Annals of Mathematical Sciences and Applications
Volume 6, Number 2, 145–172, 2021

Lyapunov functionals for some distributed delay models in epidemiology

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In this paper, we give a brief survey of Lyapunov functionals for compartmental epidemic models with distributed time delays based on the SIR model proposed by Beretta and Takeuchi (*J. Math. Biol.*, 1995). We then consider various models with an exposed compartment such as SEIR, SEIQR, SEIAR, and SEIS models. Based on the suitable construction of Lyapunov functionals, we establish the global stability of the disease free equilibrium and endemic equilibrium (if it exists) for these models.

AMS 2000 subject classifications: Primary 34K20, 34K25; secondary 92D30.
Keywords and phrases: epidemic models, distributed delay, Lyapunov functional, global stability.

1. Introduction

Since the pioneering works of Kermack and McKendrick [28, 29], a tremendous variety of epidemic models have been developed in modeling the disease transmission, analyzed mathematically, and applied to public health. Typically, continuous-time epidemic models consist of ordinary differential equations (ODEs). The population is divided into several different classes changing in time, and these models are therefore called compartment models. In a simple setting, one may assume that the population consists of three types of individuals and thus forms three classes: the susceptible class consists of individuals who are healthy but could be infected; the infective class consists of those who are sick and can transmit the disease by contact with susceptible; the removed class consists of those who have recovered with immunity. The population in these classes are denoted by $S(t)$, $I(t)$, and $R(t)$, respectively. Such a model is called an SIR model. A more complicated structure of models can be made. For example, when individuals in the removed class lose immunity, they may return to the susceptible class. This can be modeled by an SIRS model. When infected individuals do not

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become infectious immediately, one may assume that susceptible individuals undergo a latent period. Then SEIR or SEIRS models can be considered, where the label E stands for the exposed/latent stage. More complete discussions can be found in the literature [1, 5, 17, 18, 34, 38, 42] and the references cited therein.

A key notion in mathematical epidemiology is the basic reproduction number, denoted by $\mathcal{R}_0$, which defines the number of secondary cases one infectious individual will produce in a population consisting only of susceptible individuals. We refer to the seminal work of Diekmann, Heesterbeek and Metz [9]. See also [10, 48]. Intuitively, if $\mathcal{R}_0 > 1$, the disease outbreaks in the population; if $\mathcal{R}_0 < 1$, the disease disappears from the population. Mathematically, the number $\mathcal{R}_0$ plays an essential role in almost all papers regarding the dynamics of epidemic models. In general, one can establish the global stability of the disease free equilibrium when $\mathcal{R}_0 < 1$, and the locally asymptotically stability of the unique endemic equilibrium when $\mathcal{R}_0 > 1$. However, the global stability of the endemic equilibrium (it exists uniquely) for $\mathcal{R}_0 > 1$ is usually a difficult task. The direct Lyapunov method is one of the powerful methods to achieve it. This method relies on constructing an auxiliary function (called a Lyapunov function or Lyapunov functional). However, it is usually non-trivial to construct such a function. It is worth mentioning that the global stability for $\mathcal{R}_0 > 1$ for a variety of SEIR models with constant total population has been established by Li and Muldowney [35] using the monotone dynamical systems approach and an extension of Poincare-Bendixson theorem, which relies strongly on the structure of models. It was also proved by Korobeinikov [30] using the Lyapunov function technique.

The construction of Lyapunov functions for ODEs using Volterra-like functions was reported by Goh [24], and Hsu [19] for ecological models. See also the literature [25, 20, 45]. For epidemic models, Mena-Lorca and Hethcote [40] provide Lyapunov functions for SIR and SIRS models of varying size. Lyapunov functions for more general SIR and SIRS models were constructed by Korobeinikov and Wake [32]. We also refer to [23, 26, 30, 31, 44, 49, 50] (and the references therein) for a large variety of Lyapunov functions of epidemic models.

When the time delay is taken into account, Cooke [7] introduce an SIR model to model the disease transmission in a human population via a vector. This model assumes that susceptible individuals are infected by infectious vectors (such as mosquitoes), and susceptible vectors are infected by infectious individuals. The force of infection at time t is assumed to be proportional to $S(t)I(t-\tau)$ with some time delay τ. Some related works can be

found in [6, 8, 37, 51]. We also refer to [47] for a review of epidemic models with time delays.

In 1995, Beretta and Takeuchi [2] assume that τ is a distributed parameter rather than a fixed constant, and thus the incidence is proportional to $S(t)\int_0^\infty f(\tau)I(t-\tau)d\tau$, where $f(\tau)$ stands for the fraction of vector population such that the time taken to become infectious is τ. Moreover, f is assumed to non-negative and is in $L^2([0,\infty))$ with

$$\int_0^\infty f(\tau)d\tau = 1, \quad \int_0^\infty \tau f(\tau)d\tau < \infty.$$

The authors introduce a distributed delay SIR model with *constant total population size* (i.e., $S(t)+I(t)+R(t)=1$ for all t), and establish the following threshold dynamics:

- If $\mathcal{R}_0 < 1$, then the disease free equilibrium is globally asymptotically stable with respect to $\Omega := \{(S,I) \in \mathbb{R}^2_+ \,|\, S+I \le 1\}$.
- If $\mathcal{R}_0 > 1$, then the disease free equilibrium is unstable and the endemic equilibrium is locally asymptotically stable.
- If $\mathcal{R}_0 > 1$, then the endemic equilibrium is globally asymptotically stable with respect to some set $\widetilde{\Omega} \subset \Omega$.

Later, Beretta and Takeuchi [3] consider the distributed parameter to be bounded above by some positive finite time h; namely, the incidence is proportional to $S(t)\int_0^h f(\tau)I(t-\tau)d\tau$, where f satisfies

$$\int_0^h f(\tau)d\tau = 1, \quad \int_0^h \tau f(\tau)d\tau < \infty. \tag{1}$$

They improve the convergence results in [2] by working on a generalized model with *varying population size and different death rates*. Some mathematical generalizations for SIR models are obtained in [46, 4, 36]. However, the global stability of the endemic equilibrium for distributed delay SIR models when $\mathcal{R}_0 > 1$ was not fully established. In 2010, McCluskey [39] successfully solves this problem by constructing a new Lyapunov functional. Due to his work, the stability analysis for the endemic equilibrium of various kinds of delayed epidemic models has been carried out extensively. See, e.g., [11, 13, 21, 14, 41, 43] and references cited therein.

In this paper, we shall review related works and consider a variety of epidemic models with distributed delays including the exposed stage such as SEIR, SEIQR, SEIAR, and SEIS models based on the works [2, 3, 46] and

construct Lyapunov functionals (inspired by the work of McCluskey [39]) to establish the global stability of the equilibria.

The rest of the paper is organized as follows. In Section 2, we review some Lyapunov functionals used in the previous works for epidemic models with distributed time delays and prepare some known results. In Section 3, we establish the global stability of the equilibria for various models by constructing various Lyapunov functionals. Finally, we give concluding remarks in Section 4.

2. Preliminary

All initial value problems we consider in this paper are well-posed. More precisely, we consider these models with non-negative initial data belong to $\mathcal{C} := C([-h, 0], \mathbb{R}^n)$, the Banach space consisting of continuous functions mapping $[-h, 0]$ into $\mathbb{R}^n$ equipped with the supremum norm. For any $T \geq 0$, $\mathbf{x} \in C([-h, T], \mathbb{R}^n)$ and $t \in [0, T]$, we denote $\mathbf{x}_t \in \mathcal{C}$ by $\mathbf{x}_t(\theta) = \mathbf{x}(\theta + t)$ for $\theta \in [-h, 0]$. Then these models can be represented by the following delay differential equations:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}_t)$$

for some Lipschitz vector function $\mathbf{F}$. Applying the standard theory (see, e.g., [16, 33]), one can obtain the local existence and uniqueness of the solution. Moreover, one can show that the solution is non-negative and uniformly bounded, which yields the global existence of the solution. The detailed proof will not be given here.

2.1. Lyapunov functionals

We present some Lyapunov functionals for epidemic models used in the literature:

- The Volterra-like forms [24, 19, 31]:

$$V(x_1, x_2, \cdots, x_n) := \sum_{i=1}^{n} c_i \left[x_i - x_i^* - x_i^* \ln(\frac{x_i}{x_i^*}) \right],$$
$$V(x_1, x_2, \cdots, x_n) := \sum_{i=1}^{n} c_i \left[x_i - x_i^* \ln x_i \right].$$

- The quadratic forms [40, 49]:

$$V(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} c_i (x_i - x_i^*)^2,$$
$$V(x_1, x_2, \cdots, x_n) = c\Big[\sum_{i=1}^{n} (x_i - x_i^*)\Big]^2.$$

- McCluskey's form [39]:

$$V(x_t) = \int_0^h \Big[\int_\tau^h f(s)ds\Big] g\Big(\frac{x(t-\tau)}{x^*}\Big) d\tau,$$

where $g(x) = x - 1 - \ln x$ and f describes the distributed delays defined above.

2.2. A distributed delay SIR model

In this subsection, we review the following SIR model with distributed delays studied in [3, 46]:

$$\begin{cases} S' = b - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - \mu_1 S(t), \\ I' = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \kappa) I(t), \\ R' = \kappa I(t) - \mu_3 R(t), \end{cases} \tag{2}$$

where b is the recruitment rate; μ_1, μ_2 and μ_3 are the death rates for each classes, respectively; β is the transmission rate; κ is the recovery rate; $h > 0$ and $f \geq (\not\equiv) 0$ satisfying (1).

The basic reproduction number of (2) is

$$\mathcal{R}_0 = \frac{\beta b}{\mu_1(\mu_2 + \kappa)}.$$

Since R does not appear in the equations for S and I, it suffices to study the subsystem

$$\begin{cases} S' = b - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - \mu_1 S(t), \\ I' = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \kappa) I(t). \end{cases} \tag{3}$$

The system (3) always has a disease free equilibrium $E_0 = (b/\mu_1, 0)$. If $\mathcal{R}_0 > 1$, there is a unique endemic equilibrium $E_* = (S^*, I^*)$, where

$$S^* = \frac{\mu_2 + \kappa}{\beta}, \quad I^* = \frac{b - \mu_1 S^*}{\beta S^*}.$$

When $\mathcal{R}_0 < 1$, it was shown in [3] that E_0 is globally asymptotically stable. In [46], the authors showed that E_0 is globally attractive if $\mathcal{R}_0 \leq 1$. To obtain the global stability of E_* for $\mathcal{R}_0 > 1$, McCluskey [39] introduced the following Lyapunov functional:

$$V(t) = c_1\, g\Big(\frac{S(t)}{S^*}\Big) + c_2\, g\Big(\frac{I(t)}{I^*}\Big) + c_3 V_+(t), \tag{4}$$

where c_i are some constants, $i = 1, 2, 3$,

$$V_+(t) := \int_0^h \alpha(\tau) g\Big(\frac{I(t-\tau)}{I^*}\Big) d\tau, \quad \alpha(\tau) := \int_\tau^h f(s) ds, \tag{5}$$

and

$$g(x) = x - 1 - \ln x. \tag{6}$$

Note that $g(x) > g(1) = 0$ for all $x > 0$ and $x \neq 1$.

With the help of the Lyapunov functional (4), McCluskey shows that the endemic equilibrium E_* of (2) is globally asymptotically stable if $\mathcal{R}_0 > 1$. This indicates that delay cannot affect the asymptotic stability of equilibria of (2).

2.3. Some related works

Based on the works [2, 3, 46], there are many related works for various epidemic models regarding the global dynamics. We briefly summarize them as follows:

- SIR models: the authors in [4] used a Riemann-Stieltjes integral to describe the distributed delays. The work [27] considered an extension model of [3]. A class of nonlinear incidence rates was taken into account in [13]. The work [22] considered complex population networks.
- SIRS models: the global stability for the endemic equilibrium under weak delay condition was established in [52]. The works [12, 53] considered extension models of [4]. A class of nonlinear incidence rates was considered in [14, 15].

- SIS models: the work [41] considered multi-group models with nonlinear incidence rates and patch structure.

3. Global stability for various epidemic models

Based on the transmission mechanism presented in [2, 3, 46], we shall take the exposed compartment into account and consider SEIR, SEIQR, SEIAR, and SEIS models in the subsections, respectively. Lyapunov functionals for various epidemic models will be constructed to establish the global stability of equilibria.

3.1. An SEIR model

For many diseases, infected individuals may not become infectious immediately and so one may impose an additional compartment to describe the exposed stage, denoted by the class E (see, e.g., [17]). Then our SEIR model takes the following form:

$$\begin{cases} S' = \Lambda - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - \mu_1 S(t), \\ E' = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \gamma) E(t), \\ I' = \gamma E(t) - (\mu_3 + \kappa) I(t), \\ R' = \kappa I(t) - \mu_4 R(t), \end{cases} \tag{1}$$

where Λ represents the recruitment rate of susceptibles; β is the transmission rate; μ_1, μ_2, μ_3 and μ_4 are the death rates for these classes, respectively; γ is the rate of becoming infectious ($1/\gamma$ is the length of the latent period); κ is the recovery rate of infectious. All parameters are assumed to be positive.

By adding four equations of (1), we have

$$(S + E + I + R)'(t) \le \Lambda - \min\{\mu_1, \mu_2, \mu_3, \mu_4\}(S + E + I + R)(t).$$

It follows that

$$\limsup_{t\to\infty} (S + E + I + R)(t) \le \frac{\Lambda}{\min\{\mu_1, \mu_2, \mu_3, \mu_4\}}. \tag{2}$$

The basic reproduction number of (1) can be calculated as

$$\mathcal{R}_0 = \frac{\beta\gamma\Lambda}{\mu_1(\mu_2 + \gamma)(\mu_3 + \kappa)}.$$

Since R does not appear in the first three equations, it suffices to study the subsystem

$$\begin{cases} S' = \Lambda - \beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - \mu_1 S(t), \\ E' = \beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \gamma) E(t), \\ I' = \gamma E(t) - (\mu_3 + \kappa) I(t), \end{cases} \tag{3}$$

Due to (2), we can only focus on (3) in the following feasible region

$$\Omega := \Big\{ (S, E, I) \in \mathbb{R}^3_+ | S + E + I \leq \frac{\Lambda}{\min\{\mu_1, \mu_2, \mu_3, \mu_4\}} \Big\}.$$

Note that Ω is a positively invariant set. Therefore, the solution of (3) with the range of initial functions contained in Ω is ultimately bounded.

By some simple calculations, we see that the system (3) has the disease free equilibrium

$$E_0 = (S^0, 0, 0) = \Big(\frac{\Lambda}{\mu_1}, 0, 0 \Big).$$

When $\mathcal{R}_0 > 1$, there is a unique endemic equilibrium $E_* = (S^*, E^*, I^*)$, where

$$S^* = \frac{S^0}{\mathcal{R}_0}, \quad E^* = S^0 \Big(1 - \frac{1}{\mathcal{R}_0} \Big) \frac{\mu_1}{\mu_2 + \gamma}, \quad I^* = \frac{\gamma E^*}{\mu_3 + \kappa}.$$

Theorem 1. *If $\mathcal{R}_0 \leq 1$, then the disease free equilibrium E_0 of (3) is globally asymptotically stable.*

Proof. The Laypunov functional is standard (see, e.g., [3]):

$$V(t) = \frac{1}{2}(S - S^0)^2 + c_1 E(t) + c_2 I(t) + c_3 \int_0^h f(\tau) \int_{t-\tau}^t I(u) du d\tau,$$

where c_1, c_2 and c_3 are positive constants that will be determined later.

By some simple calculations and using the equations in (1), we have

$$\begin{aligned} V' &= (S - S^0) S' + c_1 E' + c_2 I' + c_3 \int_0^h f(\tau) [I(t) - I(t-\tau)] d\tau \\ &= (S - S^0) \Big[- \mu_1 (S - S^0) - \beta S \int_0^h f(\tau) I(t-\tau) d\tau \Big] \end{aligned}$$

$$
\begin{aligned}
&+c_1\Big[\beta S\int_0^h f(\tau)I(t-\tau)d\tau-(\mu_2+\gamma)E\Big]+c_2[\gamma E-(\mu_3+\kappa)I]\\
&+c_3\Big[I-\int_0^h f(\tau)I(t-\tau)d\tau\Big],\\
=\;& -\mu_1(S-S^0)^2+[-\beta S^2+\beta(S^0+c_1)S-c_3]\int_0^h f(\tau)I(t-\tau)d\tau\\
&+[c_2\gamma-c_1(\mu_2+\gamma)]E+[c_3-c_2(\mu_3+\kappa)]I,
\end{aligned}
$$

where we have used that $\Lambda = S^0\mu_1$ and $\int_0^h f(\tau)d\tau = 1$. To obtain $V' \le 0$, it suffices to find c_i, $i = 1, 2, 3$, such that

$$
\beta(S^0+c_1)^2 \le 4c_3, \quad c_2\gamma \le c_1(\mu_2+\gamma), \quad c_3 \le c_2(\mu_3+\kappa). \tag{4}
$$

For this, we choose $c_1 = S^0$ and $c_3 = \beta(S^0)^2$. Then, due to $\mathcal{R}_0 \le 1$, on can pick $c_2 > 0$ such that

$$
\frac{\beta S^0}{\mu_3+\kappa} \le \frac{c_2}{S^0} \le \frac{\mu_2+\gamma}{\gamma}. \tag{5}
$$

Then (4) holds. Consequently, we have

$$
\begin{aligned}
V' &\le -\mu_1(S-S^0)^2+\gamma S^0\Big[\frac{c_2}{S^0}-\frac{\mu_2+\gamma}{\gamma}\Big]E\\
&\quad +S^0(\mu_3+\kappa)\Big[\frac{\beta S^0}{\mu_3+\kappa}-\frac{c_2}{S^0}\Big]I\\
&\le 0.
\end{aligned} \tag{6}
$$

We next show that the largest invariant subset $\mathcal{M}$ of $\{V' = 0\}$ consists of only the disease free equilibrium. Indeed, if $\mathcal{R}_0 < 1$, the coefficient of E and I in (6) is positive, which implies $\mathcal{M} = \{(S^0, 0, 0)\}$. If $\mathcal{R}_0 = 1$, we then have $V' = -\mu_1(S-S^0)^2$ and thus $V' = 0$ if and only if $S(t) = S^0$. Since $\mathcal{M}$ is invariant, for any $(S, E, I) \in \mathcal{M}$, it follows from the equation of S that $I(t) = 0$. By the equation of I, we see that $E(t) = 0$. This implies that $\mathcal{M} = \{(S^0, 0, 0)\}$ if $\mathcal{R}_0 = 1$. Consequently, we can apply Lyapunov-LaSalle type theorem [33, p. 30, Corollary 5.2] to conclude that E_0 is globally asymptotically stable and the proof is completed. □

Next, we study the global stability of E_* when $\mathcal{R}_0 > 1$.

Theorem 2. *If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium E_* of (3) is globally asymptotically stable.*

Proof. Inspired by [39], we consider the Lyapunov functional

$$V(t) = \frac{1}{\beta I^*} g\Big(\frac{S(t)}{S^*}\Big) + \frac{E^*}{\beta S^* I^*} g\Big(\frac{E(t)}{E^*}\Big) + \frac{1}{\mu_3 + \kappa} g\Big(\frac{I(t)}{I^*}\Big) + V_+(t),$$

where $V_+(t)$ and g are defined in (5) and (6), respectively. For convenience, we adopt the following notations:

$$x = \frac{S(t)}{S^*}, \quad y = \frac{I(t)}{I^*}, \quad y_\tau = \frac{I(t-\tau)}{I^*}, \quad w = \frac{E(t)}{E^*}. \tag{7}$$

By the equation of S and using $\Lambda = \beta S^* I^* + \mu_1 S^*$, we have

$$\begin{aligned}\frac{d}{dt} g\Big(\frac{S(t)}{S^*}\Big) &= \frac{1}{S^*}(1 - \frac{S^*}{S})\Big[\beta \int_0^h f(\tau)\big[S^* I^* - S(t) I(t-\tau)\big] d\tau + \mu_1 (S^* - S)\Big] \\ &= \beta I^* \int_0^h f(\tau) \left(1 - \frac{1}{x}\right)\big(1 - x y_\tau\big) d\tau - \frac{\mu_1}{S^* S}(S - S^*)^2,\end{aligned}$$

where we also used the fact $\int_0^h f(\tau) d\tau = 1$.

By the equation of E and using $(\mu_2 + \gamma) E^* = \beta S^* I^*$, we obtain

$$\begin{aligned}\frac{d}{dt} g\Big(\frac{E(t)}{E^*}\Big) &= \frac{1}{E^*}(1 - \frac{E^*}{E}) \left(\beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \gamma) E\right) \\ &= \frac{1}{E^*}(1 - \frac{E^*}{E})\left(\beta S^* I^* \int_0^h f(\tau) \frac{S(t)}{S^*} \frac{I(t-\tau)}{I^*} d\tau - (\mu_2 + \gamma) E^* \frac{E}{E^*}\right) \\ &= \frac{\beta S^* I^*}{E^*} \int_0^h f(\tau) \left(1 - \frac{1}{w}\right)\big[x y_\tau - w\big] d\tau.\end{aligned}$$

Using $\gamma E^* = (\mu_3 + \kappa) I^*$ and the equation of I, we thus have

$$\begin{aligned}\frac{d}{dt} g\Big(\frac{I(t)}{I^*}\Big) &= \frac{1}{I^*}\left(1 - \frac{I^*}{I}\right)\left(\gamma E^* \frac{E}{E^*} - (\mu_3 + \kappa) I^* \frac{I}{I^*}\right) \\ &= (\mu_3 + \kappa) \int_0^h f(\tau) \left(1 - \frac{I^*}{I}\right)\left(\frac{E}{E^*} - \frac{I}{I^*}\right) d\tau \\ &= (\mu_3 + \kappa) \int_0^h f(\tau) \left(1 - \frac{1}{y}\right)\big(w - y\big) d\tau.\end{aligned}$$

By the definition of V_+, it follows that

$$\frac{dV_+(t)}{dt} = \int_0^h f(\tau)\big[g(y) - g(y_\tau)\big]d\tau$$

Combining the above calculations, we obtain

$$\frac{dV(t)}{dt} = -\frac{\mu_1}{\beta S^* I^*}\frac{(S-S^*)^2}{S} - \int_0^h f(\tau)C(\tau)d\tau,$$

where

$$\begin{aligned}(8)\quad C(\tau) &= -3 + \frac{1}{x} + \frac{xy_\tau}{w} + \frac{w}{y} - \ln y_\tau + \ln y \\ &= \left(\frac{1}{x} - 1 - \ln\frac{1}{x}\right) + \left(\frac{xy_\tau}{w} - 1 - \ln\frac{xy_\tau}{w}\right) + \left(\frac{w}{y} - 1 - \ln\frac{w}{y}\right) \\ &= g\left(\frac{1}{x}\right) + g\left(\frac{xy_\tau}{w}\right) + g\left(\frac{w}{y}\right) \\ &\geq 0.\end{aligned}$$

Hence, we obtain $V'(t) \leq 0$. Moreover, $V'(t) = 0$ if and only if $S(t) = S^*$ and

$$g\left(\frac{1}{x}\right) + g\left(\frac{xy_\tau}{w}\right) + g\left(\frac{w}{y}\right) = 0,$$

equivalently,

$$(9)\qquad S(t) = S^*, \quad \frac{I(t)}{I^*} = \frac{I(t-\tau)}{I^*} = \frac{E(t)}{E^*}.$$

Let $\mathcal{M}$ be the largest invariant subset of $\{V' = 0\}$. For any $(S, E, I) \in \mathcal{M}$, from (9) we see that $S(t) = S^*$ and $I(t-\tau) = I(t)$. From the equation of S, we further obtain $I(t) = I^*$. Next, from the equation of E we see that $E(t) = E^*$. Therefore, we obtain that $\mathcal{M} = \{(S^*, E^*, I^*)\}$. This allows us to apply Lyapunov-LaSalle type theorem [33, p. 30, Corollary 5.2] to conclude that E_* is globally asymptotically stable. This completes the proof. □

3.2. An SEIQR model

When infectious people could be isolated, the SEIR model (13) can be extended into an SEIQR model. The class Q represents infectious people who

are isolated to avoid further contacts. The model may take the following form:

$$
(10)\qquad \begin{cases} S' = \Lambda - \beta S(t)\displaystyle\int_0^h f(\tau)I(t-\tau)d\tau - \mu_1 S(t), \\ E' = \beta S(t)\displaystyle\int_0^h f(\tau)I(t-\tau)d\tau - (\mu_2+\gamma)E(t), \\ I' = \gamma E(t) - (\mu_3+\kappa_I+\rho)I(t), \\ Q' = \rho I(t) - (\mu_4+\kappa_Q)Q(t), \\ R' = \kappa_I I(t) + \kappa_Q Q(t) - \mu_5 R(t), \end{cases}
$$

where μ_i, $i = 1, 2, 3, 4, 5$, are the death rates for each classes; κ_I and κ_Q are the recovery rates of the infectious in I and Q class, respectively; ρ is the isolation rate.

Adding all equations in (10), we easily obtain

$$
(11)\qquad \limsup_{t\to\infty}(S+E+I+Q+R)(t) \le \frac{\Lambda}{\min\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5\}}.
$$

The basic reproduction number is

$$
\mathcal{R}_0 = \frac{\beta\Lambda\gamma}{\mu_1(\mu_3+\kappa_I+\rho)(\mu_2+\gamma)}.
$$

The system (10) has a disease free equilibrium

$$
\hat{E}_0 = (S^0, E^0, I^0, Q^0, R^0) = \left(\frac{\Lambda}{\mu_1}, 0, 0, 0, 0\right)
$$

and a unique endemic equilibrium $\hat{E}_* = (S^*, E^*, I^*, Q^*, R^*)$, where

$$
\begin{aligned}
S^* &= \frac{S^0}{\mathcal{R}_0} \\
E^* &= S^0\left(1-\frac{1}{\mathcal{R}_0}\right)\left(\frac{\mu_1}{\mu_2+\gamma}\right) \\
I^* &= S^0\left(1-\frac{1}{\mathcal{R}_0}\right)\frac{\mu_1\gamma}{(\mu_2+\gamma)(\mu_3+\kappa_I+\rho)} \\
Q^* &= S^0\left(1-\frac{1}{\mathcal{R}_0}\right)\frac{\mu_1\gamma\rho}{(\mu_2+\gamma)(\mu_3+\kappa_I+\rho)(\mu_4+\kappa_Q)} \\
R^* &= S^0\left(1-\frac{1}{\mathcal{R}_0}\right)\frac{\mu_s\gamma}{(\mu_2+\gamma)(\mu_3+\kappa_I+\rho)}\left[\kappa_I + \frac{\rho\kappa_Q}{\kappa_Q+\mu_4}\right]\frac{1}{\mu_5}.
\end{aligned}
$$

Since Q and R do not appear in the first three equations, we may reduce (10) to

$$\begin{cases} S' = \Lambda - \beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - \mu_1 S(t), \\ E' = \beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - (\mu_2 + \gamma) E(t), \\ I' = \gamma E(t) - (\mu_3 + \kappa_I + \rho) I(t). \end{cases} \tag{12}$$

The system (12) always has a disease free equilibrium E_0 and a unique endemic equilibrium E_* (if $\mathcal{R}_0 > 1$), where

$$E_0 = \Big(\frac{\Lambda}{\mu_s}, 0, 0\Big), \quad E_* = (S^*, E^*, I^*).$$

Once we obtain the global stability of E_0 and E_*, it is not hard to obtain the global stability of $\hat{E}_0$ and $\hat{E}_*$ of the full system by working on the equation of Q and R using some standard and fundamental analysis (we may transfer the differential equations into integral equations).

According to (11), we may only focus on the following feasible region (a positively invariant set):

$$\Omega := \Big\{(S, E, I) \in \mathbb{R}^3_+ | S + E + I \leq \frac{\Lambda}{\min\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}}\Big\}.$$

Since (12) has the same form as (3), the appearance of the quarantine compartment does not really complicate the analysis. One can adopt the same Lyapunov functionals with different coefficients constructed in §3.1 to conclude the following results.

Theorem 3. *If $\mathcal{R}_0 \leq 1$, then the disease free equilibrium E_0 of (12) is globally asymptotically stable.*

Theorem 4. *If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium E_* of (12) is globally asymptotically stable.*

3.3. An SEIAR model

In this subsection, we add an asymptomatic compartment into the SEIR model (1). The asymptomatic compartment, denoted by A, contains infec-

tious without symptoms. Asymptomatic infection exists for many diseases such as COVID-19, HIV, malaria, dengue, and influenza. We assume that

- the exposed individuals move to I class with probability p and to A class with probability $1-p$ for some $p>0$.
- the distribution functions of incubation times for I and A can be different.

Then the SEIR model can be extended to the following SEIAR model:

$$
\begin{cases}
S' = \Lambda - \beta_1 S(t)\displaystyle\int_0^{h_1} f_1(\tau)I(t-\tau)d\tau \\
\qquad -\beta_2 S(t)\int_0^{h_2} f_2(\tau)A(t-\tau)d\tau - \mu_1 S(t), \\
E' = \beta_1 S(t)\displaystyle\int_0^{h_1} f_1(\tau)I(t-\tau)d\tau \\
\qquad +\beta_2 S(t)\int_0^{h_2} f_2(\tau)A(t-\tau)d\tau - (\mu_2+\gamma)E(t), \\
I' = p\gamma E(t) - (\mu_3+\kappa_I)I(t), \\
A' = (1-p)\gamma E(t) - (\mu_4+\kappa_A)A(t), \\
R' = \kappa_I I(t) + \kappa_A A(t) - \mu_5 R(t),
\end{cases} \tag{13}
$$

where μ_i, $i=1,2,3,4,5$, are the death rates for each classes, κ_I and κ_A are the recovery rates of the infectious in the class I and A, respectively, $h_i>0$ and $f_i \geq (\not\equiv) 0$ are given such that

$$
\int_0^{h_i} f_i(\tau)d\tau = 1, \quad \int_0^{h_i} \tau f_i(\tau)d\tau < \infty. \quad i=1,2.
$$

By adding all equations of (13), we easily obtain

$$
\limsup_{t\to\infty}(S+E+I+A+R)(t) \leq \frac{\Lambda}{\min\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5\}}. \tag{14}
$$

The basic reproduction number can be calculated as

$$
\mathcal{R}_0 = \frac{\Lambda\gamma}{\mu_1(\mu_2+\gamma)}\Big[\frac{\beta_1 p}{\mu_3+\kappa_I} + \frac{\beta_2(1-p)}{\mu_4+\kappa_A}\Big].
$$

Since R does not appear in first three equations, (13) is thus reduced to

$$
(15)\qquad \begin{cases} S' = \Lambda - \beta_1 S(t)\displaystyle\int_0^{h_1} f_1(\tau) I(t-\tau)d\tau \\ \qquad -\beta_2 S(t)\int_0^{h_2} f_2(\tau) A(t-\tau)d\tau - \mu_1 S(t), \\ E' = \beta_1 S(t)\displaystyle\int_0^{h_1} f_1(\tau) I(t-\tau)d\tau \\ \qquad +\beta_2 S(t)\int_0^{h_2} f_2(\tau) A(t-\tau)d\tau - (\mu_2+\gamma)E(t), \\ I' = p\gamma E(t) - (\mu_3+\kappa_I)I(t), \\ A' = (1-p)\gamma E(t) - (\mu_4+\kappa_A)A(t). \end{cases}
$$

By simple calculations, the system (15) has a disease free equilibrium

$$
E_0 = (S^0, E^0, I^0, A^0) = \left(\frac{\Lambda}{\mu_1}, 0, 0, 0, 0\right)
$$

When $\mathcal{R}_0 > 1$, there is a unique endemic equilibrium $E_* = (S^*, E^*, I^*, A^*)$, where

$$
\begin{aligned} S^* &= \frac{S^0}{\mathcal{R}_0}, \quad E^* = S^0(1-\frac{1}{\mathcal{R}_0})(\frac{\mu_1}{\mu_2+\gamma}), \\ I^* &= \frac{p\gamma}{\mu_3+\kappa_I}E^*, \quad A^* = \frac{(1-p)\gamma}{\mu_4+\kappa_A}E^*. \end{aligned}
$$

Thanks to (14), we may only focus on the following feasible region

$$
\Omega := \left\{(S,E,I,A) \in \mathbb{R}^4_+ | S+E+I+A \le \frac{\Lambda}{\min\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5\}}\right\}.
$$

Theorem 5. *If $\mathcal{R}_0 \le 1$, then E_0 of (15) is globally asymptotically stable.*

Proof. Let us define

$$
\begin{aligned} V(t) \quad = \quad & \frac{1}{2}(S-S^0)^2 + c_1 E(t) + c_2 I(t) + c_3 A(I) \\ & + c_4 \int_0^{h_1} f_1(\tau)\int_{t-\tau}^{t} I(u)dud\tau + c_5 \int_0^{h_2} f_2(\tau)\int_{t-\tau}^{t} A(u)dud\tau, \end{aligned}
$$

where c_i $(i = 1, 2, 3, 4, 5)$ are positive constants that will be determined later.

By some similar calculations in the proof of Theorem 1 and using the equations in (15), we have

$$\begin{aligned} V' &= -\mu_1(S-S^0)^2 + \int_0^{h_1} f_1(\tau)I(t-\tau)d\tau[-\beta_1 S^2 + \beta_1(c_1+S^0)S - c_4] \\ &\quad + \int_0^{h_2} f_2(\tau)A(t-\tau)d\tau[-\beta_2 S^2 + \beta_2(c_1+S^0)S - c_5] \\ &\quad + \gamma\Big[c_2 p + c_3(1-p) - c_1\frac{\mu_2+\gamma}{\gamma}\Big]E \\ &\quad + [c_4 - (\mu_3+\kappa_I)c_2]I + [c_5 - (\mu_4+\kappa_A)c_3]A. \end{aligned}$$

Let us choose $c_1 = S^0$, $c_4 = \beta_1 S^0 c_1$, and $c_5 = \beta_2 S^0 c_1$. Then

$$\begin{aligned} V' &\le -\mu_1(S-S^0)^2 + \gamma\Big[c_2 p + c_3(1-p) - S^0(\frac{\mu_2+\gamma}{\gamma})\Big]E \\ &\quad + [\beta_1(S^0)^2 - (\mu_3+\kappa_I)c_2]I + [\beta_2(S^0)^2 - (\mu_4+\kappa_A)c_3]A. \end{aligned}$$

Moreover, thanks to $\mathcal{R}_0 \le 1$, or equivalently,

$$S^0\Big[\frac{\beta_1 p}{\mu_3+\kappa_I} + \frac{\beta_2(1-p)}{\mu_4+\kappa_A}\Big] \le \frac{\mu_2+\gamma}{\gamma},$$

one can choose $c_2 > 0$ and $c_3 > 0$ such that

$$\frac{c_2}{S^0} \ge S^0\Big(\frac{\beta_1}{\mu_3+\kappa_I}\Big), \quad \frac{c_3}{S^0} \ge S^0\Big(\frac{\beta_2}{\mu_4+\kappa_A}\Big),$$

and

$$\frac{c_2}{S^0}p + \frac{c_3}{S^0}(1-p) \le \frac{\mu_2+\gamma}{\gamma}.$$

This implies that $V' \le 0$. Furthermore, we can show that the largest invariant subset of $\{V'=0\}$ is $\{E_0\}$, as in the process used in the proof of Theorem 1. Therefore, by Lyapunov-LaSalle type theorem [33, p. 30, Corollary 5.2], E_0 is globally asymptotically stable and the proof is completed. □

Theorem 6. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium E_* of (15) is globally asymptotically stable.*

Proof. Let us consider the Lyapunov functional

$$\begin{aligned} V(t) &= g\Big(\frac{S(t)}{S^*}\Big) + \frac{E^*}{S^*}g\Big(\frac{E(t)}{E^*}\Big) + \frac{\beta_1 I^*}{\mu_3+\kappa_I}g\Big(\frac{I(t)}{I^*}\Big) + \frac{\beta_2 A^*}{\mu_4+\kappa_A}g\Big(\frac{A(t)}{A^*}\Big) \\ &\quad + V_{1,+}(t) + V_{2,+}(t), \end{aligned}$$

where

$$V_{1,+}(t) := \int_0^{h_1} \alpha_1(\tau) g\Big(\frac{I(t-\tau)}{I^*}\Big)d\tau, \quad V_{2,+}(t) := \int_0^{h_2} \alpha_2(\tau) g\Big(\frac{A(t-\tau)}{A^*}\Big)d\tau,$$

$$\alpha_i(\tau) := \int_\tau^{h_i} f_i(s)ds, \quad i = 1, 2.$$

For convenience, we adopt the following notations:

$$\begin{aligned} x = \frac{S(t)}{S^*}, \quad y = \frac{I(t)}{I^*}, \quad y_\tau = \frac{I(t-\tau)}{I^*}, \\ z = \frac{A(t)}{A^*}, \quad z_\tau = \frac{A(t-\tau)}{A^*}, \quad w = \frac{E(t)}{E^*}. \end{aligned}$$

By the equation of S and using $\Lambda = \beta_1 S^* I^* + \beta_2 S^* A^* + \mu_1 S^*$, we have

$$\begin{aligned} \frac{d}{dt}g\Big(\frac{S(t)}{S^*}\Big) &= \frac{1}{S^*}\left(1-\frac{S^*}{S}\right)\Big\{\beta_1\int_0^{h_1} f_1(\tau)\big[S^*I^* - S(t)I(t-\tau)\big]d\tau \\ &\quad + \beta_2\int_0^{h_2} f_2(\tau)\big[S^*A^* - S(t)A(t-\tau)\big]d\tau + \mu_1(S^*-S)\Big\} \\ &= \beta_1 I^*\int_0^{h_1} f_1(\tau)\left(1-\frac{1}{x}\right)\big(1-xy_\tau\big)d\tau \\ &\quad + \beta_2 A^*\int_0^{h_2} f_2(\tau)\left(1-\frac{1}{x}\right)\big(1-xz_\tau\big)d\tau \\ &\quad - \frac{\mu_1}{S^*S}(S-S^*)^2. \end{aligned}$$

By the equation of E and using $(\mu_2+\gamma)E^* = \beta_1 S^* I^* + \beta_2 S^* A^*$, we have

$$\begin{aligned} \frac{d}{dt}g\Big(\frac{E(t)}{E^*}\Big) &= \frac{1}{E^*}\left(1-\frac{E^*}{E}\right)\Big\{-(\mu_2+\gamma)E + \beta_1 S\int_0^{h_1} f_1(\tau)I(t-\tau)d\tau \\ &\quad + \beta_2 S\int_0^{h_2} f_2(\tau)A(t-\tau)d\tau\Big\} \\ &= \frac{1}{E^*}\left(1-\frac{E^*}{E}\right)\Big\{-(\mu_2+\gamma)E^*\frac{E}{E^*} \end{aligned}$$

$$
\begin{aligned}
&+\beta_1 S^* I^* \int_0^{h_1} f_1(\tau)\frac{S(t)}{S^*}\frac{I(t-\tau)}{I^*}d\tau \\
&+\beta_2 S^* A^* \int_0^{h_2} f_2(\tau)\frac{S(t)}{S^*}\frac{A(t-\tau)}{A^*}d\tau\Big\} \\
= \;& \frac{S^*}{E^*}\Big[\beta_1 I^* \int_0^{h_1} f_1(\tau)(1-\frac{1}{w})(xy_\tau - w)d\tau \\
&+\beta_2 A^* \int_0^{h_2} f_2(\tau)(1-\frac{1}{w})(xz_\tau - w)d\tau\Big].
\end{aligned}
$$

Using $p\gamma E^* = (\mu_3+\kappa_I)I^*$ and the equation of I, we thus have

$$
\begin{aligned}
\frac{d}{dt}g\Big(\frac{I(t)}{I^*}\Big) &= \frac{1}{I^*}\left(1-\frac{I^*}{I}\right)\left(p\gamma E^*\frac{E}{E^*} - (\mu_3+\kappa_I)I^*\frac{I}{I^*}\right) \\
&= (\mu_3+\kappa_I)\int_0^{h_1} f_1(\tau)\left(1-\frac{I^*}{I}\right)\left(\frac{E}{E^*}-\frac{I}{I^*}\right)d\tau \\
&= (\mu_3+\kappa_I)\int_0^{h_1} f_1(\tau)\left(1-\frac{1}{y}\right)\big(w-y\big)d\tau.
\end{aligned}
$$

Similarly, we have

$$
\frac{d}{dt}g\Big(\frac{A(t)}{A^*}\Big) = (\mu_4+\kappa_A)\int_0^{h_2} f_2(\tau)\left(1-\frac{1}{z}\right)\big(w-z\big)d\tau.
$$

By the definition of $V_{+,i}$, $i=1,2$, we see that

$$
\begin{aligned}
\frac{dV_{+,1}(t)}{dt} &= \int_0^{h_1} f_1(\tau)\big[g(y)-g(y_\tau)\big]d\tau, \\
\frac{dV_{+,2}(t)}{dt} &= \int_0^{h_2} f_2(\tau)\big[g(z)-g(z_\tau)\big]d\tau,
\end{aligned}
$$

Combining the above calculations, we conclude that

$$
V' = -\frac{\mu_1}{S^*S}(S-S^*)^2 - \int_0^{h_1} f_1(\tau)C_1(\tau)d\tau - \int_0^{h_2} f_2(\tau)C_2(\tau)d\tau,
$$

where

$$
\begin{aligned}
C_1(\tau) &= -3 + \frac{1}{x} + \frac{xy_\tau}{w} + \frac{w}{y} - \ln y_\tau + \ln y, \\
C_2(\tau) &= -3 + \frac{1}{x} + \frac{xz_\tau}{w} + \frac{w}{z} - \ln z_\tau + \ln z.
\end{aligned}
$$

With the same process used in the proof of Theorem 2, we have

$$C_1(\tau) = g\left(\frac{1}{x}\right) + g\left(\frac{xy_\tau}{w}\right) + g\left(\frac{w}{y}\right) \geq 0,$$
$$C_2(\tau) = g\left(\frac{1}{x}\right) + g\left(\frac{xz_\tau}{w}\right) + g\left(\frac{w}{z}\right) \geq 0.$$

Therefore, we have $V'(t) \leq 0$. Furthermore, $V'(t) = 0$ if and only if $S(t) = S^*$ and

$$g\left(\frac{1}{x}\right) = g\left(\frac{xy_\tau}{w}\right) = g\left(\frac{w}{y}\right) = g\left(\frac{xz_\tau}{w}\right) = g\left(\frac{w}{z}\right) = 0,$$

With a similar process used in the proof of Theorem 2, we see that the largest invariant subset of $\{V' = 0\}$ consists only the endemic equilibrium E^*. Therefore, one can apply Lyapunov-LaSalle type theorem [33, p. 30, Corollary 5.2] to conclude that E_* is globally asymptotically stable. This completes the proof. □

3.4. An SEIS model

If recovered individuals have no immunity and will be reinfected, the removed compartment R is not necessary and so individuals may return to S compartment immediately after recovery. Here we assume that both exposed compartment and infectious compartment have the same recovery rate. Then we consider the following SEIS model:

$$\begin{cases} S' = \Lambda - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - \mu S(t) + \phi I(t) + \phi E(t) \\ E' = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu + \gamma + \phi) E(t) \\ I' = \gamma E(t) - (\mu + \phi) I(t), \end{cases} \tag{16}$$

where μ is the death rate for each classes (we assume that each compartments have the same birth rates), and ϕ is the recovery rate.

The basic reproduction number $\mathcal{R}_0$ can be calculated as

$$\mathcal{R}_0 = \frac{\beta \Lambda \gamma}{\mu(\mu + \gamma + \phi)(\mu + \phi)}.$$

The system has a disease free equilibrium

$$E_0 = (S^0, E^0, I^0) = \Big(\frac{\Lambda}{\mu}, 0, 0\Big).$$

When $\mathcal{R}_0 > 1$, there is a unique endemic equilibrium $E_* = (S^*, E^*, I^*)$, where

$$S^* = \frac{S^0}{\mathcal{R}_0}, \quad E^* = S^0(1 - \frac{1}{\mathcal{R}_0})\Big[\frac{\mu+\phi}{\mu+\phi+\gamma}\Big], \quad I^* = \frac{\gamma}{\mu+\phi}E^*.$$

As usual, we may only focus on the following feasible region

$$\Omega := \Big\{(S, E, I) \in \mathbb{R}^3_+ | S + E + I \leq \frac{\Lambda}{\mu}\Big\}. \tag{17}$$

Theorem 7. *If $\mathcal{R}_0 < 1$, then disease free equilibrium E_0 of (16) is globally asymptotically stable.*

Proof. Let us consider the following Lyapunov functional:

$$V(t) = E + c_1 I + c_2 \int_0^h f(\tau) \int_{t-\tau}^t I(u) du d\tau$$

By some computations, we have

$$\begin{aligned} V' &= \beta S \int_0^h f(\tau) I(t-\tau) d\tau - (\mu+\gamma+\phi)E + c_1\gamma E - c_1(\mu+\phi)I \\ &\quad + c_2 \int_0^h f(\tau)[I(t) - I(t-\tau)] d\tau. \end{aligned}$$

Taking

$$c_1 = \frac{\mu+\gamma+\phi}{\gamma}, \quad c_2 = \frac{\beta\Lambda}{\mu},$$

we have

$$V' = \beta(S(t) - \frac{\Lambda}{\mu}) \int_0^h f(\tau) I(t-\tau) d\tau + \frac{1}{\gamma}(\mu+\gamma+\phi)(\mu+\phi)(\mathcal{R}_0 - 1)I \leq 0,$$

where we have used $\mathcal{R}_0 < 1$ and $S(t) \leq \Lambda/\mu$ (due to (17)). Clearly, $V' = 0$ if and only if $I = E = 0$. By Lyapunov-LaSalle type theorem [33, p. 30,

Theorem 5.3], we conclude that $\lim_{t\to\infty}(E(t), I(t)) = (0, 0)$. From the equation of S in (16), we easily see that $\lim_{t\to\infty} S(t) = S^0$. Hence E_0 is globally attractive. Moreover, following the argument used in [2] (see also [21, 14]) by considering the linearization at E_0, it is not hard to analyze the characteristic equations of the system and obtain the local stability of E_0 if $\mathcal{R}_0 < 1$. Therefore, the proof is completed. □

Finally, we give the global stability of E_* when $\mathcal{R}_0 > 1$ under some restrictions on parameters.

Theorem 8. *If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium E_* of (16) is globally asymptotically stable, provided*

$$\mu S^* \geq \phi(I^* + E^*). \tag{18}$$

Proof. We construct the following Lyapunov functional:

$$\begin{aligned} V(t) &= \frac{1}{\beta I^*} g\Big(\frac{S(t)}{S^*}\Big) + \frac{1}{\mu+\gamma+\phi} g\Big(\frac{E(t)}{E^*}\Big) + \frac{1}{\mu+\phi} g\Big(\frac{I(t)}{I^*}\Big) \\ &\quad + V_+(t) + \frac{\phi}{2\mu\beta(S^*)^2 I^*} V_Q(t) \end{aligned}$$

where V_+ is defined in (5), and

$$\begin{aligned} &V_Q(t) := \frac{1}{2}\left(N(t) - N^*\right)^2, \quad N(t) := S(t) + E(t) + I(t), \\ &N^* := S^* + E^* + I^*. \end{aligned}$$

Hereafter, we shall use the same notations as in (7).

From the equation of S, we have $\Lambda = \beta S^* I^* + \mu S^* - \phi(I^* + E^*)$. Then

$$\begin{aligned} \frac{d}{dt} g\Big(\frac{S(t)}{S^*}\Big) &= \frac{1}{S^*}\left(1 - \frac{S^*}{S}\right)\Big[\beta \int_0^h f(\tau)\big(S^* I^* - S(t) I(t-\tau)\big) d\tau + \mu(S^* - S) \\ &\quad - \phi(I^* - I) - \phi(E^* - E)\Big] \\ &= \beta I^* \int_0^h f(\tau)(1 - \frac{1}{x})\,(1 - x y_\tau)\, d\tau - \mu(1 - \frac{1}{x})(x - 1) \\ &\quad - \frac{\phi I^*}{S^*}(1 - \frac{1}{x})(1 - y) - \frac{\phi E^*}{S^*}(1 - \frac{1}{x})(1 - w). \end{aligned} \tag{19}$$

Using $\beta S^* I^* = (\mu + \gamma + \phi)E^*$, we have

$$\begin{aligned}
&\frac{d}{dt}g\Big(\frac{E(t)}{E^*}\Big) \\
&\quad = \frac{1}{E^*}(1 - \frac{E^*}{E})\left[\beta S^* I^* \int_0^h f(\tau)\frac{S(t)}{S^*}\frac{I(t-\tau)}{I^*}d\tau - (\mu+\gamma+\phi)E^*\frac{E}{E^*}\right] \\
&\quad = \frac{1}{E^*}\left(1 - \frac{E^*}{E}\right)\Big((\mu+\gamma+\phi)E^*\int_0^h f(\tau)\frac{S(t)}{S^*}\frac{I(t-\tau)}{I^*}d\tau \\
&\qquad - (\mu+\gamma+\phi)E^*\frac{E}{E^*}\Big) \\
&\quad = (\mu+\gamma+\phi)\int_0^h f(\tau)\left(1 - \frac{E^*}{E(t)}\right)\left(\frac{S(t)}{S^*}\frac{I(t-\tau)}{I^*} - \frac{E(t)}{E^*}\right)d\tau \\
&\quad = (\mu+\gamma+\phi)\int_0^h f(\tau)(1-\frac{1}{w})(xy_\tau - w)d\tau
\end{aligned} \tag{20}$$

Using $\gamma E^* = (\mu + \phi)I^*$, we conclude that

$$\begin{aligned}
\frac{d}{dt}g\Big(\frac{I(t)}{I^*}\Big) &= \frac{1}{I^*}\left(1 - \frac{I^*}{I}\right)\left(\gamma E^*\frac{E}{E^*} - (\mu+\phi)I^*\frac{I}{I^*}\right) \\
&= \left(1 - \frac{I^*}{I}\right)(\mu+\phi)\left(\frac{E}{E^*} - \frac{I}{I^*}\right) \\
&= (\mu+\phi)\int_0^h f(\tau)(1-\frac{1}{y})(w-y)d\tau
\end{aligned} \tag{21}$$

Let us calculate dV_Q/dt. Note that $\Lambda = \mu(S^* + E^* + I^*)$. Then we have

$$\begin{aligned}
\frac{dV_Q}{dt} &= (N - N^*)(\Lambda - \mu S - \mu E - \mu I) \\
&= -\mu\left[(S - S^*) + (E - E^*) + (I - I^*)\right]^2 \\
&= -\mu(S - S^*)^2 - \mu[(E - E^*) + (I - I^*)]^2 \\
&\qquad - 2\mu S^* I^*(x-1)(y-1) - 2\mu S^* E^*(x-1)(w-1).
\end{aligned} \tag{22}$$

Also, recall that

$$\frac{dV_+(t)}{dt} = \int_0^h f(\tau)\big(g(y) - g(y_\tau)\big)d\tau. \tag{23}$$

Then, combining (20), (21), (22), (19) and (23), we obtain

$$V' \le -\frac{\mu}{\beta I^*}(1-\frac{1}{x})(x-1) + \frac{\phi}{\beta S^*}(1-\frac{1}{x})(y-1) - \frac{\phi}{\beta S^*}(x-1)(y-1) + \frac{\phi E^*}{\beta S^* I^*}(1-\frac{1}{x})(w-1) - \frac{\phi E^*}{\beta S^* I^*}(x-1)(w-1) - \int_0^h f(\tau)C(\tau)d\tau$$

where $C(\tau)$ is the same as in (8) and so we have $C(\tau) \ge 0$. Also, using $y, w \ge 0$, we have

$$\begin{aligned} V' &\le -\frac{1}{\beta}(x+\frac{1}{x}-2)\Big[\frac{\mu}{I^*} + \frac{\phi}{S^*}y - \frac{\phi}{S^*} + \frac{\phi}{S^*}\frac{E^*}{I^*}w - \frac{\phi}{S^*}\frac{E^*}{I^*}\Big] \\ &\le -\frac{1}{\beta}(x+\frac{1}{x}-2)\Big[\frac{\mu}{I^*} - \frac{\phi}{S^*} - \frac{\phi}{S^*}\frac{E^*}{I^*}\Big] \\ &\le 0, \end{aligned}$$

where the last inequality holds because of (18).

Finally, it is not hard to see that the largest invariant subset of $\{V' = 0\}$ is $\{E_0\}$. By Lyapunov-LaSalle type theorem [33, p. 30, Corollary 5.2], we conclude that E_* is globally asymptotically stable. This completes the proof. □

Remark 3.1. *The condition* (18) *may be a technical assumption in constructing Lyapunov functionals, which is similar to the one used in a SIRS epidemic model with distributed delays [43] and a delayed SIS epidemic model with nonlinear incidence rates [11].*

4. Concluding remarks

In this paper, we give a brief review of Lyapunov functionals for compartmental epidemic models with distributed time delays. Then, we take exposed compartment into account and consider SEIR, SEIQR, SEIAR and SEIS models as extension of those presented in [2, 3, 46]. The global stability of the disease free equilibrium E_0 when $\mathcal{R}_0 \le 1$ (strict inequality for the SEIS model) is established by using standard Lyapunov functionals. Furthermore, based on the Lyapunov functional provided by McCluskey [39], we establish the global stability of the endemic equilibrium E_* when $\mathcal{R}_0 > 1$ for various models. In our SEIS model, we need some restriction on the parameters to guarantee the global stability of E_*. In fact, if the flowchart of epidemic models forms a cycle (e.g., SEIS, SIS, SIRS models), the global stability of E_* is still not completely established due to the lack of a better Lyapunov functional. We leave it to future studies.

Acknowledgments

We thank the reviewer for careful reading of the manuscript. This work was partly supported from the Young Scholar Fellowship Program by Ministry of Science and Technology (MOST) in Taiwan, under Grant MOST 110-2636-M-009-006.

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Received May 15, 2021