A reverse quasiconformal composition problem for $Q_{\alpha}(\mathbb{R}^n)$

Jie Xiao and Yuan Zhou

Abstract. We give a partial converse to [8, Theorem 1.3] (as a resolution of [2, Problem 8.4] for the quasiconformal Q-composition) for $Q_{0<\alpha<2^{-1}}(\mathbb{R}^{n>2})$, and yet demonstrate that if $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism then the boundedness of $u \mapsto u\circ f$ on $Q_{2^{-1}<\alpha<1}(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$ yields the quasiconformality of $f$.

1. Introduction

Recall that $Q_{-\infty<\alpha<\infty}(\mathbb{R}^n)$ is the quite-well-known Essén-Janson-Peng-Xiao’s space of all measurable functions $u$ on $\mathbb{R}^{n\geq 1}$ with

$$
\|u\|_{Q_{\alpha}(\mathbb{R}^n)} = \sup_{(x_0,r)\in \mathbb{R}^n \times (0,\infty)} \left( r^{2\alpha-n} \int_{|y-x_0|<r} \int_{|x-x_0|<r} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}} < \infty.
$$

In particular (cf. [2], [5]),

$$
Q_{0\leq \alpha<\infty}(\mathbb{R}^n) \subset Q_{-\infty<\alpha<0}(\mathbb{R}^n) = Q_{-\frac{n}{2}}(\mathbb{R}^n) = BMO(\mathbb{R}^n).
$$

As a resolution of [2, Problem 8.4] – Let $f$ be a quasiconformal self-map of $\mathbb{R}^n$. Prove or disprove that $u \mapsto C_f u = u \circ f$ is bounded on $Q_{0<\alpha<1}(\mathbb{R}^{n>2})$ (which however has an affirmative solution for $BMO(\mathbb{R}^n)$ as proved in [9, Theorem 2] – namely $-C_f$ is bounded on $BMO(\mathbb{R}^n)$ whenever $f$ is a quasiconformal self-map of $\mathbb{R}^n$), we have

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YZ is corresponding author.

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Theorem 1.1. [8, Theorem 1.3] For \( n-1 \in \mathbb{N} \) let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be quasiconformal. If there exists a closed set \( E \subseteq \mathbb{R}^n \) such that
\[ J_f, \text{ the Jacobian determinant of } f, \text{ belongs to the } E\text{-based Muckenhoupt class } A_1(\mathbb{R}^n; E); \]
\[ \dim_L E \text{ (under } E \text{ being bounded) or } \dim_{LG} E \text{ (under } E \text{ being unbounded),} \]
the local or global self-similar Minkowski dimension of \( E \) (bounded or unbounded), lies in \([0, n-2]\), i.e.,
\[ [0, n-2] \ni \begin{cases} \dim_L E & \text{as } E \text{ is bounded;} \\ \dim_{LG} E & \text{as } E \text{ is unbounded,} \end{cases} \]
then \( C_f \) is bounded on \( Q_{0<\alpha<1}(\mathbb{R}^n) \).

As a partial converse to Theorem 1.1, we here show

Theorem 1.2. For \( n-1 \in \mathbb{N} \) let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a homeomorphism. If
\[ C_f \text{ and } C_{f^{-1}} \text{ are bijective and bounded on } Q_{0<\alpha<2-1}(\mathbb{R}^n) \text{ respectively;} \]
\[ f \text{ is not only ACL (absolutely continuous on almost all lines parallel to coordinates of } \mathbb{R}^n) \text{ but also differentiable almost everywhere on } \mathbb{R}^n, \]
then \( f \) is quasiconformal.

Remark 1.3. Below are two comments on Theorem 1.2.

(i) Under the above assumptions on \( f \), we have that \( f^{-1} \) is absolutely continuous with respect to the \( n \)-dimensional Lebesgue measure. Indeed, let \( f^{-1} \) map a set \( N \) of the \( n \)-dimensional Lebesgue measure 0 to a set \( O=f^{-1}(N) \). If \( \chi_N \) and \( \chi_O \) stand for the indicators of \( N \) and \( O \) respectively, then \( k\chi_O, k\chi_N \in Q_{0<\alpha<2-1}(\mathbb{R}^n) \) for any \( k \in \mathbb{N} \), but \( k\chi_N=0 \) in \( Q_{0<\alpha<2-1}(\mathbb{R}^n) \), and hence from the first \( \triangleright \)-hypothesis in Theorem 1.2 it follows that \( k\chi_O=0 \) in \( Q_{0<\alpha<2-1}(\mathbb{R}^n) \) and so \( O=f^{-1}(N) \) is of the \( n \)-dimensional Lebesgue measure 0.

(ii) In accordance with [9, Theorem 3] (cf. [1, Theorem] & [3, Theorem 3.1] for some generalizations), we have that if the first requirement on \( C_f \) & \( C_{f^{-1}} \) in Theorem 1.2 is replaced by the condition that \( f^{-1} \) is absolutely continuous and the second requirement on \( f \) is kept the same then the boundedness of \( C_f \) on \( BMO(\mathbb{R}^n) \) derives that \( f \) is a quasiconformal self-map of \( \mathbb{R}^n \). Accordingly, this \( BMO(\mathbb{R}^n) \)-result can be naturally strengthened via Theorem 1.2 thanks to \( Q_{0<\alpha<2-1}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n) \).

In addition, while focusing on the planar situation of Theorem 1.1 and observing that the Jacobian determinant of any quasiconformal self-map of \( \mathbb{R}^n \geq 2 \) is an \( A_\infty \)-weight (cf. [4, Theorem 15.32]) we readily discover

Theorem 1.4. [8, Theorem 1.3: \( n=2 \& E=\emptyset \)] Let \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) be quasiconformal. If \( J_f \) is an \( A_1 \)-weight on \( \mathbb{R}^2 \), i.e., \( J_f \in A_1(\mathbb{R}^2; \emptyset) \), then \( C_f \) is bounded on \( Q_{0<\alpha<1}(\mathbb{R}^2) \).
On the basis of the planar cases of Theorem 1.2 and Remark 1.3(ii), a partial converse to Theorem 1.4 (under $2^{-1} < \alpha < 1$) is naturally given by

**Theorem 1.5.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism. If $C_f$ is bounded on $Q_{2^{-1} < \alpha < 1}(\mathbb{R}^2)$, then $f$ is quasiconformal.

**Remark 1.6.** Let $n \geq 2$. Recall that if a homeomorphism of $\mathbb{R}^n$ preserves either the Sobolev space $W^{1,n}(\mathbb{R}^n)$ or the Triebel-Lizorkin space $\dot{F}^{s}_{n/s,q}(\mathbb{R}^n)$ with $s \in (0,1) \& q \in [1,\infty)$, it must be quasiconformal. But any homeomorphism preserving the Besov space $\dot{B}^{s}_{n/s,q}(\mathbb{R}^n)$ with $s \in (0,1) \& q \in [1,\infty)$ or $s \in (0,1) \& q = n/s$ must be bi-Lipschitz or quasiconformal; see also [6], [7] and the references therein. By Reimann’s paper [9], a homeomorphism of $\mathbb{R}^n$ preserving the John-Nirenberg space $BMO(\mathbb{R}^n)$ and satisfying the assumptions of Theorem 1.2 must be quasiconformal.

The rest of this paper is organized as follows: §2 is employed to prove Theorem 1.2 in terms of Lemmas 2.1–2.2 & 2.4 & 2.6 as well as Corollaries 2.3 & 2.5 producing a suitable $Q_\alpha(\mathbb{R}^n)$-function. More precisely, we borrow some of Reimann’s ideas from [9] to prove Theorem 1.2, namely, prove that

$$\sup_{y \in \mathbb{R}^n \& |y|=1} \left| (Df^{-1}(x)) y \right|^n \lesssim J_{f^{-1}}(x)$$

holds for almost all $x \in \mathbb{R}^n$, where $Df^{-1}$ and $J_{f^{-1}}$ are the formal derivative and Jacobian determinant of $f^{-1}$ (cf. [4, Chapters 14-15]) – equivalently – we show that the maximal eigenvalue $\lambda_1$ of $Df^{-1}(x)$ is bounded by the minimal eigenvalue $\lambda_n$ of $Df^{-1}(x)$ – in fact – by comparing the norms of suitable scalings of some special $Q_\alpha(\mathbb{R}^n)$-functions $u_*$ (cf. Corollary 2.5 & Lemma 2.6) and their compositions with $f$, we can obtain the desired inequality $\lambda_1 \lesssim \lambda_n$. §3 is designed to demonstrate Theorem 1.5 through a $Q_\alpha(\mathbb{R}^n)$-capacity estimate given in Lemma 3.1 and a technique for reducing the space dimension shown in Lemma 2.1.

**Notation** In the above and below, $X \lesssim Y$ stands for $X \leq \varkappa Y$ with a constant $\varkappa > 0$.

2. Validation of Theorem 1.2

In order to prove the validity of Theorem 1.2, we need four lemmas and two corollaries.

**Lemma 2.1.** Let $(\alpha, n, m) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N}$ and $u: \mathbb{R}^n \to \mathbb{R}$. Then $u \in Q_\alpha(\mathbb{R}^n)$ if and only if $\mathbb{R}^n \times \mathbb{R}^m \ni (x,y) \mapsto U(x,y) = u(x)$ belongs to $Q_\alpha(\mathbb{R}^{n+m})$. 
Proof. This follows immediately from [2, Theorem 2.6] and its demonstration. □

Lemma 2.2. Let \((\alpha, n) \in [0, \min\{1, 2^{-1}n\}) \times \mathbb{N}\). Then \(x \mapsto \ln |x|\) is in \(Q_\alpha(\mathbb{R}^n)\).

Proof. For any Euclidean ball \(B = B(x_0, r)\) with centre \(x_0 \in \mathbb{R}^n\) and radius \(r \in (0, \infty)\) and a measurable function \(u\) on \(\mathbb{R}^n\) let

\[
\Phi_\alpha(u, B) = r^{2\alpha - n} \int_B \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy.
\]

So, it suffices to verify that if \(u_{\text{in}}(x) = \ln |x|\) then \(\Phi_\alpha(u_{\text{in}}, B) \lesssim 1\).

- Case \(|x_0| > 2r\). Note that there is \(\theta \in (0, 1)\) obeying

\[
0 < x, y \in B \implies r < |x|, |y| \leq 3r \implies \ln |x| - \ln |y| = \frac{||x| - |y||}{(1-\theta)|x| + \theta|y|} \leq \frac{|x - y|}{r}.
\]

So

\[
\Phi_\alpha(u_{\text{in}}, B) = r^{2\alpha - n-2} \int_B \int_B |x-y|^{2-n-2\alpha} \, dx \, dy \leq r^{2\alpha - n-2} \int_B \int_{B(x, 2r)} |x-y|^{2-n-2\alpha} \, dy \, dx \lesssim r^{2\alpha - 2} \int_0^r t^{1-2\alpha} \, dt \lesssim 1,
\]

as desired.

- Case \(|x_0| \leq 2r\). Since \(B(x_0, r) \subseteq B(0, 3r)\) – the origin-centered ball with radius \(3r\), we only need to estimate \(\Phi_\alpha(u_{\text{in}}, B)\) for \(B = B(0, r)\).

Firstly, write

\[
\begin{align*}
\Phi_\alpha(u_{\text{in}}, B) &= I_1 + I_2 + I_3; \\
I_1 &= r^{2\alpha - n} \int_B \int_{B(x, 2^{-1}|x|)} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{n+2\alpha}} \, dy \, dx; \\
I_2 &= r^{2\alpha - n} \int_B \int_{B(x, 4|x|)} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{n+2\alpha}} \, dy \, dx; \\
I_3 &= r^{2\alpha - n} \int_B \int_{B(x, 4|x|) \setminus B(x, 2^{-1}|x|)} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{n+2\alpha}} \, dy \, dx.
\end{align*}
\]

Since

\[|x - y| \leq 2^{-1}|x| \implies \ln |x| - \ln |y| \leq 2|x - y||x|^{-1},\]

one has

\[
I_1 \leq r^{2\alpha - n} \int_B |x|^{-2} \int_{B(x, 2^{-1}|x|)} |x-y|^{2-n-2\alpha} \, dy \, dx \leq 1.
\]
Secondly, write
\[
\int_{B \setminus B(x, 4|x|)} \frac{|\ln |x| - \ln |y||^2}{|x - y|^{n+2\alpha}} \, dy \leq \sum_{j \geq 3} \int_{B(x, 2^j|x|) \setminus B(x, 2^{j-1}|x|)} \frac{|\ln |x| - \ln |y||^2}{|x - y|^{n+2\alpha}} \, dy.
\]

Observe that if \(j - 2 \in \mathbb{N}\) then
\[
2^{j-1}|x| \leq |x - y| \leq 2^j|x| \implies 2^{j-2}|x| \leq |y| \leq 2^{j+1}|x|
\]
\[
\implies \int_{B(x, 2^j|x|) \setminus B(x, 2^{j-1}|x|)} \frac{|\ln |x| - \ln |y||^2}{|x - y|^{n+2\alpha}} \, dy \lesssim 2^{j(2 - 2\alpha)} |x|^{2\alpha}.
\]

Thus
\[
I_2 \lesssim r^{2\alpha - n} \int_B |x|^{-2\alpha} \sum_{j=3}^\infty (\ldots) \, dy \lesssim r^{2\alpha - n} \int_B |x|^{-2\alpha} \, dx \lesssim 1.
\]

Thirdly, note that
\[
y \in B(x, 4|x|) \setminus B(x, 2^{-1}|x|) \implies |y| \leq 5|x|.
\]

So
\[
I_3 \lesssim r^{2\alpha - n} \int_B \int_{B(x, 4|x|) \setminus B(x, 2^{-1}|x|)} \frac{|\ln |x| - \ln |y||^2}{|x - y|^{n+2\alpha}} \, dy \, dx
\]
\[
\lesssim r^{2\alpha - n} \int_B |x|^{-(n+2\alpha)} \int_{B(0, 5|x|)} \left( \frac{|x|}{|y|} \right)^2 \, dy \, dx
\]
\[
\lesssim r^{2\alpha - n} \int_B |x|^{-(n+2\alpha)} \sum_{i=1}^\infty (2^{-i}5|x|)^n i^2 \, dx
\]
\[
\lesssim r^{2\alpha - n} \int_B |x|^{-2\alpha} \, dx
\]
\[
\lesssim 1. \quad \square
\]

**Corollary 2.3.** Let \((n-1, c) \in \mathbb{N} \times \mathbb{R}\). Then

(i) 
\[x = (x_1, x_2, \ldots, x_n) \mapsto \max \{c, \ln(x_1^2)\}\]

is in \(Q_{0 \leq \alpha < 2^-(\mathbb{R}^n)}\).

(ii) 
\[x = (x_1, x_2, \ldots, x_n) \mapsto \max \{c, \ln(x_1^2 + x_2^2)^{-1}\}\]

is in \(Q_{0 \leq \alpha < 1(\mathbb{R}^n)}\).
Proof. This follows from  
\[
\max\{u, v\} = 2^{-1} (u + v + |u - v|) = u + \max\{v - u, 0\},
\]
the basic fact that \( Q_\alpha(\mathbb{R}^n) \) is a linear space with 
\[
w \in Q_\alpha(\mathbb{R}^n) \implies |w| \in Q_\alpha(\mathbb{R}^n),
\]
and Lemmas 2.1–2.2. \( \square \)

Lemma 2.4. Let \((\alpha, n-1) \in (0, 1) \times \mathbb{N}\). If
\[
\begin{align*}
||u||_{Q_\alpha} &= ||u||_{Q_\alpha(\mathbb{R}^n)} + \sup_{(x_0, r) \in \mathbb{R}^n \times [1, \infty)} \left( r^{2\alpha-n} \int_{B(x_0, r)} |u(x)|^2 \, dx \right)^{1/2} < \infty; \\
||g||_{\infty, Lip} &= ||g||_{\mathcal{L}^\infty(\mathbb{R})} + \sup_{z_1, z_2 \in \mathbb{R}, z_1 \neq z_2} |g(z_1) - g(z_2)| |z_1 - z_2|^{-1} < \infty,
\end{align*}
\]
then \( \mathbb{R}^n \times \mathbb{R} \ni (x, z) \mapsto u(x)g(z) \) belongs to \( Q_\alpha(\mathbb{R}^n \times \mathbb{R}) \).

Proof. For any 
\[(x_0, z_0, \rho, r, k+2) \in \mathbb{R}^n \times \mathbb{R} \times (0, \infty) \times (0, \infty) \times \mathbb{N},\]
set 
\[
\begin{align*}
C(x_0, z_0, \rho) &= \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : |(x-x_0, z-z_0)| \leq \rho\}; \\
A(k, x_0, z_0, r) &= C(x_0, z_0, 2^{-k}r) \setminus C(x_0, z_0, 2^{-k-1}r); \\
a_{k,r}(x_0, z_0) &= u_{A(k, x_0, z_0, r)} g(z_0).
\end{align*}
\]
Here and henceforth, for a given set \( E \subset \mathbb{R}^{m \geq 1} \) with the m-dimensional Lebesgue measure \( |E| > 0 \), the symbol
\[
u_{E} = \int_{E} u(x) \, dx = |E|^{-1} \int_{E} u(x) \, dx
\]
stands for the average of \( u \) over \( E \). We make the following claim
\[
\Psi_\alpha(u g, C(x_0, z_0, r)) := \sum_{k \geq -1} 2^{2k\alpha} \iint_{C(x_0, z_0, r)} \int_{A(k, x, z, r)} |u(\tilde{x})g(\tilde{z}) - a_{k,r}(x, z)|^2 \, d\tilde{z} \, d\tilde{x} \, dx
\]
\[
\lesssim \left( ||g||_{\infty, Lip} ||u||_{Q_\alpha} \right)^2.
\]
Assume that the last estimation holds for the moment. Then an application of the basic fact that
\[
\begin{align*}
C(x, z, 2r) &= \bigcup_{k \geq -1} A(k, x, z, r); \\
A(k, x, z, r) \cap A(l, x, z, r) &= \emptyset \quad \forall k \neq l; \\
\{(x, z), (y, w)\} \in C(x_0, z_0, r) \times C(x_0, z_0, r) \implies (y, w) \in C(x, z, 2r) \subset C(x_0, z_0, 3r),
\end{align*}
\]
the Hölder inequality and Lemma 2.1 gives
\[
\begin{align*}
 r^{2\alpha-1-2} & \int_{C(x_0,z_0,r)} \int_{C(x_0,z_0,r)} \frac{|u(x)g(z) - u(y)g(w)|^2}{|(x,z) - (y,w)|^{n+2\alpha}} \, dx \, dz \\
& \leq r^{2\alpha} \int_{C(x_0,z_0,r)} \int_{C(x,z,2r)} \frac{|u(x)g(z) - u(y)g(w)|^2}{|(x,z) - (y,w)|^{n+2\alpha}} \, dy \, dw \, dx \\
& \leq \int_{C(x_0,z_0,r)} \sum_{k \geq -1} (2^{2k\alpha})^{n+1} \int_{A(k,x,z,r)} \frac{2^{2k\alpha} dy \, dz}{|u(x)g(z) - u(y)g(w)|^{-2}} \\
& \leq \Psi_{\alpha}(ug, C(x_0,z_0,r)) + \|g\|_{\infty,Lip}^{2} \sum_{k \geq -1} 2^{2k\alpha} \int_{C(x_0,z_0,r)} |u(x) - u(y)|^{2} \, dy \, dz \\
& \leq \Psi_{\alpha}(ug, C(x_0,z_0,r)) + \|g\|_{\infty,Lip}^{2} \int_{C(x_0,z_0,3r)} \int_{C(x,z,2r) \subset C(x_0,z_0,3r)} \frac{|u(x) - u(y)|^{2} \, dy \, dw \, dx \, dz}{r^{n+1-2\alpha}|(x,y,z,w)|^{1+n+2\alpha}} \\
& \leq \Psi_{\alpha}(ug, C(x_0,z_0,r)) + \left(\|g\|_{\infty,Lip} \, \|u\|_{Q_{\alpha}}\right)^2.
\end{align*}
\]

This, plus the foregoing claim, yields
\[
\|ug\|_{Q_{\alpha}(\mathbb{R}^{n+1})}^{2} = \sup_{(x_0,z_0,r) \in \mathbb{R}^{n} \times (0,\infty)} \Psi_{\alpha}(ug, C(x_0,z_0,r)) + \left(\|g\|_{\infty,Lip} \, \|u\|_{Q_{\alpha}}\right)^2 \\
\leq \left(\|g\|_{\infty,Lip} \, \|u\|_{Q_{\alpha}}\right)^2,
\]

Now, it remains to verify the above claim.
First of all, we have
\[
\int_{A(k,x,x,r)} |u(\tilde{x})g(\tilde{z}) - a_{k,r}(x,z)|^{2} \, d\tilde{x} \, d\tilde{z}
\]
\[
\begin{align*}
\lesssim & \int_{A(k,x,z,r)} |u(\tilde{x}) - u_A(k,x,z,r)|^2 |g(\tilde{z})|^2 \, d\tilde{x} \, d\tilde{z} + \int_{A(k,x,z,r)} |g(\tilde{z}) - g(z)|^2 \, u_A(k,x,z,r) \, d\tilde{x} \, d\tilde{z} \\
\lesssim & \|g\|^2_{\infty, \text{Lip}} \left( \sum_{k \geq -1} 2^{2k\alpha} \int_{C(x_0, z_0, r)} \int_{A(k,x,z,r)} |u(\tilde{x}) - u_A(k,x,z,r)|^2 \, d\tilde{x} \, d\tilde{z} \, dx \, dz \right) \\
& + I(u, \alpha) \\
\lesssim & \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right) \\
& + \int_{C(x_0, z_0, r)} \sum_{k \geq -1} \int_{A(k,x,z,r)} \frac{2^{2k\alpha} |u(\tilde{x}) - u(x)|^2 \, d\tilde{x} \, d\tilde{z} \, dx \, dz}{|\tilde{x}, \tilde{z} - (x, z)|^{1 + n + 2\alpha (2-k\alpha) - n - 1 - 2\alpha}} \\
& + I(u, \alpha) \\
\lesssim & \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right) \\
& + \int_{C(x_0, z_0, r)} \sum_{k \geq -1} \int_{A(k,x,z,r)} \frac{|u(\tilde{x}) - u(x)|^2 \, d\tilde{x} \, d\tilde{z} \, dx \, dz}{|\tilde{x}, \tilde{z} - (x, z)|^{1 + n + 2\alpha r (1+n-2\alpha)}} + I(u, \alpha) \\
\lesssim & \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right)
\end{align*}
\]
thereby finding that if

\[
I(u, \alpha) = \sum_{k \geq -1} 2^{2k\alpha} \min\{2^{-k\alpha}, 1\}^2 \int_{C(x_0, z_0, r)} |u_A(k,x,z,r)|^2 \, dx \, dz
\]
then an application of the triangle inequality, the Hölder inequality and Lemma 2.1 derives

\[
\Psi_\alpha(ug, C(x_0, z_0, r)) \leq \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right) \\
\lesssim \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right) \\
+ \int_{C(x_0, z_0, r)} \sum_{k \geq -1} \int_{A(k,x,z,r)} \frac{2^{2k\alpha} |u(\tilde{x}) - u(x)|^2 \, d\tilde{x} \, d\tilde{z} \, dx \, dz}{|\tilde{x}, \tilde{z} - (x, z)|^{1 + n + 2\alpha (2-k\alpha) - n - 1 - 2\alpha}} \\
+ I(u, \alpha) \\
\lesssim \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right) \\
+ \int_{C(x_0, z_0, r)} \sum_{k \geq -1} \int_{A(k,x,z,r)} \frac{|u(\tilde{x}) - u(x)|^2 \, d\tilde{x} \, d\tilde{z} \, dx \, dz}{|\tilde{x}, \tilde{z} - (x, z)|^{1 + n + 2\alpha r (1+n-2\alpha)}} + I(u, \alpha) \\
\lesssim \|g\|^2_{\infty, \text{Lip}} \left( \|u\|^2_{Q, \alpha} \right)
\]
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$$+ \int_{C(x_0,z_0,3r)} \int_{C(x,z,2r) \subset C(x_0,z_0,3r)} \frac{|u(\tilde{x}) - u(x)|^2 d\tilde{x} d\tilde{z} dx dz}{|x - \tilde{x}|^{1+n+2\alpha} + |z - \tilde{z}|^{1+n+2\alpha} + I(u, \alpha)}$$

$$\lesssim \|g\|_{\infty, \text{Lip}}^2 \left( \|u\|_{Q_\alpha}^2 + I(u, \alpha) \right).$$

Next, we handle $I(u, \alpha)$ according to the following two cases.

- **Case $r < 2$.** By the hypothesis on $u$ and the inclusion $Q_\alpha(\mathbb{R}^n) \subseteq BMO(\mathbb{R}^n)$ we obtain that if $k+2 \in \mathbb{N}$ then Lemma 2.1 yields

$$\left| u_A(k,x,z,r) \right| \lesssim (2^{-k}r)^{-n-1} \left| \int_{C(x,z,2^{-k}r)} u(y) dy dw - \int_{C(x,z,2^{-k-1}r)} u(y) dy dw \right|$$

$$\lesssim |u_{C(x,z,2^{-k}r)}| + |u_{C(x,z,2^{-k-1}r)}|$$

$$\lesssim |u_{C(x,z,2)}| + |u_{C(x,z,2^{-k}r)} - u_{C(x,z,2^{-k-1}r)}| + |u_{C(x,z,1)}|$$

$$+ |u_{C(x,z,1)} - u_{C(x,z,2^{-k-1}r)}|$$

$$\lesssim \left( |u|^2_{B(x,2)} \right)^{2^{-1}} + \left( |u|^2_{B(x,1)} \right)^{2^{-1}} + \left( k + 1 + \ln \frac{4}{r} \right) \|u\|_{Q_\alpha(\mathbb{R}^n)}$$

and hence

$$I(u, \alpha) \lesssim \|u\|_{Q_\alpha}^2 \sum_{k \geq -1} 2^{2k-2k} \left( k + 2 + \ln \frac{4}{r} \right)^2 \lesssim \|u\|_{Q_\alpha}^2.$$

- **Case $r \geq 2$.** An application of the hypothesis on $u$, the Hölder inequality and the Fubini theorem gives that if $k+2 \in \mathbb{N}$ then

$$\int_{C(x_0,z_0,r)} |u_A(k,x,z,r)|^2 dx dz$$

$$\lesssim \int_{C(x_0,z_0,r)} \left( |u|_{C(x,z,2^{-k}r)} \right)^2 dx dz$$

$$\lesssim \int_{C(x_0,z_0,r)} \int_{C(x,z,2^{-k}r)} |u(y)|^2 dy dw dx dz$$

$$\lesssim \int_{C(x_0,z_0,r)} \int_{C(0,0,2^{-k}r)} |u(x+z)|^2 dx dz dy dw$$

$$\lesssim r^{-2\alpha} \|u\|_{Q_\alpha}^2$$
and hence
\[
I(u, \alpha) \lesssim \|u\|_{Q_\alpha}^2 \left( \sum_{k \geq \ln r} 2^{2k\alpha - 2k r^2 - 2\alpha} + \sum_{-1 \leq k \leq \ln r} 2^{2k\alpha r^2 - 2\alpha} \right) \lesssim \|u\|_{Q_\alpha}^2.
\]

Finally, upon putting the previous two cases together, we achieve the desired estimation
\[
\Psi_\alpha(ug, C(x_0, z_0, r)) \lesssim \|g\|_{\infty, \text{Lip}}^2 \left( \|u\|_{Q_\alpha}^2 + I(u, \alpha) \right) \lesssim \left( \|g\|_{\infty, \text{Lip}} \|u\|_{Q_\alpha} \right)^2.
\]

\[ \Box \]

\textbf{Corollary 2.5.} For \( n - 1 \in \mathbb{N} \) let
\[
\phi(t) = \begin{cases} 
0 & \text{as } t \in (-\infty, -2]; \\
1 - |1 + t| & \text{as } t \in [-2, 0]; \\
1 - |1 - t| & \text{as } t \in [0, 2]; \\
0 & \text{as } t \in [2, \infty),
\end{cases}
\]
and
\[
\psi(t) = \begin{cases} 
1 & \text{as } |t| \leq 1; \\
2 - |t| & \text{as } 1 \leq |t| \leq 2; \\
0 & \text{as } |t| \geq 2.
\end{cases}
\]
If
\[
u_*(x_1, \ldots, x_n) = \begin{cases} 
\max \{0, \ln(x_1^{-2})\} \phi(x_2) & \text{for } n=2; \\
\left( \max \{0, \ln(x_1^{-2})\} \right) \psi(x_2) \cdots \psi(x_{n-1}) \phi(x_n) & \text{for } n \geq 3,
\end{cases}
\]
then \( u_* \in Q_{0<\alpha<2-1}(\mathbb{R}^n) \).

\textbf{Proof.} Note that
\[
\|\phi\|_{\infty, \text{Lip}} + \|\psi\|_{\infty, \text{Lip}} < \infty
\]
holds and (via Corollary 2.3(i))
\[
u(x_1, \ldots, x_n) = \max \{0, \ln(x_1^{-2})\} \quad \text{enjoys} \quad \|\nu\|_{Q_{0<\alpha<2-1}} < \infty.
\]
So, the assertion \( u_* \in Q_{0<\alpha<2-1}(\mathbb{R}^n) \) follows from Lemma 2.4. \[ \Box \]
Lemma 2.6. For \( n-1 \in \mathbb{N} \) let \( a=(a_1, \ldots, a_n) \) be with \( 0 < a_1 \leq a_2 \leq \ldots \leq a_n = 1 \). Given \( r > 0 \) set
\[
\begin{align*}
\left\{ (u_*)_r(x) &= u_*(r^{-1}x) ; \\
P_{a,r} &= \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1| \leq a_1r , \ldots , |x_n| \leq a_nr \} ; \\
(u_*)_{a,r} &= \frac{(u_*)_r \chi_{P_{a,r}}}{|P_{a,r}|} = \frac{(u_*)_r \chi_{P_{a,r}}}{(2r)^{n}a_1 \ldots a_n} ; \\
c_a &= \int_{\mathbb{R}^n} |(u_*)_{a,r}(x)| \, dx = f_{P_{a,r}} |(u_*)_r(x)| \, dx = f_{P_{a,r}} |u_*(x)| \, dx.
\end{align*}
\]
If \( h \in L^1(\mathbb{R}^n) \), then there exists a subsequence \( \{r_j\} \) converging to 0 such that for any rational point \( a \in \mathbb{R}^n \) one has that
\[
\begin{align*}
\left\{ (u_*)_{a,r_j} \ast h(y) &= \int_{\mathbb{R}^n} (u_*)_{a,r_j}(z) h(y-z) \, dz \to 0 ; \\
|(u_*)_{a,r_j}| \ast h(y) &= \int_{\mathbb{R}^n} (u_*)_{a,r_j}(z) |h(y-z)| \, dz \to c_a h(y),
\end{align*}
\]
holds for almost all \( y \in \mathbb{R}^n \).

Proof. The argument is similar to the proof of [9, Lemma 8]. □

Proof of Theorem 1.2. We are about to use Reimann’s procedure in [9]. Rather than showing that \( f \) is quasiconformal, we prove that \( f^{-1} \) (the inverse of \( f \)) is quasiconformal. It suffices to verify that
\[
\sup_{y \in \partial B(0,1)} \left| (Df^{-1}(x)) y \right|^n \lesssim J_{f^{-1}}(x)
\]
holds for almost all \( x \in \mathbb{R}^n \) where \( Df^{-1} \) and \( J_{f^{-1}} \) are the formal derivative and Jacobian determinant of \( f^{-1} \) (cf. [4, p.250]). Since \( f^{-1} \) is absolutely continuous with respect to the \( n \)-dimensional Lebesgue measure, one has
\[
J_{f^{-1}}(x) = \lim_{r \to 0} \frac{|f^{-1}(B(x,r))|}{|B(x,r)|}
\]
amost everywhere and \( J_{f^{-1}} \in L^1_{\text{loc}}(\mathbb{R}^n) \) where the absolute values right after \( \lim_{r \to 0} \) stand for the \( n \)-dimensional Lebesgue measures of the sets \( f^{-1}(B(x,r)) \) and \( B(x,r) \) respectively. Also our hypothesis implies that \( f^{-1} \) is (totally) differentiable almost everywhere, and \( J_{f^{-1}} > 0 \) holds almost everywhere. We may assume \( J_{f^{-1}}(0) > 0 \) and \( h=\chi_{B(0,1)} J_{f^{-1}} \) in Lemma 2.6. Up to some rotation, translation and scaling which preserve the \( Q_\alpha(\mathbb{R}^n) \)-norm, we may also assume
\[
\begin{align*}
f^{-1}(0) &= 0 ; \\
Df^{-1}(0) &= \text{diag}\{\lambda_1, \ldots, \lambda_n\} ; \\
\lambda_1 \geq \ldots \geq \lambda_n &= 1.
\end{align*}
\]
and so are required to verify

\[ \lambda_1^n \lesssim \lambda_1 \ldots \lambda_n. \]

Given any sufficiently small \( \varepsilon > 0 \), we choose

\[ a_m = (a_{m1}, \ldots, a_{mn}) \]

rationally such that

\[ 0 < a_{m1} \leq a_{m2} \leq \ldots \leq a_{mn} = 1 \quad \& \quad \sum_{k=1}^{n} |a_{mk} \lambda_k - 1| < \varepsilon. \]

Let

\[
\begin{align*}
P_r &= \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : |z_1|, \ldots, |z_n| \leq r\}; \\
P_{a_m, r} &= \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : |z_1| \leq a_{m1} r, \ldots, |z_n| \leq a_{mn} r\}.
\end{align*}
\]

Upon using Lemma 2.6 with \( a = a_m \), we write

\[ c_{a_m} = \int_{P_{a_m, 1}} |u_\ast(x)| \, dx. \]

By the definition of \( u_\ast \) as in Corollary 2.5 with \( a = a_m \) we have

\[ (\dagger) \quad c_{a_m} \gtrsim -\ln a_{m1}. \]

Indeed, if \( n = 2 \), then

\[ 0 < a_{m1} \leq 1 = a_{m2} \]

derives

\[
\begin{align*}
\int_{P_{a_m, 1}} |u_\ast(x)| \, dx &= (4a_{m1}a_{m2})^{-1} \int_{-a_{m1}}^{a_{m1}} \int_{-a_{m2}}^{a_{m2}} \max \{0, \ln(x_1^{-2})\} |\phi(x_2)| \, dx_1 \, dx_2 \\
&\gtrsim (a_{m1}a_{m2})^{-1} \int_{0}^{a_{m2}} \left( \int_{0}^{a_{m1}} \ln(x_1^{-2}) \, dx_1 \right) x_2 \, dx_2 \\
&\gtrsim -\ln a_{m1}.
\end{align*}
\]

Furthermore, if \( n \geq 3 \), then a similar argument, along with

\[ \psi(t) = 1 \quad \forall \ |t| \leq 1, \]

will also ensure \((\dagger)\).

In this way, for a sufficiently small \( r < \delta_1 \) we have that \( f^{-1}(P_{a_m, r}) \) contains

\[ R = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : |z_1|, \ldots, |z_n| \leq r(1 - \varepsilon)\} \]
A reverse quasiconformal composition problem for $Q_\alpha(\mathbb{R}^n)$ and is contained in

$$S = \{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n : |z_1|, \ldots, |z_n| \leq r(1+\varepsilon) \}.$$ 

In fact, this can be obtained by the differentiability of $f^{-1} \& f$ at 0, and

$$Df^{-1}(0) = \text{diag}\{\lambda_1, \ldots, \lambda_n\} \quad \& \quad Df(0) = \text{diag}\{\lambda_1^{-1}, \ldots, \lambda_n^{-1}\}.$$ 

By virtue of the assumption on $f$ and the function $u_\ast$ constructed in Corollary 2.5, we have

$$\|Cf u_\ast\|_{Q_{-2}(\mathbb{R}^n)} \lesssim \|Cf u_\ast\|_{Q_{0<\alpha<2-1}(\mathbb{R}^n)} \lesssim \|(u_\ast)_r\|_{Q_{0<\alpha<2-1}(\mathbb{R}^n)} \lesssim 1.$$ 

Since

$$Q_{0<\alpha<2-1}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n) = Q_{-2}(\mathbb{R}^n)$$

we are required to control

$$\|Cf u_\ast\|_{BMO(\mathbb{R}^n)} = \|Cf u_\ast\|_{Q_{-2}(\mathbb{R}^n)}$$

via

$$\|Cf u_\ast\|_{BMO(\mathbb{R}^n)} \gtrsim \int_{f^{-1}(P_{a_m,r})} |Cf u_\ast(x) - \int_{f^{-1}(P_{a_m,r})} Cf u_\ast(y) dy| \, dx.$$ 

Note that if

$$h(x) = \begin{cases} Jf(x) & \text{for } x \in B(0,1); \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B(0,1), \end{cases}$$

then

$$\int_{f^{-1}(P_{a_m,r})} Cf u_\ast(x) \, dx = \frac{|P_{a_m,r}|}{|f^{-1}(P_{a_m,r})|} \int_{P_{a_m,r}} (u_\ast)_r(z) Jf^{-1}(z) \, dz$$

$$= \frac{|P_{a_m,r}|}{|f^{-1}(P_{a_m,r})|} (u_\ast)_{a_m,r} \ast h(0).$$

So, upon applying Lemma 2.6, we obtain a constant $\delta_2 \in (0, \delta_1)$ and a sequence $r_j < \delta_2$ such that

$$\left| \int_{f^{-1}(P_{a_m,r})} (Cf(u_\ast)_{r_j})(x) \, dx \right| \leq \varepsilon \quad \forall \; a_m.$$
Accordingly,
\[ \| C_f u_* \|_{BMO(\mathbb{R}^n)} \geq \int_{f^{-1}(P_{a_m,r})} |C_f u_*(x)| \, dx - \varepsilon \quad \forall \ r \in (0, \infty). \]

Similarly, we have
\[ \int_{f^{-1}(P_{a_m,r})} |C_f u_*(x)| \, dx = \left( \frac{|P_{a_m,r}|}{|f^{-1}(P_{a_m,r})|} \right) \int_{P_{a_m,r}} \|(u_*)_r(z)\| J_{f^{-1}}(z) \, dz \]
\[ \quad = \left( \frac{|P_{a_m,r}|}{|f^{-1}(P_{a_m,r})|} \right) (u_*)_{a_m,r} \ast h(0), \]

thereby using Lemma 2.6 to discover
\[ \liminf_{r_j \to 0} \int_{f^{-1}(P_{a_m,r_j})} \left| (C_f(u_*))_{a_m,r_j}(x) \right| \, dx = \left( \liminf_{r_j \to 0} \frac{|P_{a_m,r_j}|}{|f^{-1}(P_{a_m,r_j})|} \right) c_{a_m} h(0). \]

For \( r_j < \delta_1 \), we utilize
\[ 1 - \varepsilon \leq a_{mk} \lambda_k \leq 1 + \varepsilon \quad \forall \ k \in \{1, \ldots, n\}, \]

to deduce
\[ \left( \frac{|P_{a_m,r_j}|}{|f^{-1}(P_{a_m,r_j})|} \right) h(0) \geq (1 + \varepsilon)^{-n} (a_{m1} \ldots a_{mn})(\lambda_1 \ldots \lambda_n) \geq \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^n, \]
whence
\[ \liminf_{r_j \to 0} \int_{f^{-1}(P_{a,r})} \left| (C_f(u_*))_{r_j}(x) \right| \, dx \geq \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^n c_{a_m}, \]
which in turn implies
\[ \| C_f u_* \|_{BMO(\mathbb{R}^n)} \geq \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^n c_{a_m} - \varepsilon. \]

Upon combining this with (†)–(‡), we achieve a constant \( \kappa > 0 \) (independent of \( a_m \)) such that
\[ - \ln a_{m1} \leq \kappa \quad \& \quad a_{m1} \geq e^{-\kappa}. \]

Consequently, we gain
\[ 1 = \lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_1 \leq 2e^{\kappa}, \]
thereby reaching (♯). \( \square \)
3. Validation of Theorem 1.5

In order to prove Theorem 1.5, we need the concept of a $Q_\alpha(\mathbb{R}^n)$-capacity. For $(\alpha, n) \in (-\infty, 1) \times \mathbb{N}$ and any pair of disjoint continua $E, F \subset \mathbb{R}^n$, let

$$\text{Cap}_{Q_\alpha(\mathbb{R}^n)}(E, F) = \inf \left\{ \|u\|_{Q_\alpha(\mathbb{R}^n)}^2 : u \in \Delta_\alpha(E, F) \right\}$$

be the $Q_\alpha(\mathbb{R}^n)$-capacity of the pair $(E, F)$, where $\Delta_\alpha(E, F)$ is the class of all continuous functions $u \in Q_\alpha(\mathbb{R}^n)$ enjoying

$$\begin{cases} 
0 \leq u \leq 1 & \text{on } \mathbb{R}^n; \\
u = 0 & \text{on } E; \\
u = 1 & \text{on } F.
\end{cases}$$

Obviously, if $\tilde{E} \& \tilde{F}$ are disjoint continua satisfying $E \subset \tilde{E} \& F \subset \tilde{F}$, then

$$\text{Cap}_{Q_\alpha(\mathbb{R}^n)}(E, F) \leq \text{Cap}_{Q_\alpha(\mathbb{R}^n)}(\tilde{E}, \tilde{F}).$$

Moreover, we have

**Lemma 3.1.** Given a constant $\delta \in (0, \infty)$ let $n=1$ & $\alpha \in (0, 2^{-1}]$ or $n=2$ & $\alpha \in (2^{-1}, 1)$. If $E \& F$ are disjoint continua in $\mathbb{R}^n$ such that their diameters $\text{diam } E \& \text{diam } F$ and Euclidean distance $\text{dist } (E, F)$ obey

$$\min\{\text{diam } E, \text{diam } F\} \geq \delta \text{ dist } (E, F) > 0,$$

then

$$\text{Cap}_{Q_\alpha(\mathbb{R}^n)}(E, F) \gtrsim 1.$$

**Proof.** Without loss of generality we may assume

$$\text{diam } E = \text{diam } F \geq \delta \text{ dist } (E, F).$$

If

$$x_0 \in E \& r = (2+\delta^{-1}) \text{ diam } E,$$

then

$$E, F \subset B(x_0, r).$$

Thanks to either $n=1$ & $\alpha \in (0, 2^{-1}]$ or $n=2$ & $\alpha \in (2^{-1}, 1)$, we may assume

$$\begin{cases} 
\nu \in \Delta_\alpha(E, F); \\
u_B(x_0, r) \geq 2^{-1}; \\
0 < \varepsilon \leq 1 - n + 2\alpha.
\end{cases}$$
For every $x \in E$ and $\rho > 0$ we utilize
\[
\Phi_\alpha(u, B(x, \rho)) = \rho^{2\alpha-n} \int_{B(x, \rho)} \int_{B(x, \rho)} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}} \, dz \, dw
\]
\[
\gtrsim \int_{B(x, \rho)} \int_{B(x, \rho)} |u(z) - u(w)| \, dz \, dw
\]
to estimate
\[
2^{-1} \leq |u(x) - u_{B(x_0, r)}|
\]
\[
\leq \sum_{i=1}^{\infty} |u_{B(x, 2^{-i}r)} - u_{B(x, 2^{-i-1}r)}| + |u_{B(x, 2r)} - u_{B(x_0, r)}|
\]
\[
\lesssim \sum_{i=1}^{\infty} \left( \int_{B(x, 2^{-i}r)} \int_{B(x, 2^{-i}r)} |u(z) - u(w)|^2 \, dz \, dw \right)^{2^{-1}}
\]
\[
\lesssim \sum_{i=1}^{\infty} \left( \Phi_\alpha(u, B(x, 2^{-i}r)) \right)^{2^{-1}}
\]
\[
\lesssim \sum_{i=1}^{\infty} (2^{-i}r)^{\frac{\epsilon}{ \alpha}} \sup_{t \leq 2^r} t^{-\frac{\epsilon}{\alpha}} \left( \Phi_\alpha(u, B(x, t)) \right)^{2^{-1}}
\]
\[
\lesssim r^{\frac{\epsilon}{\alpha}} \sup_{t \leq 2^r} t^{-\frac{\epsilon}{\alpha}} \left( \Phi_\alpha(u, B(x, t)) \right)^{2^{-1}}.
\]
Accordingly, for each $x \in E$ there exists a $t_x \in (0, 2r]$ such that
\[
\begin{cases}
1 \leq r^\epsilon t_x^{-\frac{\epsilon}{\alpha}} \Phi_\alpha(u, B(x, t_x)); \\
t_x^{-n-2\alpha+\epsilon} \lesssim r^\epsilon \int_{B(x, t_x)} \int_{B(x, t_x)} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}} \, dz \, dw.
\end{cases}
\]
By the Vitali covering lemma, we can find points $x_i \in E$ and radii $r_i > 0$ as above such that ball $B(x_i, t_i)$ are mutually disjoint and $E \subseteq \bigcup_i B(x_i, 5t_i)$. Hence,
\[
\text{diam } E \lesssim \sum_{i=1}^{\infty} t_i \lesssim r^{\frac{\epsilon}{n-2\alpha+\epsilon}} \sum_{i=1}^{\infty} \left( \int_{B(x_i, t_i)} \int_{B(x_i, t_i)} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}} \, dz \, dw \right)^{\frac{1}{n-2\alpha+\epsilon}}.
\]
Upon noticing $1/(n-2\alpha+\epsilon) \geq 1$, we obtain
\[
\frac{r}{2+\delta-1} \lesssim r^{\frac{\epsilon}{n-2\alpha+\epsilon}} \left( \sum_{i=1}^{\infty} \int_{B(x_i, t_i)} \int_{B(x_i, t_i)} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}} \, dz \, dw \right)^{\frac{1}{n-2\alpha+\epsilon}}
\]
\[
\lesssim r^{\frac{\epsilon}{n-2\alpha+\epsilon}} \left( \int_{B(x_0, 4r)} \int_{B(x_0, 4r)} \frac{|u(z) - u(w)|^2}{|z-w|^{n+2\alpha}} \, dz \, dw \right)^{\frac{1}{n-2\alpha+\epsilon}},
\]
whence
\[ \Phi_\alpha(u, B(x_0, 4r)) \gtrsim 1, \]
which yields
\[ \text{Cap}_{\mathcal{Q}_\alpha(\mathbb{R}^n)}(E, F) \gtrsim 1. \]

**Proof of Theorem 1.5.** By the metric characterization of a quasiconformal mapping (cf. [7]), it is enough to validate that if
\[ \left\{ \begin{array}{l}
\ell(f, r) = \inf \{ |f(x) - f(x_0)| : |x - x_0| \geq r \}; \\
L(f, r) = \sup \{ |f(x) - f(x_0)| : |x - x_0| \leq r \}; \\
(x_0, r) \in \mathbb{R}^2 \times (0, \infty),
\end{array} \right. \]
then
\[ L(f, r) \leq c(f) \ell(f, r), \]
where \( c(f) \) is a positive constant depending on \( f \).

To this end, if
\[ v(y) = \begin{cases} 
1 & \text{as } |y - x_0| \leq \ell(f, r); \\
\frac{\ln L(f, r) - \ln |y - x_0|}{\ln L(f, r) - \ln \ell(f, r)} & \text{as } \ell(f, r) \leq |y - x_0| \leq L(f, r); \\
0 & \text{as } |y - x_0| \geq L(f, r),
\end{cases} \]
then
\[ |\nabla v(y)| = \begin{cases} 
0 & \text{as } |y - x_0| \leq \ell(f, r); \\
\frac{|y - x_0|^{-1}}{\ln L(f, r) - \ln \ell(f, r)} & \text{as } \ell(f, r) \leq |y - x_0| \leq L(f, r); \\
0 & \text{as } |y - x_0| \geq L(f, r),
\end{cases} \]
and hence
\[ \|v\|_{W^{1,2}(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |\nabla v(y)|^2 dy = \left( \ln \frac{L(f, r)}{\ell(f, r)} \right)^{-2} \int_{l \leq |y - x_0| \leq L} \frac{dy}{|y - x_0|^2} \leq \left( \ln \frac{L(f, r)}{\ell(f, r)} \right)^{-1}. \]

This last estimation, along with [10, Theorem 4.1] under \( n = 2 \) & \( \alpha < 1 \), implies
\[ \|v\|_{Q_{2-1<\alpha<1}(\mathbb{R}^2)} \leq \|v\|_{W^{1,2}(\mathbb{R}^2)} \lesssim \left( \ln \frac{L(f, r)}{\ell(f, r)} \right)^{-2}. \]
Let
\[ E = f^{-1}\left( B(f(x_0), \ell) \right), \]
i.e., the preimage of \( B(f(x_0), \ell) \) under \( f \). Then \( E \) is connected and enjoys
\[ E \subseteq B(x_0, r) \quad \& \quad \text{diam } E \geq r. \]
Moreover, observe that as the connected preimage of \( \mathbb{R}^2 \setminus B(f(x_0), L) \) under \( f \),
\[ f^{-1}\left( \mathbb{R}^2 \setminus B(f(x_0), L) \right) \]
joins
\[ \overline{B}(x_0, r) = \{ x \in \mathbb{R}^2 : |x - x_0| \leq r \} \quad \& \quad \mathbb{R}^2 \setminus B(x_0, 2r). \]
So we can find a connected continuum \( F \) such that it is contained in
\[ f^{-1}(\mathbb{R}^2 \setminus B(f(x_0), L)) \]
and joins \( \overline{B}(x_0, r) \) and \( \mathbb{R}^2 \setminus B(x_0, 2r) \), and consequently we may assume
\[ F \subseteq \overline{B}(x_0, 2r) \setminus B(x_0, r). \]
Obviously, we have
\[ \text{diam } F \geq r \quad \& \quad 0 < \text{dist } (E, F) \leq 5r \leq 10 \min \{ \text{diam } E, \text{diam } F \}. \]
Upon applying Lemma 3.1 under \( n=2 \) & \( 2^{-1} < \alpha < 1 \) we discover
\[ \text{Cap}_{\alpha, \mathbb{R}^2}(E, F) \gtrsim 1, \]
thereby arriving at the required inequality
\[ \ln \frac{L(f, r)}{\ell(f, r)} \lesssim 1. \]

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References


Jie Xiao
Department of Mathematics and Statistics
Memorial University
St. John’s NL A1C 5S7
Canada
jxiao@mun.ca

Yuan Zhou
Department of Mathematics
Beihang University
Beijing 100191
China
yuanzhou@buaa.edu.cn

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