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Pull-back of singular Levi-flat hypersurfaces

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Abstract. We study singular real analytic Levi-flat subsets invariant by singular holomorphic foliations in complex projective spaces. We give sufficient conditions for a real analytic Levi-flat subset to be the pull-back of a semianalytic Levi-flat hypersurface in a complex projective surface under a rational map or to be the pull-back of a real algebraic curve under a meromorphic function. In particular, we give an application to the case of a singular real analytic Levi-flat hypersurface. Our results improve previous ones due to Lebl and Bretas–Fernández-Pérez–Mol.

1. Introduction and statement of the results

Let M be a complex manifold of $\dim_{\mathbb{C}} M=N\geq 2$, a closed subset $H\subset M$ is a *real analytic subvariety* if for every $p\in H$, there are real analytic functions with real values $\varphi_1,...,\varphi_k$ defined in a neighborhood $U\subset M$ of p, such that $H\cap U$ is equal to the set where all $\varphi_1,...,\varphi_k$ vanish. A complex subvariety is precisely the same notion, considering holomorphic functions instead of real analytic functions. We say that a real analytic subvariety H is *irreducible* if whenever we write $H=H_1\cup H_2$ for two subvarieties H_1 and H_2 of M, then either $H_1=H$ or $H_2=H$. If H is irreducible, it has a well-defined dimension $\dim_{\mathbb{R}} H$. Let H_{reg} denote its *regular part*, i.e., the subset of points near which H is a real analytic submanifold of dimension equal to $\dim_{\mathbb{R}} H$. A set is *semianalytic* if it is locally constructed from real analytic sets by finite union, finite intersection, and complement. For a real analytic subvariety H, the set $\overline{H_{reg}}$ is a semianalytic subset where the closure is with the standard topology. In general, the inclusion $\overline{H_{reg}}\subset H$ is proper, which happens, for instance in the

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Whitney umbrella. We really only study the set $\overline{H_{reg}}$, in this sense, we consider $\mathsf{Sing}(H){:=}\overline{H_{reg}}\backslash H_{reg}$ as the *singular set* of H, this is not the usual definition of the singular set in the literature, see for instance [15].

If $H{\subset}M$ is a real analytic hypersurface i.e., a real analytic subvariety of real codimension one, then for each $p{\in}H_{reg}$, there is a unique complex hyperplane $\mathscr{L}_p{\subset}T_pH_{reg}$. This defines a real analytic distribution $p{\mapsto}\mathscr{L}_p$ of complex hyperplanes in TH_{reg}. When this distribution is *integrable* in the sense of Frobenius, we say that H is *Levi-flat*. Here, H_{reg} is foliated by codimension one immersed complex submanifolds. This foliation, denoted by $\mathscr{L}$, is known as *Levi foliation*. According to Cartan [4], $\mathscr{L}$ can be extended to a non-singular holomorphic foliation in a neighborhood of H_{reg} in M, but in general, it is not possible to extend $\mathscr{L}$ to a singular holomorphic foliation in a neighborhood of H. There are examples of singular Levi-flat hypersurfaces whose Levi foliations extend to singular *holomorphic webs* in the ambient space, see for instance [8] and [21]. When there is a singular holomorphic foliation $\mathscr{F}$ in the ambient space M that coincide with the Levi foliation on H_{reg}, we say either that H is *invariant* by $\mathscr{F}$ or that $\mathscr{F}$ is *tangent* to H. Cerveau and Lins Neto [6] proved that germs of singular foliations of codimension one at $(\mathbb{C}^N,0)$ tangent to real analytic Levi-flat hypersurfaces have meromorphic (possibly holomorphic) first integrals. We recall that a non-constant function f is the *first integral* for a foliation $\mathscr{F}$ if each leaf of $\mathscr{F}$ is contained in a level set of f. In the global context, the same problem has been studied in [1] and [9].

The aim of this paper is to study holomorphic foliations tangent to real analytic Levi-flat subsets in complex manifolds. An irreducible real analytic subvariety $H{\subset}M$, where M is an N-dimensional complex manifold, $N{\geq}2$, is a *Levi-flat subset* if it has real dimension $2n{+}1$ and its regular part H_{reg} is foliated by immersed complex manifolds of complex dimension n. Similarly to the case of hypersurfaces, this foliation is called *Levi foliation* of H and will be denoted by $\mathscr{L}$. The number n is the *Levi dimension* of H. We use the qualifier "*Levi*" for the foliation, its leaves, and its dimension. Since we deal with real analytic Levi-flat subsets in complex manifolds we shall consider that H is *coherent*. Coherence implies that H admits a global complexification [11, p. 40]. Here coherent means that its ideal sheaf $\mathcal{I}(H)$ in $\mathcal{A}_{\mathbb{R},M}$, the sheaf of germs of real analytic functions with real values in M, is a coherent sheaf of $\mathcal{A}_{\mathbb{R},M}$-modules. It follows from Oka's theorem [17, p. 94, Proposition 5] that H is coherent if the sheaf $\mathcal{I}(H)$ is *locally finitely generated*, the latter means that for every point $p{\in}H$ there exists an open neighborhood $U{\subset}M$ and a finite number of functions φ_j, real analytic in U and vanishing on H, such that for any $q{\in}U$, the germs of φ_j at q generate the ideal $\mathcal{I}(H_q)$, where H_q is the germ of H at q. We remark that not every real analytic subset is coherent as we shall see in Section 3 of this paper.

In [3], singular Levi-flat subsets appear in the result of the lifting of a real analytic Levi-flat hypersurface to the projectivized cotangent bundle of the ambient space through the Levi foliation and in [20], the authors gave a complete characterization of dicritical singularities of local Levi-flat subsets in terms of their Segre varieties.

Let Y be a complex projective surface, $T\subset Y$ be a real analytic Levi-flat hypersurface, $X\subset\mathbb{P}^N$, $N\geq 3$, be a complex projective subvariety of complex dimension $k<N$ and $\rho: X \dashrightarrow Y$ be a dominant rational map. Then it is easy to show that $H=\overline{\rho^{-1}(T)}$ is a real analytic Levi-flat subset in $\mathbb{P}^N$ and so H is a Levi-flat subset defined via pull-back. Therefore, one natural question is:

Given a real analytic Levi-flat subset $H\subset\mathbb{P}^N$. Under what condition, H is given by the pull-back of a Levi-flat hypersurface in a projective complex surface via a rational map?

In [14], Lebl gave sufficient conditions for a real analytic Levi-flat hypersurface in $\mathbb{P}^N$ to be a pull-back of a real algebraic curve in $\mathbb{C}$ via a meromorphic function. In [2], Bretas et al. proved an analogous result for real analytic Levi-flat subsets in $\mathbb{P}^N$. The main hypothesis in these articles is that the Levi foliation has infinitely many *algebraic leaves*. In this paper, we give an answer to the question, assuming that H is invariant by a singular holomorphic foliation on $\mathbb{P}^N$ with *quasi-invariant subvarieties* (see Section 2). An irreducible complex subvariety $S\subset X$ of complex dimension n is *quasi-invariant* by a global n-dimensional foliation $\mathscr{F}$ on a complex projective manifold X if it is not $\mathscr{F}$-invariant, but the restriction to the foliation $\mathscr{F}$ to S is an *algebraically integrable foliation* of dimension $n-1$, i.e. every leaf of $\mathscr{F}|_S$ is algebraic. The concept of *quasi-invariant subvarieties* was introduced by Pereira-Spicer [19] for codimension one holomorphic foliations on complex projective manifolds to prove a variant of the classical Darboux-Jouanolou Theorem. Here we shall use this concept for Levi foliations to prove our main result:

Theorem 1.1. *Let $H\subset\mathbb{P}^N$, $N\geq 3$, be an irreducible real analytic Levi-flat subset of Levi dimension n invariant by an n-dimensional singular holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^N$. Suppose that H is coherent and $n>N/2$. If the Levi foliation has infinitely many quasi-invariant subvarieties of complex dimension n, then there exists a unique projective subvariety X of complex dimension $n+1$ containing H such that either there exists a rational map $R: X\dashrightarrow\mathbb{P}^1$, and real algebraic curve $C\subset\mathbb{P}^1$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ or there exists a dominant rational map $\rho: X\dashrightarrow Y$ on a projective surface Y and a semianalytic Levi-flat subset $T\subset Y$ such that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$.*

We emphasize that the hypothesis $n>N/2$ implies that H is necessarily a real analytic subvariety with singularities. In fact, Ni-Wolfson [18, Theorem 2.4] proved that no nonsingular real analytic Levi-flat subset of the Levi dimension n exist in $\mathbb{P}^N$, $n>N/2$.

Applying Theorem 1.1 to $n=N-1$, we get the following corollary:

Corollary 1.2. *Let $H\subset\mathbb{P}^N$, $N\geq 3$, be an irreducible coherent real analytic Levi-flat hypersurface invariant by a codimension one holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^N$. If the Levi foliation has infinitely many quasi-invariant complex hypersurfaces, then either there exists a rational map $R:\mathbb{P}^N \dashrightarrow \mathbb{P}^1$, and real algebraic curve $C\subset\mathbb{P}^1$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ or there exists a dominant rational map $\rho:\mathbb{P}^N\dashrightarrow Y$ on a projective surface Y and a semianalytic Levi-flat subset $T\subset Y$ such that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$.*

When H is a real analytic hypersurface, the above corollary gives a nice characterization of coherent real analytic Levi-flat hypersurfaces in $\mathbb{P}^N$, $N\geq 3$, invariant by codimension one holomorphic foliations which admit infinitely many quasi-invariant complex hypersurfaces. Observe that, in order to improve our results, we need to extend the Levi foliation of a Levi-flat subset to a holomorphic foliation in the ambient space. Therefore, another interesting question is:

Given a real analytic Levi-flat subset $H\subset\mathbb{P}^N$ with Levi foliation $\mathcal{L}$. Under what condition, $\mathcal{L}$ extend to a singular holomorphic foliation on $\mathbb{P}^N$?

When H is a local real analytic Levi-flat hypersurface, Lebl solved the above question in the non-dicritical case in [15].

The paper is organized as follows: in Section 2, we define the concept of quasi-invariant subvarieties of a foliation with complex leaves and state the main result of [19], such a result is key to prove Theorem 1.1. Section 3 is devoted to the study of real analytic Levi-flat subset in complex manifolds, using some results of [3] and [2], we prove the algebraic extension of the intrinsic complexification of H. In Section 4, we prove Theorem 1.1 and in Section 5 we prove Corollary 1.2. Finally, in Section 6, we give two examples. The first is an example of a Levi-flat hypersurface where Theorem 1.1 applies. In the second example, we construct a Levi-flat hypersurface in $\mathbb{P}^3$ that is not a pull-back of a Levi-flat hypersurface of $\mathbb{P}^2$ under a rational map. Moreover, this example also is not a pull-back of a real algebraic curve under a meromorphic function.

2. Foliations with complex leaves and quasi-invariant subvarieties

2.1. Foliations with complex leaves

A *foliation with complex leaves of complex dimension n* is a smooth foliation $\mathcal{G}$ of dimension $2n$ whose local models are domains $U=W\times B$ of $\mathbb{C}^n\times\mathbb{R}^k$, $W\subset\mathbb{C}^n$,

$B\subset\mathbb{R}^k$ and whose local transformations are of the form

$$\varphi(z,t)=(f(z,t),h(t)), \tag{1}$$

where f is holomorphic with respect to z. A domain U as above is said to be a *distinguished coordinate domain* of $\mathcal{G}$ and $z=(z_1,...,z_n)$, $t=(t_1,...,t_k)$ are said to be *distinguished local coordinates*. As examples of such foliations we have the Levi foliations of Levi-flat hypersurfaces of $\mathbb{C}^n$, see for instance [5] and [10].

If we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and in (1) we assume $t\in\mathbb{C}^k$ and that f,h are holomorphic with respect to z, t then we get the notion of *holomorphic foliation* of *complex codimension* k.

Now we define foliations with singularities. Let M be a complex manifold. A *singular foliation with complex leaves* $\mathcal{G}$ *of dimension* n on M is a foliation with complex leaves of dimension n on $M\setminus E$, where E is a real analytic subvariety of M of real dimension $<2n$. A point $p\in E$ is called a *removable singularity* of $\mathcal{G}$ of there is a chart (U,φ) around p, compatible with the atlas $\mathcal{A}$ of $\mathcal{G}$ restricted to $M\setminus E$, in the sense that $\varphi\circ\varphi_i^{-1}$ and $\varphi_i\circ\varphi^{-1}$ have the form (1) for all $(U_i,\varphi_i)\in\mathcal{A}$ with $U\cap U_i\neq\varnothing$. The set of non-removable singularities of $\mathcal{G}$ in E is called the *singular set* of $\mathcal{G}$, and is denoted by $\mathsf{Sing}(\mathcal{G})$.

2.2. Quasi-invariant subvarieties

Let Z be a projective manifold of complex dimension $N\geq 2$ and let $\mathscr{G}$ be a foliation with complex leaves of dimension n on Z.

Definition 2.1. We say that $\mathscr{G}$ is an *algebraically integrable foliation* on Z if every leaf of $\mathscr{G}$ is algebraic, i.e. every leaf of $\mathscr{G}$ is a projective complex subvariety in Z.

Motivated by [19], we define the concept of a *subvariety quasi-invariant* by a real analytic foliation with complex leaves.

Definition 2.2. An irreducible subvariety $S\subset Z$ of complex dimension n is *quasi-invariant* by a foliation $\mathscr{G}$ if it is not $\mathscr{G}$-invariant, but the restriction of the foliation $\mathscr{G}$ to S is an algebraically integrable foliation.

We note that the restriction foliation $\mathscr{G}|_S$ is a codimension one foliation on S and when $\mathscr{G}|_S$ is an algebraically integrable foliation, we have that every leaf of $\mathscr{G}|_S$ are projective complex hypersurfaces in S. Codimension one holomorphic foliations on Z which admit infinitely many quasi-invariant hypersurfaces have been studied in [19] and its main result is the following.

Theorem 2.3. (Pereira-Spicer [19]) *Let $\mathscr{F}$ be a codimension one holomorphic foliation on a projective manifold Z. If $\mathscr{F}$ admits infinitely many quasi-invariant hypersurfaces then either $\mathscr{F}$ is an algebraically integrable foliation, or $\mathscr{F}$ is a pull-back of a foliation of dimension one on a projective surface under a dominant rational map.*

3. Real analytic subsets

3.1. Coherent real analytic subsets.

We present some of the fundamental results concerning coherent real analytic subsets.

Let H be a real analytic subset in an open set $U\subset\mathbb{C}^n$ and let $\mathcal{I}(H)$ be its ideal sheaf, it is the sheaf of germs of real analytic functions with real values vanishing on H.

Definition 3.1. H is said to be coherent if $\mathcal{I}(H)$ is a coherent sheaf of $\mathcal{A}_{\mathbb{R},U}$-modules, where $\mathcal{A}_{\mathbb{R},U}$ is the sheaf of germs of real analytic functions with real values in U.

Proposition 3.2. ([17, p. 95]) *If H is a coherent real analytic subset and the germ H_p of H at p is irreducible, then for q near p, we have*

$$\dim_{\mathbb{R}} H_p = \dim_{\mathbb{R}} H_q.$$

It is well known that locally, a real analytic subset always admits a complexification (see for instance [11, p. 40]) and it is not true for global real analytic subsets. It is shown in [11, p. 54] that the global complexification of a coherent real analytic subset in a complex manifold always exists.

Theorem 3.3. ([11, p. 54]) *A real analytic subset in a complex manifold is coherent if and only if it admits a global complexification.*

Now we build an irreducible real analytic hypersurface in $\mathbb{P}^3$ which is not coherent. Let $[z_0:z_1:z_2:z_3]$ be the homogeneous coordinates in $\mathbb{P}^3$ and set $H\subset\mathbb{P}^3$ be the complex cone whose equation is

$$H=\{(z_3\bar{z}_0+\bar{z}_3z_0)\left((z_1\bar{z}_0+\bar{z}_1z_0)^2+(z_2\bar{z}_0+\bar{z}_2z_0)^2\right)-(z_1\bar{z}_0+\bar{z}_1z_0)^3=0\}.$$

The germ H_p of H at $p=[1:0:0:0]$ is irreducible and of real dimension 5 at p. However, in a neighborhood of $[1:0:0:z]$, $z\neq 0$, H reduces to the complex line $z_1=z_2=0$, which is of real dimension 2. By Proposition 3.2, it follows that H is not coherent.

3.2. Levi-flat subset in complex manifolds.

We give a brief resume of definitions and some known results about real analytic Levi-flat subsets in complex manifolds. Let H be an irreducible real analytic Levi-flat subset of Levi dimension n in an N-dimensional complex manifold M. The notion of Levi-flat subset germifies and, in general, we do not distinguish a germ at $(\mathbb{C}^N,0)$ from its realization in some neighborhood U of $0\in\mathbb{C}^N$. If $p\in H_{reg}$ then, according to [2, Proposition 3.1], there exists a holomorphic coordinate system $z=(z',z'')\in\mathbb{C}^{n+1}\times\mathbb{C}^{N-n-1}$ such that $z(p)=0\in\mathbb{C}^N$ and the germ of H at p is defined by

$$H=\{z=(z',z'')\in\mathbb{C}^{n+1}\times\mathbb{C}^{N-n-1}:\ \mathrm{Im}(z_{n+1})=0,\quad z''=0\},\tag{2}$$

where $z'=(z_1,...,z_{n+1})$ and $z''=(z_{n+2},...,z_N)$ and the Levi foliation is given by

$$\{z=(z',z'')\in\mathbb{C}^{n+1}\times\mathbb{C}^{N-n-1}:\ z_{n+1}=c,\quad z''=0,\ \text{with } c\in\mathbb{R}\}.$$

This trivial model is, in fact, a local form for a non-singular real analytic Levi-flat subset. Note that in the local form (2), $\{z''=0\}$ corresponds to the unique local $(n+1)$-dimensional complex subvariety of the ambient space containing the germ of H_{reg} at p. These local subvarieties glue together forming a complex variety defined in a whole neighborhood of H_{reg}. It is analytically extendable to a neighborhood of $\overline{H_{reg}}$ by the following theorem:

Theorem 3.4. (Brunella [3]) *Let M be an N-dimensional complex manifold and $H\subset M$ be a real analytic Levi-flat subset of Levi dimension n. Then, there exists a neighborhood $V\subset M$ of $\overline{H_{reg}}$ and a unique complex variety $X\subset V$ of dimension $n+1$ containing H.*

The variety X is the realization in the neighborhood V of a germ of complex analytic variety around H. We denote it — or its germ — by $H^\imath$ and call it *intrinsic complexification* or *$\imath$-complexification* of H. It plays a central role in the theory of real analytic Levi-flat subsets. The notion of intrinsic complexification also appears in [22] with the name of the *Segre envelope*. If H is invariant by a holomorphic foliation on M, the same holds for its $\imath$-complexification, see for instance [2, Proposition 3.3].

Proposition 3.5. *Let $H\subset M$ be a real analytic Levi-flat subset of Levi dimension n, where M is a complex manifold of dimension N. If H is invariant by an n-dimensional holomorphic foliation $\mathcal{F}$ on M, then its $\imath$-complexification $H^\imath$ is also invariant by $\mathcal{F}$.*

As a consequence, if we denote by $\mathscr{F}^\imath := \mathscr{F}|_{H^\imath}$ (the restriction of $\mathscr{F}$ to $H^\imath$), we have $\mathscr{F}^\imath$ has codimension one in $H^\imath$. The following proposition shows the importance of the assumption of the *coherence* of a Levi-flat subset.

Proposition 3.6. ([2, Proposition 3.6]) *Let M be an N-dimensional complex manifold and $H \subset M$ be an irreducible real analytic Levi-flat subset of Levi dimension n. Suppose that H is coherent. Then, there exist an open neighborhood $V \subset M$ of H and a unique irreducible complex subvariety X of V of complex dimension $n+1$ containing H.*

The variety X is the small variety of complex dimension $n+1$ that contains H. Again, let us denote this variety by $H^\imath$, the intrinsic complexification of H.

3.3. Levi-flat subsets in complex projective spaces

In this subsection, we state some results of real analytic Levi-flat subset in $\mathbb{P}^N$. Let $\sigma: \mathbb{C}^{N+1} \to \mathbb{P}^N$ be the natural projection. Suppose that H is a real-analytic subvariety of $\mathbb{P}^N$. Define the set $\tau(H)$ to be the set of points $z \in \mathbb{C}^{N+1}$ such that $\sigma(z) \in H$ or $z=0$. A real analytic subvariety $H \subset \mathbb{P}^N$ is said to be *algebraic* if $H = \sigma(V)$ for some real algebraic complex cone V in $\mathbb{C}^{N+1}$. A set V is a complex cone when $p \in V$ implies $\lambda p \in V$ for all $\lambda \in \mathbb{C}$.

The following construction offers several examples of Levi-flat subsets in $\mathbb{P}^N$.

Proposition 3.7. [2, *Proposition* 6.1] *Let $X \subset \mathbb{P}^N$ be an irreducible $(n+1)$-dimensional algebraic variety, R be a rational function in X and $C \subset \mathbb{P}^1$ be a real algebraic one-dimensional subvariety. Then the set $\overline{R^{-1}(C)}$ is a real algebraic Levi-flat subset of Levi dimension n whose $\imath$-complexification is X.*

When we add the hypothesis that the Levi-flat subset is invariant by a singular holomorphic foliation in the ambient space, we can state a reciprocal result.

Proposition 3.8. ([2, Proposition 6.3]) *Let $\mathscr{F}$ be a singular holomorphic foliation in $\mathbb{P}^N$ tangent to a real analytic Levi-flat subset H of Levi dimension n. Suppose that H is coherent and its $\imath$-complexification extends to an algebraic subvariety $H^\imath$ in $\mathbb{P}^N$. If $\mathscr{F}^\imath$ has a rational first integral R, then there exists a real algebraic one-dimensional subvariety $C \subset \mathbb{P}^1$ such that $\overline{H_{reg}} \subset \overline{R^{-1}(C)}$.*

Now, since H is coherent, the intrinsic complexification $H^\imath$ is well-defined as a complex subvariety in a neighborhood of H. Our aim is to extend $H^\imath$ to an algebraic subvariety in $\mathbb{P}^N$. To get this, we use the following extension theorem.

Theorem 3.9. (Chow [7]) *Let $Z \subset \mathbb{P}^N$ be a complex algebraic subvariety of dimension k and V be a connected neighborhood of Z in $\mathbb{P}^N$. Then any complex* *analytic subvariety of dimension higher than $N-k$ in V that intersects Z extends algebraically to $\mathbb{P}^N$.*

Under certain hypotheses, we can prove that the $\imath$-complexification $H^\imath$ can be extended to $\mathbb{P}^N$.

Proposition 3.10. *Let $H\subset\mathbb{P}^N$, $N\geq 3$, be an irreducible coherent real analytic Levi-flat subset of Levi dimension n such that $n>N/2$. If the Levi foliation $\mathscr{L}$ has a quasi-invariant complex algebraic subvariety of complex dimension n, then $H^\imath$ extends algebraically to $\mathbb{P}^N$.*

Proof. Denote by L such quasi-invariant algebraic complex subvariety with $\dim_{\mathbb{C}} L=n-1$. Since L algebraic with $L\subset H^\imath$ and $\dim_{\mathbb{C}} H^\imath=n+1>N-(n-1)$, we can apply Theorem 3.9 to prove that $H^\imath$ extends algebraically to $\mathbb{P}^N$. $\square$

To end this section, we shall prove the following proposition.

Proposition 3.11. *Let $H\subset\mathbb{P}^N$ be an irreducible coherent real analytic Levi-flat subset of Levi dimension n invariant by an n-dimensional singular holomorphic foliation $\mathscr{F}$ in $\mathbb{P}^N$. Suppose that the $\imath$-complexification $H^\imath$ extends to an algebraic variety in $\mathbb{P}^N$. If the Levi-foliation $\mathcal{L}$ has infinitely many quasi-invariant algebraic subvarieties of complex dimension $n-1$. Then, either the foliation $\mathscr{F}^\iota=\mathscr{F}|_{H^\imath}$ has a rational first integral in $H^\imath$, or $\mathscr{F}^\iota$ is a pull-back of a foliation on a projective surface under a dominant rational map.*

Proof. First of all, we need to desingularize the $\imath$-complexification $H^\imath$. According to Hironaka desingularization theorem, there exist a complex manifold $\widetilde{H^\imath}$ and a proper bimeromorphic morphism $\pi:\widetilde{H^\imath}\to H^\imath$ such that

1. $\pi:\widetilde{H^\imath}\backslash(\pi^{-1}(\mathsf{Sing}(H^\imath))\to H^\imath\backslash\mathsf{Sing}(H^\imath)$ is a biholomorphism,
2. $\pi^{-1}(\mathsf{Sing}(H^\imath))$ is a simple normal crossing divisor.

Since $H^\imath$ is compact then $\widetilde{H^\imath}$ is too. We lift $\mathscr{F}^\imath$ to an n-dimensional singular holomorphic foliation $\widetilde{\mathscr{F}^\imath}$ on $\widetilde{H^\imath}$. Since $\dim_{\mathbb{C}}\widetilde{H^\imath}=n+1$, we have $\widetilde{\mathscr{F}^\imath}$ has codimension one on $\widetilde{H^\imath}$ and the tangency condition between $\mathscr{F}^\imath$ and H implies that $\mathscr{F}^\imath$ has infinitely many quasi-invariant closed subvarieties (these are algebraic and of codimension one in $\widetilde{H^\imath}$). Thus the same holds for $\widetilde{\mathscr{F}^\imath}$. By Theorem 2.3, either $\widetilde{\mathscr{F}^\imath}$ has a rational first integral or there exist a dominant rational map $\tilde{\rho}:\widetilde{H^\imath}\dashrightarrow Y$, where Y is a projective complex surface, $\mathcal{G}$ is a foliation by curves on Y and $\widetilde{\mathscr{F}^\imath}=\tilde{\rho}^*(\mathcal{G})$. If $\widetilde{\mathscr{F}^\imath}$ admits a rational first integral in $\widetilde{H^\imath}$, then all leaves of $\widetilde{\mathscr{F}^\imath}$ are compact and so their π-images are compact leaves of $\mathscr{F}^\imath$ in $H^\imath$. Applying Gómez-Mont's theorem [12], we have that there exists a one-dimensional projective manifold S and a rational map $f:H^\imath\dashrightarrow S$ whose fibers contain the leaves of $\mathscr{F}^\imath$. A rational first integral is obtained by composing f with any non-constant rational map $r:S\dashrightarrow\mathbb{P}^1$. If $\widetilde{\mathscr{F}^\imath}$ is a pull-back of a foliation $\mathcal{G}$ on a projective complex surface Y under a dominant rational map $\tilde{\rho}:\widetilde{H^\imath} \dashrightarrow Y$ then $\mathscr{F}^\imath$ is the pull-back of $\mathcal{G}$ under $\rho:=\tilde{\rho}\circ\pi^{-1}:H^\imath \dashrightarrow Y$, since π is a birational map. □

4. Proof of Theorem 1.1

With all the above results, we can prove Theorem 1.1.

Theorem 4.1. *Let $H\subset\mathbb{P}^N$, $N\geq 3$, be an irreducible real analytic Levi-flat subset of Levi dimension n invariant by an n-dimensional singular holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^N$. Suppose that H is coherent and $n>N/2$. If the Levi foliation has infinitely many quasi-invariant subvarieties of complex dimension n, then there exists a unique projective subvariety X of complex dimension $n+1$ containing H such that either there exists a rational map $R:X\dashrightarrow\mathbb{P}^1$, and real algebraic curve $C\subset\mathbb{P}^1$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ or there exists a dominant rational map $\rho:X\dashrightarrow Y$ on a projective surface Y and a semianalytic Levi-flat subset $T\subset Y$ such that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$.*

Proof. By Proposition 3.6, there exist an open neighborhood $V\subset\mathbb{P}^N$ of H and a unique irreducible complex subvariety $H^\imath$ of V of complex dimension $n+1$ containing H. The Proposition 3.5 implies that $H^\imath$ is invariant by $\mathscr{F}$ and moreover it extends algebraically to $\mathbb{P}^N$ by Proposition 3.10. We denote $\mathscr{F}^\imath:=\mathscr{F}|_{H^\imath}$ the restrict foliation to $H^\imath$. Observe now that $\mathscr{F}^\imath$ is a foliation of codimension one on $H^\imath$ which admit infinitely many quasi-invariant subvarieties of complex dimension $n-1$. Therefore, either $\mathscr{F}^\imath$ has a rational first integral in $H^\imath$, or $\mathscr{F}^\imath$ is a pull-back of a foliation on a projective surface under a dominant rational map by Proposition 3.11.

If $\mathscr{F}^\imath$ has a first integral R then there exists a real algebraic curve $C\subset\mathbb{P}^1$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ by Proposition 3.8. Now if we assume that $\mathscr{F}^\imath$ is a pull-back of a foliation $\mathcal{G}$ on a projective complex surface Y under a dominant rational map $\rho:H^\imath\dashrightarrow Y$. Then we can take $X=H^\imath$. Let us prove that there exists a semianalytic Levi-flat subset $T\subset Y$. Indeed, let $z\in H_{reg}\backslash Ind(\rho)$ (here $Ind(\rho)$ denotes the indeterminacy set of ρ). Then there exists a neighborhood $U\subset H^\imath\backslash Ind(\rho)$ of z and a non-singular real analytic curve $\gamma:(-\varepsilon,\varepsilon)\to U$ such that $\gamma(0)=z$, $\{\gamma\}\subset H_{reg}$, and such that γ is transverse to the Levi foliation $\mathscr{L}$ on H_{reg}. Let $L_{\gamma(t)}$ be the leaf of $\mathscr{L}$ through $\gamma(t)$. Since $L_{\gamma(t)}$ is also a leaf of $\mathscr{F}^\imath$ and $\mathscr{F}^\imath=\rho^*(\mathcal{G})$, then $\rho(L_{\gamma(t)})$ is a leaf of $\mathcal{G}$. Let us denote $A_t=\overline{\rho(L_{\gamma(t)})}\subset Y$ and define

$$T_z:=\bigcup_{t\in(-\varepsilon,\varepsilon)} A_t\subset V_z,$$

where V_z is a neighborhood of T_z on Y. Note that T_z is a union of complex subvarieties parametrized by t such that each A_t contains leaves of $\mathcal{G}$, thus T_z is a semianalytic Levi-flat subset on V_z. These local constructions are sufficiently canonical to be patched together when z varies on H_{reg}: if $T_{z_1}\subset V_{z_1}$ and $T_{z_2}\subset V_{z_2}$ are as above, with $V_{z_1}\cap V_{z_2}\neq\varnothing$, then $T_{z_1}\cap V_{z_1}\cap V_{z_2}$ and $T_{z_2}\cap V_{z_1}\cap V_{z_2}$ have some common leaves of $\mathcal{G}$ because $\mathcal{G}$ is a global foliation defined on Y, so T_{z_1} and T_{z_2} can be glued by identifying these leaves. In this way, we get a semianalytic Levi-flat subset T in Y.

Finally, we assert that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$. In fact, let $w\in\overline{H_{reg}}$, then there exists a sequence $z_k\to w$, $z_k\in H_{reg}$, so $\rho(z_k)\in T$ which imply that $z_k\in\rho^{-1}(T)$ and $w\in\overline{\rho^{-1}(T)}$. This finishes the proof. □

5. Proof of Corollary 1.2

Corollary 5.1. *Let $H\subset\mathbb{P}^N$, $N\geq 3$, be an irreducible coherent real analytic Levi-flat hypersurface invariant by a codimension one holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^N$. If the Levi foliation has infinitely many quasi-invariant complex hypersurfaces, then either there exists a rational map $R:\mathbb{P}^N \dashrightarrow \mathbb{P}^1$, and real algebraic curve $C\subset\mathbb{P}^1$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ or there exists a dominant rational map $\rho:\mathbb{P}^N\dashrightarrow Y$ on a projective complex surface Y and a semianalytic Levi-flat subset $T\subset Y$ such that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$.*

Proof. If H is an irreducible real analytic Levi-flat hypersurface in $\mathbb{P}^N$, $N\geq 3$, then the Levi dimension of H is $N-1$. Moreover

$$N-1>N/2 \quad\Longleftrightarrow\quad N>2.$$

Thus, we can apply Theorem 1.1 to H, so there exist a unique projective subvariety X of complex dimension N containing H such that either there exists a rational map $R:X\dashrightarrow\mathbb{C}$, and real algebraic curve $C\subset\mathbb{C}$ such that $\overline{H_{reg}}\subset\overline{R^{-1}(C)}$ or there exists a dominant rational map $\rho:X\dashrightarrow Y$ on a projective complex surface Y and a semianalytic Levi-flat subset $T\subset Y$ such that $\overline{H_{reg}}\subset\overline{\rho^{-1}(T)}$. Since $X\subset\mathbb{P}^N$ has complex dimension N, we must have $X=\mathbb{P}^N$ and hence we conclude the proof. □

6. Examples

Example 6.1. We give an example of a real analytic Levi-flat hypersurface in $\mathbb{P}^3$ where Theorem 1.1 applies. Let

$$H=\{[z_0:z_1:z_2:z_3]\in\mathbb{P}^3 : z_0z_1\bar z_2\bar z_3-z_2z_3\bar z_0\bar z_1=0\},$$

then H is Levi-flat because it is foliated by the complex hypersurfaces

$$z_0z_1=cz_2z_3,\quad\text{where}\quad c\in\mathbb{R}. \tag{3}$$

Let $\mathscr{F}$ be the codimension one holomorphic foliation on $\mathbb{P}^3$ of degree two defined by

$$\omega = z_1 z_2 z_3 dz_0 + z_0 z_2 z_3 dz_1 - z_0 z_1 z_3 dz_2 - z_0 z_1 z_2 dz_3,$$

then $\mathscr{F}$ has a rational first integral $R{:}\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ given by

$$R[z_0 : z_1 : z_2 : z_3] = [z_0 z_1 : z_2 z_3].$$

Since the leaves of $\mathscr{F}|_H$ coincide with the leaves of the Levi foliation (3), H must be invariant by $\mathscr{F}$. On the other hand, note that $H = \overline{R^{-1}(C)}$, where

$$C = \{[t:u] \in \mathbb{P}^1 : t\bar{u} - u\bar{t} = 0\}.$$

Example 6.2. In the following example, we construct a real analytic Levi-flat hypersurface H in $\mathbb{P}^3$ that is not a pull-back of a Levi-flat hypersurface of $\mathbb{P}^2$ under a rational map, furthermore, H also is not a pull-back of a real algebraic curve under a meromorphic function.

Consider $z = (z_0, z_1, z_2, z_3)$, $\bar{z} = (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ and

$$F(z,\bar{z}) = \det \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & 0 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 & 0 \\ 0 & 0 & z_0 & z_1 & z_2 & z_3 \\ \bar{z}_0 & \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & 0 & 0 \\ 0 & \bar{z}_0 & \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & 0 \\ 0 & 0 & \bar{z}_0 & \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{pmatrix}$$

Define $H = \{[z_0{:}z_1{:}z_2{:}z_3] \in \mathbb{P}^3 {:} F(z,\bar{z}) = 0\}$, H is a real analytic hypersurface well defined since F is a bihomogeneous polynomial of bi-degree $(3,3)$. Moreover, H is Levi-flat, because it is foliated by the complex hyperplanes

$$z_0 + c z_1 + c^2 z_2 + c^3 z_3 = 0, \quad \text{where} \quad c \in \mathbb{R}. \tag{4}$$

Let $\mathscr{W}$ be the codimension one holomorphic 3-web on $\mathbb{P}^3$ given by the implicit differential equation $\Omega = 0$,

$$\Omega = \det \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & 0 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 & 0 \\ 0 & 0 & z_0 & z_1 & z_2 & z_3 \\ dz_0 & dz_1 & dz_2 & dz_3 & 0 & 0 \\ 0 & dz_0 & dz_1 & dz_2 & dz_3 & 0 \\ 0 & 0 & dz_0 & dz_1 & dz_2 & dz_3 \end{pmatrix}$$

Since the leaves of $\mathscr{W}|_H$ and $\mathscr{L}$ are the same, we get H is invariant by $\mathscr{W}$.

Now, we prove that H is not a pull-back of a Levi-flat hypersurface of $\mathbb{P}^2$. To prove this fact, we use the following result of [13, Proposition 4.4]:

Proposition 6.3. *Let ω_1, ω_2 and ω_3 be independent germs of integrable 1-forms at $(\mathbb{C}^3, 0)$ with singular sets of codimension at least two. Suppose that there exists a non-zero holomorphic 2-form η, locally decomposable outside its singular set, that is tangent to each ω_i, for $i{=}1, 2, 3$. Then ω_1, ω_2 and ω_3 define foliations that are in a pencil. Furthermore, η is integrable, defining the axis foliation of this pencil.*

Suppose by contradiction that H is a pull-back of a Levi-flat hypersurface under a dominant rational map $\rho{:}\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Then pick a point $p{\in}U_0$, where U_0 is an open subset in $\mathbb{P}^3$ such that $\rho|_{U_0}{:}U_0{\subset}\mathbb{C}^3{\to}\mathbb{C}^2$ is a holomorphic submersion. We may have needed to perhaps move to yet another point $p'{\in}U_0$ such that U_0 does not intersect the discriminant set of the web $\mathscr{W}$. We set $p{=}p'$ and works in a neighborhood of U_0. Therefore, the germ of $\mathscr{W}$ at p is a decomposable 3-web, defined by the superposition of three independent foliations $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$. We can assume that these foliations are defined by independent germs of integrable 1-forms ω_1, ω_2, and ω_3 respectively. Since H is given by a pull-back, all the leaves of $\mathscr{L}$ and, hence the leaves of $\mathscr{W}$ in $H{\cap}U_0$ are tangent to the fibers of $\rho|_{U_0}$, these fibers define a non-zero holomorphic 2-form η_ρ that is tangent to each ω_i, for $i{=}1, 2, 3$. Then, according to Proposition 6.3, ω_1, ω_2, and ω_3 define foliations that are in a pencil, an absurd. Hence, the assertion is proved.

Now we assert that H is not a pull-back of a real algebraic curve under a meromorphic function. In fact, H is a Levi-flat hypersurface in $\mathbb{P}^3$ such that there does not exist a point contained in infinitely many leaves of $\mathscr{L}$, because, the leaves of $\mathscr{L}$ are given by the equation (4) and through at a point only pass three leaves. If H is defined by a pull-back of a meromorphic function, there has to exist a point p of indeterminacy since the dimension is at least 2. Then through at p pass infinitely many leaves of $\mathscr{L}$. Since H does not satisfy this property, we finish the proof of the assertion.

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