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NORMAL BUNDLES ON THE EXCEPTIONAL SETS OF SIMPLE SMALL RESOLUTIONS*

RONG DU† AND XINYI FANG‡

Abstract. We study the normal bundles of the exceptional sets of higher dimensional isolated simple small singularities when the Picard groups of the exceptional sets are of rank one and their normal bundles have certain good filtrations. In particular, we prove numerical inequalities satisfied by normal bundles of exceptional sets. Moreover, we generalize Laufer's results on rationality and embedding dimension of singularities to higher dimension.

Key words. normal bundle, small singularity, exceptional set, embedding dimension.

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1. Introduction. Let X be a complex manifold of dimension n containing a compact smooth projective manifold Z of dimension p. We keep this assumption (i.e., Z being smooth in addition to X) intact throughout the paper. The submanifold Z is called exceptional or contractible, if there exists a birational proper surjective morphism $\varphi : X \to Y$ whose exceptional set is Z, where Y is an analytic space with normal singularities. If the subset Z of X is of codimension greater than or equal to 2, the morphism φ is called a small contraction, and the pair $(Y, \varphi(Z))$ is called a small singularity. Moreover, if dim $\varphi(Z) = 0$, then the singularity $(Y, \varphi(Z))$ is called isolated simple small singularity. Here, the adjective 'simple' reflects the condition (X, φ) exists with φ small and X smooth.

When X is a smooth projective surface, an algebraic curve C on X is exceptional if and only if its self-intersection number $(C.C)$ is negative. In particular, if C is a smooth curve, then C is exceptional if and only if its normal bundle $N_{C/X}$ is negative. When dim $X \geq 3$, Grauert showed in [Gra] that, if the normal bundle $N_{Z/X}$ of a compact smooth analytic subset Z is anti-ample, i.e., the conormal bundle $N^{\vee}_{Z/X}$ of Z is ample (in particular, Z is projective), then Z is exceptional. However, the converse is not true. For example, we know many exceptional curves whose normal bundles are not anti-ample (cf. [La2], [Re]). Nakayama and Ando have some very interesting works on the normal bundles of exceptional curves in higher dimensions, under the assumption the exceptional set is a projective line $\mathbb{P}^1$ (cf. [An1, 2], [Na]).

Let dim $X \geq 3$ and the exceptional set be a projective line $\mathbb{P}^1$. Ando proved the following interesting theorem.

Theorem 1.1 ([An1]). *Let X be a nonsingular projective variety of dimension $N \geq 3$ over $\mathbb{C}$, and let $\mathbb{P}^1 \subset X$. Assume that surjective morphism $f : X \to Y$ is a contraction map with the $\mathbb{P}^1$ as the exceptional set. Suppose that the normal bundle*

$$N_{\mathbb{P}^1/X} = \oplus_{i=1}^{N-1} \mathscr{O}_{\mathbb{P}^1}(-a_i), \ (a_1 \leq \cdots \leq a_{N-1}).$$

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†School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Rm. 312, Math. Bldg, No. 500, Dongchuan Road, Shanghai, 200241, P. R. China (rdu@math.ecnu.edu.cn).

‡Department of mathematics, Nanjing University, No. 22 Hankou Road, Nanjing, 210093, P. R. China (2315885681@qq.com).

Then we have

$$2(a_1 + \cdots + a_h) + (a_{h+1} + \cdots + a_{N-1}) \geq 0,$$

for any $1 \leq h \leq N-1$.

From another perspective, when X is a smooth projective variety and the canonical bundle K_X is not numerically effective, according to Mori's theory, there exists a contraction morphism $\varphi : X \to Y$ such that the fibers of φ are connected and the relative Picard number $\rho(X/Y)$ equals 1 ([Ka1],[Mo1]). The major goal of Mori Theory is to construct a minimal model for each nonuniruled birational equivalence class of varieties. Studying the structure of small contraction maps, as well as finding the associated "surgery operation" (flip), is of great importance for the minimal model program. To this end, it is essential to analyze the exceptional loci and their normal sheaves ([Mo2], [Ka2]). When $n = 4$, Kawamata ([Ka2]) proved that the exceptional locus E of φ is a disjoint union of its irreducible components E_i such that $E_i \cong \mathbb{P}^2$. In general, when $n = 2k$, Zhang ([Zh]) proved that if each irreducible component E_i of the exceptional locus E of φ is a smooth subvariety of dimension k, then $E_i \cong \mathbb{P}^k$. He also gave the same conclusion when $n = 5$ and the dimension of the exceptional set is 3. Later, Su-Zhao ([S-Z]) showed that when $n = 2k-1$, if each irreducible component E_i of the exceptional locus E of φ is a smooth subvariety of dimension k, then E_i is isomorphic to $\mathbb{P}^k$, quadratic hypersurface $Q^k \subseteq \mathbb{P}^{k+1}$, or a linear $\mathbb{P}^{k-1}$-bundle over a smooth curve. Moreover, if dim $\varphi(E) = 0$, the third case never happens. Since dim $E \geq \frac{1}{2}$ dim X by [Wi], we know that the structure of E is very simple when the dimension of E is the minimum for a small contraction $\varphi : X \to Y$ in this case. Therefore studying the projective spaces as the exceptional sets is a basic and important research direction, especially for the normal bundles of the exceptional sets. Kachi ([Kac]) studied the normal bundle of $\mathbb{P}^2$ as the exceptional set of some special flip contraction. However, as far as we know, such kind of result is very few for higher dimensional exceptional sets. One major setback that hinders this line of research is the absence of a sensible classification of vector bundles on $\mathbb{P}^k$.

The purpose of this paper is to study the normal bundles of the exceptional sets of higher dimensional isolated simple small singularities when the Picard groups of the exceptional sets are $\mathbb{Z}$ and their normal bundles have certain good filtrations. The most archetypal examples are those whose exceptional sets are projective spaces with splitting normal bundles. In this paper, we generalize Nakayama and Ando's results([Na],[An1]) to higher dimensions along these lines.

Definition 1.2. Let E be a holomorphic vector bundle on a projective variety Z of rank r and Pic $(Z) \cong \mathbb{Z}$. If there exists a filtration

$$E = \mathscr{F}_0 \supset \mathscr{F}_1 \supset \cdots \supset \mathscr{F}_r = 0$$

with all $\mathscr{F}_{i-1}/\mathscr{F}_i$ $(1 \leq i \leq r)$ are invertible sheaves and $\mathscr{F}_{i-1}/\mathscr{F}_i = L^{a_i}$, where L is an ample generator of Pic (Z) and $a_i \in \mathbb{Z}$, we say that E has a good filtration of type $(a_1, a_2, \cdots, a_r)$. We also denote $\mathscr{O}_Z(a_i) := L^{a_i}$ if no confusion is likely.

Our main theorem is as follows.

Theorem 1.3. *Let X be a complex manifold of dimension n and Z be its smooth projective subset of dimension p $(n - p \geq 2, p$ is odd$)$ which is exceptional, such that Pic $Z \cong \mathbb{Z}$. Let I_Z be the ideal sheaf of $Z \subseteq X$. If the conormal bundle I_Z/I_Z^2 has a* *good filtration of type* $(a_1, a_2, \cdots, a_{n-p})$*, then we have*

$$\sum_{\substack{t_1+t_2+\cdots+t_{n-p}=p \\ t_i \geq 0 \ (1 \leq i \leq n-p)}} 2^{t_1+\cdots+t_h} \cdot a_1^{t_1} a_2^{t_2} \cdots a_{n-p}^{t_{n-p}} \geq 0,$$

for any $1 \leq h \leq n-p$.

In particular, when $h = n-p$*, the corresponding inequality has a simple form, i.e.*

$$\sum_{\substack{t_1+t_2+\cdots+t_{n-p}=p \\ t_i \geq 0 \ (1 \leq i \leq n-p)}} a_1^{t_1} a_2^{t_2} \cdots a_{n-p}^{t_{n-p}} \geq 0.$$

Remark 1.4.

(1) These inequalities are not sharp even for dim $Z = 1$ by Ando's result ([An2]).
(2) When dim Z is even, those inequalities in Theorem 1.3 do not give any new information because they always hold by the positive definiteness of complete symmetric polynomials of even degree (see Theorem 1 in [Hun]).
(3) If dim Z is small, then we can write out those inequalities more explicitly. For example, if dim $Z = 3$, then we have

$$\sum_{i=1}^{n-3} a_i^3 + \sum_{\substack{1 \leq i,j \leq n-3 \\ i \neq j}} a_i^2 a_j + \sum_{1 \leq i < j < k \leq n-3} a_i a_j a_k \geq 0$$

and

$$\sum_{i=1}^{n-3} b_i^3 + \sum_{\substack{1 \leq i,j \leq n-3 \\ i \neq j}} b_i^2 b_j + \sum_{1 \leq i < j < k \leq n-3} b_i b_j b_k \geq 0,$$

for any $1 \leq h \leq n-3$, where

$$b_m = \begin{cases} 2a_m, & 1 \leq m \leq h; \\ a_m, & \text{otherwise.} \end{cases}$$

Corollary 1.5. *Let* X *be a complex manifold of dimension* n *and* $\mathbb{P}^p$ *be the exceptional set of an isolated simple small singularity,* $(n-p \geq 2,\ p$ *is odd*$)$*. If the normal bundle* $N_{\mathbb{P}^p/X} \cong \oplus_{i=1}^{n-p} \mathscr{O}_{\mathbb{P}^p}(-a_i)$ $(a_1 \leq \cdots \leq a_{n-p})$*, then we have*

$$\sum_{\substack{t_1+t_2+\cdots+t_{n-p}=p \\ t_i \geq 0 \ (1 \leq i \leq n-p)}} 2^{t_1+\cdots+t_h} \cdot a_1^{t_1} a_2^{t_2} \cdots a_{n-p}^{t_{n-p}} \geq 0,$$

for any $1 \leq h \leq n-p$.

Set $p = 1$, then we can get Ando's result (Theorem 1.1).

Corollary 1.6. *Fix the same assumptions as Theorem 1.3 and suppose that the dimension of* Z *is of codimension* 2 *in* X*, then* $a_1 + a_2 \geq 0$ *and* $2a_1 + a_2 \geq 0$.

In [La2], Laufer gave a sufficient condition for the rationality of an isolated singularity when the exceptional set is $\mathbb{P}^1$. Moreover, he also calculated the Hilbert function of the singularity and gave the embedding dimension of it especially. We generalize Laufer's results to higher dimensions as follows.

Theorem 1.7. *Let $Z \cong \mathbb{P}^p$ be an exceptional set in the n-dimensional manifold X. Suppose the normal bundle $N_{\mathbb{P}^p/X} \cong \oplus_{i=1}^{n-p} \mathscr{O}_{\mathbb{P}^p}(-a_i)$ and $a_i \geq 0$ for $1 \leq i \leq n-p$. Let $\varphi : (X, Z) \to (Y, y)$ be the contraction morphism, then (Y, y) is a rational singularity. Let m_y be the maximal ideal of Y at y. Let $h(r, y) = dim(m_y^r / m_y^{r+1})$ be the Hibert function for Y at y, then*

$$h(r,y) = \sum_{\substack{m_1+m_2+\cdots+m_{n-p}=r \\ m_i \geq 0 \ (1\leq i \leq n-p)}} \binom{p + m_1 a_1 + \cdots + m_{n-p} a_{n-p}}{p}.$$

In particular, at y the embedding dimension of Y is $\sum_{i=1}^{n-p} \binom{p+a_i}{p}$.

2. Normal bundles of the exceptional sets. Let X be a complex manifold of dimension n and Z its smooth projective subset of dimension p which is exceptional, such that Pic $Z \cong \mathbb{Z}$. Let I_Z be the ideal sheaf of $Z \subseteq X$ and I_Z/I_Z^2 the conormal bundle of Z in X.

Definition 2.1. Two vector bundles $\mathscr{F}$ and $\mathscr{G}$ over a smooth projective manifold Z of dimension p is called to be numerical equivalent (denoted $\mathscr{F} \equiv \mathscr{G}$), if rank $\mathscr{F}$ = rank $\mathscr{G}$ and $c_i(\mathscr{F}) = c_i(\mathscr{G})$ for all i $(1 \leq i \leq p)$.

If the conormal bundle I_Z/I_Z^2 has a good filtration of type $(a_1, a_2, \cdots, a_{n-p})$, then we have a corresponding filtration $I_Z = \mathscr{F}_0 \supset \mathscr{F}_1 \supset \cdots \supset \mathscr{F}_{n-p} = {I_Z}^2$ with all $\mathscr{F}_{i-1}/\mathscr{F}_i$ are invertible sheaves and $\mathscr{F}_{i-1}/\mathscr{F}_i \cong \mathscr{O}_Z(a_i)$.

For any $1 \leq h \leq n-p-1$, let $J = \mathscr{F}_h$. Put

$$\begin{aligned} B(h,r) = \{ &(m_1, m_2, \cdots, m_{n-p}) \in \mathbb{Z}^{n-p} \bigm| m_i \geq 0 \ (1 \leq i \leq n-p), \\ &(m_1 + m_2 + \cdots + m_h) + 2(m_{h+1} + \cdots + m_{n-p}) = r \}. \end{aligned}$$

$$Z(i,j) = \{ (m_1, m_2, \cdots, m_i) \in \mathbb{Z}^i \bigm| m_t \geq 0 \ (1 \leq t \leq i), \ m_1 + m_2 + \cdots + m_i = j \}.$$

By mimicking Ando's argument ([An2], Proposition 2.5), for any non-negative integer r, we obtain the following results. (By convention we set $I_Z^0 = \mathscr{O}_X$.)

Proposition 2.2.

$$J^r / I_Z J^r \equiv \bigoplus_{(m_1, m_2, \cdots, m_{n-p}) \in B(h, 2r)} \mathscr{O}_Z(m_1 a_1 + \cdots + m_{n-p} a_{n-p}),$$

$$I_Z J^r / J^{r+1} \equiv \bigoplus_{(m_1, m_2, \cdots, m_{n-p}) \in B(h, 2r+1)} \mathscr{O}_Z(m_1 a_1 + \cdots + m_{n-p} a_{n-p}),$$

and

$$I_Z^r / I_Z^{r+1} \equiv \bigoplus_{(m_1, m_2, \cdots, m_{n-p}) \in Z(n-p, r)} \mathscr{O}_Z(m_1 a_1 + \cdots + m_{n-p} a_{n-p}).$$

The following result from combinatorial theory will be used in the proof of the main theorem.

THEOREM 2.3 (Euler's Finite Difference Theorem, cf. [Qu-Go], (6.16), Page 68).

$$\sum_{t=0}^{k}(-1)^t\binom{k}{t}(k+i-t)^k = k!\ (i \geq 0).$$

LEMMA 2.4. *For any positive integers i, j, k and s $(1 \leq s \leq i)$, there exist integers u_i $(1 \leq i \leq k-1)$ such that*

$$\sum_{(m_1,m_2,\cdots,m_i)\in Z(i,j)} m_s^k = \binom{i+j+k-2}{i+k-1} + u_1\binom{i+j+k-3}{i+k-1} + \cdots + u_{k-1}\binom{i+j-1}{i+k-1},$$

and

$$1 + u_1 + \cdots + u_{k-1} = k!.$$

Proof. By calculation and induction, we can get

$$\begin{aligned}
\sum_{(m_1,m_2,\cdots,m_i)\in Z(i,j)} m_s^k &= 1^k\binom{i+j-3}{i-2} + \cdots + k^k\binom{i+j-k-2}{i-2} + \cdots + j^k\binom{i-2}{i-2} \\
&= \binom{i+j-2}{i-1} + \cdots + \left(k^k - (k-1)^k\right)\binom{i+j-k-1}{i-1} + \cdots \\
&\quad + \left(j^k - (j-1)^k\right)\binom{i-1}{i-1} \\
&= \binom{i+j-1}{i} + \cdots + \left(k^k - 2(k-1)^k + (k-2)^k\right)\binom{i+j-k}{i} \\
&\quad + \cdots + \left(j^k - 2(j-1)^k + (j-2)^k\right)\binom{i}{i} \\
&= \cdots\cdots \\
&= \binom{i+j+k-3}{i+k-2} + \cdots + \sum_{t=0}^{k}(-1)^t\binom{k}{t}(k-t)^k\binom{i+j-2}{i+k-2} + \\
&\quad \sum_{t=0}^{k}(-1)^t\binom{k}{t}(k+1-t)^k\binom{i+j-3}{i+k-2} + \cdots \\
&\quad + \sum_{t=0}^{k}(-1)^t\binom{k}{t}(j-t)^k\binom{i+k-2}{i+k-2}.
\end{aligned}$$

Due to Theorem 2.3, the above equality can be simplified to

$$\begin{aligned}
\sum_{(m_1,m_2,\cdots,m_i)\in Z(i,j)} m_s^k &= \binom{i+j+k-3}{i+k-2} + \cdots + k!\binom{i+j-2}{i+k-2} + k!\binom{i+j-3}{i+k-2} \\
&\quad + \cdots + k!\binom{i+k-2}{i+k-2} \\
&= \binom{i+j+k-2}{i+k-1} + u_1\binom{i+j+k-3}{i+k-1} + \cdots + u_{k-1}\binom{i+j-1}{i+k-1},
\end{aligned}$$

where $1 + u_1 + \cdots + u_{k-1} = k!$. □

Proof of Theorem 1.3. By Proposition 2.2, we have

$$ch(\mathscr{O}_X/I_Z^r) = \binom{n+r-p-1}{n-p} + \sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} (m_1a_1 + \cdots + m_{n-p}a_{n-p}) +$$

$$\frac{1}{2!}\sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} (m_1a_1 + \cdots + m_{n-p}a_{n-p})^2 + \cdots$$

$$+\frac{1}{p!}\sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} (m_1a_1 + \cdots + m_{n-p}a_{n-p})^p,$$

where $ch(\mathscr{O}_X/I_Z^r)$ is the exponential Chern character of $\mathscr{O}_X/I_Z^r$. By Lemma 2.4, we know that, for any $1 \le j \le p$,

$$\sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} (m_1a_1 + \cdots + m_{n-p}a_{n-p})^j$$

is a polynomial in a single indeterminate r of degree $n - p + j$. Therefore

$$\frac{1}{p!}\sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} (m_1a_1 + \cdots + m_{n-p}a_{n-p})^p$$

$$= \frac{1}{p!} \sum_{(t_1,t_2,\cdots,t_{n-p})\in T} \binom{p}{t_1} \cdots \binom{p-t_1-\cdots-t_{n-p-1}}{t_{n-p}}$$

$$\cdot a_1^{t_1}a_2^{t_2}\cdots a_{n-p}^{t_{n-p}} \sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} m_1^{t_1}\cdots m_{n-p}^{t_{n-p}} \qquad (*)$$

is a polynomial in a single indeterminate r of degree n, where

$$T = \{(t_1, t_2, \cdots, t_{n-p}) \in \mathbb{Z}^{n-p} \mid t_1 \ge 0, t_2 \ge 0, \cdots t_{n-p} \ge 0,\ t_1 + t_2 + \cdots + t_{n-p} = p\}.$$

By Lemma 2.4 and induction, for any $(t_1, t_2, \cdots, t_{n-p}) \in T$, we know the leading coefficient of

$$\sum_{i=0}^{r-1} \sum_{(m_1,m_2,\cdots,m_{n-p})\in Z(n-p,i)} m_1^{t_1}\cdots m_{n-p}^{t_{n-p}} \in \mathbb{Q}[r]$$

is $\frac{1}{n!}t_1!t_2!\cdots t_{n-p}!$.

Thus the leading coefficient of the above polynomial $(*)$ is

$$\sum_{(t_1,t_2,\cdots,t_{n-p})\in T} a_1^{t_1}a_2^{t_2}\cdots a_{n-p}^{t_{n-p}}$$

up to a positive constant.

By generalized Grothendieck-Hirzebruch Riemann-Roch theorem ([O-T-T]), we can express the Euler characteristic $\chi(\mathscr{O}_X/I_Z^r)$ as a polynomial in a single indeterminate r , the leading coefficient of which is

$$\sum_{(t_1,t_2,\cdots,t_{n-p})\in T} a_1^{t_1}a_2^{t_2}\cdots a_{n-p}^{t_{n-p}},$$

up to a positive constant.

On the other hand, we have

$$\chi(\mathscr{O}_X/I_Z^r) \geq \sum_{i=1}^{p}(-1)^i \dim H^i(\mathscr{O}_X/I_Z^r).$$

However, since Z is the exceptional set, for all i $(1 \leq i \leq p)$, we have

$$\varprojlim_r H^i(\mathscr{O}_X/I_Z^r) \cong (R^i\varphi_*\mathscr{O}_X^\wedge)_y$$

by the holomorphic functions theorem (cf. [Knu], Chapter 5, Theorem 3.1), where $\varphi : (X,Z) \to (Y,y)$ is the contraction morphism. Therefore,

$$\dim \varprojlim_r H^i(\mathscr{O}_X/I_Z^r) < +\infty.$$

Thus there exists some constant M which is independent of r such that dim $H^i(\mathscr{O}_X/I_Z^r) \leq M$ for all i $(1 \leq i \leq p)$.
Thus

$$\sum_{(t_1,t_2,\cdots,t_{n-p})\in T} a_1^{t_1}a_2^{t_2}\cdots a_{n-p}^{t_{n-p}} \geq 0.$$

Moreover, for any $1 \leq h \leq n-p-1$, put $J = \mathscr{F}_h$, we can consider the sheaf $\mathscr{O}_X/J^r$, then by Proposition 2.2 and the similar argument as above, we only need to focus on the leading coefficient of the polynomial $\chi(\mathscr{O}_X/J^r)$. Similarly, we have

$$\sum_{(t_1,t_2,\cdots,t_{n-p})\in T} b_1^{t_1}b_2^{t_2}\cdots b_{n-p}^{t_{n-p}} \geq 0,$$

where

$$b_m = \begin{cases} 2a_m, & 1 \leq m \leq h; \\ a_m, & \text{otherwise.} \end{cases}$$

□

Corollary 2.5. *Let X be a complex manifold of dimension n and $\mathbb{P}^p$ be the exceptional set of an isolated simple small singularity, $(n-p \geq 2,\ p$ is odd$)$. If the normal bundle $N_{\mathbb{P}^p/X} \cong \oplus_{i=1}^{n-p}\mathscr{O}_{\mathbb{P}^p}(-a_i)$ $(a_1 \leq \cdots \leq a_{n-1})$, then we have a system of inequalities*

$$\sum_{(t_1,t_2,\cdots,t_{n-p})\in T} 2^{t_1+\cdots+t_h}\cdot a_1^{t_1}a_2^{t_2}\cdots a_{n-p}^{t_{n-p}} \geq 0,$$

where

$$T = \{(t_1, t_2, \cdots, t_{n-p}) \in \mathbb{Z}^{n-p} \mid t_1 \geq 0, t_2 \geq 0, \cdots t_{n-p} \geq 0,\ t_1 + t_2 + \cdots + t_{n-p} = p\}.$$

REMARK 2.6. Set $p = 1$, we can get Ando's result (Theorem 1.1).

Proof of Corollary 1.6. We may assume dim $X = 2n+1$, by Theorem 1.3, we know that

$$\sum_{i=0}^{2n-1} a_1^i a_2^{2n-1-i} \geq 0$$

and

$$\sum_{i=0}^{2n-1} (2a_1)^i a_2^{2n-1-i} \geq 0.$$

Since

$$\sum_{i=0}^{n-1} a_1^i a_2^{2n-1-i} = (a_1 + a_2)(\sum_{i=0}^{n-1} a_1^{2i} a_2^{2n-2-2i})$$

and

$$\sum_{i=0}^{n-1} a_1^{2i} a_2^{2n-2-2i} \geq 0,$$

we have

$$a_1 + a_2 \geq 0.$$

Similarly,

$$2a_1 + a_2 \geq 0.$$

□

3. Isolated small singularities. Employ the same notations as in §§1 − 2. When dim $X \geq 3$, Grauert showed that if the normal bundle $N_{Z/X}$ of algebraic set Z is negative, then Z is exceptional. However, the converse is not true (see [La2] Example 2.3). Nevertheless, if we consider a local Stein neighborhood Y of an isolated Gorenstein singularity which admits a simple small resolution $\varphi : X \to Y$ with a smooth exceptional locus Z, such that the canonical divisor of Z is anti-ample, then $c_1(N_{Z/X})$ is anti-ample. Indeed, under the assumption Y is Gorenstein, it follows that φ is crepant, namely, the canonical divisor ω_X is relatively trivial over Y. By adjunction formula, we have

$$\omega_Z \cong \omega_X \otimes \wedge^r(N_{Z/X}).$$

So $c_1(N_{Z/X}) = K_Z$ is negative.

Laufer studied the embedding dimension of a simple small singularity when the exceptional set is $\mathbb{P}^1$ ([La2]). If the exceptional set is $\mathbb{P}^p$, we can have similar result.

THEOREM 3.1. *Let $Z \cong \mathbb{P}^p$ be an exceptional set in the n-dimensional manifold X. Suppose the normal bundle $N_{\mathbb{P}^p/X} \cong \oplus_{i=1}^{n-p}\mathscr{O}_{\mathbb{P}^p}(-a_i)$ and $a_i \geq 0$ for $1 \leq i \leq n-p$. Let $\varphi : (X,Z) \to (Y,y)$ be the contraction morphism. Then (Y,y) is a rational singularity. Let m_y be the maximal ideal of Y at y. Let $h(r,y) = dim(m_y^r/m_y^{r+1})$ be the Hibert function for Y at y, then*

$$h(r,y) = \sum_{\substack{m_1+m_2+\cdots+m_{n-p}=r \\ m_i\geq 0\ (1\leq i\leq n-p)}} \binom{p+m_1a_1+\cdots+m_{n-p}a_{n-p}}{p}.$$

In particular,at y the embedding dimension of Y is $\sum\limits_{i=1}^{n-p}\binom{p+a_i}{p}$.

Proof. Let I_Z be the ideal sheaf of Z in $\mathscr{O}_X$. We formally put $I_Z^0 = \mathscr{O}_X$. Consider the exact sheaf sequence

$$0 \to I_Z^{r+1} \to I_Z^r \to I_Z^r/I_Z^{r+1} \to 0.$$

Since $a_i \geq 0$ for $1 \leq i \leq n-p$, $H^j(Z, I_Z^r/I_Z^{r+1}) = 0$ for all $r \geq 0$ and $1 \leq j \leq p$. Hence $H^j(X, I_Z^{r+1}) \to H^j(X, I_Z^r)$ is onto for all $r \geq 0$ and $1 \leq j \leq p$. By [Gra], Satz 4.2, $H^j(X, I_Z^r) = 0$ for all $r \geq 0$ and $1 \leq j \leq p$. In particular, $H^j(X, \mathscr{O}_X) = 0$ for all $1 \leq j \leq p$. Hence (Y,y) is a rational singularity.

Then, as in [La1], $m_y^r \cong \Gamma(Z, I_Z^r)$ and $m_y^r/m_y^{r+1} \cong \Gamma(Z, I_Z^r/I_Z^{r+1})$, hence

$$\begin{aligned} h(r,y) &= \dim(m_y^r/m_y^{r+1}) = \dim\Gamma(Z, I_Z^r/I_Z^{r+1}) = \dim\Gamma\big(Z, S^r(I_Z/I_Z^2)\big) \\ &= \sum_{\substack{m_1+m_2+\cdots+m_{n-p}=r \\ m_i\geq 0\ (1\leq i\leq n-p)}} \binom{p+m_1a_1+\cdots+m_{n-p}a_{n-p}}{p}. \end{aligned}$$

In particular, at y the embedding dimension of Y is $h(1,y) = \sum\limits_{i=1}^{n-p}\binom{p+a_i}{p}$. □

EXAMPLE 3.2. Let U, V and W be copies of $\mathbb{C}^4$ with coordinates (t_1,t_2,x_1,x_2), (s_1,s_2,y_1,y_2) and (w_1,w_2,z_1,z_2). We construct X by patching U, V and W satisfying the following transition functions:

$$\begin{cases} x_1 = \frac{z_2}{z_1} = \frac{1}{y_1} \\ x_2 = \frac{1}{z_1} = \frac{y_2}{y_1} \\ t_1 = z_1^5 w_1 = y_1^5 s_1 \\ t_2 = z_1 w_2 = y_1 s_2. \end{cases}$$

The exceptional set $Z \cong \mathbb{P}^2$ is contracted by

$$\begin{cases} v_1 = t_1 = z_1^5 w_1 = y_1^5 s_1 \\ v_2 = t_2^2 = w_2^2 z_1^2 = s_2^2 y_1^2 \\ v_3 = x_2^5 t_1 = w_1 = s_1 y_2^5 \\ v_4 = x_2^2 t_2^2 = w_2^2 = s_2^2 y_2^2 \\ v_5 = x_1^5 t_1 = w_1 z_2^5 = s_1 \\ v_6 = x_1^2 t_2^2 = w_2^2 z_2^2 = s_2^2. \end{cases}$$

It is easy to check $I_Z/I_Z^2 \cong \mathscr{O}_Z(5) \oplus \mathscr{O}_Z(1)$. So (Y,y) is a rational singularity with embedding dimension 24. Here $\varphi : (X,Z) \to (Y,y)$ is the contraction morphism of Z (see [An2] Section 3).

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