

MEROMORPHIC CONNECTIONS, DETERMINANT LINE BUNDLES AND THE TYURIN PARAMETRIZATION*

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Abstract. We develop a holomorphic equivalence between on one hand the space of pairs (stable bundle, flat connection on the bundle) and the “sheaf of holomorphic connections” (the sheaf of holomorphic splittings of the one-jet sequence) for the determinant (Quillen) line bundle over the moduli space of vector bundles on a compact connected Riemann surface. This equivalence is shown to be holomorphically symplectic. The equivalences, both holomorphic and symplectic, are rather quite general, for example, they extend to other general families of holomorphic bundles and holomorphic connections, in particular those arising from “Tyurin families” of stable bundles over the surface. These families generalize the Tyurin parametrization of stable vector bundles E over a compact connected Riemann surface, and one can build above them spaces of (equivalence classes of) holomorphic connections, which are again symplectic. These spaces are also symplectically biholomorphically equivalent to the sheaf of holomorphic connections for the determinant bundle over the Tyurin family. The last portion of the paper shows how this extends to moduli of framed bundles.

Key words. Tyurin parametrization, meromorphic connection, framing, stable bundle, symplectic form, Atiyah bundle.

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1. Introduction. We will address in this paper an equivalence which seems to hold in a certain generality both in the holomorphic and holomorphically symplectic category between two objects, defined over various moduli space of stable bundles on a Riemann surface X . The equivalence is somewhat surprising, as we do not have a direct map between the two spaces, but instead obtain the equivalence by showing that certain cohomology classes are the same in both cases, and indeed have the same representative.

Our basic examples of the first set of spaces are moduli spaces of pairs (E, ∇) with E a stable bundle and ∇ a holomorphic connection on E . Let X be a compact connected Riemann surface. A stable vector bundle over X of degree zero admits holomorphic connections. Once we fix a point $x_0 \in X$, a stable vector bundle E on X of degree d and rank r admits logarithmic connections on $X \setminus \{x_0\}$ whose residue at the point x_0 is $-\frac{d}{r}\text{Id}_{E_{x_0}}$. Let \mathcal{C} denote the moduli space of pairs of the form (E, D) , where E is a stable vector bundle on X of rank r and degree d , and D is a holomorphic or logarithmic connection on E of the above type depending on whether d is zero or not. This moduli space \mathcal{C} is equipped with a natural holomorphic symplectic structure constructed by Goldman and Atiyah–Bott [Go], [AB]. Let \mathcal{M} denote the moduli space of stable vector bundles on X of rank r and degree d . The projection $\mathcal{C} \rightarrow \mathcal{M}, (E, D) \mapsto E$, has the structure of a holomorphic $T^*\mathcal{M}$ -torsor.

The more general family in the first set of spaces corresponding to pairs of a vector bundle and a holomorphic connection on it is developed from the Tyurin parametrization. This parametrization of vector bundles on curves was introduced in [Ty1], [Ty2]. To explain it briefly, fixing a holomorphic vector bundle V of rank r on a compact con-

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nected Riemann surface X , consider all torsion quotients of V of degree d' . This way a holomorphic family of vector bundles on X of rank r and degree $d := \text{degree}(V) - d'$ is obtained, as the kernels of natural projection to the torsion quotients. Choosing d' correctly gives a parametrization of some suitable open set of the moduli space of stable vector bundles of rank r and degree d . This Tyurin parametrization has turned out to be very useful; see [Kr], [KN], [Hu], [She] and references therein. We now allow more general families by letting the degree d' vary. Let \mathcal{Q}_s be a Tyurin parameter space of stable vector bundles on X of rank r and degree d , built from a fixed vector bundle of rank r and degree d' . For a point $z \in \mathcal{Q}_s$, the corresponding stable vector bundle on X will be denoted by \mathcal{K}^z . Our space $\mathcal{C}(\mathcal{Q}_s)$ will be built from the pull-back from \mathcal{M} of the earlier mentioned holomorphic fiber bundle \mathcal{C} , taking the fiber product with $T^*\mathcal{Q}_s$ and quotienting by an equivalence relation. The resulting variety $\mathcal{C}(\mathcal{Q}_s)$ is a $T^*\mathcal{Q}_s$ -torsor and has a holomorphic symplectic structure (Lemma 4.2).

Our second set of spaces will be the sheaves of connections on a determinant line bundle. Given any holomorphic line bundle L on a complex manifold Z , let $\rho : \text{Conn}(L) \rightarrow Z$ be the holomorphic fiber bundle given by the “sheaf” of holomorphic connections on L . More precisely, $\text{Conn}(L) \subset \text{At}(L)^* = J^1(L) \otimes L^*$ is the inverse image, for the natural projection $\text{At}(L)^* \rightarrow \mathcal{O}_Z$, of the image of the section of \mathcal{O}_Z given by the constant function 1 on Z ; here $\text{At}(L)$ denote the Atiyah bundle for L (it contains \mathcal{O}_Z in a natural way with the quotient being TZ). Then $\text{Conn}(L)$ is a holomorphic torsor over Z for the holomorphic cotangent bundle T^*Z , and it is equipped with a holomorphic symplectic structure given by the curvature of the tautological holomorphic connection on the holomorphic line bundle $\rho^*L \rightarrow \text{Conn}(L)$. In our cases the line bundles L will be the natural determinant line bundles over our moduli spaces of vector bundles.

The main theorems are that the spaces of the first set are equivalent torsors over the various moduli of vector bundles to the spaces in the second set; for the natural symplectic forms, this equivalence is also a holomorphic symplectic equivalence (Proposition 2.3, Theorem 3.1, Theorem 4.1). Note that the Tyurin families step outside of the family of stable bundles and, since the Quillen metric is uniformly defined, this provides symplectic “extensions” of the space \mathcal{C} . In some sense, the determinant bundle is a much more robust object, and the equivalence of the torsors allows us to better understand the symplectic geometry of the “connection space” $\mathcal{C}(\mathcal{Q}_s)$. These latter spaces, with their torsor structures, have proved extremely useful in understanding the symplectic and Hamiltonian aspects of isomonodromic deformations; see [Kr], [KN], [Hu].

For the framed version, fix a nonzero effective divisor \mathbb{S} on X . Let \mathcal{D}^F be the space corresponding to the triples of the form (z, σ, D) , where $z \in \mathcal{Q}_s$ and σ is a framing on \mathcal{K}^z over \mathbb{S} while D is a meromorphic connection on \mathcal{K}^z whose polar part has support contained in \mathbb{S} . This moduli space \mathcal{D}^F has a holomorphic symplectic structure (Corollary 6.2). This holomorphic symplectic structure on \mathcal{D}^F is constructed by identifying \mathcal{D}^F with $\text{Conn}(L')$, where L' is the determinant line bundle on the moduli space of pairs of the form (z, σ) , where $z \in \mathcal{Q}_s$ and σ is a framing on the vector bundle \mathcal{K}^z over \mathbb{S} (Theorem 6.1).

2. Isomorphism of torsors for the cotangent bundle.

2.1. Moduli spaces of vector bundles and connections. The holomorphic cotangent bundle of Z will be denoted by T^*Z . The real tangent bundle of Z will be denoted by $T^{\mathbb{R}}Z$.

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. Let K_X denote the holomorphic cotangent bundle of X . The holomorphic tangent bundle of a complex manifold Z will be denoted by TZ .

For fixed integers $r \geq 2$ and d , let

$$\mathcal{M}_X(r, d) =: \mathcal{M} \tag{2.1}$$

denote the moduli space of all stable vector bundles on X of rank r and degree d . This \mathcal{M} is an irreducible smooth quasiprojective complex variety of dimension $r^2(g-1)+1$ (see [Ne]). It is projective if and only if d is coprime to r .

If $d \neq 0$, fix a point $x_0 \in X$. Let

$$\mathcal{C}_X(r, d) =: \mathcal{C} \tag{2.2}$$

denote the moduli space of isomorphism classes of logarithmic connections (E, D) , where

- (1) $E \in \mathcal{M}$ (defined in (2.1)), and
- (2) D is a logarithmic connection on E , nonsingular over $X \setminus \{x_0\}$, such that the residue $\text{Res}(D, x_0)$ of D at x_0 is $-\frac{d}{r}\text{Id}_{E_{x_0}}$. (See [De] for logarithmic connections and their residues.)

If $d = 0$, then

$$\mathcal{C}_X(r, d) =: \mathcal{C}$$

denotes the moduli space of isomorphism classes of holomorphic connections (E, D) , where

- (1) $E \in \mathcal{M}$, and
- (2) D is a holomorphic connection on E . (See [At] for holomorphic connections.)

It is known that for every $E \in \mathcal{M}$, there is a logarithmic/holomorphic connection D of the above type.

The moduli space \mathcal{C} is an irreducible smooth quasiprojective complex variety of dimension $2(r^2(g-1)+1)$ (see [Si1], [Si2], [Ni]). It is equipped with an algebraic symplectic form

$$\Omega_{\mathcal{C}} \in H^0(\mathcal{C}, \bigwedge^2 T^*\mathcal{C}) \tag{2.3}$$

[AB], [Go]; the symplectic form on \mathcal{C} constructed in [AB], [Go] is, a priori, only holomorphic, however in [Bi, p. 331, Theorem 3.2] it is shown that this holomorphic symplectic form coincides with a natural algebraic form on \mathcal{C} .

Let

$$\phi : \mathcal{C} \longrightarrow \mathcal{M} \tag{2.4}$$

be the forgetful map $(E, D) \longmapsto E$ that forgets the logarithmic/holomorphic connection D on E .

Given any $(E, D) \in \mathcal{C}$, and any $\theta \in H^0(X, \text{End}(E) \otimes K_X)$, note that

$$(E, D + \theta) \in \mathcal{C}.$$

Conversely, if $(E, D') \in \mathcal{C}$, then we have

$$D' - D \in H^0(X, \text{End}(E) \otimes K_X).$$

On the other hand, $H^0(X, \text{End}(E) \otimes K_X)$ is the fiber $T_E^*\mathcal{M} = (T_E\mathcal{M})^*$ of the holomorphic cotangent bundle of \mathcal{M} at the point E . The projection ϕ in (2.4) and the above fiberwise action of $T^*\mathcal{M}$ on the moduli space \mathcal{C} in (2.2) actually make \mathcal{C} a complex algebraic torsor over \mathcal{M} for the holomorphic cotangent bundle $T^*\mathcal{M} \rightarrow \mathcal{M}$.

We note that the holomorphic isomorphism classes of the holomorphic $T^*\mathcal{M}$ -torsors on \mathcal{M} are parametrized by $H^1(\mathcal{M}, T^*\mathcal{M})$; this aspect is elaborated in Section 2.3.

2.2. The determinant line bundle and a torsor over \mathcal{M} . As before, fix a point $x_0 \in X$.

Let $\mathcal{L} \rightarrow \mathcal{M}$ be the determinant line bundle whose fiber over any $E \in \mathcal{M}$ is

$$\left(\bigwedge^{\text{top}} H^0(X, E)^* \otimes \bigwedge^{\text{top}} H^1(X, E)\right)^{\otimes r} \otimes \left(\bigwedge^r E_{x_0}\right)^\chi,$$

where $\chi := \chi(E) = h^0(E) - h^1(E) = d - r(g - 1)$ is the Euler characteristic. We shall briefly recall the construction of \mathcal{L} . Given any holomorphic family of stable vector bundles V of rank r and degree d on X

$$V \rightarrow X \times T \xrightarrow{p} T$$

parametrized by a complex manifold T , we have the holomorphic line bundle

$$(\det R^0 p_* V)^{\otimes -r} \otimes (\det R^1 p_* V)^{\otimes r} \otimes \det(s_{x_0}^* V)^{\otimes \chi} \rightarrow T, \tag{2.5}$$

where $\det \mathcal{W} \rightarrow T$ is the determinant line bundle for a coherent analytic sheaf \mathcal{W} on T (see [Ko, Ch. V, § 6] for the construction of determinant bundle) and

$$s_{x_0} : T \rightarrow X \times T, \quad t \mapsto (x_0, t)$$

is the section of the projection p , while χ as before is $d - r(g - 1) \in \mathbb{Z}$. The holomorphic line bundle on T in (2.5) does not change if the vector bundle V is replaced by $V \otimes p^*L_0$, where L_0 is a holomorphic line bundle on T . Therefore, the construction in (2.5) produces a holomorphic line bundle

$$\mathcal{L} \rightarrow \mathcal{M} \tag{2.6}$$

which is in fact algebraic.

It should be mentioned that the determinant line bundle \mathcal{L} exists even when there is no Poincaré bundle over $X \times \mathcal{M}$. See also [Qu], [KM] for the construction of the determinant line bundle in a general context.

Consider the Atiyah exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow \text{At}(\mathcal{L}) \xrightarrow{\zeta} T\mathcal{M} \rightarrow 0 \tag{2.7}$$

for the holomorphic line bundle \mathcal{L} in (2.6), where $\text{At}(\mathcal{L}) = \mathcal{L} \otimes J^1(\mathcal{L})^*$ is the Atiyah bundle for \mathcal{L} with $J^1(\mathcal{L})$ being the first jet bundle for \mathcal{L} (see [At]). Let

$$0 \rightarrow T^*\mathcal{M} = \Omega_{\mathcal{M}}^1 \rightarrow \text{At}(\mathcal{L})^* \xrightarrow{q} \mathcal{O}_{\mathcal{M}} \rightarrow 0 \tag{2.8}$$

be the dual sequence of (2.7); note that $\text{At}(\mathcal{L})^* = J^1(\mathcal{L}) \otimes \mathcal{L}^*$, and $J^1(\mathcal{L})$ fits in the short exact sequence

$$0 \rightarrow \Omega_{\mathcal{M}}^1 \rightarrow J^1(\mathcal{L}) \rightarrow J^0(\mathcal{L}) = \mathcal{L} \rightarrow 0.$$

The section of $\mathcal{O}_{\mathcal{M}}$ given by the constant function 1 on \mathcal{M} will be denoted by $1_{\mathcal{M}}$. Define

$$\text{At}(\mathcal{L})^* \supset q^{-1}(1_{\mathcal{M}}) =: \text{Conn}(\mathcal{L}) \xrightarrow{q_0} \mathcal{M}, \tag{2.9}$$

where q is the projection in (2.8), and q_0 is the restriction of q to the subvariety $\text{Conn}(\mathcal{L}) \subset \text{At}(\mathcal{L})^*$. From (2.8) it follows immediately that $\text{Conn}(\mathcal{L})$ is a complex algebraic torsor for the holomorphic cotangent bundle $T^*\mathcal{M} \rightarrow \mathcal{M}$.

2.3. Cohomological invariants for holomorphic torsors. Let Z be a complex manifold and V a holomorphic vector bundle over Z . The isomorphism classes of holomorphic torsors $\mathbb{V} \rightarrow Z$ for V are parametrized by $H^1(Z, V)$. To see this, choose local holomorphic sections of $s_i : U_i \rightarrow \mathbb{V}|_{U_i}$, where $\{U_i\}_{i \in I}$ is an open covering of Z . For any ordered pair $(i, j) \in I \times I$, consider $s_i - s_j$ on $U_i \cap U_j$, which is in fact a holomorphic section of $V|_{U_i \cap U_j}$. This 1-cocycle $\{s_i - s_j\}_{i, j \in I}$ gives the element of $H^1(Z, V)$ corresponding to \mathbb{V} .

There is also a Dolbeault type construction of the above cohomology class in $H^1(Z, V)$ associated to the holomorphic V -torsor \mathbb{V} . For this first note that since the fibers of $\mathbb{V} \rightarrow Z$ are contractible, there are C^∞ sections of this fiber bundle (however there is a holomorphic section if and only if the torsor \mathbb{V} for V is trivial). Take a C^∞ section s of the fiber bundle $\mathbb{V} \rightarrow Z$. The obstruction for s to be holomorphic is clearly the failure of the differential ds of the map s to intertwine the almost complex structures of Z and \mathbb{V} . More precisely, consider the homomorphism

$$(ds)' : T^{\mathbb{R}}Z \rightarrow s^*T^{\mathbb{R}}\mathbb{V}, \quad v \mapsto \frac{1}{2}(ds(v) + J_{\mathbb{V}}(ds(J_Z(v)))) , \tag{2.10}$$

where J_Z and $J_{\mathbb{V}}$ are the almost complex structures on Z and \mathbb{V} respectively, and

$$ds : T^{\mathbb{R}}Z \rightarrow s^*T^{\mathbb{R}}\mathbb{V}$$

is the differential of the map s . It is straight-forward to check that

- $(ds)' = 0$ if and only if the map s is holomorphic,
- $(ds)'$ is a C^∞ section of $\Omega_Z^{0,1} \otimes V$ over Z , and
- $\bar{\partial}_V((ds)') = 0$, so $(ds)'$ defines an element of the Dolbeault cohomology $H^1(Z, V)$.

The element of $H^1(Z, V)$ defined by $(ds)'$ coincides with the Čech cohomology class constructed earlier using locally defined holomorphic sections of \mathbb{V} . Note that when V is the holomorphic cotangent bundle T^*Z , the above section $(ds)'$ is a $\bar{\partial}$ -closed $(1, 1)$ -form on Z .

The class in $H^1(\mathcal{M}, T^*\mathcal{M})$ corresponding to the $T^*\mathcal{M}$ -torsor $\text{Conn}(\mathcal{L})$ constructed in (2.9) is $2\pi\sqrt{-1} \cdot c_1(\mathcal{L})$, where $c_1(\mathcal{L})$ is the rational first Chern class of the holomorphic line bundle \mathcal{L} in (2.6).

In Section 2.1 we saw that \mathcal{C} is a torsor over \mathcal{M} for $T^*\mathcal{M}$, and in Section 2.2 we saw that $\text{Conn}(\mathcal{L})$ is a torsor over \mathcal{M} for $T^*\mathcal{M}$. We shall compare the isomorphism classes of these two torsors.

First consider the projection ϕ in (2.4). Given any $E \in \mathcal{M}$, by a theorem of Narasimhan and Seshadri, [NS], there is a unique logarithmic connection D_E on E such that

- (1) D_E is nonsingular on $X \setminus \{x_0\}$,
- (2) the monodromy of D_E lies in $U(r)$, and
- (3) the residue $\text{Res}(D_E, x_0)$ of D_E at x_0 is $-\frac{d}{r}\text{Id}_{E_{x_0}}$.

If $d = 0$, then D_E is a holomorphic connection on E whose monodromy lies in $U(r)$. Therefore, the projection ϕ in (2.4) has a canonical section

$$\beta : \mathcal{M} \longrightarrow \mathcal{C} \tag{2.11}$$

given by $E \mapsto D_E$. This section β is C^∞ , however it is not holomorphic.

The moduli space \mathcal{M} is equipped with a natural Kähler form [AB], [Go]; this Kähler form on \mathcal{M} will be denoted by $\omega_{\mathcal{M}}$. We briefly recall the construction of $\omega_{\mathcal{M}}$. Just as in the construction of β in (2.11), identify \mathcal{M} with the equivalence classes of unitary representations of $\pi_1(X \setminus \{x_0\})$ such that the monodromy around x_0 is $\exp(2\pi\sqrt{-1}d/r) \cdot \text{Id}$. On the other hand, such a representation space is equipped with the Goldman symplectic form. This symplectic form coincides with the Kähler form $\omega_{\mathcal{M}}$. It should be mentioned that

$$\omega_{\mathcal{M}} = \beta^* \Omega_{\mathcal{C}}, \tag{2.12}$$

where $\Omega_{\mathcal{C}}$ is the holomorphic symplectic form on \mathcal{C} in (2.3).

The following lemma is proved in [BR, p. 308, Theorem 2.11].

LEMMA 2.1. *For the C^∞ section β in (2.11), the corresponding $(1,1)$ -form $(d\beta)'$ on \mathcal{M} constructed in (2.10) coincides with $\omega_{\mathcal{M}}/2$.*

Proof. In [BR, p. 308, Theorem 2.11] it was proved that the $(1,1)$ -form $(d\beta)'$ coincides with $\frac{1}{2}\beta^*\Omega_{\mathcal{C}}$, where $\Omega_{\mathcal{C}}$ is the symplectic form on \mathcal{C} in (2.3). As noted in (2.12), the pulled back form $\beta^*\Omega_{\mathcal{C}}$ coincides with $\omega_{\mathcal{M}}$. \square

Next consider the projection q_0 in (2.9). Quillen in [Qu] using analytic torsion constructed a Hermitian structure on the holomorphic line bundle \mathcal{L} constructed in (2.6); he also computed the curvature of the corresponding Chern connection on \mathcal{L} . Let

$$\gamma : \mathcal{M} \longrightarrow \text{Conn}(\mathcal{L}) \tag{2.13}$$

be the C^∞ section of the projection q_0 in (2.9) given by the Chern connection associated to the Quillen metric on \mathcal{L} .

The following lemma is proved in [BR, p. 320, Theorem 4.20].

LEMMA 2.2. *For the C^∞ section γ in (2.13), the corresponding $(1,1)$ -form $(d\gamma)'$ on \mathcal{M} constructed in (2.10) coincides with $r \cdot \omega_{\mathcal{M}}$.*

It may be clarified that in our case the integer N in [BR, p. 320, Theorem 420] is 1.

Let

$$\delta : \mathcal{C} \times_{\mathcal{M}} T^*\mathcal{M} \longrightarrow \mathcal{C} \quad \text{and} \quad \eta : \text{Conn}(\mathcal{L}) \times_{\mathcal{M}} T^*\mathcal{M} \longrightarrow \text{Conn}(\mathcal{L}) \tag{2.14}$$

be the $T^*\mathcal{M}$ -torsor structures on \mathcal{C} and $\text{Conn}(\mathcal{L})$ respectively. Let

$$\mathbf{m} : T^*\mathcal{M} \longrightarrow T^*\mathcal{M}, \quad v \mapsto 2r \cdot v \tag{2.15}$$

be the multiplication by $2r$.

PROPOSITION 2.3. *There is a unique holomorphic isomorphism*

$$F : \mathcal{C} \longrightarrow \text{Conn}(\mathcal{L})$$

such that

- (1) $\phi = q_0 \circ F$, where ϕ and q_0 are the projections in (2.4) and (2.9) respectively,
- (2) $F \circ \beta = \gamma$, where β and γ are the sections in (2.11) and (2.13) respectively, and
- (3) $F \circ \delta = \eta \circ (F \times \mathbf{m})$, where δ , η and \mathbf{m} are constructed in (2.14) and (2.15).

Proof. It is straight-forward to check that there is a unique C^∞ diffeomorphism

$$F : \mathcal{C} \longrightarrow \text{Conn}(\mathcal{L}) \tag{2.16}$$

that satisfies the above three conditions. To prove the proposition we need to show that this map F in (2.16) is actually holomorphic.

From Lemma 2.1 and Lemma 2.2 it can be deduced that for point every $y \in \mathcal{M}$, the differential of F takes the almost complex structure on $T_{\beta(y)}\mathcal{C}$ to the almost complex structure on $T_{\gamma(y)}\text{Conn}(\mathcal{L})$, where β and γ are the sections in (2.11) and (2.13) respectively. Indeed, this follows from the constructions of $(d\beta)'$ and $(d\gamma)'$ (see (2.10)), and the fact that they differ by multiplication by $2r$ (which follows from Lemma 2.1 and Lemma 2.2).

Consider the diffeomorphisms

$$\alpha_1 : T^*\mathcal{M} \longrightarrow \mathcal{C} \quad \text{and} \quad \alpha_2 : T^*\mathcal{M} \longrightarrow \text{Conn}(\mathcal{L})$$

that send any $v \in T_y^*\mathcal{M}$ to $\beta(y) + v$ and $\gamma(y) + 2r \cdot v$ respectively. From the three properties of the map F in (2.16) it follows that

$$F \circ \alpha_1 = \alpha_2. \tag{2.17}$$

Let J_1 (respectively, J_2) denote the almost complex structure on the total space of $T^*\mathcal{M}$ obtained by pulling back the almost complex structure on \mathcal{C} (respectively, $\text{Conn}(\mathcal{L})$) using this diffeomorphism α_1 (respectively, α_2).

From the above observation that the differential of F takes the almost complex structure on $T_{\beta(y)}\mathcal{C}$ to the almost complex structure on $T_{\gamma(y)}\text{Conn}(\mathcal{L})$ for every $y \in \mathcal{M}$, it follows that J_1 and J_2 coincide for every point $(y, 0) \in T^*\mathcal{M}$, $y \in \mathcal{M}$. Also, the restrictions of J_1 and J_2 to the fiber $T_y^*\mathcal{M}$ coincide for every $y \in \mathcal{M}$.

Let J_0 denote the natural almost complex structure on $T^*\mathcal{M}$ given by the complex structure on \mathcal{M} . Both the almost complex structures J_1 and J_2 on $T^*\mathcal{M}$ have the property that they coincide with the tautological almost complex structure J_0 . Using these properties of J_1 and J_2 it follows that J_1 coincides with J_2 . In view of (2.17), this implies that the map F is holomorphic. \square

REMARK 2.4. There is an algebro-geometric construction of an isomorphism of the form in Proposition 2.3. To simplify, let us suppose that we are in the case where the degree is $r(g - 1)$; this means that $H^0(X, E) = 0 = H^1(X, E)$ for all E in a nonempty Zariski open subset of $\mathcal{M}_X(r, r(g - 1))$. The locus where this does not hold is the theta divisor $\Theta \subset \mathcal{M}_X(r, r(g - 1))$. There is a natural section of the determinant line bundle, which vanishes on Θ , so we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}_X(r, r(g-1))} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_\Theta \longrightarrow 0$$

For any $E \in \mathcal{M}_X(r, r(g - 1)) \setminus \Theta$, on $X \times X$, the Künneth formula and Serre duality together give

$$H^0(X \times X, E \boxtimes (E^* \times K_X)) = 0 = H^1(X \times X, E \boxtimes (E^* \otimes K_X)),$$

so from the long exact sequence of cohomologies for the short exact sequence of sheaves

$$0 \longrightarrow E \boxtimes (E^* \times K_X) \longrightarrow E \boxtimes (E^* \otimes K_X)(\Delta) \longrightarrow (E \boxtimes (E^* \otimes K_X)(\Delta))|_\Delta \longrightarrow 0,$$

where $\Delta \subset X \times X$ is the reduced diagonal and $W(\Delta) = W \otimes \mathcal{O}_{X \times X}(\Delta)$ for any holomorphic vector bundle W on $X \times X$, we obtain an isomorphism

$$H^0(X \times X, E \boxtimes (E^* \otimes K_X)(\Delta)) = H^0(\Delta, (E \boxtimes (E^* \otimes K_X)(\Delta))|_\Delta) = H^0(X, \text{End}(E)).$$

Now choose a theta-characteristic $K_X^{1/2}$ on X , and write $E = V \otimes K_X^{1/2}$, so that V is of degree zero. Consequently, we have

$$H^0(X \times X, (V \boxtimes V^*) \otimes (K_X^{1/2} \boxtimes K_X^{1/2})(\Delta)) = H^0(X, \text{End}(V)).$$

Let $s_V \in H^0(X \times X, (V \boxtimes V^*) \otimes (K_X^{1/2} \boxtimes K_X^{1/2})(\Delta))$ be the section corresponding to $\text{Id}_V \in H^0(X, \text{End}(V))$. Note that $(K_X^{1/2} \boxtimes K_X^{1/2})(\Delta)|_\Delta = \mathcal{O}_\Delta$, and the isomorphism extends to 2Δ , in a unique way if one imposes anti-symmetry under involution of $X \times X$. Thus, s_V gives a section of $H^0(2\Delta, V \boxtimes V^*)$ that extends the identity automorphism of V over Δ . But such a section defines a holomorphic connection on V .

Now let \mathbb{U} be the Zariski open dense subset of the moduli space $\mathcal{M}_X(r, 0)$ parametrizing all V such that $H^i(X, V \otimes K_X^{1/2}) = 0$ for $i = 0, 1$. The above construction of holomorphic connection produces a section of \mathcal{C} over \mathbb{U} . On the other hand the pullback of the theta line bundle has a canonical trivialization over \mathbb{U} . Consequently, we get an isomorphism of two $T^*\mathbb{U}$ -torsors on \mathbb{U} as in Proposition 2.3.

A natural question is whether the above isomorphism of $T^*\mathbb{U}$ -torsors coincides with the isomorphism constructed in Proposition 2.3.

3. Holomorphic symplectic forms. Consider the holomorphic line bundle $q_0^*\mathcal{L} \longrightarrow \text{Conn}(\mathcal{L})$, where q_0 is the projection in (2.9). Since $\text{Conn}(\mathcal{L})$ is defined by the sheaf of holomorphic connections on \mathcal{L} , there is a tautological holomorphic connection $\widehat{\nabla}$ on the pulled back holomorphic line bundle $q_0^*\mathcal{L}$. To briefly describe $\widehat{\nabla}$, first note that there is a homomorphism

$$\beta_0 : q_0^*\text{At}(\mathcal{L}) \longrightarrow q_0^*\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\text{Conn}(\mathcal{L})}$$

given by the tautological splitting of the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow q_0^*\mathcal{O}_{\mathcal{M}} \longrightarrow q_0^*\text{At}(\mathcal{L}) \longrightarrow q_0^*T\mathcal{M} \longrightarrow 0$$

on $\text{Conn}(\mathcal{L})$. On the other hand, there is a tautological projection

$$\beta'_0 : \text{At}(q_0^*\mathcal{L}) \longrightarrow q_0^*\text{At}(\mathcal{L})$$

such that the diagram

$$\begin{array}{ccc} \text{At}(q_0^*\mathcal{L}) & \xrightarrow{\beta'_0} & q_0^*\text{At}(\mathcal{L}) \\ \downarrow \zeta' & & \downarrow q_0^*\zeta \\ T\text{Conn}(\mathcal{L}) & \xrightarrow{dq_0} & q_0^*T\mathcal{M} \end{array}$$

is commutative, where dq_0 is the differential of the projection q_0 in (2.9), ζ is the projection in (2.7) and ζ' is the natural projection of $\text{At}(q_0^*\mathcal{L})$ to $T\text{Conn}(\mathcal{L})$ (asin(2.7)). Now the composition

$$\beta_0 \circ \beta'_0 : \text{At}(q_0^*\mathcal{L}) \longrightarrow \mathcal{O}_{\text{Conn}(\mathcal{L})}$$

gives a splitting of the Atiyah exact sequence for the holomorphic line bundle $q_0^*\mathcal{L}$. This splitting $\beta_0 \circ \beta'_0$ defines the tautological connection $\widehat{\nabla}$ on $q_0^*\mathcal{L}$.

The curvature $\Omega_{\mathcal{L}}$ of the above holomorphic connection $\widehat{\nabla}$ is a closed holomorphic 2-form on $\text{Conn}(\mathcal{L})$. This holomorphic 2-form $\Omega_{\mathcal{L}}$ is symplectic. To see this, choose a local holomorphic trivialization of \mathcal{L} over an open subset $U \subset \mathcal{M}$. Using the trivial connection of a trivial line bundle, the inverse image $q_0^{-1}(U)$ gets identified with T^*U ; this identification sends the zero section of T^*U to the section of $q_0^{-1}(U)$ given by the trivial connection on $\mathcal{L}|_U$. In terms of this identification, the connection $\widehat{\nabla}|_{q_0^{-1}(U)}$ becomes the Liouville 1-form on T^*U . Therefore, the 2-form $\Omega_{\mathcal{L}}|_{q_0^{-1}(U)}$ coincides with the exterior derivative of the Liouville 1-form, and hence it is nondegenerate. So $\Omega_{\mathcal{L}}$ is a symplectic form as it is closed.

We shall describe some properties of the above symplectic form $\Omega_{\mathcal{L}}$ on $\text{Conn}(\mathcal{L})$. Take any C^∞ section

$$s_0 : \mathcal{M} \longrightarrow \text{Conn}(\mathcal{L})$$

of the projection q_0 in (2.9), so $q_0 \circ s_0 = \text{Id}_{\mathcal{M}}$. This s_0 defines a C^∞ complex connection on \mathcal{L} ; we shall denote this complex connection by ∇^{s_0} . This connection ∇^{s_0} coincides with the pulled back connection $s_0^*\widehat{\nabla}$ after invoking the natural identification of $s_0^*q_0^*\mathcal{L}$ with \mathcal{L} . This implies that the curvature of the connection ∇^{s_0} is the pulled back form $s_0^*\Omega_{\mathcal{L}}$.

Note that $s_0^*\Omega_{\mathcal{L}}$ need not be holomorphic, because the map s_0 need not be holomorphic. Let

$$\theta \in C^\infty(U, T^*U)$$

be a smooth $(1, 0)$ -form defined on an open subset $U \subset \mathcal{M}$. Then $s_0 + \theta$ is a C^∞ section of q_0 over the open subset U , which is constructed using the $T^*\mathcal{M}$ -torsor structure of $\text{Conn}(\mathcal{L})$. Since the curvature of the connection ∇^{s_0} on \mathcal{L} , given by a section s_0 , is $s_0^*\Omega_{\mathcal{L}}$, it follows immediately that

$$((s_0 + \theta)^*\Omega_{\mathcal{L}})|_U = (s_0^*\Omega_{\mathcal{L}})|_U + d\theta. \tag{3.1}$$

The fibers of the projection q_0 are Lagrangian with respect to the symplectic form $\Omega_{\mathcal{L}}$, because the fibers of the cotangent bundle $T^*\mathcal{M}$ are Lagrangian with respect to the Liouville symplectic form on $T^*\mathcal{M}$.

As in (2.3), let $\Omega_{\mathcal{C}}$ denote the holomorphic symplectic form on \mathcal{C} .

THEOREM 3.1. *For the biholomorphism F in Proposition 2.3,*

$$F^*\Omega_{\mathcal{L}} = 2r \cdot \Omega_{\mathcal{C}}.$$

Proof. We shall first show that the symplectic form $\Omega_{\mathcal{C}}$ on \mathcal{C} is compatible with the $T^*\mathcal{M}$ -torsor structure of \mathcal{C} . The compatibility condition in question says that for every locally defined holomorphic section

$$\mathcal{M} \supset U \xrightarrow{\sigma} \mathcal{C}$$

of the projection ϕ in (2.4), where $U \subset \mathcal{M}$ is an open subset, and a holomorphic 1-form θ on U ,

$$(\sigma + \theta)^*\Omega_{\mathcal{C}} = \sigma^*\Omega_{\mathcal{C}} + d\theta. \tag{3.2}$$

The set-up of [AB] will be used for proving (3.2): we compute on the infinite dimensional space of connections, and then quotient by the gauge group. Fix a C^∞ complex vector bundle V on X of rank r and degree d . Let

$$\mathcal{A}^{0,1} := C^\infty(X, \text{End}(V) \otimes (T^{0,1}X)^*) \quad \text{and} \quad \mathcal{A}^{1,0} := C^\infty(X, \text{End}(V) \otimes (T^{1,0}X)^*)$$

be respectively the spaces of all smooth $(0, 1)$ -forms and $(1, 0)$ -forms on X with values in $\text{End}(V)$. Using the nondegenerate pairing on $\mathcal{A}^{0,1} \oplus \mathcal{A}^{1,0}$

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) \mapsto \int_X \text{trace}(\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1), \tag{3.3}$$

identify $\mathcal{A}^{0,1} \times \mathcal{A}^{1,0}$ with a subset of the holomorphic cotangent bundle $(T^{1,0}\mathcal{A}^{0,1})^*$. Therefore, the restriction to $\mathcal{A}^{0,1} \oplus \mathcal{A}^{1,0}$ of the Liouville symplectic form on $(T^{1,0}\mathcal{A}^{0,1})^*$ coincides with the one given by the pairing in (3.3). The two-form on $\mathcal{A}^{0,1} \times \mathcal{A}^{1,0}$ given by the pairing in (3.3) will be denoted by Ω'_{AB} .

A Dolbeault operator on V is a differential operator

$$\bar{\partial}_1 : V \longrightarrow V \otimes (T^{0,1}X)^*$$

of order one satisfying the Leibniz condition which says that

$$\bar{\partial}_1(fs) = f \cdot \bar{\partial}_1(s) + s \otimes \bar{\partial}f,$$

where s is any locally defined smooth section of V and f is any locally defined smooth function on X . Let \mathcal{B} denote the space of all Dolbeault operators on V . So \mathcal{B} is an affine space for the complex vector space $\mathcal{A}^{0,1}$. Let \mathcal{G} denote the space of all differential operators

$$\partial_1 : V \longrightarrow V \otimes (T^{1,0}X)^*$$

of order one satisfying the condition that

$$\partial_1(fs) = f \cdot \partial_1(s) + s \otimes \partial f,$$

where s is any locally defined smooth section of V and f is any locally defined smooth function. So \mathcal{G} is an affine space for the complex vector space $\mathcal{A}^{1,0}$ defined above.

Therefore, as before, the pairing in (3.3) produces a 2-form on $\mathcal{B} \times \mathcal{G}$; this 2-form on $\mathcal{B} \times \mathcal{G}$ will be denoted by Ω_{AB} . This Ω_{AB} actually coincides with the 2-form Ω'_{AB} on $\mathcal{A}^{0,1} \times \mathcal{A}^{1,0}$, once we identify $\mathcal{B} \times \mathcal{G}$ with $\mathcal{A}^{0,1} \times \mathcal{A}^{1,0}$ by fixing a point of $\mathcal{B} \times \mathcal{G}$.

The symplectic form $\Omega_{\mathcal{C}}$ on \mathcal{C} is constructed from the above form Ω_{AB} as follows.

For any $(\bar{\partial}_1, \partial_1) \in \mathcal{B} \times \mathcal{G}$, note that $\partial_1 + \bar{\partial}_1$ is a C^∞ complex connection on the vector bundle V . The curvature of the connection $\partial_1 + \bar{\partial}_1$ will be denoted by $(\partial_1 + \bar{\partial}_1)^2$. Define

$$\mathcal{F} := \{(\bar{\partial}_1, \partial_1) \in \mathcal{B} \times \mathcal{G} \mid (\partial_1 + \bar{\partial}_1)^2 = 0\}.$$

Restrict Ω_{AB} to \mathcal{F} . Let $\text{Aut}(V)$ denote the group of all C^∞ automorphisms of the vector bundle V over the identity map of X . The group $\text{Aut}(V)$ acts on \mathcal{F} by inducing

connections on V from given ones via automorphism of V . The above restricted form $\Omega_{AB}|_{\mathcal{F}}$ descends to a 2-form to the quotient under this action. The symplectic manifold $(\mathcal{C}, \Omega_{\mathcal{C}})$ is given by this quotient of \mathcal{F} and the descended 2-form on the quotient.

Since Ω_{AB} is given by the Liouville symplectic form Ω'_{AB} on $\mathcal{A}^{0,1} \times \mathcal{A}^{1,0}$, we conclude that the identity in (3.2) holds.

In view of (3.1) and (3.2), the theorem follows from Lemma 2.1, Lemma 2.2 and Proposition 2.3. \square

4. Tyurin parametrization. Fix a holomorphic vector bundle \mathcal{E}_0 on X of rank r and degree $d + d_0$. Let

$$\mathcal{Q} := \mathcal{Q}(\mathcal{E}_0, d_0) \tag{4.1}$$

denote the quot scheme parametrizing all torsion quotients of \mathcal{E}_0 of degree d_0 . On $X \times \mathcal{Q}$, there is a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow p_X^* \mathcal{E}_0 \longrightarrow Q \longrightarrow 0, \tag{4.2}$$

where $p_X : X \times \mathcal{Q} \longrightarrow X$ is the natural projection and Q is the tautological torsion quotient on $X \times \mathcal{Q}$. We note that \mathcal{K} is a holomorphic family of vector bundles of rank r degree d on X parametrized by \mathcal{Q} . For any $z \in \mathcal{Q}$, let $\mathcal{K}^z := \mathcal{K}|_{X \times \{z\}}$ be the vector bundle on X in this family corresponding to the point z . Let

$$\mathcal{Q}_s := \{z \in \mathcal{Q} \mid \mathcal{K}^z \text{ is stable}\} \subset \mathcal{Q} \tag{4.3}$$

be the subset that parametrizes all the stable vector bundles in this family parametrized by \mathcal{Q} . It is known that \mathcal{Q}_s is a Zariski open subset of \mathcal{Q} [Ma, p. 635, Theorem 2.8(B)] (see also [Sha]). For a generic choice of \mathcal{E}_0 one can ensure that this Zariski open subset \mathcal{Q}_s is non-empty.

We shall construct two holomorphic $T^*\mathcal{Q}_s$ -torsors over \mathcal{Q}_s . We note that the dimension of \mathcal{Q}_s does not necessarily match that of the space \mathcal{M} , so it will not be merely a question of pulling back our two torsors over \mathcal{M} .

Let

$$\xi : \mathcal{Q}_s \longrightarrow \mathcal{M} \tag{4.4}$$

be the classifying morphism for the family of stable vector bundles \mathcal{K} in (4.2). So for any $z \in \mathcal{Q}_s$, the point of \mathcal{M} corresponding to the stable vector bundle \mathcal{K}^z is $\xi(z)$. Let

$$(d\xi)^* : \xi^* \Omega_{\mathcal{M}} = \xi^* T^* \mathcal{M} \longrightarrow T^* \mathcal{Q}_s \tag{4.5}$$

be the homomorphism of cotangent bundles given by the dual of the differential $d\xi : T\mathcal{Q}_s \longrightarrow \xi^* T\mathcal{M}$ of the map ξ constructed in (4.4). Let

$$\xi^* \mathcal{C} \longrightarrow \mathcal{Q}_s$$

be the pull-back to \mathcal{Q}_s of the holomorphic fiber bundle $\phi : \mathcal{C} \longrightarrow \mathcal{M}$ in (2.4). Consider the action of $\xi^* T^* \mathcal{M}$ on the fiber product

$$T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}$$

under which each $v \in T_{\xi(y)}^* \mathcal{M}$, $y \in \mathcal{Q}_s$, sends every $(w, u) \in T_y^* \mathcal{Q}_s \times \phi^{-1}(\xi(y))$ to

$$(w - (d\xi)^*(v), u + v) \in T_y^* \mathcal{Q}_s \times \phi^{-1}(\xi(y)).$$

Let

$$\mathcal{C}(\mathcal{Q}_s) := (T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}) / \xi^* T^* \mathcal{M} \tag{4.6}$$

be the quotient for this action of $\xi^* T^* \mathcal{M}$ on $T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}$. Let

$$\psi : \mathcal{C}(\mathcal{Q}_s) \longrightarrow \mathcal{Q}_s \tag{4.7}$$

be the natural projection given by the projections $T^* \mathcal{Q}_s \longrightarrow \mathcal{Q}_s$ and $\xi^* \mathcal{C} \longrightarrow \mathcal{Q}_s$.

REMARK: This quotienting appears in the paper [Hu] in a different guise.

The translation action of $T^* \mathcal{Q}_s$ on itself and the trivial action of $T^* \mathcal{Q}_s$ on $\xi^* \mathcal{C}$ together produce an action of $T^* \mathcal{Q}_s$ on $T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}$. This action of $T^* \mathcal{Q}_s$ on $T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}$ clearly descends to an action of $T^* \mathcal{Q}_s$ on the quotient space $\mathcal{C}(\mathcal{Q}_s)$ constructed in (4.6). It is straightforward to check that this action of $T^* \mathcal{Q}_s$ on $\mathcal{C}(\mathcal{Q}_s)$ makes $(\mathcal{C}(\mathcal{Q}_s), \psi)$ in (4.7) a holomorphic torsor on \mathcal{Q}_s for $T^* \mathcal{Q}_s$.

Note that there is a natural map to $\mathcal{C}(\mathcal{Q}_s)$ from the moduli space of pairs of the form (z, D) , where

- $z \in \mathcal{Q}_s$, and
- D is a logarithmic connection on \mathcal{K}^z (see (4.3)) nonsingular on $X \setminus \{x_0\}$ such that the residue of D at x_0 is $-\frac{d}{r} \text{Id}_{E_{x_0}}$.

This map sends (z, D) to the point of $\psi^{-1}(z)$ given by the pair $(0, D) \in T^* \mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^* \mathcal{C}$ (see (4.6)). It may be mentioned that this map to $\mathcal{C}(\mathcal{Q}_s)$ from the moduli space of pairs need not be injective or surjective in general. This map is injective if the differential $d\xi(z)$ is surjective for all $z \in \mathcal{Q}_s$, and it is surjective if $d\xi(z)$ is injective for all $z \in \mathcal{Q}_s$.

To construct the second torsor for $T^* \mathcal{Q}_s$, let

$$\tilde{f} : X \times \mathcal{Q} \longrightarrow \mathcal{Q}$$

be the projection to the second factor. For the family of holomorphic vector bundles \mathcal{K} in (4.2) parametrized by \mathcal{Q} , consider the holomorphic line bundle constructed in (2.5). Let

$$L := (\det R^0 \tilde{f}_* \mathcal{K})^{\otimes -r} \otimes (\det R^1 \tilde{f}_* \mathcal{K})^{\otimes r} \otimes \det(s_{x_0}^* \mathcal{K})^{\otimes X} \longrightarrow \mathcal{Q} \tag{4.8}$$

be this line bundle, where $s_{x_0} : \mathcal{Q} \longrightarrow X \times \mathcal{Q}$ as in (2.5) is the section $y \longmapsto (x_0, y)$. Construct

$$\tilde{\varphi} : \text{Conn}(L) \longrightarrow \mathcal{Q}$$

as in (2.9), so $\text{Conn}(L) \subset \text{At}(L)^*$, where $\text{At}(L) \longrightarrow \mathcal{Q}$ is the Atiyah bundle for L . We note that $(\text{Conn}(L), \tilde{\varphi})$ is a torsor over \mathcal{Q} for $T^* \mathcal{Q}$. Let

$$\text{Conn}(L)_s := \tilde{\varphi}^{-1}(\mathcal{Q}_s) \subset \text{Conn}(L) \tag{4.9}$$

be the Zariski open subset, where \mathcal{Q}_s is the Zariski open subset in (4.3). Let

$$\varphi : \text{Conn}(L)_s \longrightarrow \mathcal{Q}_s \tag{4.10}$$

be the restriction of the map $\tilde{\varphi}$ to $\text{Conn}(L)_s$. Consequently, $(\text{Conn}(L)_s, \varphi)$ is a torsor over \mathcal{Q}_s for $T^* \mathcal{Q}_s$.

Let

$$\begin{aligned} \tilde{\delta} &: \mathcal{C}(\mathcal{Q}_s) \times_{\mathcal{Q}_s} T^*\mathcal{Q}_s \longrightarrow \mathcal{C}(\mathcal{Q}_s) \\ \text{and } \tilde{\eta} &: \text{Conn}(L)_s \times_{\mathcal{Q}_s} T^*\mathcal{Q}_s \longrightarrow \text{Conn}(L)_s \end{aligned} \tag{4.11}$$

be the $T^*\mathcal{Q}_s$ -torsor structures on $\mathcal{C}(\mathcal{Q}_s)$ and $\text{Conn}(L)_s$ constructed in (4.7) and (4.10) respectively. Let

$$\mathbf{m}_1 : T^*\mathcal{Q}_s \longrightarrow T^*\mathcal{Q}_s, \quad v \longmapsto 2r \cdot v \tag{4.12}$$

be the multiplication by $2r$.

THEOREM 4.1. *There is a canonical biholomorphic map*

$$\Phi : \mathcal{C}(\mathcal{Q}_s) \longrightarrow \text{Conn}(L)_s$$

such that

- (1) $\varphi \circ \Phi = \psi$, where ψ and φ are the projections in (4.7) and (4.10) respectively, and
- (2) $\Phi \circ \tilde{\delta} = \tilde{\eta} \circ (\Phi \times \mathbf{m}_1)$, where $\tilde{\delta}$, $\tilde{\eta}$ and \mathbf{m}_1 are constructed in (4.11) and (4.12).

Proof. From the constructions of the map ξ in (4.4) and the line bundles \mathcal{L} and L (in (2.6) and (4.8)) it follows that the line bundle $\xi^*\mathcal{L}$ is holomorphically identified with $L|_{\mathcal{Q}_s}$. In view of this identification of $L|_{\mathcal{Q}_s}$ with $\xi^*\mathcal{L}$, we conclude that the fiber bundle $\text{Conn}(L)_s$ in (4.9) is constructed from $\text{Conn}(\mathcal{L})$ (see (2.9)) in the following way.

Consider the holomorphic fiber bundle $q_0 : \text{Conn}(\mathcal{L}) \longrightarrow \mathcal{M}$ in (2.9). Let

$$\xi^*\text{Conn}(\mathcal{L}) \longrightarrow \mathcal{Q}_s$$

be the pull-back of it by the map ξ in (4.4). Next consider the homomorphism $(d\xi)^*$ in (4.5). The holomorphic vector bundle $\xi^*T^*\mathcal{M}$ acts on the fiber product

$$T^*\mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^*\text{Conn}(\mathcal{L})$$

as follows: for every $y \in \mathcal{Q}_s$ and every $v \in T^*_{\xi(y)}\mathcal{M}$, the action of v sends any $(w, u) \in T^*_y\mathcal{Q}_s \times q_0^{-1}(\xi(y))$ to

$$(w - (d\xi)^*(v), u + v) \in T^*_y\mathcal{Q}_s \times q_0^{-1}(\xi(y)).$$

Let

$$\mathcal{Z} := (T^*\mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^*\text{Conn}(\mathcal{L})) / \xi^*T^*\mathcal{M} \tag{4.13}$$

be the quotient for this action. The projections $T^*\mathcal{Q}_s \longrightarrow \mathcal{Q}_s$ and $\xi^*\text{Conn}(\mathcal{L}) \longrightarrow \mathcal{Q}_s$ together produce a projection of \mathcal{Z} to \mathcal{Q}_s . The translation action of $T^*\mathcal{Q}_s$ on itself and the trivial action of $T^*\mathcal{Q}_s$ on $\xi^*\text{Conn}(\mathcal{L})$ together produce an action of $T^*\mathcal{Q}_s$ on $T^*\mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^*\text{Conn}(\mathcal{L})$. This action in turn produces an action of $T^*\mathcal{Q}_s$ on the quotient space \mathcal{Z} constructed in (4.13). This action of $T^*\mathcal{Q}_s$ on \mathcal{Z} makes \mathcal{Z} a holomorphic torsor on \mathcal{Q}_s for $T^*\mathcal{Q}_s$. It should be emphasized that this $T^*\mathcal{Q}_s$ -torsor \mathcal{Z} is holomorphically identified, in a natural way, with the $T^*\mathcal{Q}_s$ -torsor $\text{Conn}(L)_s$ constructed in (4.9).

The biholomorphism Φ in the statement of the theorem is constructed by comparing the above description of $\text{Conn}(L)_s$ with the construction of $\mathcal{C}(\mathcal{Q}_s)$ in (4.6). To see this, consider the map

$$\tilde{F} : T^*\mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^*\mathcal{C} \longrightarrow T^*\mathcal{Q}_s \times_{\mathcal{Q}_s} \xi^*\text{Conn}(\mathcal{L}), \quad (a, b) \longmapsto (a, \xi^*F(b)),$$

where F is the biholomorphism in Proposition 2.3; note that F induces a map of fiber bundles over \mathcal{Q}_s

$$\xi^*F : \xi^*\mathcal{C} \longrightarrow \xi^*\text{Conn}(\mathcal{L})$$

which is uniquely determined by the following commutative diagram:

$$\begin{array}{ccc} \xi^*\mathcal{C} & \xrightarrow{\xi^*F} & \xi^*\text{Conn}(\mathcal{L}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \text{Conn}(\mathcal{L}) \end{array}$$

(the vertical maps are the canonical ones). This map \tilde{F} descends to a map between the quotient spaces

$$\Phi : \mathcal{C}(\mathcal{Q}_s) \longrightarrow \text{Conn}(L)_s = \mathcal{Z}$$

in (4.13) and (4.6). From the properties of F in Proposition 2.3 it follows that Φ satisfies the two conditions in the theorem. \square

The pulled back holomorphic line bundle $\varphi^*L \longrightarrow \text{Conn}(L)_s$, where φ is the projection in (4.10), has a tautological holomorphic connection; this tautological holomorphic connection on φ^*L will be denoted by D_{φ^*L} . The curvature of D_{φ^*L} , which will be denoted by $\tilde{\Theta}$, is a holomorphic symplectic form on $\text{Conn}(L)_s$. Consider the biholomorphism Φ in Theorem 4.1. Let

$$\Theta := \frac{1}{2r}\Phi^*\tilde{\Theta} \tag{4.14}$$

be the holomorphic symplectic form on $\mathcal{C}(\mathcal{Q}_s)$.

The above construction is summarized in the following lemma.

LEMMA 4.2. *The complex manifold $\mathcal{C}(\mathcal{Q}_s)$ is equipped with a natural holomorphic symplectic form Θ constructed in (4.14). The fibers of the projection ψ in (4.7) are Lagrangian with respect to Θ . The form Θ is compatible with the $T^*\mathcal{Q}_s$ -torsor structure on $\mathcal{C}(\mathcal{Q}_s)$ in the following way: If $s : U \longrightarrow \mathcal{C}(\mathcal{Q}_s)$ is a holomorphic section of the fibration ψ over an open subset $U \subset \mathcal{Q}_s$, and ω is a holomorphic 1-form on U , then for the holomorphic section*

$$s_\omega : U \longrightarrow \mathcal{C}(\mathcal{Q}_s), \quad z \longmapsto s(z) + \omega(z),$$

the equation

$$s_\omega^*\Theta = s^*\Theta + d\omega$$

holds.

Proof. For the symplectic form $\tilde{\Theta}$ on $\text{Conn}(L)_s$, the fibers of φ (see (4.10)) are Lagrangian.

If $s : U \longrightarrow \text{Conn}(L)_s$ is a holomorphic section of the fibration φ over an open subset $U \subset \mathcal{Q}_s$, and ω is a holomorphic 1-form on U , then for the holomorphic section

$$s_\omega : U \longrightarrow \text{Conn}(L)_s, \quad z \longmapsto s(z) + \omega(z),$$

the equation

$$s_\omega^* \tilde{\Theta} = s^* \tilde{\Theta} + d\omega$$

holds; see (3.1).

Therefore, from Theorem 4.1 we conclude that Θ has the properties stated in the lemma. \square

REMARK 4.3. Note that the $T^*\mathcal{Q}_s$ -torsor $\mathcal{C}(\mathcal{Q}_s)$ does not have any natural extension to a $T^*\mathcal{Q}$ -torsor over the larger variety \mathcal{Q} in (4.3). Indeed, for some $z \in \mathcal{Q} \setminus \mathcal{Q}_s$ the vector bundle \mathcal{K}^z in (4.3) may not have any logarithmic/holomorphic connection satisfying the residue condition in the definition on \mathcal{C} (see (2.2)). However, the other $T^*\mathcal{Q}_s$ -torsor, namely $\text{Conn}(L)_s$, has a canonical extension to a $T^*\mathcal{Q}$ -torsor over the larger variety \mathcal{Q} . Indeed, $\text{Conn}(L)$ is the canonical extension of $\text{Conn}(L)_s$ (see (4.9)). Note that the symplectic structure on $\text{Conn}(L)_s$ also extends along this extension.

5. Framed bundles and meromorphic connections.

5.1. Framed bundles. Fix a nonzero effective divisor

$$\mathbb{S} = \sum_{i=1}^n n_i x_i \tag{5.1}$$

on X ; so $x_i \in X$, $n \geq 1$ and $n_i \geq 1$ for all $1 \leq i \leq n$. For notational convenience, $V \otimes \mathcal{O}_X(-\mathbb{S})$ and $V \otimes \mathcal{O}_X(\mathbb{S})$, where V is any coherent analytic sheaf on X , will be denoted by $V(-\mathbb{S})$ and $V(\mathbb{S})$ respectively.

A framed vector bundle is a holomorphic vector bundle E on X together with an isomorphism of \mathcal{O}_X -modules $\sigma : E|_{\mathbb{S}} \rightarrow \mathcal{O}_{\mathbb{S}}^{\oplus r}$, where $r = \text{rank}(E)$. The space of infinitesimal deformations of a framed bundle (E, σ) are parametrized by $H^1(X, \text{End}(E)(-\mathbb{S}))$. Consider the short exact sequence of coherent analytic sheaves

$$0 \rightarrow \text{End}(E)(-\mathbb{S}) \rightarrow \text{End}(E) \rightarrow \text{End}(E)|_{\mathbb{S}} \rightarrow 0$$

on X . Let

$$\rightarrow H^0(X, \text{End}(E)|_{\mathbb{S}}) \xrightarrow{h_1} H^1(X, \text{End}(E)(-\mathbb{S})) \xrightarrow{h_2} H^1(X, \text{End}(E)) \rightarrow 0$$

be the corresponding long exact sequence of cohomologies. The homomorphism h_1 in this long exact sequence corresponds to deforming the framing σ keeping the vector bundle E fixed, while the other homomorphism h_2 is the forgetful map that sends an infinitesimal deformation of (E, σ) to the infinitesimal deformation of E given by it by simply forgetting the framing. Note that by Serre duality,

$$H^1(X, \text{End}(E)(-\mathbb{S}))^* = H^0(X, \text{End}(E) \otimes K_X(\mathbb{S})). \tag{5.2}$$

A meromorphic connection on E is a holomorphic differential operator of order one

$$D : E \rightarrow E \otimes K_X(\mathbb{S})$$

that satisfies the Leibniz identity which says that

$$D(fs) = f \cdot D(s) + s \otimes \partial f,$$

where s is any locally defined holomorphic section of E and f is any locally defined holomorphic function on X .

A *framed meromorphic connection* is a triple of the form (E, σ, D) , where (E, σ) is a framed bundle and D is a meromorphic connection on E .

Let

$$\mathcal{N}_X(r, d) =: \mathcal{N} \tag{5.3}$$

be the moduli space of all isomorphism classes framed vector bundle (E, σ) of rank r and degree d such that the underlying vector bundle E is stable. Let

$$\mathcal{D}_X(r, d) =: \mathcal{D} \tag{5.4}$$

denote the moduli space of isomorphism classes of framed meromorphic connections (E, σ, D) such that

- (1) E is a stable vector bundle of rank r and degree d ,
- (2) σ is a framing on E over \mathbb{S} , and
- (3) D is a meromorphic connection on E whose polar part has support contained in \mathbb{S} .

Let

$$\widehat{\phi} : \mathcal{D} \longrightarrow \mathcal{N}, \quad (E, \sigma, D) \longmapsto (E, \sigma) \tag{5.5}$$

be the forgetful map that simply forgets the meromorphic connection. Since the divisor \mathbb{S} is nonzero, it can be shown that any stable vector bundle admits a meromorphic connection. Indeed, if E is a stable vector bundle on X of rank r and degree d , and $x_1 \in \mathbb{D}$, then E admits a logarithmic connection nonsingular over $X \setminus \{x_1\}$ whose monodromy is unitary and its residue at x_1 is $-\frac{d}{r}\text{Id}_{E_{x_1}}$ [NS].

The space of all meromorphic connections on E is an affine space for the vector space $H^0(X, \text{End}(E) \otimes K_X(\mathbb{S}))$. Hence using (5.2) it follows that $(\mathcal{D}, \widehat{\phi})$ is a holomorphic $T^*\mathcal{N}$ -torsor over \mathcal{N} , where \mathcal{D} and $\widehat{\phi}$ are constructed in (5.4) and (5.5) respectively.

Let

$$\varpi : \mathcal{N} \longrightarrow \mathcal{M} \tag{5.6}$$

be the projection defined by $(E, \sigma) \longmapsto E$. Consider the line bundle \mathcal{L} in (2.6) constructed by setting the base point x_0 to be the point x_1 in (5.1). Let

$$\varpi^*\mathcal{L} \longrightarrow \mathcal{N}$$

be its pullback to \mathcal{N} by the map in (5.6). Construct

$$q_1 : \text{Conn}(\varpi^*\mathcal{L}) \longrightarrow \mathcal{N} \tag{5.7}$$

as in (2.9) from the Atiyah bundle $\text{At}(\varpi^*\mathcal{L})$. We note that $(\text{Conn}(\varpi^*\mathcal{L}), q_1)$ is a torsor over \mathcal{N} for $T^*\mathcal{N}$.

Let

$$\widehat{\delta} : \mathcal{D} \times_{\mathcal{N}} T^*\mathcal{N} \longrightarrow \mathcal{D} \quad \text{and} \quad \widetilde{\eta} : \text{Conn}(\varpi^*\mathcal{L}) \times_{\mathcal{N}} T^*\mathcal{N} \longrightarrow \text{Conn}(\varpi^*\mathcal{L}) \tag{5.8}$$

be the $T^*\mathcal{N}$ -torsor structures on \mathcal{D} and $\text{Conn}(\varpi^*\mathcal{L})$ respectively. Let

$$\mathbf{m}_2 : T^*\mathcal{N} \longrightarrow T^*\mathcal{N}, \quad v \longmapsto 2r \cdot v \tag{5.9}$$

be the multiplication by $2r$.

THEOREM 5.1. *There is a canonical biholomorphic map*

$$\Psi : \mathcal{D} \longrightarrow \text{Conn}(\varpi^*\mathcal{L})$$

such that

- (1) $q_1 \circ \Psi = \widehat{\phi}$, where $\widehat{\phi}$ and q_1 are the projections in (5.5) and (5.7) respectively, and
- (2) $\Psi \circ \widehat{\delta} = \widehat{\eta} \circ (\Psi \times \mathbf{m}_2)$, where $\widehat{\delta}$, $\widehat{\eta}$ and \mathbf{m}_2 are constructed in (5.8) and (5.9).

Proof. Let $(d\varpi)^* : \varpi^*T^*\mathcal{M} \longrightarrow T^*\mathcal{N}$ be the dual of the differential

$$d\varpi : T\mathcal{N} \longrightarrow \varpi^*T\mathcal{M}$$

of the map ϖ in (5.6). Using it, the pullback, to \mathcal{N} , of a $T^*\mathcal{M}$ -torsor on \mathcal{M} produces a $T^*\mathcal{N}$ -torsor on \mathcal{N} . To give more details of this construction, let \mathcal{T} be a $T^*\mathcal{M}$ -torsor on \mathcal{M} . Consider the fiber product

$$(T^*\mathcal{N}) \times_{\mathcal{N}} \varpi^*\mathcal{T} \longrightarrow \mathcal{N}.$$

Now $\varpi^*T^*\mathcal{M}$ acts on it as follows: for any $z \in \mathcal{N}$ and $w \in T^*_{\varpi(z)}\mathcal{M}$, the action of w sends $(a, b) \in (T^*_z\mathcal{N}) \times \mathcal{T}_{\varpi(z)}$ to $(a - (d\varpi)^*(w), b + w) \in (T^*_z\mathcal{N}) \times \mathcal{T}_{\varpi(z)}$. The quotient

$$((T^*\mathcal{N}) \times_{\mathcal{N}} \varpi^*\mathcal{T}) / \varpi^*T^*\mathcal{M} \longrightarrow \mathcal{N}$$

is a $T^*\mathcal{N}$ -torsor.

Now, \mathcal{D} is identified with the $T^*\mathcal{N}$ -torsor on \mathcal{N} given by the $T^*\mathcal{M}$ -torsor \mathcal{C} on \mathcal{M} in (2.4), while $\text{Conn}(\varpi^*\mathcal{L})$ is identified with the $T^*\mathcal{N}$ -torsor on \mathcal{N} given by the $T^*\mathcal{M}$ -torsor $\text{Conn}(\mathcal{L})$ on \mathcal{M} in (2.9). Consequently, the biholomorphism F in Proposition 2.3 produces the isomorphism Ψ in the statement of the theorem. \square

We recall that the pulled back holomorphic line bundle

$$q_1^*\varpi^*\mathcal{L} \longrightarrow \text{Conn}(\varpi^*\mathcal{L}),$$

where q_1 and ϖ are the maps in (5.7) and (5.6) respectively, has a tautological holomorphic connection whose curvature is a holomorphic symplectic form on $\text{Conn}(\varpi^*\mathcal{L})$. Let $\widetilde{\Theta}_{\mathcal{N}}$ denote this holomorphic symplectic form on $\text{Conn}(\varpi^*\mathcal{L})$.

Theorem 5.1 gives the following:

COROLLARY 5.2. *The pulled back form*

$$\Theta_{\mathcal{N}} := \frac{1}{2r} \Psi^* \widetilde{\Theta}_{\mathcal{N}}$$

is a holomorphic symplectic structure on \mathcal{D} . The fibers of the projection $\widehat{\phi}$ (5.5) are Lagrangian with respect to this symplectic form $\Theta_{\mathcal{N}}$. The form $\Theta_{\mathcal{N}}$ is compatible with the $T^*\mathcal{N}$ -torsor structure on \mathcal{D} in the following way: If $s : U \longrightarrow \mathcal{D}$ is any holomorphic section of the fibration $\widehat{\phi}$ over an open subset $U \subset \mathcal{N}$, and ω is any holomorphic 1-form on U , then for the holomorphic section

$$s_{\omega} : U \longrightarrow \mathcal{D}, \quad z \longmapsto s(z) + \omega(z),$$

the equation

$$s_\omega^* \Theta_{\mathcal{N}} = s^* \Theta_{\mathcal{N}} + d\omega$$

holds.

Proof. Since $\tilde{\Theta}_{\mathcal{N}}$ has the above two properties, it follows from Theorem 5.1 that $\Theta_{\mathcal{N}}$ also has these two properties. \square

6. Tyurin parametrization with framings and meromorphic connections. Consider \mathcal{Q}_s constructed in (4.3). Let \mathcal{Q}_s^F be the moduli space of pairs (z, σ) , where

- $z \in \mathcal{Q}_s$, and
- σ is a framing, over \mathbb{S} , on the holomorphic vector bundle \mathcal{K}^z (see (4.3)).

Let

$$\varpi' : \mathcal{Q}_s^F \longrightarrow \mathcal{Q}_s, \quad (z, \sigma) \longmapsto z \tag{6.1}$$

be the projection.

We shall construct two $T^* \mathcal{Q}_s^F$ -torsors over \mathcal{Q}_s^F .

To construct the first $T^* \mathcal{Q}_s^F$ -torsor, note that there is a natural morphism

$$\varpi^F : \mathcal{Q}_s^F \longrightarrow \mathcal{N}, \quad (z, \sigma) \longmapsto (\mathcal{K}^z, \sigma), \tag{6.2}$$

so $\varpi \circ \varpi^F = \xi \circ \varpi'$, where ϖ and ξ are constructed in (5.6) and (4.4) respectively. Let

$$(d\varpi^F)^* : (\varpi^F)^* T^* \mathcal{N} \longrightarrow T^* \mathcal{Q}_s^F$$

be the dual of the differential

$$d\varpi^F : T\mathcal{Q}_s^F \longrightarrow (\varpi^F)^* T\mathcal{N}$$

of the map ϖ^F in (6.2). Consider the fiber product

$$(T^* \mathcal{Q}_s^F) \times_{\mathcal{Q}_s^F} ((\varpi^F)^* \mathcal{D}) \longrightarrow \mathcal{Q}_s^F,$$

where $\hat{\phi} : \mathcal{D} \longrightarrow \mathcal{N}$ is the $T^* \mathcal{N}$ -torsor in (5.5). Now $(\varpi^F)^* T^* \mathcal{N}$ acts on it as follows. For any $x \in \mathcal{Q}_s^F$, the action of $w \in T_{\varpi^F(x)}^* \mathcal{N}$ on $T_x^* \mathcal{Q}_s^F \times \hat{\phi}^{-1}(\varpi^F(x))$ sends any $(v, D) \in T_x^* \mathcal{Q}_s^F \times \hat{\phi}^{-1}(\varpi^F(x))$ to $(v - (d\varpi^F)^*(w), D + w)$. Let

$$\mathcal{D}^F := ((T^* \mathcal{Q}_s^F) \times_{\mathcal{Q}_s^F} ((\varpi^F)^* \mathcal{D})) / ((\varpi^F)^* T^* \mathcal{N}) \tag{6.3}$$

be the corresponding quotient. Let

$$\hat{\phi}^F : \mathcal{D}^F \longrightarrow \mathcal{Q}_s^F \tag{6.4}$$

be the natural map given by the projections $T^* \mathcal{Q}_s^F \longrightarrow \mathcal{Q}_s^F$ and $(\varpi^F)^* \mathcal{D} \longrightarrow \mathcal{Q}_s^F$.

The translation action of $T^* \mathcal{Q}_s^F$ on itself and the trivial action of $T^* \mathcal{Q}_s^F$ on $(\varpi^F)^* \mathcal{D}$ together produce an action of $T^* \mathcal{Q}_s^F$ on $(T^* \mathcal{Q}_s^F) \times_{\mathcal{Q}_s^F} ((\varpi^F)^* \mathcal{D})$. This action descends to an action of $T^* \mathcal{Q}_s^F$ on the quotient \mathcal{D}^F in (6.3). Now $(\mathcal{D}^F, \hat{\phi}^F)$ gets the structure of a $T^* \mathcal{Q}_s^F$ -torsor on \mathcal{Q}_s^F using this action of $T^* \mathcal{Q}_s^F$ on \mathcal{D}^F .

We note that there is a natural map to \mathcal{D}^F from the moduli space triples of the form (z, σ, D) , where

- $z \in \mathcal{Q}_s$,
- σ is a framing over \mathbb{S} on the vector bundle \mathcal{K}^z in (4.3), and
- D is a meromorphic connection on \mathcal{K}^z .

More precisely, any triple (z, σ, D) of the above form is sent to the point of $(\widehat{\phi}^F)^{-1}(z, \sigma)$ (the map $\widehat{\phi}^F$ is defined in (6.4)) corresponding to $(0, D) \in (T^*\mathcal{Q}_s^F) \times_{\mathcal{Q}_s^F} ((\varpi^F)^*\mathcal{D})$. This map to \mathcal{D}^F from the moduli space of triples need not be injective or surjective in general. This map is injective if the differential $d\varpi^F(z)$ is surjective for every $z \in \mathcal{Q}_s^F$, and it is surjective if $d\varpi^F(z)$ is injective for every $z \in \mathcal{Q}_s^F$.

To construct the second $T^*\mathcal{Q}_s^F$ -torsor, first consider the holomorphic line bundle

$$(\varpi \circ \varpi^F)^*\mathcal{L} \longrightarrow \mathcal{Q}_s^F,$$

where ϖ is the projection in (5.6), and \mathcal{L} is the holomorphic line bundle constructed in (2.6). We note that $(\varpi \circ \varpi^F)^*\mathcal{L}$ coincides with the determinant line bundle over \mathcal{Q}_s^F for the family of vector bundles $(\varpi')^*\mathcal{K}$ over X parametrized by \mathcal{Q}_s^F , where \mathcal{K} and ϖ' are constructed in (4.2) and (6.1) respectively. Construct the holomorphic fiber bundle

$$q_1^F : \text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L}) \longrightarrow \mathcal{Q}_s^F \tag{6.5}$$

using the Atiyah bundle $\text{At}((\varpi \circ \varpi^F)^*\mathcal{L})$ that corresponds to the sheaf of holomorphic connections on $(\varpi \circ \varpi^F)^*\mathcal{L}$. It is a $T^*\mathcal{Q}_s^F$ -torsor over \mathcal{Q}_s^F . As noted before, $\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})$ is equipped with a holomorphic symplectic structure given by the curvature of the tautological holomorphic connection on the line bundle $(\varpi \circ \varpi^F \circ q_1^F)^*\mathcal{L}$. Let

$$\widetilde{\Theta}_{\mathcal{Q}_s^F} \in H^0(\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L}), \bigwedge^2 T^*\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})) \tag{6.6}$$

be this holomorphic symplectic form on $\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})$.

Let

$$\widehat{\delta}^F : \mathcal{D}^F \times_{\mathcal{Q}_s^F} T^*\mathcal{Q}_s^F \longrightarrow \mathcal{D}^F \tag{6.7}$$

and

$$\widehat{\eta}^F : \text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L}) \times_{\mathcal{Q}_s^F} T^*\mathcal{Q}_s^F \longrightarrow \text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L}) \tag{6.8}$$

be the $T^*\mathcal{Q}_s^F$ -torsor structures on \mathcal{D}^F and $\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})$ constructed in (6.4) and (6.5) respectively. Let

$$\mathbf{m}^F : T^*\mathcal{Q}_s^F \longrightarrow T^*\mathcal{Q}_s^F, \quad v \longmapsto 2r \cdot v \tag{6.9}$$

be the multiplication by $2r$.

Now we have the following analog of Theorem 5.1.

THEOREM 6.1. *There is a canonical biholomorphic map of fiber bundles*

$$\Psi^F : \mathcal{D}^F \longrightarrow \text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})$$

such that

- (1) $q_1^F \circ \Psi^F = \widehat{\phi}^F$, where $\widehat{\phi}^F$ and q_1^F are the projections in (6.4) and (6.5) respectively, and
- (2) $\Psi^F \circ \widehat{\delta}^F = \widehat{\eta}^F \circ (\Psi \times \mathbf{m}^F)$, where $\widehat{\delta}^F$, $\widehat{\eta}^F$ and \mathbf{m}^F are constructed in (6.7), (6.8) and (6.9) respectively.

Proof. A proof of Theorem 6.1 can be constructed from the proof of Theorem 5.1. We omit the details. \square

An analogue of Remark 4.3 persists in this set-up with framings. To elaborate, let \mathcal{Q}^F be the moduli space of pairs (z, σ) , where

- $z \in \mathcal{Q}$, and
- σ is a framing, over \mathbb{S} , on the holomorphic vector bundle \mathcal{K}^z (see (4.3)).

So \mathcal{Q}_s^F is a Zariski open subset of \mathcal{Q}^F (see (4.3)). The $T^*\mathcal{Q}_s^F$ -torsor \mathcal{D}^F over \mathcal{Q}_s^F does not have natural extension to a $T^*\mathcal{Q}^F$ -torsor over \mathcal{Q}^F . But the $T^*\mathcal{Q}_s^F$ -torsor $\text{Conn}((\varpi \circ \varpi^F)^*\mathcal{L})$ over \mathcal{Q}_s^F has a natural extension to a $T^*\mathcal{Q}^F$ -torsor over \mathcal{Q}^F , because the determinant line bundle $(\varpi \circ \varpi^F)^*\mathcal{L}$ over \mathcal{Q}_s^F has a natural extension to \mathcal{Q}^F .

Theorem 6.1 gives the following.

COROLLARY 6.2. *For the biholomorphic map Ψ^F in Theorem 6.1, the pulled back form*

$$\Theta_{\mathcal{D}^F} := \frac{1}{2r}(\Psi^F)^*\tilde{\Theta}_{\mathcal{Q}_s^F}$$

(see (6.6)) defines a holomorphic symplectic structure on \mathcal{D}^F . The fibers of the projection $\widehat{\phi}^F$ (6.4) are Lagrangian with respect to this symplectic form $\Theta_{\mathcal{D}^F}$. The form $\Theta_{\mathcal{D}^F}$ is compatible with the $T^*\mathcal{Q}_s^F$ -torsor structure on \mathcal{D}^F in the following way: If $s : U \rightarrow \mathcal{D}^F$ is any holomorphic section of the fibration $\widehat{\phi}^F$ over an open subset $U \subset \mathcal{Q}_s^F$, and ω is a holomorphic 1-form on U , then for the holomorphic section

$$s_\omega : U \rightarrow \mathcal{D}^F, \quad z \mapsto s(z) + \omega(z),$$

the equation

$$s_\omega^*\Theta_{\mathcal{D}^F} = s^*\Theta_{\mathcal{D}^F} + d\omega$$

holds.

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