_{\mathfrak{M}}(S)$.

Reduction of the general case to lemma 3.4.6. Using reduction B, we may assume that Y is smooth over k. By proposition 2.3.1, there exist an object $(\overline{p} : \overline{Y} \to \mathbf{G}_{m,S}, \overline{\sigma})$ and an equivariant open immersion $j : Y \hookrightarrow \overline{Y}$ such that $\overline{p}$ is proper and $\overline{Y}$ is smooth over k. By the generic smoothness theorem (see [18, III Corollary 10.7]), and reduction A, we may further assume that $\overline{Y}$ is smooth over S. In that case, the theorem follows from lemma 3.4.6 and reduction A. □

4. The monodromic nearby motivic sheaf functor. Let X be a smooth k-variety endowed with a morphism of k-schemes $f\colon X \to \mathbf{A}^1_k$. In this section, given a motive A on the generic fiber X_η, we prove that the motivic nearby sheaf $\Psi_f(A)$ defined by Ayoub in [2] can be canonically lifted to the category of quasi-unipotent motives $\mathbf{QUSH}_{\mathfrak{M}}(X_\sigma)$. We make precise and prove this assertion in theorem 4.1.1 and theorem 4.2.1.

4.1. Relation with the nearby motivic sheaf functor. The key ingredient of our approach is theorem 4.1.1, which is inspired from [28, Proposition 7.1]. Let $f : X \to \mathbf{A}^1_k$ be a morphism of k-varieties. We denote by $f^{\mathbf{G}_m} : \mathbf{G}_{m,X} \to \mathbf{A}^1_k$ the morphism obtained as the composition

$$X \times_k \mathbf{G}_{m,k} \xrightarrow{f\times_k \mathrm{Id}} \mathbf{A}^1_k \times_k \mathbf{G}_{m,k} \xrightarrow{m_k} \mathbf{A}^1_k,$$

that is the morphism $\mathsf{Spec}(\mathscr{O}_X[T,T^{-1}]) \to \mathsf{Spec}(k[t])$ defined by $t \mapsto Tf$. With this definition, diagram (1.1) admits the following factorization

$$\begin{array}{ccccc}
X_\sigma & \longrightarrow & X & \longleftarrow & X_\eta \\
\downarrow{\scriptstyle 1_{X_\sigma}} & \square & \downarrow{\scriptstyle 1_X} & \square & \downarrow{\scriptstyle 1_{X_\eta}} \\
\mathbf{G}_{m,X_\sigma} & \longrightarrow & \mathbf{G}_{m,X} & \longleftarrow & \mathbf{G}_{m,X_\eta} \\
\downarrow{\scriptstyle f^{\mathbf{G}_m}_\sigma} & \square & \downarrow{\scriptstyle f^{\mathbf{G}_m}} & \square & \downarrow{\scriptstyle f^{\mathbf{G}_m}_\eta} \\
\mathsf{Spec}(k) & \longrightarrow & \mathbf{A}^1_k & \longleftarrow & \mathbf{G}_{m,k}
\end{array}$$

and **(SPE2)** of [2, Définition 3.1.1] provides a canonical morphism, for every object B in $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,X_\eta})$,

$$1^*_{X_\sigma}\Psi_{f^{\mathbf{G}_m}}(B) \to \Psi_f(1^*_{X_\eta}B) \tag{4.1}$$

which is functorial in B. Let $p : \mathbf{G}_{m,X_\eta} \to X_\eta$ be the canonical projection. Defining the *monodromic nearby motivic sheaf functor* as the composition

$$\boxed{\Psi_f^{\mathrm{mon}} := \Psi_{f\mathbf{G}_m} p^*,}$$

we obtain from (4.1), for an object $A \in \mathbf{SH}_{\mathfrak{M}}(X_\eta)$ a canonical morphism

$$1^*_{X_\sigma} \Psi_f^{\mathrm{mon}}(A) \to \Psi_f(A).$$

Our first main result is the following theorem.

THEOREM 4.1.1. *For every object $A \in \mathbf{SH}_{\mathfrak{M}}(X_\eta)$, the canonical morphism*

$$1^*_{X_\sigma} \Psi_f^{\mathrm{mon}}(A) \to \Psi_f(A) \tag{4.2}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(X_\sigma)$.

To prove this theorem we will need two results taken from [1, 7, 8]. The first one is [1, Proposition 2.2.27] (see also [8, Lemma 3.3]) that we recall:

LEMMA 4.1.2. *Let S be a k-variety. The triangulated category $\mathbf{SH}_{\mathfrak{M}}(S)$ is generated by motives of the form $h_* \mathbb{1}_Y(r)$ where $h : Y \to S$ is a projective morphism, Y is smooth over k and $r \in \mathbf{Z}$ is an integer.*

Note that in [8, Lemma 3.3], the above lemma is stated for étale motives and proved for perfect field of arbitrary characteristic. In this generality, the proof uses de Jong's resolution of singularities by alteration. Using instead Hironaka's resolution of singularities in characteristic zero, there is no need for [1, Lemme 2.1.165] and the simplified proof carries out also not only for étale motives but more generally in $\mathbf{SH}_{\mathfrak{M}}(S)$.

The second one is the following reformulation of [7, Theorem 6.1] (see the end of the introduction for the notation):

THEOREM 4.1.3. *Assume that X_σ is a strict normal crossing divisor in X and let $(D_i)_{i\in I}$ be its set of irreducible components. Denote by $D_J^\circ \xrightarrow{v_J} D_J \xrightarrow{u_J} X_\sigma$ the corresponding open and closed immersions. Then, for every nonempty subset $J \subseteq I$, the canonical morphism*

$$(v_J)_!(v_J)^!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \to (u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \tag{4.3}$$

is an isomorphism.

Proof. Consider the duality functors

$$\mathsf{D}_\sigma(-) := \underline{\mathsf{Hom}}(-, f_\sigma^! \mathbb{1}_k) \quad \mathsf{D}_\eta(-) := \underline{\mathsf{Hom}}(-, f_\eta^! \mathbb{1}_{\mathbf{G}_{m,k}}).$$

Then, it is enough to prove that the image of (4.3) by D_σ is an isomorphism. Using [1, Théorème 2.3.75], we have to prove that

$$(u_J)^*\mathsf{D}_\sigma\Psi_f(\mathbb{1}_{X_\eta}) \to (v_J)_*(v_J)^*(u_J)^*\mathsf{D}_\sigma\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism. By [2, Théorème 3.5.20], we have a canonical isomorphism $\Psi_f(\mathsf{D}_\eta\mathbb{1}_{X_\eta}) \simeq \mathsf{D}_\sigma\Psi_f(\mathbb{1}_{X_\eta})$. Hence, we have to prove that

$$(u_J)^*\Psi_f(\mathsf{D}_\eta\mathbb{1}_{X_\eta}) \to (v_J)_*(v_J)^*(u_J)^*\Psi_f(\mathsf{D}_\eta\mathbb{1}_{X_\eta})$$

is an isomorphism. Let $a : X \to \mathsf{Spec}(k)$, $p : \mathbf{G}_{m,k} \to \mathsf{Spec}(k)$ be the structural morphisms and $j : X_\eta \to X$ be the open immersion. Since p is smooth and the $\mathscr{O}$-module of relative differentials Ω_p is free, by definition (see [1, §1.5.3.1]), we have $p^!\mathbb{1}_k = \mathsf{Th}(\Omega_p)p^*\mathbb{1}_k \simeq \mathbb{1}_{\mathbf{G}_{m,k}}(1)[2]$. Hence

$$f_\eta^!\mathbb{1}_{\mathbf{G}_{m,k}} \simeq f_\eta^! p^! \mathbb{1}_k(-1)[-2] \simeq j^! a^! \mathbb{1}_k(-1)[-2] \simeq j^* a^! \mathbb{1}_k(-1)[-2]$$

since, j being an open immersion, we have $j^! = j^*$. Since a is smooth, by definition (see [1, §1.5.3.1]), $a^! := \mathsf{Th}(\Omega_a)a^*$ where Ω_a is the locally free $\mathscr{O}_X$-module of relative differentials. The assertion being local on X for the Zariski topology, we may assume that Ω_a free of rank d. In that case, we obtain an isomorphism $f_\eta^!\mathbb{1}_{\mathbf{G}_{m,k}} \simeq \mathbb{1}_{X_\eta}(r)[2r]$ where $r := d-1$. This shows that

$$\mathsf{D}_\eta \mathbb{1}_{X_\eta} \simeq \underline{\mathsf{Hom}}(\mathbb{1}_{X_\eta}, \mathbb{1}_{X_\eta}(r)[2r]) \simeq f_\eta^* A$$

where $A = \mathbb{1}_k(r)[2r]$. Then, the statement follows from [7, Theorem 6.1]. □

Proof of theorem 4.1.1. We divide the proof in several steps.

Step one. Let us first show that we may assume $A = \mathbb{1}_{X_\eta}$. Observe that, if (4.2) in an isomorphism for some object A, then, it is an isomorphism for any Tate twist $A(n)$, $n \in \mathbf{Z}$, of A. By lemma 4.1.2 the triangulated category $\mathbf{SH}_{\mathfrak{M}}(X)$ is generated by the Tate twists of motives of the form $h_*\mathbb{1}_Y$ where $h : Y \to X$ is a projective morphism and Y is smooth over k.

Let $j : X_\eta \to X$ be the open immersion. Since the counit $j^*j_* \to \mathrm{Id}$ is invertible, the triangulated category $\mathbf{SH}_{\mathfrak{M}}(X_\eta)$ is generated by the Tate twists of motives of the form $j^*h_*\mathbb{1}_Y \simeq (h_\eta)_*\mathbb{1}_{Y_\eta}$.

Hence, we may assume $A = (h_\eta)_*\mathbb{1}_{Y_\eta}$. Then, $p^*(h_\eta)_*\mathbb{1}_{Y_\eta} \simeq (\breve{h}_\eta)_*\mathbb{1}_{\mathbf{G}_{m,Y_\eta}}$ and we consider the commutative diagram

$$\begin{array}{ccc} 1_{X_\sigma}^*\Psi_f^{\mathrm{mon}}((h_\eta)_*\mathbb{1}_{Y_\eta}) & \longrightarrow & \Psi_f((h_\eta)_*\mathbb{1}_{Y_\eta}) \\ \downarrow & & \\ 1_{X_\sigma}^*(\breve{h}_\sigma)_*\Psi_{f\circ h}^{\mathrm{mon}}(\mathbb{1}_{Y_\eta}) & & \downarrow \\ \downarrow & & \\ (h_\sigma)_*1_{Y_\sigma}^*\Psi_{f\circ h}^{\mathrm{mon}}(\mathbb{1}_{Y_\eta}) & \longrightarrow & (h_\sigma)_*\Psi_{f\circ h}(\mathbb{1}_{Y_\eta}). \end{array} \tag{4.4}$$

The property **SPE2** of [2, Définition 3.1.1], applied to the morphism h, ensures that the vertical morphism on the right in (4.4) is an isomorphism. Similarly, by the proper base change theorem [1, page 208], applied to the cartesian square

$$\begin{array}{ccc} Y_\sigma & \xrightarrow{1_{Y_\sigma}} & \mathbf{G}_{m,Y_\sigma} \\ \downarrow{\scriptstyle h_\sigma} & & \downarrow{\scriptstyle \breve{h}_\sigma} \\ X_\sigma & \xrightarrow{1_{X_\sigma}} & \mathbf{G}_{m,X_\sigma} \end{array}$$

and the property **SPE2** of [2, Définition 3.1.1], applied to the morphism $\breve{h}$, the vertical morphism on the left in (4.4) is also an isomorphism. Hence, it is enough to show that

$$1_{Y_\sigma}^*\Psi_{f\circ h}^{\mathrm{mon}}(\mathbb{1}_{Y_\eta}) \to \Psi_{f\circ h}(\mathbb{1}_{Y_\eta})$$

is an isomorphism. Therefore, we may assume $A = \mathbb{1}_{X_\eta}$.

Step two. By applying Hironaka's resolution of singularities [19] (see also [13, Grand théorème]) to the pair (X, X_σ), we may find a smooth k-scheme Y and and proper morphism $h : Y \to X$ such that the base change h_η of h to X_η is an isomorphism and Y_σ is a strict normal crossings divisor in Y. Then, $\mathbb{1}_{X_\eta} \simeq (h_\eta)_*\mathbb{1}_{Y_\eta}$ and by considering (4.4) and the same arguments as in step one, we may replace f by $f \circ h$ and assume that X_σ is a strict normal crossing divisor in X.

Step three. From now on, we assume that X_σ is a strict normal crossing divisor. Let I be the set of irreducible components of X_σ and D_i be the irreducible components associated with $i \in I$. We use the conventions in the introduction and denote by $D_J^\circ \xrightarrow{v_J} D_J \xrightarrow{u_J} X_\sigma$ the corresponding open and closed immersions. Since the counit $1_{X_\sigma}^*(1_{X_\sigma})_* \to \mathrm{Id}$ is invertible, we may as well prove that the image by the functor $1_{X_\sigma}^*$ of the morphism

$$\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (1_{X_\sigma})_*\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism. Let C be a cone of this morphism in $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,X_\sigma})$. Then, using the Mayer-Vietoris property, we see that C belongs to the triangulated subcategory generated by the motives $(\breve{u}_J)_!(\breve{u}_J)^!C$, for $J \subseteq I$ nonempty (see [1, Lemme 2.2.31]). Hence, $1_{X_\sigma}^*C$ belongs to the triangulated subcategory generated by the motives $1_{X_\sigma}^*(\breve{u}_J)_!(\breve{u}_J)^!C$, for $J \subseteq I$ nonempty, and we are reduce to prove that the image by the functor $1_{X_\sigma}^*$ of the morphism

$$(\breve{u}_J)_!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (\breve{u}_J)_!(\breve{u}_J)^!(1_{X_\sigma})_*\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism for any nonempty subset $J \subseteq I$. By the smooth base change theorem, applied to the cartesian square

$$\begin{array}{ccc} D_J & \xrightarrow{1_{D_J}} & \mathbf{G}_{m,D_J} \\ \downarrow{\scriptstyle u_J} & & \downarrow{\scriptstyle \breve{u}_J} \\ X_\sigma & \xrightarrow{1_{X_\sigma}} & \mathbf{G}_{m,X_\sigma} \end{array} \tag{4.5}$$

the canonical morphism $(\breve{u}_J)^!(1_{X_\sigma})_* \to (1_{D_J})_*(u_J)^!$ is invertible. Since both u_J and $\breve{u}_J$ are closed immersions, we have $(\breve{u}_J)_! = (\breve{u}_J)_*$ and $(u_J)_! = (u_J)_*$ and by functoriality, we have to prove that the image by the functor $1_{X_\sigma}^*$ of the morphism

$$(\breve{u}_J)_!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (1_{X_\sigma})_*(u_J)_!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism, that is we have to prove that

$$(1_{X_\sigma})^*(\breve{u}_J)_!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (u_J)_!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism. This morphism admits a factorization

$$\begin{array}{ccc} (1_{X_\sigma})^*(\breve{u}_J)_!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) & \longrightarrow & (u_J)_!(1_{D_J})^*(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \\ & \searrow & \downarrow \\ & & (u_J)_!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \end{array}$$

in which the horizontal morphism is an isomorphism by the proper base change theorem applied to the cartesian square (4.5). Hence, it is enough to prove that for a nonempty subset $J \subseteq I$, the morphism

$$(1_{D_J})^*(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (u_J)^!\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism.

Step four. Let us show that we may assume $|J| = 1$. Suppose $|J| \geqslant 2$. Consider the blow-up $\varepsilon(J) : X(J) \to X$ in X with center D_J and denote by $E(J)$ its exceptional divisor. Its special fiber $X(J)_\sigma$ is again a strict normal crossing divisor of $X(J)$, whose (reduced) irreducible components are exactly the closed subscheme $E(J)$ and the strict transforms of D_i, for all $i \in I$.

Set $f(J) := f \circ \varepsilon(J)$ and let $e : E(J) \to X(J)_\sigma$ be the closed immersion. We have a canonical commutative diagram

$$\begin{array}{ccc}
(1_{D_J})^*(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) & \longrightarrow & (u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \\
\downarrow & & \downarrow \\
(1_{D_J})^*(\breve{u_J})^!(\breve{\varepsilon(J)}_\sigma)_*\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) & & (u_J)^!(\varepsilon(J)_\sigma)_*\Psi_{f(J)}(\mathbb{1}_{X(J)_\eta}) \\
\downarrow & & \downarrow \\
(1_{D_J})^*(\breve{\varepsilon(J)}_\sigma)_*\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) & & \\
\downarrow & & \\
(\varepsilon(J)_\sigma)_*(1_{E(J)})^*\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) & \longrightarrow & (\varepsilon(J)_\sigma)_*e^!\Psi_{f(J)}(\mathbb{1}_{X(J)_\eta}).
\end{array} \tag{4.6}$$

The property **SPE2** of [2, Définition 3.1.1], applied to the morphism $\varepsilon(J)$, and the smooth base change theorem, applied to the cartesian square

$$\begin{array}{ccc}
E(J) & \xrightarrow{e} & X(J)_\sigma \\
{\scriptstyle \varepsilon(J)_\sigma}\downarrow & \square & \downarrow{\scriptstyle \varepsilon(J)_\sigma} \\
D_J & \xrightarrow{u_J} & X_\sigma,
\end{array} \tag{4.7}$$

ensure that the vertical morphism on the right in (4.6) is an isomorphism. Similarly, using the property **SPE2** of [2, Définition 3.1.1], applied to the morphism $\breve{\varepsilon(J)}$, the smooth base change theorem applied to the base change of (4.7) along $\mathbf{G}_{m,k} \to \mathsf{Spec}\, k$ and the proper base change theorem [1, page 208] applied to the cartesian squares (4.7) and

$$\begin{array}{ccc}
E(J) & \xrightarrow{1_{E(J)}} & \mathbf{G}_{m,E(J)} \\
\downarrow{\scriptstyle \varepsilon(J)_\sigma} & & \downarrow{\scriptstyle \breve{\varepsilon(J)}_\sigma} \\
D_J & \xrightarrow{1_{D_J}} & \mathbf{G}_{m,D_J},
\end{array}$$

we see that the vertical morphism on the left in (4.6) is an isomorphism. Hence, it is enough to show that

$$1_{E(J)}^*\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) \to e^!\Psi_{f(J)}(\mathbb{1}_{X(J)_\eta})$$

is an isomorphism. We may therefore assume $|J| = 1$.

Final step. Consider the commutative square

$$\begin{array}{ccc} (1_{D_J})^*(\breve{v}_J)_!(\breve{v}_J)^!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{\mathbf{G}_{m,X_\eta}}) & \longrightarrow & (v_J)_!(v_J)^!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \\ \downarrow & & \| \\ (v_J)_!(v_J)^!(1_{D_J})^*(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{\mathbf{G}_{m,X_\eta}}) & \longrightarrow & (v_J)_!(v_J)^!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta}) \\ \downarrow & & \downarrow \\ (1_{D_J})^*(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{\mathbf{G}_{m,X_\eta}}) & \longrightarrow & (u_J)^!\Psi_f(\mathbb{1}_{X_\eta}). \end{array} \tag{4.8}$$

By theorem 4.1.3, the vertical morphism on the right in (4.8) is an isomorphism. Since both v_J and $\breve{v}_J$ are open immersions, we have $(\breve{v}_J)^! = (\breve{v}_J)^*$ and $(v_J)^! = (v_J)^*$. By theorem 4.1.3, and the proper base change theorem [1, page 208] applied to the cartesian square

$$\begin{array}{ccc} D_J^\circ & \xrightarrow{1_{D_J^\circ}} & \mathbf{G}_{m,D_J^\circ} \\ \downarrow{\scriptstyle v_J} & & \downarrow{\scriptstyle \breve{v}_J} \\ D_J & \xrightarrow{1_{D_J}} & \mathbf{G}_{m,D_J} \end{array}$$

we see that the vertical morphism on the left in (4.8) is also an isomorphism. Hence, it remains to show that

$$(1_{D_J})^*(\breve{v}_J)_!(\breve{v}_J)^!(\breve{u}_J)^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (v_J)_!(v_J)^!(u_J)^!\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism.

Let U be the open subscheme of X complement of $D(I \setminus J)$ and $u : U \hookrightarrow X$ be the corresponding open immersion. Note that $(U_\sigma)_{\mathrm{red}} = D_J^\circ$ and the morphism $u_J \circ v_J$ is equal to

$$D_J^\circ \xrightarrow{r} U_\sigma \xrightarrow{u_\sigma} X_\sigma$$

where r is the canonical morphism. Since $u_J \circ v_J$ is an open immersion, we want to show that the morphism

$$(1_{D_J})^*(\breve{v}_J)_!\breve{r}^*(\breve{u}_\sigma)^*\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to (v_J)_!r^*(u_\sigma)^*\Psi_f(\mathbb{1}_{X_\eta})$$

is an isomorphism. Let $f|_U := f \circ u$ be the restriction of f to the open subscheme U. By applying the property **SPE2** of [2, Définition 3.1.1], to the open immersions $\breve{u}$ and u, we see that the canonical morphisms

$$(\breve{u}_\sigma)^*\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to \Psi_{f|_U}^{\mathrm{mon}}(\mathbb{1}_{U_\eta})$$

and

$$u_\sigma^*\Psi_f(\mathbb{1}_{X_\eta}) \to \Psi_{f|_U}(\mathbb{1}_{U_\eta})$$

are isomorphisms. Since the assertion is local on X_σ for the Zariski topology, we may further assume that $f|_U = v\tau^m$ where $\tau \in \mathscr{O}(U)$ is a generator of the ideal of definition of $D_J^\circ = D_J \cap U$, $v \in \mathscr{O}(U)^\times$ is a unit and $m \geqslant 1$ is an integer. Let $v_0 \in \mathscr{O}(D_i^\circ)$ be the restriction of v and consider the following finite étale covers

$$r_m : D_m := \mathsf{Spec}(\mathscr{O}_{D_J^\circ}[S]/\langle S^m - v_0\rangle) \to D_J^\circ$$

and

$$r_m^{\mathrm{mon}} : Q_m^{gm}(D_J^\circ, v_0) = \mathsf{Spec}(\mathscr{O}_{D_J^\circ}[T, T^{-1}, S]/\langle S^m - v_0 T\rangle) \to \mathsf{Spec}\,\mathscr{O}_{D_J^\circ}[T, T^{-1}] = \mathbf{G}_{m, D_J^\circ}.$$

By [7, Proposition 3.4] (see also [3, Théorème 10.6]), there are canonical isomorphisms

$$\breve{r}^*\Psi^{\mathrm{mon}}_{f|_U}(\mathbb{1}_{U_\eta}) \simeq (r_m^{\mathrm{mon}})_*\mathbb{1}_{Q_m^{gm}(D_J^\circ, v_0)}, \quad r^*\Psi_{f|_U}(\mathbb{1}_{U_\eta}) \simeq (r_m)_*\mathbb{1}_{D_m}.$$

Hence, we have to show that the morphism

$$1^*_{D_J}(\breve{v_J})_!(r_m^{\mathrm{mon}})_*\mathbb{1}_{Q_m^{gm}(D_J^\circ, v_0)} \to (v_J)_!(r_m)_*\mathbb{1}_{D_m}$$

is an isomorphism. This morphism admits a canonical factorization

$$\begin{array}{ccc} 1^*_{D_J}(\breve{v_J})_!(r_m^{\mathrm{mon}})_*\mathbb{1}_{Q_m^{gm}(D_J^\circ, v_0)} & \longrightarrow & (v_J)_!(r_m)_*\mathbb{1}_{D_m} \\ \big\downarrow & & \big\| \\ (v_J)_! 1^*_{D_J^\circ}(r_m^{\mathrm{mon}})_*\mathbb{1}_{Q_m^{gm}(D_J^\circ, v_0)} & \longrightarrow & (v_J)_!(r_m)_*\mathbb{1}_{D_m}. \end{array} \tag{4.9}$$

The proper base change theorem applied to the cartesian squares

$$\begin{array}{ccc} D_m & \longrightarrow & Q_m^{gm}(D_J^\circ, v_0) \\ \big\downarrow{\scriptstyle r_m} & \square & \big\downarrow{\scriptstyle r_m^{\mathrm{mon}}} \\ D_i^\circ & \xrightarrow{1_{D_J^\circ}} & \mathbf{G}_{m, D_J^\circ} \end{array} \qquad \begin{array}{ccc} D_J^\circ & \xrightarrow{1_{D_J^\circ}} & \mathbf{G}_{m, D_J^\circ} \\ \big\downarrow{\scriptstyle v_J} & & \big\downarrow{\scriptstyle \breve{v_J}} \\ D_J & \xrightarrow{1_{D_J}} & \mathbf{G}_{m, D_J} \end{array}$$

ensures that the lower horizontal morphism and the vertical morphism on the left in (4.9) are isomorphisms. This concludes the proof. □

4.2. Quasi-unipotence and monodromic nearby motivic sheaf functor. Let us prove now our second main theorem, which asserts that the monodromic nearby motivic sheaf functor produces quasi-unipotent motives.

THEOREM 4.2.1. *For every object $A \in \mathbf{SH}_{\mathfrak{M}}(X_\eta)$, the motive $\Psi_f^{\mathrm{mon}}(A)$ is quasi-unipotent, that is, belongs to the category $\mathbf{QUSH}_{\mathfrak{M}}(X_\sigma)$.*

Proof. As in the proof of theorem 4.1.1, we may assume that $A = (h_\eta)_*\mathbb{1}_{Y_\eta}$ where $h : Y \to X$ is a projective morphism and Y is smooth over k. Then, the canonical morphism $\Psi_f^{\mathrm{mon}}((h_\eta)_*\mathbb{1}_{Y_\eta}) \to (\breve{h}_\sigma)_*\Psi^{\mathrm{mon}}_{f\circ h}(\mathbb{1}_{Y_\eta})$ is an isomorphism by the property **SPE2** of [2, Définition 3.1.1]. Hence, by proposition 3.2.3, it is enough to prove that $\Psi_{f\circ h}(\mathbb{1}_{Y_\eta})$ is quasi-unipotent and we may assume $A = \mathbb{1}_{X_\eta}$.

Similarly by considering a log-resolution of singularities $h : Y \to X$ of the pair (X, X_σ), we may further assume that X_σ is a strict normal crossing divisor. Let I be the set of irreducible components of X_σ and D_i be the irreducible components associated with $i \in I$. We use the conventions in the introduction and denote by $D_J^\circ \xrightarrow{v_J} D_J \xrightarrow{u_J} X_\sigma$ the corresponding open and closed immersions. Using the Mayer-Vietoris property, we see that $\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$ belongs to the triangulated subcategory generated by the motives $(\breve{u_J})_!(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$, for $J \subseteq I$ nonempty (see [1, Lemme 2.2.31]). Hence, it is enough to prove that these motives are quasi-unipotent.

Let us show that we may assume $|J| = 1$. Suppose $|J| \geqslant 2$. Consider the blow-up $\varepsilon(J) : X(J) \to X$ in X with center D_J and denote by $E(J)$ its exceptional divisor. Its special fiber $X(J)_\sigma$ is again a strict normal crossing divisor of $X(J)$, whose (reduced) irreducible components are exactly the closed subscheme $E(J)$ and the strict transforms of D_i, for all $i \in I$. Set $f(J) := f \circ \varepsilon(J)$ and let $e : E(J) \to X(J)_\sigma$ be the closed immersion. We have a canonical morphism

$$\begin{array}{ccc}
(\breve{u_J})_!(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) & \longrightarrow & (\breve{u_J})_!(\breve{u_J})^!(\varepsilon(\breve{J})_\sigma)_*\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) \\
 & & \downarrow \\
 & & (\breve{u_J})_!(\varepsilon(\breve{J})_\sigma)_*\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta}) \\
 & & \downarrow \simeq \\
 & & (\varepsilon(\breve{J})_\sigma)_*\breve{e}_!\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta})
\end{array}$$

which is an isomorphism, by the property **SPE2** of [2, Définition 3.1.1] applied to the morphism $\varepsilon(J)$, and the smooth base change theorem, applied to the base change of (4.7) along $\mathbf{G}_{m,k} \to \mathsf{Spec}\, k$. Hence, by proposition 3.2.3, it is enough to prove that $\breve{e}_!\breve{e}^!\Psi_{f(J)}^{\mathrm{mon}}(\mathbb{1}_{X(J)_\eta})$ is quasi-unipotent and we may assume $|J| = 1$.

By theorem 4.1.3 and proposition 3.2.3, to prove that $(\breve{u_J})_!(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$ is quasi-unipotent, it suffices to show that $(\breve{v_J})^!(\breve{u_J})^!\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$ is quasi-unipotent. Let U be the open subscheme of X complement of $D(I \setminus J)$ and $u : U \hookrightarrow X$ be the corresponding open immersion. Note that $(U_\sigma)_{\mathrm{red}} = D_J^\circ$ and the morphism $u_J \circ v_J$ is equal to

$$D_J^\circ \xrightarrow{r} U_\sigma \xrightarrow{u_\sigma} X_\sigma$$

where r is the canonical morphism. Since $u_J \circ v_J$ is an open immersion, we want to show that $r^*(\breve{u_\sigma})^*\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})$ is quasi-unipotent. Let $f|_U := f \circ u$ be the restriction of f to the open subscheme U. By applying the property **SPE2** of [2, Définition 3.1.1], to the open immersion $\breve{u}$, we see that the canonical morphism

$$(\breve{u}_\sigma)^*\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta}) \to \Psi_{f|_U}^{\mathrm{mon}}(\mathbb{1}_{U_\eta})$$

is an isomorphism. Since the assertion is local on X_σ for the Zariski topology, we may further assume that $f|_U = v\tau^m$ where $\tau \in \mathscr{O}(U)$ is a generator of the ideal of definition of $D_J^\circ = D_J \cap U$, $v \in \mathscr{O}(U)^\times$ is a unit and $m \geqslant 1$ is an integer. Let $v_0 \in \mathscr{O}(D_i^\circ)$ be the restriction of v and consider the following finite étale cover $r_m^{\mathrm{mon}} : Q_m^{gm}(D_J^\circ, v_0) \to \mathbf{G}_{m,D_J^\circ}$. By [7, Proposition 3.4] (see also [3, Théorème 10.6]), there is a canonical isomorphism

$$\breve{r}^*\Psi_{f|_U}^{\mathrm{mon}}(\mathbb{1}_{U_\eta}) \simeq (r_m^{\mathrm{mon}})_*\mathbb{1}_{Q_m^{gm}(D_J^\circ, v_0)}.$$

This concludes the proof since this motive is quasi-unipotent by definition. □

5. Monodromic Grothendieck rings and quasi-unipotent motives. In this section, we compare the theory of motivic nearby sheaves, as introduced in [2,

§3.5], with that of motivic nearby cycles studied in [9, 10, 11, 23, 24]. Proposition 5.2.1 extends [7, Lemma 8.5] to the monodromic context; its corollary gives rise to a complete picture of the relations between the different motivic Euler characteristics with compact support. In the end, theorem 5.3.1 provides a monodromic version of the comparison of nearby cycles theories, which generalizes [7, Corollary 8.7].

5.1. Monodromic Grothendieck rings of varieties. Let us recall two different variants of monodromic Grothendieck rings of varieties employed in [17, §2.3]. We refer to *loc. cit.* for details. Let $n \in \mathbf{N}^{\times}$.

• The Grothendieck group $\mathrm{K}_0(\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n})$ is defined as the free abelian group on the isomorphism classes of objects $(Y \to \mathbf{G}_{m,S}, \sigma)$ in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n}$, modulo the relations

(1) for every $\mathbf{G}_m$-invariant closed subscheme Z of Y, we have $[(Y \to \mathbf{G}_{m,S}, \sigma)] = [(Z \to \mathbf{G}_{m,S}, \sigma_{|Z})] + [(Y \setminus Z) \to \mathbf{G}_{m,S}, \sigma_{|Y\setminus Z})]$;

(2) for every $(p : Y \to \mathbf{G}_{m,S}, \sigma)$ in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n}$, for every liftings σ' and σ'' of the $\mathbf{G}_{m,k}$-action σ on Y to affine actions, i.e., its restriction to all fibers is affine, where the morphism $Y \times_k \mathbf{A}^n_k \to \mathbf{G}_{m,S}$ is the composition of p with the projection on the first factor, we have

$$[(Y \times_k \mathbf{A}^n_k \to \mathbf{G}_{m,S}, \sigma')] = [(Y \times_k \mathbf{A}^n_k \to \mathbf{G}_{m,S}, \sigma'')].$$

It has ring structure induced by the fiber product over $\mathbf{G}_{m,S}$ with diagonal action whose unit is

$$\mathbf{1}_{S\times\mathbf{G}_m} = [(\mathbf{G}_{m,S} \xrightarrow{\mathrm{Id}} \mathbf{G}_{m,S}, m_S \circ (\mathrm{Id}_{\mathbf{G}_{m,S}} \times e_n))].$$

If $\mathrm{K}_0(\mathbf{Var}_k)$ denotes the standard Grothendieck ring of varieties, there is a natural structure of $\mathrm{K}_0(\mathbf{Var}_k)$-algebra on $\mathrm{K}_0(\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n})$ given by the fiber product over k. We denote by $\mathbf{L}$ the element $[\mathbf{A}^1_k] \cdot \mathbf{1}_{S\times\mathbf{G}_m}$ in this module, and we set $\mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n} = \mathrm{K}_0(\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n})[\mathbf{L}^{-1}]$.

• The Grothendieck ring $\mathrm{K}_0^{\boldsymbol{\mu}_n}(\mathbf{Var}_S)$ is defined as the free abelian group on the isomorphism classes of objects $(p : Y \to S, \sigma)$ in $\mathbf{Var}_S^{\boldsymbol{\mu}_n}$, modulo the relations

(1) for every $\boldsymbol{\mu}_{n,k}$-invariant closed subscheme Z of Y, we have $[(Y \to S, \sigma)] = [(Z \to S, \sigma_{|Z})] + [(Y \setminus Z) \to S, \sigma_{|Y\setminus Z})]$;

(2) for every $(p\colon Y \to S, \sigma)$ in $\mathbf{Var}_S^{\boldsymbol{\mu}_n}$, for every affine bundle A, A' of Y of same rank endowed with any lifting σ' and σ'' of the $\boldsymbol{\mu}_{n,k}$-action σ on Y to affine actions, where the morphism $A, A' \to S$ is the composition of p with the projection on the first factor, we have

$$[(A \to S, \sigma')] = [(A' \to S, \sigma'')].$$

It has ring structure induced by the fiber products over S with diagonal action whose unit is the class of S endowed with the trivial action. We denote by $\mathbf{L}$ the class of the affine line with the trivial action, and set $\mathscr{M}_S^{\boldsymbol{\mu}_n} = \mathrm{K}_0(\mathbf{Var}_S^{\boldsymbol{\mu}_n})[\mathbf{L}^{-1}]$.

As remark in *loc.cit.*, the rings $\mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n}$ (resp. $\mathscr{M}_S^{\boldsymbol{\mu}_n}$) form inductive systems indexed by $\mathbf{N}^{\times}$ ordered by the division relation. We denote by $\mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m}$ (resp. $\mathscr{M}_S^{\hat{\boldsymbol{\mu}}}$) the colimit. For every pair of nonzero integers $(m, n) \in \mathbf{N}^2$ with $n = \ell m$, in case of $\mathbf{G}_{m,k}$-actions (and similarly in case of $\hat{\boldsymbol{\mu}}$-actions), one defines a functor

$$\theta^n_m : \mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},m} \to \mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_{m},n}$$

which sends the object $(Y \to \mathbf{G}_{m,S}, \sigma)$ to $(Y \to \mathbf{G}_{m,S}, \sigma')$ where the action σ' is given by $(y, \lambda) \mapsto \sigma((y, \lambda^{\ell}))$; these functors induce the transition maps in the colimits. The equivalence recalled in proposition 2.1.1 implies the following result (see [17, Proposition 2.6]).

Proposition 5.1.1. *For every integer $n \in \mathbf{N}^{\times}$, the ring morphism $1_S^* \colon \mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m,n} \to \mathscr{M}_S^{\boldsymbol{\mu}_n}$, which sends the class $[(Y \to \mathbf{G}_{m,S}, \sigma)]$ to $[(Y_1 \to S, \sigma_1)]$, is an isomorphism. In particular, the induced ring morphism $1_S^* \colon \mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m} \to \mathscr{M}_S^{\hat{\boldsymbol{\mu}}}$ also is an isomorphism.*

5.2. Monodromic Euler characteristic. Proposition 5.2.1 is a monodromic analog of [7, Lemma 8.5] (see also [20, Lemma 2.1] for a formulation in the category of étale motives).

Proposition 5.2.1. *Let $n \in \mathbf{N}^{\times}$. There exists a unique morphism of rings*

$$\chi_{S,\mathrm{c}}^{\mathbf{G}_m} : \mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m} \to \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S))$$

such that, for every object $(p \colon Y \to \mathbf{G}_{m,k}, \sigma)$ in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m,n}$

$$\chi_{S,\mathrm{c}}^{\mathbf{G}_m}([(Y \xrightarrow{p} \mathbf{G}_{m,S}, \sigma)]) = [p_! p^* \mathbb{1}_{\mathbf{G}_{m,S}}].$$

By proposition 5.1.1, the morphism $\chi_{S,\mathrm{c}}^{\mathbf{G}_m}$ also defines a ring morphism

$$\chi_{S,\mathrm{c}}^{\hat{\boldsymbol{\mu}}} \colon \mathscr{M}_S^{\hat{\boldsymbol{\mu}}} \to \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S))$$

such that the following diagram

$$\begin{array}{ccc}
\mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m} & \xrightarrow{\chi_{S,c}^{\mathbf{G}_m}} & \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)) \\
\downarrow{\scriptstyle 1_S^*} & & \downarrow{\scriptstyle =} \\
\mathscr{M}_S^{\hat{\boldsymbol{\mu}}} & \xrightarrow{\chi_{S,c}^{\hat{\boldsymbol{\mu}}}} & \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)) \\
\downarrow{\scriptstyle \mathrm{Forget}} & & \downarrow{\scriptstyle 1_S^*} \\
\mathscr{M}_S & \xrightarrow{\chi_{S,c}} & \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S))
\end{array} \tag{5.1}$$

commutes.

Proof of proposition 5.2.1. Let $(p : Y \to \mathbf{G}_{m,S}, \sigma)$ be an object in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m}$. Let Z be a closed subvariety of Y invariant under the $\mathbf{G}_{m,k}$-action σ and let U be its open complement. Denote by $z : Z \hookrightarrow Y$ and $u : U \hookrightarrow Y$ the corresponding immersion. By theorem 3.4.1, we have the following localization triangle in $\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)$

$$p_! u_! \mathbb{1}_U \to p_! \mathbb{1}_Y \to p_! z_! \mathbb{1}_Z \xrightarrow{+1} .$$

Therefore, we have the equality

$$[p_! \mathbb{1}_Y] = [p_! u_! \mathbb{1}_U] + [p_! z_! \mathbb{1}_Z]$$

in $\mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S))$. Moreover, the second relation is verified by homotopy invariance. This shows that there exists a unique group morphism

$$\chi_{S,\mathrm{c}}^{\mathbf{G}_m} : \mathrm{K}_0(\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m,n}) \to \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)) \tag{5.2}$$

which associates with every object $(p\colon Y \to S, \sigma)$ in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m}$ the element

$$\chi_{S,\mathrm{c}}^{\mathbf{G}_m}([(p : Y \to \mathbf{G}_{m,S}, \sigma)]) := [p_!\mathbb{1}_Y] \in \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)).$$

Since the image of $\mathbf{L}$ by this morphism is $[\mathbb{1}_{\mathbf{G}_{m,S}}(-1)]$ which is invertible, it is enough to check that the morphism (5.2) is a morphism of unitary rings. Let $(Y \to \mathbf{G}_{m,S}, \sigma)$ and $(Y' \to \mathbf{G}_{m,S}, \sigma')$ be two objects in $\mathbf{Var}_{S\times\mathbf{G}_m}^{\mathbf{G}_m}$. Let $a : Y \to \mathbf{G}_{m,S}$ and $a : Y' \to \mathbf{G}_{m,S}$ be the structural morphisms. By applying the proper base change theorem of [1] to the cartesian square

$$\begin{array}{ccc} Y \times_{\mathbf{G}_{m,S}} Y' & \xrightarrow{q} & Y' \\ {\scriptstyle p}\downarrow & \searrow^{r} & \downarrow{\scriptstyle a'} \\ Y & \xrightarrow{a} & \mathbf{G}_{m,S} \end{array}$$

and using [1, Theorem 2.3.40], we have

$$\begin{aligned} \chi_{S,\mathrm{c}}^{\mathbf{G}_m}([(p : Y \to\mathbf{G}_{m,S}, \sigma)] \cdot [(p' : Y' \to \mathbf{G}_{m,S}, \sigma')]) &= \left[r_!(\mathbb{1}_{Y\times_{\mathbf{G}_{m,S}}Y'})\right] \\ &= [a_!\,(p_!q^*\mathbb{1}_{Y'})] = [a_!\,(a^*a'_!\mathbb{1}_{Y'})] \\ &= [a_!\,(\mathbb{1}_Y \otimes_Y a^*a'_!\mathbb{1}_{Y'})] = [a_!\mathbb{1}_X \otimes_S a'_!\mathbb{1}_{Y'}] \\ &= \chi_{S,\mathrm{c}}^{\mathbf{G}_m}([(p : Y \to \mathbf{G}_{m,S}, \sigma)]) \cdot \chi_{S,\mathrm{c}}^{\mathbf{G}_m}([(p' : Y' \to \mathbf{G}_{m,S}, \sigma')]) \end{aligned}$$

as desired. The assertions on $\chi_{S,\mathrm{c}}^{\hat{\mu}}$ then follows from proposition 5.1.1. □

Remark 5.2.2. Let $q_S : \mathbf{G}_{m,S} \to S$ be the projection. Let us note that the symmetric monoidal functor

$$q_S^* : \mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S) \to \mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S)$$

defines a structure of $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S))$-algebra on $\mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S))$. The square

$$\begin{array}{ccc} \mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m} & \xrightarrow{\chi_{S,c}^{\mathbf{G}_m}} & \mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(S) \\ \uparrow & & \uparrow \\ \mathscr{M}_S & \xrightarrow{\chi_{S,\mathrm{c}}} & \mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S) \end{array}$$

being commutative, this structure is compatible with the structure of $\mathscr{M}_S$-algebra on $\mathscr{M}_{S\times\mathbf{G}_m}^{\mathbf{G}_m}$

5.3. Relation with the work of Denef-Loeser. In the context of diagram (1.1), for every point $x \in X_\sigma(k)$, Denef and Loeser have introduced the *motivic Milnor fiber* $\psi_{f,x}$ of x by the formula $\psi_{f,x} = x^*\psi_f$ (see [9, 10, 11, 23, 24]). We can now prove the main result of this section, which extends to a monodromic context [7, Corollary 8.7] and [20, Theorem 5.1]:

Theorem 5.3.1. *Let k be a field of characteristic zero. Let X be a smooth k-scheme of finite type. Let $f : X \to \mathbf{A}_k^1$ be a flat morphism of k-schemes.*

(1) Then, we have the following equality, in the group $\mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathrm{ct}}(X_\sigma))$,

$$[\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X\eta})] = \chi_{X_\sigma,\mathrm{c}}^{\mathbf{G}_m}(\psi_f) = \chi_{X_\sigma,\mathrm{c}}^{\hat{\boldsymbol{\mu}}}(\psi_f). \tag{5.3}$$

(2) Let $x \in X_\sigma(k)$ *be a rational point. Then, we have in* $\mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathrm{ct}}(k))$

$$[\breve{x}^*\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X\eta})] = \chi_{k,\mathrm{c}}^{\mathbf{G}_m}(\psi_{f,x}) = \chi_{X_\sigma,\mathrm{c}}^{\hat{\boldsymbol{\mu}}}(\psi_{f,x}). \tag{5.4}$$

Proof. We only have to check formula (5.3) since it implies (5.4). If S is a k-variety and $p : Y \to S$ is a S-variety, it will be useful to denote, as in [7], by $\mathrm{M}_{S,c}(Y)$ the motive with compact support defined to be $p_!\mathbb{1}_Y$. Let $h : Y \to X$ be a log-resolution of singularities of the pair (X, X_σ), I be the set of irreducible components of Y_σ and D_i be the irreducible components associated with $i \in I$. We use the conventions in the introduction. For $\emptyset \neq J \subset I$, let

$$\rho_J : \tilde{D}_J^\circ \to D_J^\circ \quad \text{and} \quad \rho_J^{\mathrm{mon}} : \widetilde{\mathbf{G}_{m,D_J^\circ}} \to \mathbf{G}_{m,D_J^\circ}$$

be the finite étale covers defined as in [20, §3.1.3]. By theorem 4.2.1 and [7, Theorem 8.6] (see also [20, Theorem 3.1]), we have in $\mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M}}(X_\sigma))$

$$[\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})] = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} \left[\mathrm{M}_{\mathbf{G}_{m,X_\sigma},c}\left(\widetilde{\mathbf{G}_{m,D_J^\circ}} \times_k \mathbf{G}_{m,k}^{|J|-1}\right)\right]$$

Using the module structure described in remark 5.2.2, we may rewrite this equality as follows

$$\Psi_f^{\mathrm{mon}}(\mathbb{1}_{X_\eta})] = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} \left[\mathrm{M}_{X_\sigma,c}(\mathbf{G}_{m,X_\sigma})\right]^{|J|-1} \cdot \left[\mathrm{M}_{\mathbf{G}_{m,X_\sigma},c}\left(\widetilde{\mathbf{G}_{m,D_J^\circ}}\right)\right].$$

Since we have the equality in $\mathscr{M}_{X_\sigma}$

$$\chi_{X_\sigma,\mathrm{c}}(\mathbf{L} - \mathbf{1}) = \left[\mathrm{M}_{X_\sigma,c}(\mathbf{G}_{m,X_\sigma})\right],$$

using (1.2), we only have to check that

$$\chi_{X_\sigma,\mathrm{c}}^{\hat{\boldsymbol{\mu}}}[\tilde{D}_J^\circ, \mu_{n_J}] = \left[\mathrm{M}_{\mathbf{G}_{m,X_\sigma},c}\left(\widetilde{\mathbf{G}_{m,D_J^\circ}}\right)\right]$$

but this follows from the local description of the étale covers and lemma 2.2.1. □

Remark 5.3.2. In [7, Remarks 8.14, 8.15], with Ayoub, the authors have suggested that it could be possible to combine the theory of rigid motives and the theory of motivic integration of E. Hrushovski and D. Kazhdan to obtain analogs of our proposition 5.2.1, diagram (5.1) and assertion (2) of our theorem 5.3.1. This has been positively answered by A. Forey in a recent preprint (see [12]). Let us stress that this relies on Ayoub's equivalence between quasi-unipotent motives over k and rigid motives over $k(\!(t)\!)$ and on an important presentation of the monodromic Grothendieck ring of varieties obtained by Hrushovski-Kazhdan using their theory of motivic integration. To the best of our knowledge, each of these results requires the base scheme S to be a field of characteristic zero. Their generalizations to a relative setting are unknown and seem to be a difficult challenging problem. In the present paper, our approach broadly differs from the one suggested in [7] and avoid these issues.

REMARK 5.3.3. In [9, 10, 11, 23, 24], by analogy to the works of J.-I. Igusa, J. Denef and F. Loeser have associated with diagram (1.1) the *motivic zeta function* $\mathscr{Z}_f(T)$ defined as follows (see also [25]). For every integer $n \in \mathbf{N}$, we denote by $\mathscr{L}_n(X)$ the jet scheme of level n of X. We set $\mathscr{X}_{n,1}^f = \{\varphi(t) \in \mathscr{L}_n(X),\ f\circ\varphi = t^n + \mathrm{O}(t^{n+1})\}$ for every integer $n \in \mathbf{N}^\times$. This (reduced) subscheme of $\mathscr{L}_n(X)$ naturally is endowed with a good $\boldsymbol{\mu}_{n,k}$-action induced by the reparametrization jets $t \mapsto \lambda t$. Then, Denef and Loeser introduced the power series

$$\mathscr{Z}_f(T) = \sum_{n\geq 1} \mathbf{L}_{X_\sigma}^{-nd}[\mathscr{X}_{n,1}^f; \boldsymbol{\mu}_{n,k}]T^n \in \mathscr{M}_{X_\sigma}^{\hat{\mu}}[[T]]. \tag{5.5}$$

By *loc. cit.*, one can show that this power series are rational and that its limit when $T \to \infty$ equals $-\psi_f$. Let us stress that theorem 4.2.1 has an analog in this context. Precisely, one has the formula

$$1_{X_\sigma}^* \mathscr{Z}_{f^{\mathbf{G}_m}}(T) = \mathscr{Z}_f(T). \tag{5.6}$$

Let us justify this formula. For every integer $n \in \mathbf{N}^\times$, the variety $\mathscr{X}_{n,1}^{f^{\mathbf{G}_m}}$ equals the set

$$\{(\varphi(t), \gamma(t)) \in \mathscr{L}_n(X) \times \mathscr{L}_n(\mathbf{G}_{m,k}),\ \gamma(t)f(\varphi(t)) = t^n + O(t^{n+1})\}.$$

which can be identified with

$$\{(\varphi(t), \gamma(t)) \in \mathscr{L}_n(X) \times \mathscr{L}_n(\mathbf{G}_{m,k}),\ f(\varphi(t)) = \gamma(0)^{-1}t^n + O(t^{n+1})\}. \tag{5.7}$$

We observe that he projection $\mathscr{L}_n(\mathbf{G}_{m,X}) \to \mathscr{L}_0(\mathbf{G}_{m,X})$ induces the structure of $\mathbf{G}_{m,X_\sigma}$-scheme. Formulas (5.5) and (5.7) then implies that

$$\begin{aligned} 1_{X_\sigma}^* \mathscr{Z}_{f^{\mathbf{G}_m}}(T) &= 1_{X_\sigma}^*\left(\textstyle\sum_{n\geq 1} \mathbf{L}_{\mathbf{G}_{m,X_\sigma}}^{-n(d+1)}[\mathscr{X}_{n,1}^{f^{\mathbf{G}_m}}; \boldsymbol{\mu}_{n,k}]T^n\right) \\ &= \textstyle\sum_{n\geq 1} \mathbf{L}_{X_\sigma}^{-n(d+1)} 1_{X_\sigma}^*[\mathscr{X}_{n,1}^{f^{\mathbf{G}_m}}; \boldsymbol{\mu}_{n,k}]T^n \\ &= \textstyle\sum_{n\geq 1} \mathbf{L}_{X_\sigma}^{-n(d+1)} \mathbf{L}_{X_\sigma}^n[\mathscr{X}_{n,1}^f; \boldsymbol{\mu}_{n,k}]T^n \\ &= \mathscr{Z}_f(T). \end{aligned}$$

In particular, at the level of Grothendieck rings, formula (5.6) implies that $1_{X_\sigma}^*\psi_{f^{\mathbf{G}_m}} = \psi_f$, which is, by theorem 5.3.1, a direct specialization of theorem 4.2.1.

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