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Fundamental solutions of generalized non-local Schrodinger operators

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Abstract. Let $d\in\{1,2,3,...\}$ and $s\in(0,1)$ be such that $d>2s$. We consider a generalized non-local Schrodinger operator of the form

$$L=L_K+\nu,$$

where L_K is a non-local operator with kernel K that includes the fractional Laplacian $(-\Delta)^s$ for $s\in(0,1)$ as a special case. The potential ν is a doubling measure subjected to a certain constraint. We show that the fundamental solution of L exists, is positive and possesses extra decaying properties.

1. Introduction

The idea of fundamental solutions lies at the core of partial differential equations. The well-known Malgrange–Ehrenpreis theorem essentially states that a non-zero linear differential operator with constant coefficients always has a fundamental solution. Nevertheless the situation becomes much more complicated for differential operators with variable coefficients. A satisfactory answer is obtained in the framework of Schrodinger operator with non-negative potential in the reverse Holder class, cf. [She95] and also a related work [She99]. Specifically, the fundamental solutions under such circumstances exist and enjoy a further decaying property. Generalizations in this spirit include [MP19] for magnetic Schrodinger operators, [KS00b] for uniformly elliptic operators and [CW88], [KS00a] for degenerate elliptic operators. Recently [CK18a] provided a counterpart of [She95, Theorem 2.7] in a non-local setting which covers the fractional Laplacian as a special case. The non-local term in such a setting was in turn inspired by [DCKP14], [DCKP16] and [KMS15].

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Apart from these extensions, fundamental solutions for parabolic differential equations and for elliptic systems are also studied. For instances, cf. [Gue66], [Kur00], [HK07] and the references therein.

Motivated by the works of [She99] and [CK18a], in this paper we aim to investigate the existence of a non-local Schrodinger-type operator whose potential is a measure together with its decaying estimates. This in particular takes part in the ongoing study of non-local elliptic equations with measure data. In this realm, we refer the readers to important papers such as [CV14], [CQ18], [CW21], etc., for further discussions.

Back to our setting, the details are as follows. Let $d \in \{1,2,3,\ldots\}$ and $s \in (0,1)$ be such that $d > 2s$. Consider the operator

$$L_K = \frac{1}{2}\,\text{p.v.}\int_{\mathbb{R}^d} \big(2u(x) - u(x+y) - u(x-y)\big)\, K(y)\, dy,$$

where $K : \mathbb{R}^d \setminus \{0\} \longrightarrow (0,\infty)$ satisfies there exists $c_{d,s} > 0$ and $\lambda, \Lambda > 0$ such that

$$c_{d,s}\,\frac{\lambda}{|y|^{d+2s}} \le K(y) = K(-y) \le c_{d,s}\,\frac{\Lambda}{|y|^{d+2s}}$$

for all $y \in \mathbb{R}^d \setminus \{0\}$. Here $c_{d,s}$ is the normalizing constant given by

$$c_{d,s}\int_{\mathbb{R}^d} \frac{1-\cos(x_1)}{|x|^{d+2s}}\, dx = 1. \tag{1}$$

In particular, this notion of L_K is general enough to include the fractional Laplacian $(-\Delta)^s$.

Next define

$$L = L_K + \nu,$$

where ν is a doubling measure on $\mathbb{R}^d$ such that there exist constants $C_0 > 0$ and

$$\delta > 2s - \frac{ds}{d-s} \tag{2}$$

such that

$$\nu(B(x,r)) \le C_0\left(\frac{r}{R}\right)^{d-2s+\delta} \nu(B(x,R)) \tag{3}$$

for all $x \in \mathbb{R}^d$ and $R > r > 0$. Hereafter, by a doubling measure ν we mean a non-negative Radon measure such that there exists a constant $D_0 > 1$ satisfying

$$\nu(2B) \le D_0\, \nu(B) \tag{4}$$

for all ball $B \subset \mathbb{R}^d$. We call D_0 the doubling constant of ν.

A more general version μ of such a ν first appeared in [She99], in which the author investigated the fundamental solution of the generalized Schrodinger operator $-\Delta+\mu$. However μ does not fit well into our non-local framework, which leads us to consider the doubling measure ν instead. It is worth mentioning that this general family of potentials strictly extends the reverse Holder class previously studied in [She95] so that the fundamental solution's estimate [She95, Theorem 2.7] remains valid. In fact, it was pointed out in [She99, Remark 0.10] that μ and also our ν need not be absolutely continuous with respect to the Lebesgue measure. To be specific, we have the following remark.

Remark 1. We provide three examples below to illustrate the measure ν in our setting. In fact, the measures in these examples are taken from [She99, Remark 0.10], in which the author verified that (3) holds for them. We emphasize that our ν is required to be doubling. Hence we focus and discuss more on the doubling property of the measures in these examples.

(i) Let $d\in\{1,2,3,...\}$ and V belong to the reverse Holder class RH_q with $q\geq\frac{d}{2}$, in the sense that

$$\left(\frac{1}{|B|}\int_B V^q\right)^{1/q}\leq\frac{C(q,V)}{|B|}\int_B V$$

holds for every ball $B\subset\mathbb{R}^d$. Define

$$d\nu=V(x)\,dx.$$

Then it follows from [She95, (1.1) and Lemma 1.2] ν is a doubling measure which satisfies (3).

(ii) Let $d\in\{3,4,5,...\}$ and σ be a doubling measure on $\mathbb{R}^2$. Set

$$d\nu=d\sigma(x_1,x_2)\,dx_3...dx_d.$$

Then ν is a doubling measure which satisfies (3). Note that σ, and hence ν, may not be absolutely continuous with respect to the Lebesgue measure.

(iii) Let $d\in\{2,3,4,...\}$, $\varphi:\mathbb{R}^{d-1}\longrightarrow\mathbb{R}$ be a Lipschitz function and σ be the surface measure on

$$S=\left\{(x',\varphi(x'))\in\mathbb{R}^d : x'\in\mathbb{R}^{d-1}\right\}.$$

Set

$$d\nu=\sigma(A\cap S)$$

for each open subset A of $\mathbb{R}^d$. Then ν satisfies (3) but is not doubling. Indeed, take a ball $B\subset\mathbb{R}^d$ such that $\sigma(2B\cap S)\neq 0$ and $B\cap S=\varnothing$. Then $\nu(2B)\neq 0$, whereas $\nu(B)=0$. As such (4) can not hold. Also ν need not be absolutely continuous with respect to the Lebesgue measure.

A transparent technical difficulty arises when a measure potential is employed. That is, pointwise estimates concerning such a potential is no longer available. Despite this we will show that under the condition (2) the existence of the fundamental solution and some of its properties persist. We note that in the case when a non-negative potential V in the reverse Holder class RH_q for some $q>\frac{d}{2}$ is considered, [She95, Lemma 1.2] reveals that $\delta=2s-\frac{d}{q}$ and so (2) reads $q>\frac{d}{s}-1$, which is stronger in comparison with the condition $q>\frac{d}{2s}$ in [CK18a] for the operator L_K+V. This compensates the aforementioned fact that ν can merely be a measure.

Back to our setting, for all $x\in\mathbb{R}^d$ define the *critical function*

$$\rho(x,\nu):=\frac{1}{m(x,\nu)}:=\sup\left\{r>0:\frac{\nu(B(x,r))}{r^{d-2s}}\le D_0\right\}, \tag{5}$$

where D_0 is the doubling constant of ν. This is an indispensable tool in our analysis of the generalized non-local Schrodinger operator L.

Before stating the main result, we need one more definition. For each $p\in[1,\infty)$ let

$$L^p_s(\mathbb{R}^d):=\left\{u\in L^p_{\text{loc}}(\mathbb{R}^d):\int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+|x|)^{d+2s}}\,dx<\infty\right\}$$

be endowed with the norm

$$\|u\|_{L^p_s(\mathbb{R}^d)}:=\left(\int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+|x|)^{d+2s}}\,dx\right)^{1/p}.$$

As noticed in [CK18b, (1.6)], the chain of inclusions

$$L^p_s(\mathbb{R}^d)\subset L^1_s(\mathbb{R}^d)\subset\mathcal{S}'_s(\mathbb{R}^d)\subset\mathcal{S}'(\mathbb{R}^d)\subset\mathcal{D}'(\mathbb{R}^d) \tag{6}$$

hold for all $p\in[1,\infty)$, where $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the spaces of distributions and tempered distributions on $\mathbb{R}^d$ respectively and $\mathcal{S}'_s(\mathbb{R}^d)$ is the dual space of

$$\mathcal{S}_s(\mathbb{R}^d):=\left\{f\in C^\infty(\mathbb{R}^d):\sup_{x\in\mathbb{R}^d}(1+|x|)^{d+2s}\,|D^\alpha f(x)|<\infty\text{ for all }\alpha\in\mathbb{N}^d\right\}.$$

The main result of this paper is as follows.

Theorem 1.1. *Let $d\in\{1,2,3,...\}$ and $s\in(0,1)$ be such that $d>2s$. Let ν be a doubling measure which satisfies (3). Then there exists a fundamental solution Γ_ν of L such that $\Gamma_\nu\in L^p_s(\mathbb{R}^d)$ for all $p\in(1,\frac{d}{d-2s})$ and*

$$L\Gamma_\nu=\delta_0\quad\textit{in the sense of }\mathcal{D}'(\mathbb{R}^d),$$

where δ_0 is the Dirac delta function concentrated at 0.

Moreover, for all $k\in\mathbb{N}$ *there exists a* $C=C(d,s,\lambda,\Lambda,k)>0$ *such that*

$$0\leq\Gamma_\nu(x-y)\leq\frac{C}{\big(1+|x-y|\,m(x_0,\nu)\big)^k}\,\frac{1}{|x-y|^{d-2s}}$$

for all $x,y\in\mathbb{R}^d$ *such that* $x\neq y$.

We emphasize that the extra decaying property so derived is due to ν which is a doubling measure satisfying (3). According to [CK18b, Theorem 1.1], if the potential is only an element of $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, then the fundamental solution can at best be bounded above by the principal term $\frac{1}{|x-y|^{d-2s}}$. For similar results to ours, cf. [She95, Theorem 2.7], [She99, Theorem 0.8] and [CK18a, Theorem 1.1].

As a by-product we obtain the following off-diagonal estimates.

Proposition 1.2. *Let* $d\in\{1,2,3,...\}$ *and* $s\in(0,1)$ *be such that* $d>2s$. *Let* ν *be a doubling measure which satisfies* (3). *Let* $\theta\in[0,d)$ *and define*

$$\Delta_\theta=\left\{(p,q)\in(1,\infty)^2:p\leq q\ \text{and}\ \frac{1}{p}-\frac{1}{q}=\frac{\theta}{d}\right\}.$$

Then the following statements hold.

(a) *If* $\theta\in[0,2s)$ *and* $(p,q)\in\Delta_\theta\cup(\infty,\infty)$, *then there exists a* $C=C(d,s,\lambda,p)$ *such that*

$$\|m(\cdot,\nu)^{2s-\theta}\,L^{-1}f\|_{L^q(\mathbb{R}^d)}\leq C\,\|f\|_{L^p(\mathbb{R}^d)}.$$

(b) *If* $p=1$ *then there exists a* $C=C(d,s,\lambda,\theta)>0$ *such that*

$$\|m(\cdot,\nu)^{2s-\theta}\,L^{-1}f\|_{L^{q,\infty}(\mathbb{R}^d)}\leq C\,\|f\|_{L^1(\mathbb{R}^d)},$$

where $L^{q,\infty}(\mathbb{R}^d)$ *is the usual Lorentz space on* $\mathbb{R}^d$.

(c) *If* $(p,q)\in\Delta_{2s}$, *then there exists a* $C=C(d,s,\lambda,p)$ *such that*

$$\|L^{-1}f\|_{L^{q,\infty}(\mathbb{R}^d)}\leq C\,\|f\|_{L^p(\mathbb{R}^d)}.$$

The paper is outlined as follows. In Section 2 we provide essential facts about the critical functions. In the following section we derive Fefferman-Phong, a weak Harnack's and Caccioppoli's inequalities. With these we are in a position to prove Theorem 1.1 and Proposition 1.2 in Section 4.

Notations. Throughout the paper the following set of notation is used without mentioning. Set $\mathbb{N}=\{0,1,2,3,...\}$ and $\mathbb{N}^*=\{1,2,3,...\}$. Given a $\lambda>0$ and a ball $B=B(x,r)$, we let $\lambda B=B(x,\lambda r)$. For all $a,b\in\mathbb{R}$, $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$. For all ball $B\subset\mathbb{R}^d$ we write $\nu(B):=\int_B d\nu$ for a given measure ν. The constants C and c are always assumed to be positive and independent of the main parameters whose values change from line to line. For any two functions f and

g, we write $f \lesssim g$ and $f \sim g$ to mean $f \le Cg$ and $cg \le f \le Cg$ respectively. Given a $p \in [1,\infty)$, the conjugate index of p is denoted by p'. We write $L^2(\mathbb{R}^d)$ to mean the space of square-integrable function with respect to the Lebesgue measure dx. When a different measure ν is used, we will use the notation $L^2_\nu(\mathbb{R}^d) = L^2(\mathbb{R}^d, d\nu)$.

Throughout assumptions. In the whole paper let $d \in \mathbb{N}^*$ and $s \in (0,1)$ be such that $d > 2s$. The domain $\Omega \subset \mathbb{R}^d$ is open bounded with Lipschitz boundary. The potential ν is a doubling measure which satisfies (3).

2. Critical functions

In this section we explore several basic estimates on the critical function which are useful for later development.

Recall from (3) that $\delta > 2s - \frac{ds}{d-s}$. By continuity it is possible to choose a sufficiently small $\varepsilon_0 > 0$, which will be fixed from here onward, such that

$$\delta > 2s - \left(\frac{d}{d-s} - \varepsilon_0 \right) (s - \varepsilon_0). \tag{7}$$

Let $a \in [s-\varepsilon_0, s]$, $b \in \left[\frac{d}{d-s} - \varepsilon_0, 2 \right]$. Define

$$\rho_{a,b}(x,\nu) := \frac{1}{m_{a,b}(x,\nu)} := \sup \left\{ r > 0 : \frac{\nu(B(x,r))}{r^{d-ab}} \le D_0 \right\}$$

for all $x \in \mathbb{R}^d$, where D_0 is the doubling constant of ν. When $a = s$ and $b = 2$ we simply write $m(\cdot,\nu)$ in place of $m_{s,2}(\cdot,\nu)$, which agrees with (5).

It is important to observe that

$$\delta' := \delta - 2s + ab > 0 \tag{8}$$

as a consequence of (7).

Proposition 2.1. *The following statements hold.*

(i) *The function $\rho_{a,b}(\cdot,\nu)$ is well-defined, i.e., $\rho_{a,b}(x,\nu) \in (0,\infty)$ for every $x \in \mathbb{R}^d$.*

(ii) *For every $x \in \mathbb{R}^d$ one has*

$$r^{d-ab} < \nu(B(x,r)) \le D_0\, r^{d-ab}$$

with $r = \rho_{a,b}(x,\nu)$.

(iii) *If $|x-y| \lesssim \rho_{a,b}(x,\nu)$, then $\rho_{a,b}(x,\nu) \sim \rho_{a,b}(y,\nu)$.*

(iv) *There exist* $k_0>0$ *and* $C>1$ *such that*

$$C^{-1} m_{a,b}(y,\nu)\left(1+|x-y|\, m_{a,b}(y,\nu)\right)^{-k_0/(k_0+1)}$$
$$\leq m_{a,b}(x,\nu)$$
$$\leq C m_{a,b}(y,\nu)\left(1+|x-y|\, m_{a,b}(y,\nu)\right)^{k_0}$$

for all $x,y\in\mathbb{R}^d$.

Proof. Let $x,y\in\mathbb{R}^d$, $r=\rho_{a,b}(x,\nu)$ and $R=\rho_{a,b}(y,\nu)$.

(i) It follows from (3) that

$$\lim_{t\to 0}\frac{1}{t^{d-ab}}\,\nu(B(x,t))=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{1}{t^{d-ab}}\,\nu(B(x,t))=\infty.$$

This, in combination with (3), implies $\rho_{a,b}(x,\nu)\in(0,\infty)$.

(ii) By definition we have

$$\nu(B(x,r))=\lim_{t\to r^-}\nu(B(x,t))\leq D_0\, r^{d-ab}.$$

Also

$$D_0\,(2r)^{d-ab}\leq \nu(B(x,2r))\leq D_0\,\nu(B(x,r)),$$

where we used the definition of $\rho_{a,b}(\cdot,\nu)$ in the first step and the doubling property of ν in the second step. Hence we deduce that

$$\nu(B(x,r))>r^{d-ab}.$$

(iii) Suppose that $|x-y|<Cr$ for some $C>0$. Then $B(y,r)\subset B(x,(C+1)r)$. Using the doubling property of ν and (ii) we obtain

$$\nu(B(x,(C+1)r))\lesssim \nu(B(x,r))\lesssim r^{d-ab}.$$

Consequently it follows from (3) that

$$\frac{1}{(tr)^{d-ab}}\,\nu(B(y,tr))\leq C_0\, t^{\delta'}\,\frac{1}{r^{d-ab}}\,\nu(B(y,r))$$
$$\lesssim t^{\delta'}\,\frac{1}{r^{d-ab}}\,\nu(B(x,(C+1)r))$$
$$\lesssim t^{\delta'}<D_0,$$

where δ' is given by (8) and t is chosen to be sufficiently small. Therefore $R\geq tr$ by definition, where we recall that $R=\rho_{a,b}(y,\nu)$. This in turn implies $|x-y|\lesssim R$. By swapping the roles of x and y in the above argument, we then obtain $R\lesssim r$.

(iv) The case $|x-y|<R$ is clear from (iii). So we assume that $|x-y|\geq R$. Let $j\in\mathbb{N}^*$ be such that $2^{j-1}R\leq|x-y|\leq 2^jR$. Then $B(x,R)\subset B(y,(2^j+1)R)$. By virtue of (ii) and the doubling property of ν one has

$$\nu(B(x,R))\leq D_0^{j+2}\,R^{d-ab}.$$

It follows from (3) that

$$\begin{aligned}\frac{1}{(tR)^{d-ab}}\,\nu(B(y,tR))&\leq C_0\,t^{\delta'}\,\frac{1}{R^{d-ab}}\,\nu(B(y,R))\\&\lesssim t^{\delta'}\,\frac{1}{R^{d-ab}}\,\nu(B(x,(C+1)R))\\&\lesssim t^{\delta'}<D_0,\end{aligned}$$

where δ' is given by (8) and t is chosen to be sufficiently small. So the definition of ρ gives $r\geq tR$ or equivalently

$$m(x,\nu)\leq\frac{m(y,\nu)}{t}\lesssim m(y,\nu)\,\big(1+|x-y|\,m(y,\nu)\big)^{k_0} \tag{9}$$

for some $k_0>0$.

For the remaining inequality, using (9) we obtain that

$$1+|x-y|\,m(x,\nu)\lesssim\big(1+|x-y|\,m(y,\nu)\big)^{k_0+1}.$$

With this in mind we apply (9) once more to obtain

$$m(y,\nu)\gtrsim m(x,\nu)\,\big(1+|x-y|\,m(x,\nu)\big)^{-k_0/(k_0+1)}.$$

The proof is complete. □

Lemma 2.2. *There exist a sequence $(x_j)_{j\in\mathbb{N}}\subset\mathbb{R}^d$ and a family of functions $(\psi_j)_{j\in\mathbb{N}}$ such that the following hold.*

(i) $\bigcup_{j\in\mathbb{N}}B_j=\mathbb{R}^d$, *where $\rho_j=\rho_{a,b}(x_j,\nu)$ and $B_j=B(x_j,\rho_j)$ for all $j\in\mathbb{N}$.*

(ii) *For all $\tau\geq 1$ there exist constants $C,\zeta_0>0$ such that*

$$\sum_{j\in\mathbb{N}}\chi_{B(x_j,\tau\rho_j)}\leq C\,\tau^{\zeta_0}.$$

(iii) $\operatorname{supp}\psi_j\subset B(x_j,\rho_j)$ *and* $0\leq\psi_j\leq 1$.

(iv) $|\nabla\psi_j(x)|\lesssim 1/\rho_j$ *for all $x,y\in\mathbb{R}^d$.*

(v) $\sum_{j\in\mathbb{N}}\psi_j=1$.

Proof. We note that $\rho_{a,b}(\cdot,\nu)$ acquires all the properties analogous to those of the critical functions given in [She99]. Hence the proof for this lemma is done verbatim as in [She99, Proof of Lemma 3.3]. □

3. Inequalities

We devote this section to deriving three crucial inequalities: Fefferman-Phong inequality, a weak Harnack's inequality and Caccioppoli's inequality.

3.1. Fefferman-Phong inequality

Let $a\in[s-\varepsilon_0, s]$, $b\in\left[\frac{d}{d-s}-\varepsilon_0, 2\right]$, where ε_0 is given by (7). We start with an embedding result that is a consequence of [BBM02, Theorem 1] and [MS02, Corollary 2] together.

Proposition 3.1. *Let $p\geq 1$ be such that $sp<d$. Then there exists a $C=C(d)>0$ such that*

$$\|u-u_A\|^p_{L^p(B)}\leq C\,\frac{(1-s)}{(d-sp)^{p-1}}\,|A|^{sp/d}\int_A\int_A\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dx\,dy$$

for all ball (or cube) $A\subset\mathbb{R}^d$ and $u\in W^{s,p}(A)$.

In what follows, we denote $W^{a,b}_c(\mathbb{R}^d)$ to be the set of functions in $W^{a,b}(\mathbb{R}^d)$ with compact supports. The Fefferman-Phong inequality is as follows.

Proposition 3.2. *Let $u\in W^{a,b}_c(\mathbb{R}^d)$. Then the following statements hold.*
(i) *If $u\in L^b(\mathbb{R}^d,d\nu)$ then $m_{a,b}(\cdot,\nu)^a\,u\in L^b(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d}|u|^b\,m_{a,b}(x,\nu)^{ab}\,dx\leq C\left(\|u\|_{W^{a,b}(\mathbb{R}^d)}+\|u\|_{L^b(\mathbb{R}^d,d\nu)}\right)$$

for some $C=C(d,a)>0$.
(ii) *If $m_{a,b}(\cdot,\nu)^a\,u\in L^b(\mathbb{R}^d)$ then $u\in L^b(\mathbb{R}^d,d\nu)$ and*

$$\|u\|_{L^b(\mathbb{R}^d,d\nu)}\leq C\left(\|u\|_{W^{a,b}(\mathbb{R}^d)}+\int_{\mathbb{R}^d}|u|^b\,m_{a,b}(x,\nu)^{ab}\,dx\right)$$

for some $C=C(d,a)>0$.

Proof. Let $x_0\in\mathbb{R}^d$ and $r_0=\rho_{a,b}(x_0,\nu)$. Set $B=B(x_0,r_0)$.
(i) Let $u\in W^{a,b}_c(\mathbb{R}^d)\cap L^b(\mathbb{R}^d,d\nu)$. By Proposition 2.1 (ii) we have

$$I:=\int_B\left(r_0^{d-ab}\wedge\nu(B)\right)|u|^b\,dx\geq r_0^{d-ab}\int_B|u|^b\,dx.$$

Also it follows from Proposition 3.1 that

$$I\lesssim\int_B\int_B\frac{1}{r_0^{ab}}\,|u(x)-u(y)|^b\,dx\,dy+|B|\int_B|u(y)|^b\,d\nu(y)$$

$$\lesssim r_0^d \left(\int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}} \, dx \, dy + \int_B |u(x)|^b \, d\nu(x) \right).$$

Hence

$$\frac{1}{r_0^{ab}} \int_B |u|^b \, dx \lesssim \int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}} \, dx \, dy + \int_B |u|^b \, d\nu \tag{10}$$

or equivalently

$$\int_B |u|^b \, m_{a,b}(\cdot,\nu)^{d+ab} \, dx \lesssim \int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}} \, m_{a,b}(x,\nu)^d \, dx \, dy + \int_B |u|^b \, m_{a,b}(\cdot,\nu)^d \, d\nu,$$

as $m_{a,b}(x,\nu) \sim 1/r_0$ for all $x \in B$ by Proposition 2.1(iii).

Integrating both sides with respect to x_0 on $\mathbb{R}^d$, keeping in mind that for each $x \in B$ one has

$$\int_{|x-x_0|<\rho_{a,b}(x_0,\nu)} dx_0 \sim \int_{|x-x_0|<\rho_{a,b}(x,\nu)} dx_0 \sim m_{a,b}(x,\nu)^{-d}$$

and then applying Fubini's theorem, we arrive at the conclusion.

(ii) The proof is similar to (i). Let $u \in W_c^{a,b}(\mathbb{R}^d)$ and $m_{a,b}(\cdot,\nu)^a \, u \in L^b(\mathbb{R}^d)$. The main idea is to establish the counterpart of (10) in this case. The rest follows the same argument as in (i).

First observe that (3) holds if we replace a ball B with a closed cube Q. That is,

$$\nu(Q(x,r)) \le C_0 \left(\frac{r}{R} \right)^{d-2s+\delta} \nu(Q(x,R)) \tag{11}$$

for all $x \in \mathbb{R}^d$ and $R > r > 0$, where $Q(x,r)$ denotes the closed cube centered at x whose side length is r (cf. [She99, Proof of Lemma 2.24]).

Secondly, let $\varkappa > 0$ be sufficiently small such that $a - \varkappa \in [s - \varepsilon_0, s]$. Using [KS00b, Theorem 2.3] (also cf. [VW95, Theorem A] and [SWZ96, Theorem 1.3]) we deduce that

$$\int_Q \left(\int_Q \frac{|f(y)|}{|x-y|^{d-a+\varkappa}} \, dy \right)^b d\mu(x) \lesssim \int_Q |f(y)|^b \, dy$$

for all $f \in L^b(Q)$, provided that

$$\int_A \left(\int_A \frac{d\mu(x)}{|x-y|^{d-a+\varkappa}} \right)^{b'} dy \lesssim \mu(A) \tag{12}$$

for all cube $A \subset Q$.

In view of (11) we may choose

$$d\mu = \frac{r_Q^{d-(a-\varkappa)b}}{\nu(2Q)}\, d\nu \quad \text{with } Q := Q(x_Q, r_Q).$$

Then μ satisfies (12). Explicitly we have

$$\int_Q \left(\int_Q \frac{|f(y)|}{|x-y|^{d-a+\varkappa}}\, dy \right)^b d\nu(x) \lesssim \frac{\nu(2Q)}{r_Q^{d-(a-\varkappa)b}} \int_Q |f(y)|^b\, dy \tag{13}$$

for all $f \in L^b(Q)$.

With the above tools in mind, we now have

$$\begin{aligned}
\int_Q |u(x)|^b\, d\nu(x) &\lesssim \int_Q |u(x)-u_Q|^b\, d\nu(x) + \int_Q |u_Q|^b\, d\nu(x)\\
&\lesssim \int_Q \left(\int_Q \frac{|g(y)|}{|x-y|^{d-a+\varkappa}}\, dy \right)^b d\nu(x) + r_Q^{-d}\, \nu(Q) \int_Q |u(y)|^b\, dy\\
&\lesssim \frac{\nu(2Q)}{r_Q^{d-(a-\varkappa)b}} \int_Q |g(x)|^b\, dx + r_Q^{-d}\, \nu(Q) \int_Q |u(y)|^b\, dy\\
&\lesssim \frac{\nu(2Q)}{r_Q^{d-ab}} \int_Q \int_Q \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\, dx\, dy + r_Q^{-d}\, \nu(Q) \int_Q |u(y)|^b\, dy,
\end{aligned}$$

where we used [DIV16, Theorem 2.5] with

$$g(y) = \int_{Q(y,r_y)} \frac{|u(y)-u(z)|}{|y-z|^{d+a-\varkappa}}\, dz \quad \text{and} \quad r_y := \operatorname{dist}(y, \partial Q)$$

in the second step and then applied (13) in the third step as well as Holder's inequality in the fourth step. Hence

$$\int_Q |u|^b\, d\nu \lesssim \frac{\nu(2Q)}{r_Q^{d-ab}} \int_Q \int_Q \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\, dx\, dy + \frac{\nu(Q)}{r_Q^d} \int_Q |u|^b\, dx. \tag{14}$$

Lastly, we take a closed cube $Q \subset \mathbb{R}^d$ such that

$$\frac{1}{2} B \subset Q \subset B,$$

where $B = B(x_0, r_0)$ and $r_0 = \rho_{a,b}(x_0, \nu)$. Then (14) reads

$$\int_{\frac{1}{2}B} |u|^b\, d\nu \lesssim \frac{\nu(2B)}{r_0^{d-ab}} \int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\, dx\, dy + \frac{\nu(B)}{r_0^d} \int_B |u|^b\, dx$$

$$\lesssim \frac{\nu(B)}{r_0^{d-ab}} \int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\, dx\, dy + \frac{\nu(B)}{r_0^d} \int_B |u|^b\, dx$$
$$\lesssim \int_B \int_B \frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\, dx\, dy + \frac{1}{r_0^{ab}} \int_B |u|^b\, dx,$$

where we used the doubling property of ν in the second step and Proposition 2.1(ii) in the third step. This is the counterpart of (10) in (i). □

As a consequence, the following embedding result is available.

Lemma 3.3. *Let $B \subset \mathbb{R}^d$ be a ball. Then the embedding*

$$W_c^{a,b}(B) \hookrightarrow L^b(B, d\nu)$$

is compact.

Proof. Let $\{x_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ be as in Lemma 2.2. Since B is compact we can cover it by a finite number of balls $B_j := B(x_j, \rho_j)$. Without loss of generality assume that $B \subset \cup_{j=1}^{j_0} B_j$ for some $j_0 \in \mathbb{N}^*$.

Therefore using Proposition 3.2(ii) one has

$$\begin{aligned}\int_B |u|^b\, d\nu &\lesssim \|u\|_{W^{a,b}(\mathbb{R}^d)} + \int_B |u|^b\, m_{a,b}(\cdot,\nu)^{ab}\, dx \\ &\le \|u\|_{W^{a,b}(B)} + \sum_{j=1}^{j_0} \int_{B\cap B_j} |u|^b\, m_{a,b}(\cdot,\nu)^{ab}\, dx \\ &\lesssim \|u\|_{W^{a,b}(B)} + \sum_{j=1}^{j_0} m_{a,b}(x_j,\nu)^{ab} \int_{B\cap B_j} |u|^b\, dx \\ &\le \left(1 \vee \sum_{j=1}^{j_0} m_{a,b}(x_j,\nu)^{ab}\right) \|u\|_{W^{a,b}(B)} < \infty \end{aligned} \tag{15}$$

for all $u \in W_c^{a,b}(B)$, where we used [BRS16, Lemma 1.3] in the second step and Proposition 2.1(iii) in the third step.

The compactness of the embedding follows from (15) using a standard argument as in [She95, Lemma 2.24]. For the sake of clarity, we present a detailed proof.

Let Q be a closed cube containing B. It suffices to show that

$$W_c^{a,b}(Q) \hookrightarrow L^b(Q, d\nu) \tag{16}$$

is compact.

Denote R to be the side length of Q. We partition Q into finite closed subcubes $\{Q_j\}_{j=1}^{j_0}$ whose side lengths are identically $r\in(0,R)$, where $j_0\in\mathbb{N}^*$. We apply (14) to each Q_j to obtain

$$\begin{aligned}\int_{Q_j}|u|^b\,d\nu &\lesssim \frac{\nu(2Q_j)}{r^{d-ab}}\int_{Q_j}\int_{Q_j}\frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\,dx\,dy+\frac{\nu(Q_j)}{r^d}\int_{Q_j}|u|^b\,dx\\ &\lesssim \frac{\nu(3Q)}{R^{d-ab}}\left[\left(\frac{r}{R}\right)^{\delta'}\int_{Q_j}\int_{Q_j}\frac{|u(x)-u(y)|^b}{|x-y|^{d+ab}}\,dx\,dy+\frac{1}{r^{ab}}\left(\frac{r}{R}\right)^{\delta'}\int_{Q_j}|u|^b\,dx\right]\\ &\lesssim \frac{\nu(3Q)}{R^{d-ab}}\left[\left(\frac{r}{R}\right)^{\delta'}\|u\|^b_{W^{a,b}(Q_j)}+\frac{1}{r^{ab}}\left(\frac{r}{R}\right)^{\delta'}\int_{Q_j}|u|^b\,dx\right],\end{aligned}$$

where in the second step we used (3) and the fact that $\frac{R}{r}2Q_j\subset 3Q$ for all $j\in\{1,\ldots,j_0\}$. Here λQ means the dilated cube with the same center as Q whose side length is λR. Summing this estimate over j yields

$$\int_Q|u|^b\,d\nu\lesssim\frac{\nu(2Q)}{R^{d-ab}}\left[\left(\frac{r}{R}\right)^{\delta'}\|u\|^b_{W^{a,b}(Q)}+\frac{1}{r^{ab}}\left(\frac{r}{R}\right)^{\delta'}\int_Q|u|^b\,dx\right].$$

If r is chosen to be sufficiently small, we arrive at the statement: For each $\varepsilon>0$ there exists a $C_\varepsilon>0$ such that

$$\int_Q|u|^b\,d\nu\lesssim\frac{\nu(Q)}{R^{d-ab}}\left[\varepsilon\,\|u\|^b_{W^{a,b}(Q)}\,dx\,dy+C_\varepsilon\int_Q|u|^b\,dx\right]. \tag{17}$$

This can be considered as a fractional version of the Friedrich-type inequality [She95, (2.26) in Lemma 2.24]. To obtain the compactness of the embedding (16), we argue as follows.

Let $\{u_n\}_{n\in\mathbb{N}}\subset W_c^{a,b}(Q)$ be bounded (in norm) by $K>0$. Observe that the embedding $W_c^{a,b}(Q)\hookrightarrow L^b(Q)$ is compact. Hence $\{u_n\}_{n\in\mathbb{N}}$ has a strongly convergent subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ in $L^b(Q)$. At the same time, $\{u_{n_j}\}_{j\in\mathbb{N}}\subset L^b(Q,\nu)$ due to (15). Then (17) applied to $\{u_{n_j}\}_{j\in\mathbb{N}}$ reads

$$\begin{aligned}\int_Q|u_{n_j}-u_{n_{j'}}|^b\,d\nu&\lesssim\frac{\nu(2Q)}{R^{d-ab}}\left[\varepsilon\,\|u_{n_j}-u_{n_{j'}}\|^b_{W^{a,b}(Q)}+C_\varepsilon\int_Q|u_{n_j}-u_{n_{j'}}|^b\,dx\right]\\ &\lesssim\frac{\nu(2Q)}{R^{d-ab}}\left[2\varepsilon K^b+C_\varepsilon\int_Q|u_{n_j}-u_{n_{j'}}|^b\,dx\right]\end{aligned}$$

for all $j\in\mathbb{N}$. Hence by the strong convergence of $\{u_{n_j}\}_{j\in\mathbb{N}}$ in $L^b(Q)$, we may choose $n_0\in\mathbb{N}$ such that

$$C_\varepsilon\int_Q|u_{n_j}-u_{n_{j'}}|^b\,dx<\varepsilon$$

for all $j, j' \geq n_0$. This leads to

$$\int_Q |u_{n_j} - u_{n_{j'}}|^b \, d\nu \lesssim (2K^b + 1)\, \varepsilon$$

for all $j, j' \geq n_0$. Since $\varepsilon > 0$ is arbitrary, this last display implies that $\{u_{n_j}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^b(Q, d\nu)$. Hence the embedding (16) is compact.

This verifies our claim. $\square$

3.2. Weak Harnack's inequality

In what follows, let $M(\mathbb{R}^d)$ be the set of measurable functions on $\mathbb{R}^d$. Denote

$$\mathbb{R}^{2d}_\Omega = \mathbb{R}^{2d} \setminus (\Omega^C \times \Omega^C).$$

The following spaces are significant in subsequent analysis:

- $X(\Omega) = \left\{ u \in M(\mathbb{R}^d) : u|_\Omega \in L^2(\Omega) \text{ and } \int\int_{\mathbb{R}^{2d}_\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx\, dy < \infty \right\}$.
- $X_0(\Omega) = \{ v \in X(\Omega) : v = 0 \text{ a.e. in } \Omega^C \}$.
- $X_g^\pm(\Omega) = \{ v \in X(\Omega) : (g - v)^\pm \in X_0(\Omega) \}$, where $g \in H^s(\mathbb{R}^d)$.
- $X_g(\Omega) = X_g^+(\Omega) \cap X_g^-(\Omega)$, where $g \in H^s(\mathbb{R}^d)$.

When dealing with these spaces, it is useful to keep the following relations in mind.

Lemma 3.4. ([CK18a, Lemma 2.1]) *Let $u \in X_0(\Omega)$. Then the following hold.*

(i) *One has*

$$\frac{1}{r^2} \int_{|x-y|<r} |x-y|^2 \, K(x-y)\, dy + \int_{|x-y|\geq r} K(x-y)\, dy \leq \frac{\Lambda\, \omega_d}{s} \, \frac{1}{r^{2s}}$$

for all $x \in \mathbb{R}^d$, where ω_d denotes the surface measure of the unit sphere in $\mathbb{R}^d$.

(ii) *One has*

$$\frac{1}{\Lambda\, c_{d,s}} \int_\Omega \int_\Omega |u(x) - u(y)|^2 \, K(x-y)\, dx\, dy \leq \|u\|_{H^s(\Omega)}$$
$$\leq \frac{1}{\lambda\, c_{d,s}} \int_\Omega \int_\Omega |u(x) - u(y)|^2 \, K(x-y)\, dx\, dy,$$

where $c_{d,s}$ is given by (1).

Now let $g \in H^s(\mathbb{R}^d)$. Consider the problem

$$(NSE_0) \quad \begin{cases} L_K u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Definition 1. A function $u\in X_g(\Omega)$ is called a *weak solution* of (NSE_0) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x)-u(y)\big)\,\big(\phi(x)-\phi(y)\big)\,K(x-y)\,dx\,dy=0$$

for all $\phi\in X_0(\Omega)$.

Next a function $u\in X_g^-(\Omega)$ is called a *sub-solution* of (NSE_0) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x)-u(y)\big)\,\big(\phi(x)-\phi(y)\big)\,K(x-y)\,dx\,dy\leq 0$$

for all $0\leq\phi\in X_0(\Omega)$.

Similarly a function $u\in X_g^+(\Omega)$ is called a *super-solution* of (NSE_0) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x)-u(y)\big)\,\big(\phi(x)-\phi(y)\big)\,K(x-y)\,dx\,dy\geq 0$$

for all $0\leq\phi\in X_0(\Omega)$.

Following [DCKP14] we take into account the *tail* $\mathcal{T}(u;x_0,R)$ defined by

$$\mathcal{T}(u;x_0,R):=R^{2s}\int_{(B(x_0,R))^C}\frac{|v(x)|}{|x-x_0|^{n+2s}}dx$$

for all function $u\in H^s(\mathbb{R}^d)$ and $B(x_0,R)\subset\mathbb{R}^d$. It turns out that this notion plays a significant role in a non-local setting.

The next two results provide Harnack-type inequalities for a non-negative sub-solution of (NSE_0).

Lemma 3.5. ([CK18a, Theorem 4.4]) *Let $g\in H^s(\mathbb{R}^d)$ and $u\in X_g^-(\Omega)$ be a sub-solution of (NSE_0). Set $B=B(x_0,r)\subset\Omega$. Then there exists a $c=c(d,s,\lambda,\Lambda)$ such that*

$$\sup_{\frac{1}{2}B}u\leq\delta\,\mathcal{T}(u^+;x_0,r/2)+c\delta^{-d/4s}\left(\frac{1}{|B|}\int\big(u^+(x)\big)^2\,dx\right)^{1/2},$$

for all $\delta\in(0,1]$.

Moreover, if $u\geq 0$ in $B(x_0,R)$ with $R>r$ then there is a $C=C(d,s,\lambda,\Lambda)$ such that

$$\mathcal{T}(u^+;x_0,r)\leq c\sup_{B(x_0,r)}u+C\left(\frac{r}{R}\right)^{2s}\mathcal{T}(u^-;x_0,R).$$

Lemma 3.6. ([CK18a, Proposition 2]) *Let $g\in H^s(\mathbb{R}^d)$ and $u\in X_g(\Omega)$ be a non-negative sub-solution of (NSE_0). Set $B=B(x_0,r)\subset\Omega$ to be a ball. Then there exists a constant $C=C(d,s,\lambda,\Lambda)$ such that*

$$\sup_{\frac{1}{2}B}u\leq C\left(\frac{1}{|B|}\int_B u(y)^2\,dy\right)^{1/2}.$$

3.3. Caccioppoli's estimate

Recall that $L = L_K + \nu$. Consider the non-local Schrodinger equation

$$(NSE) \qquad \begin{cases} Lu = 0 \text{ in } \Omega, \\ u = g \quad \text{in } \mathbb{R}^d \setminus \Omega, \end{cases}$$

where $g \in H^s(\mathbb{R}^d)$.

The analysis of this problem requires the following function spaces:

- $Y(\Omega) = \{u \in X_g(\Omega) : \int_{\mathbb{R}^d} u^2 \, d\nu < \infty\}$.
- $Y_g^{\pm}(\Omega) = \{v \in Y(\Omega) : (g - v)^{\pm} \in X_0(\Omega)\}$.
- $Y_g(\Omega) = Y_g^{+}(\Omega) \cap Y_g^{-}(\Omega)$.

Definition 2. A function $u \in Y_g(\Omega)$ is called a *weak solution* of (NSE) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x) - u(y)\big) \big(\phi(x) - \phi(y)\big) K(x - y) \, dx \, dy + \int_{\mathbb{R}^d} u(x) \, \phi(x) \, d\nu(x) = 0$$

for all $\phi \in X_0(\Omega)$.

Next a function $u \in Y_g^{-}(\Omega)$ is called a *sub-solution* of (NSE) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x) - u(y)\big) \big(\phi(x) - \phi(y)\big) K(x - y) \, dx \, dy + \int_{\mathbb{R}^d} u(x) \, \phi(x) \, d\nu(x) \le 0 \tag{18}$$

for all $0 \le \phi \in X_0(\Omega)$.

Similarly a function $u \in Y_g^{+}(\Omega)$ is called a *super-solution* of (NSE) if

$$\int\int_{\mathbb{R}^{2d}} \big(u(x) - u(y)\big) \big(\phi(x) - \phi(y)\big) K(x - y) \, dx \, dy + \int_{\mathbb{R}^d} u(x) \, \phi(x) \, d\nu(x) \ge 0$$

for all $0 \le \phi \in X_0(\Omega)$.

We also need the following cut-off function for later use. Given $R > r > 0$ and $x_0 \in \mathbb{R}^d$, denote

$$\phi_{r,R,x_0}(x) := \left(\frac{R - |x - x_0|}{R - r} \vee 0 \right) \wedge 1 \tag{19}$$

for all $x \in \mathbb{R}^d$. Note that $\phi_{r,R,x_0} \in W_0^{1,\infty}(B(x_0, R))$.

One can construct the Caccioppoli's inequality for a solution of (NSE) as shown below.

Lemma 3.7. *Let* $x_0\in\Omega$ *and* u *be a non-negative sub-solution of* (NSE)*. Then there exists a* $C=C(d,s,\lambda,\Lambda)>0$ *such that*

$$\|\phi\, u\|^2_{H^s(\mathbb{R}^d)}+\int_{B(x_0,r)}|u|^2\,d\nu\le\frac{C}{(R-r)^{2s}}\left(\frac{R}{R-r}\right)^d\int_{B(x_0,R)}|u|^2\,dx$$

for all $r\in\big(0,\operatorname{dist}(x_0,\partial\Omega)/2\big)$*,* $R\in(r,2r]$*, where* $\phi=\phi_{r,\sigma,x_0}$ *and* $\sigma=\frac{r+R}{2}$.

Proof. Let $r\in\big(0,\operatorname{dist}(x_0,\partial\Omega)/2\big)$, $R\in(r,2r]$. Set $\psi=\phi^2\,u$ to be a test function in (18). Then

$$\int\int_{\mathbb{R}^{2d}}\big(u(x)-u(y)\big)\,\big(\psi(x)-\psi(y)\big)\,K(x-y)\,dx\,dy+\int_{\mathbb{R}^d}u(x)\,\psi(x)\,d\nu(x)\le 0.$$

Observe that

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2d}}\big(u(x)-u(y)\big)\,\big(\psi(x)-\psi(y)\big)\,K(x-y)\,dx\,dy\\
&=\int\int_{\mathbb{R}^{2d}_\Omega}\big(u(x)-u(y)\big)\,\big(\psi(x)-\psi(y)\big)\,K(x-y)\,dx\,dy\\
&=\int\int_{B(x_0,r)^2}\big(u(x)-u(y)\big)^2\,K(x-y)\,dx\,dy\\
&\quad+\int\int_{\mathbb{R}^{2d}_\Omega\setminus B(x_0,r)^2}\big(\phi(x)\,u(x)-\phi(y)\,u(y)\big)^2\,K(x-y)\,dx\,dy\\
&\quad-\int\int_{\mathbb{R}^{2d}_\Omega\setminus B(x_0,r)^2}\big(\phi(x)-\phi(y)\big)^2\,u(x)\,u(y)\,K(x-y)\,dx\,dy,
\end{aligned}$$

where $B(x_0,r)^2:=B(x_0,r)\times B(x_0,r)$.

Consequently we obtain

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2d}_\Omega}\big(\phi(x)\,u(x)-\phi(y)\,u(y)\big)^2\,K(x-y)\,dx\,dy+\int_{B(x_0,r)}u(x)\,\psi(x)\,d\nu(x)\\
&\le\int\int_{\mathbb{R}^{2d}_\Omega\setminus B(x_0,r)^2}\big(\phi(x)-\phi(y)\big)^2\,u(x)\,u(y)\,K(x-y)\,dx\,dy\\
&=:I.
\end{aligned}$$

Next

$$I \le \frac{1}{2} \int\int_{B(x_0,R)^2\setminus B(x_0,r)^2} \big(\phi(x)-\phi(y)\big)^2 \big(u(x)+u(y)\big)^2 K(x-y)\,dx\,dy$$

$$+2 \int\int_{B(x_0,R)\times B(x_0,R)^C} \phi(x)^2\, u(x)\, u(y)\, K(x-y)\,dx\,dy$$

$$\le \int\int_{B(x_0,R)^2} \big(\phi(x)-\phi(y)\big)^2 u(x)^2 K(x-y)\,dx\,dy$$

$$+2 \int_{B(x_0,R)} \phi^2(x)\, u(x) \left(\int_{B_{2R}(x_0)^C} u(y)\, K(x-y)\,dy\right) dx$$

$$\le \frac{C}{(R-r)^{2s}} \|u\|^2_{L^2(B(x_0,R))} + C\Lambda \left(\frac{2R}{R-r}\right)^{d+2s} \|u\|_{L^1(B(x_0,R))}$$

$$\times \int_{B(x_0,R)^C} \frac{|u(y)|}{|y-x_0|^{d+2s}}\,dy$$

for some $C=C(d,s)>0$, where we used Lemma 3.4(i) and the fact that

$$\sup_{x,y\in\mathbb{R}^d} \frac{\big(\phi(x)-\phi(y)\big)^2}{|x-y|^2} \le \left(\frac{1}{\sigma-r}\right)^2 \le \frac{4}{(R-r)^2}$$

and

$$|x-y| \ge |x_0-y| - |x_0-x| \ge \frac{(R-r)\,|x_0-y|}{2R}$$

for all $(x,y)\in B(x_0,\sigma)\times B(x_0,R)^C$ in the last step.

The non-negativity of u implies $\mathcal{T}(u^-,x_0,R)=0$ and whence

$$\left(\frac{2R}{R-r}\right)^{d+2s} \|u\|_{L^1(B(x_0,R))} \int_{B(x_0,R)^C} \frac{|u(y)|}{|y-x_0|^{d+2s}}\,dy$$

$$\le C \left(\frac{R}{R-r}\right)^{d+2s} |B(x_0,R)|^{1/2} \|u\|_{L^2(B(x_0,R))} \left(\frac{R}{2}\right)^{-2s} \mathcal{T}(u,x_0,R/2)$$

$$\le \frac{C}{(R-r)^{2s}} \left(\frac{R}{R-r}\right)^{d} |B(x_0,R)|^{1/2} \|u\|_{L^2(B(x_0,R))} \sup_{B_{R/2}(x_0)} u$$

$$\le \frac{C}{(R-r)^{2s}} \left(\frac{R}{R-r}\right)^{d} |B(x_0,R)|^{1/2} \|u\|_{L^2(B(x_0,R))} \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} u(y)^2 dy\right)^{1/2}$$

$$= \frac{C}{(R-r)^{2s}} \left(\frac{R}{R-r}\right)^{d} \|u\|^2_{L^2(B(x_0,R))}$$

for some $C=C(d,s,\lambda,\Lambda)$, where we used Lemma 3.5 in the first and second steps and Lemma 3.6 in the third step.

Combining the above estimates together gives

$$\int\int_{\mathbb{R}^{2d}_\Omega} \big(\phi(x)\,u(x)-\phi(y)\,u(y)\big)^2\,K(x-y)\,dx\,dy+\int_{B(x_0,r)} u(x)\,\psi(x)\,d\nu(x)$$

$$\leq \frac{C}{(R-r)^{2s}}\left(\frac{R}{R-r}\right)^d \|u\|^2_{L^2(B(x_0,R))}$$

for some $C=C(d,s,\lambda,\Lambda)$, as required. □

4. Proof of main result

We are now ready to prove the main theorem. For convenience we first prove an auxiliary result.

Lemma 4.1. *Let $x_0\in\mathbb{R}^d$, $R>0$ and $B=B(x_0,R)$. Let u be a solution of $Lu=0$ in $4B$. Then for all $k\in\mathbb{N}$ there exists a $C=C(d,s,k)>0$ such that*

$$\sup_B |u| \leq \frac{C}{\big(1+R\,m_w(x_0,\nu)\big)^k}\left(\frac{1}{|2B|}\int_{2B}|u|^2\,dx\right)^{1/2}.$$

Proof. Let $k\in\mathbb{N}$, $B=B(x_0,R)$ and $B_k=B(x_0,R_k):=(1+2^{-k})B$. Then Lemma 3.6 gives

$$\sup_B |u| \lesssim \left(\frac{1}{|B_k|}\int_{B_k}|u|^2\,dx\right)^{1/2}.$$

Hence the claim is clear if $k=0$.

Next suppose $k\geq 1$. Let $\eta=\phi_{R_k,R_{k-1},x_0}$, where ϕ_{R_k,R_{k-1},x_0} is given by (19). Applying Proposition 3.2(i) to $u\,\eta$ and then using Lemma 3.7 we arrive at

$$\int_{B_k} m(\cdot,\nu)^{2s}\,|u|^2\,dx \lesssim \|u\,\eta\|_{H^s(\mathbb{R}^d)}+\int_{B_{k-1}}|u|^2\,d\nu \lesssim \frac{2^{kd}}{R^{2s}}\int_{B_{k-1}}|u|^2\,dx.$$

Combining this with Proposition 2.1(iv) we yield

$$\int_{B_k}|u|^2\,dx \lesssim \frac{1}{\big(1+R\,m_w(x_0,\nu)\big)^{2s/(k_0+1)}}\int_{B_{k-1}}|u|^2\,dx.$$

Iterating the above estimate k times and using Lemma 3.6 we arrive at the conclusion. □

Proof of Theorem 1.1. We divide the proof into two parts: Existence of fundamental solution Γ_ν and its decaying property.

Existence: Choose a radial function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\varphi \geq 0, \quad \operatorname{supp} \varphi \subset B_1(0) \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi = 1.$$

Let $r>0$. For each $t\in(0,r)$ define

$$\varphi_t = \frac{1}{t^d}\,\varphi\left(\frac{x}{t}\right) \quad \text{and} \quad V_t = \varphi_t * \nu.$$

Then $V_t \in C^\infty(\mathbb{R}^d)$ for all $t\in(0,r)$.

Now fix $t\in(0,r)$ and $\psi\in C_c^\infty(\mathbb{R}^d)$. Suppose that $\operatorname{supp}\psi\subset B$. It follows from [CK18b, Proof of Theorem 1.1] that there exists a fundamental solution $\Gamma_{V_t}\in L_s^p(\mathbb{R}^d)\cap W_{\mathrm{loc}}^{\gamma,q}(\mathbb{R}^d)$ for all $p\in[1,\frac{d}{d-2s})$, $\gamma\in(0,s)$ and $q\in[1,\frac{d}{d-s})$ such that

$$\int_B \Gamma_{V_t}(x)\, L\psi(x)\, dx = \int_B \Gamma_{V_t}(x)\, L_K\psi(x)\, dx + \int_B \Gamma_{V_t}(x)\, V_t(x)\, \psi(x)\, dx = \psi(0) \tag{20}$$

and

$$0 \leq \Gamma_{V_t}(x) \leq \frac{C}{|x|^{d-2s}} \tag{21}$$

for all $x\in\mathbb{R}^d\setminus\{0\}$, where $C=C(d,s,\lambda,\Lambda)$.

Also [CK18b, Lemma 5.8 and Proof of Theorem 1.1] imply

$$\|\Gamma_{V_t}\|_{W^{\gamma,q}(2B)} \leq C(d,s,\lambda,q,r)$$

for all $t\in(0,r)$, $\gamma\in(0,s)$ and $q\in[1,\frac{d}{d-s})$.

Now fix $a\in[s-\varepsilon_0,s)$, $b\in\left[\frac{d}{d-s}-\varepsilon_0,\frac{d}{d-s}\right)$, where ε_0 is given by (7). By the Sobolev compact embedding, there exists a sequence $\{t_j\}$ and $v\in W^{a,b}(2B)$ such that

$$\begin{cases} \Gamma_{V_{t_j}} \longrightarrow v & \text{weakly in } W^{a,b}(2B), \\ \Gamma_{V_{t_j}} \longrightarrow v & \text{strongly in } L^b(2B) \text{ and} \\ \Gamma_{V_{t_j}} \longrightarrow v & \text{a.e. in } 2B. \end{cases} \tag{22}$$

Observe that (21) and the pointwise convergence above yield

$$0 \leq v(x) \leq \frac{C(d,s)}{|x|^{d-2s}}$$

for all $x\in\mathbb{R}^d\setminus\{0\}$. This in turn implies

$$v \in L_s^p(\mathbb{R}^d) \subset L_s^1(\mathbb{R}^d) \subset \mathcal{S}_s'(\mathbb{R}^d)$$

for all $p\in[1,\frac{d}{d-2s})$, where we made use of (6).

Next we apply Lebesgue's dominated convergence theorem to obtain

$$\Gamma_{V_{t_j}} \longrightarrow v \quad \text{in } L^1_s(\mathbb{R}^d).$$

It follows from [Buc16, p.4] that $L_K\psi\in\mathcal{S}_s(\mathbb{R}^d)$. Therefore

$$\lim_{j\to\infty}\int_B \Gamma_{V_{t_j}}(x)\, L_K\psi(x)\, dx = \int_B v\, L_K\psi(x)\, dx.$$

Next we write

$$\begin{aligned}
&\int_B \Gamma_{V_{t_j}}(x)\, V_{t_j}(x)\, \psi(x)\, dx - \int_B v(x)\, \psi(x)\, d\nu\\
&= \int_B \big(\Gamma_{V_{t_j}}(x) - v(x)\big)\, V_{t_j}(x)\, \psi(x)\, dx + \left(\int_B v(x)\, V_{t_j}(x)\, \psi(x)\, dx - \int_B v(x)\, \psi(x)\, d\nu\right)\\
&=: I + II.
\end{aligned}$$

We have

$$\begin{aligned}
|I| &= \left|\int_B \Big(\big(\Gamma_{V_{t_j}}(x) - v(x)\big)\, \psi\Big) * \varphi_{t_j}\, d\nu\right|\\
&\le \nu(2B)^{1/b'} \left(\int_{2B} \left|\Big(\big(\Gamma_{V_{t_j}}(x) - v(x)\big)\, \psi\Big) * \varphi_{t_j}\right|^b d\nu\right)^{1/b}\\
&\le \nu(2B)^{1/b'} \left(\int_{2B} \left|\big(\Gamma_{V_{t_j}}(x) - v(x)\big)\, \psi\right|^b d\nu\right)^{1/b} \longrightarrow 0
\end{aligned}$$

where the last step follows from Lemma 3.3 and (22).

Also by the same token,

$$\begin{aligned}
|II| &\le \nu(B)^{1/b'} \left(\int_{2B} |(v\,\psi) * \varphi_{t_j} - v\,\psi|^b\, d\nu\right)^{1/b}\\
&\le C_B\, \|(v\,\psi) * \varphi_{t_j} - v\,\psi\|_{W^{a,b}(2B)} \longrightarrow 0
\end{aligned}$$

as $j \longrightarrow \infty$.

Hence

$$\lim_{j\to\infty}\int_B \Gamma_{V_{t_j}}(x)\, V_{t_j}(x)\, \psi(x)\, dx = \int_B v(x)\, \psi(x)\, d\nu.$$

Combining the above estimates together we deduce from (20) that

$$\int_B \Gamma_\nu(x)\, L\psi(x)\, dx := \int_B v(x)\, L\psi(x)\, dx = \psi(0).$$

Decaying property: Let $x, y \in \mathbb{R}^d$ be such that $x \neq y$. Then the previous consideration gives

$$0 \leq \Gamma_\nu(x-y) \leq \frac{C(d,s)}{|x-y|^{d-2s}}. \tag{23}$$

For the extra decaying term, set $R = |x-y|$ and $B = B(x, R/4)$. Observe that $u(\cdot) := \Gamma_\nu(\cdot - y)$ is a weak solution of L in $2B$. It follows that for all $k \in \mathbb{N}$ one has

$$\begin{aligned} \sup_B |u| &\lesssim \frac{1}{\big(1 + R\, m(x,\nu)\big)^k} \left(\frac{1}{|2B|} \int_{2B} |u|^2 \, dz \right)^{1/2} \\ &\lesssim \frac{1}{\big(1 + R\, m(x,\nu)\big)^k} \frac{1}{R^{d-2s}}, \end{aligned}$$

where we used Lemma 4.1 in the first step as well as (23) and the fact that $|z-y| \geq |x-y|/2$ for all $z \in 2B$ in the last step. □

Proof of Proposition 1.2. As shown in Proposition 2.1, the critical function $\rho(\cdot, \nu)$ acquires all the required properties stated in [CK18a, Lemma 3.1]. Therefore [CK18a, Proof of Theorem 1.4] extends verbatim to our setting to derive Proposition 1.2. □

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