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Notes on $H^{\log}$: structural properties, dyadic variants, and bilinear H^1-BMO mappings

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Abstract. This article is devoted to a study of the Hardy space $H^{\log}(\mathbb{R}^d)$ introduced by Bonami, Grellier, and Ky. We present an alternative approach to their result relating the product of a function in the real Hardy space H^1 and a function in BMO to distributions that belong to $H^{\log}$ based on dyadic paraproducts. We also point out analogues of classical results of Hardy-Littlewood, Zygmund, and Stein for $H^{\log}$ and related Musielak-Orlicz spaces.

1. Introduction

There are many situations where the classical L^p spaces, especially with $p=1$, do not fully capture finer properties of functions or operators acting on functions. In such instances, it may be necessary to consider substitutes for L^1, as is the case when studying endpoint bounds for operators on L^p as $p\to 1^+$. The behaviour of the ubiquitous Hardy-Littlewood maximal function near $p=1$ is a typical example. The maximal function is defined for a locally integrable function $f\colon \mathbb{R}^d\to\mathbb{C}$ by setting

$$M(f)(x):=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy \quad x\in\mathbb{R}^d,$$

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where $B(x,r)$ denotes the open ball in $\mathbb{R}^d$ centred at x with radius $r>0$, and $|A|$ denotes the Lebesgue measure of $A\subseteq\mathbb{R}^d$. It is a basic fact that the mapping $f\mapsto M(f)$ is bounded on $L^p(\mathbb{R}^d)$ for $1<p\leq\infty$. The maximal operator is also bounded from $L^1(\mathbb{R}^d)$ to weak-L^1, but does not map $L^1(\mathbb{R}^d)$ to itself (see, for instance, [28] for an in-depth discussion).

However, $M(f)$ is locally integrable provided f is compactly supported and satisfies the $L\log L$ condition

$$\int_{\mathbb{R}^d} |f(x)|\log^+|f(x)|dx<\infty,$$

where, as usual, $\log^+|x|=\max\{\log|x|,0\}$. In a 1969 paper, E. M. Stein [26] proved that this $L\log L$ condition is both sufficient and necessary for integrability of the Hardy-Littlewood maximal function, in the following sense: if f is supported in some finite ball $B=B(r)$ of radius $0<r<\infty$, then

$$\int_B M(f)dx<\infty \quad \text{if, and only if,} \quad \int_B |f(x)|\log^+|f(x)|dx<\infty.$$

Thus, $L\log L$ is a natural substitute for L^1 for the purposes of studying the boundedness of the Hardy-Littlewood maximal function in the scale of L^p spaces.

Another classical result that involves the space $L\log L$ is due to Zygmund, and asserts that the periodic Hilbert transform H maps $L\log L(\mathbb{T})$ to(1) $L^1(\mathbb{T})$; see e.g. Theorem 2.8 in Chapter VII of [37]. Zygmund's result implies that $L\log L(\mathbb{T})$ is contained in the real Hardy space $H^1(\mathbb{T})$ consisting of integrable functions on the torus whose Hilbert transforms are integrable. Moreover, as shown by Stein in [26], Zygmund's theorem has a partial converse, namely if $f\in H^1(\mathbb{T})$ and f is non-negative, then f necessarily belongs to $L\log L(\mathbb{T})$. Therefore, in view of the aforementioned results of Zygmund and Stein, the Hardy space $H^1(\mathbb{T})$ is naturally associated with the Orlicz space $L\log L(\mathbb{T})$.

In several problems in harmonic analysis it is natural to consider Hardy-Orlicz spaces, for instance when one studies certain problems related to endpoint mapping properties of operators; see e.g. [36, Theorem 8], Théorème 2 in Chapitre II and Théorème 1 (c) in Chapitre IV of [22] as well as [16], [18], [29], [31] or, even more generally, Musielak-Orlicz Hardy spaces [35]. In this paper, we shall mainly focus on certain structural aspects of the space $H^{\log}(\mathbb{R}^d)$ appearing in the work of A. Bonami, S. Grellier, and L. D. Ky [4].

Before we proceed with the outline of our paper, let us recall some definitions and properties of Orlicz-type spaces, and in particular that of the space $H^{\log}(\mathbb{R}^d)$.

(1) Notice that the aforementioned endpoint bounds for M and H can be regarded as special cases of the fact that if T is any sublinear operator that is bounded on L^{p_0} for some $p_0>1$ and maps L^1 to weak-L^1, then T locally maps $L\log L$ to L^1.

A function $\Phi\colon[0,\infty)\to[0,\infty)$ is called an Orlicz function if it is strictly positive on $(0,\infty)$, non-decreasing, unbounded and satisfies $\Phi(0)=0$.

A measurable function $\Psi\colon \mathbb{R}^d\times[0,\infty)\to[0,\infty)$ is a Musielak-Orlicz function if for all $x\in\mathbb{R}^d$, $\Psi(x,\cdot)$ is Orlicz. We say that such a function is of *uniformly lower type* (resp., *upper type*) p for $p\in\mathbb{R}$ if there exists a positive constant C so that

$$\Psi(x,st)\leq Cs^p\Psi(x,t), \tag{1.1}$$

for all $x\in\mathbb{R}^d$ and $t\geq 0, s\in(0,1)$(resp., $s\in[1,\infty)$). Note that if Ψ is of upper type p, it satisfies the doubling Δ_2 condition uniformly on x, since (1.1) yields

$$\Psi(x,2t)\leq C2^p\Psi(x,t),$$

for all $x\in\mathbb{R}^d$ and $t\geq 0$. Also, if Ψ is of upper type 1 and of lower type $p\in(0,1)$ it follows that for all $t,c>0$,

$$\Psi(x,ct)\sim_c \Psi(x,t) \tag{1.2}$$

with constants independent of $x\in\mathbb{R}^d$.

Given $p\in[1,\infty)$, we say that Ψ satisfies a uniform Muckenhoupt condition for q, and we write $\Psi\in\mathbb{A}_q(\mathbb{R}^d)$, if

$$\sup_{t>0}\sup_{B\subset\mathbb{R}^d}\frac{1}{|B|}\int_B\Psi(x,t)dx\Big(\frac{1}{|B|}\int_B\Psi(x,t)^{-1/(q-1)}dx\Big)^{q-1}<\infty,\quad \text{if}\, 1<q<\infty,$$

and

$$\sup_{t>0}\sup_{B\subset\mathbb{R}^d}\frac{1}{|B|}\int_B\Psi(x,t)dx\Big(\operatorname*{ess\,sup}_{x\in B}\Psi(x,t)\Big)^{-1}<\infty,\quad \text{if}\, q=1,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. We say that $\Psi\in\mathbb{A}_\infty(\mathbb{R}^d)$, if there exists an index $p\in[1,\infty)$ such that $\Psi\in\mathbb{A}_p(\mathbb{R}^d)$.

A Musielak-Orlicz function Ψ is called a growth function, if it is uniformly lower type $p\in(0,1)$, uniformly upper type 1, and belongs to the class $\mathbb{A}_\infty(\mathbb{R}^d)$.

If B is a measurable subset of $\mathbb{R}^d$, and Ψ is a Musielak-Orlicz function, the Musielak-Orlicz-type space $L_\Psi(B)$ is the set of all measurable functions f such that $\int_B\Psi(x,|f(x)|/\lambda)dx<\infty$ for some $\lambda>0$, endowed with the Luxembourg (quasi-)norm

$$\|f\|_{L_\Psi}:=\inf\Big\{\lambda>0:\int_B\Psi(x,|f(x)|/\lambda)dx\leq 1\Big\}.$$

It is immediate to see that if Ψ is an Orlicz function, independent of x, it belongs to $\mathbb{A}_1(\mathbb{R}^d)$, and so the associated space L_Ψ is indeed a classical Orlicz space.

In this paper, we will mainly focus our attention on the Musielak-Orlicz function

$$\Psi(x,t):=\frac{t}{\log(e+t)+\log(e+|x|)},\quad (x,t)\in\mathbb{R}^d\times[0,\infty),$$

and the Orlicz function

$$\Psi_0(t) := t \cdot [\log(e+t)]^{-1}, \quad t \in [0,\infty).$$

The proof that these are such functions is discussed in [35, Example 1.1.5 (i)].

From now on, except when stated otherwise, Ψ and Ψ_0 denote the two functions above.

Let $K \subset \mathbb{R}^d$ be a given compact set. Note that for $x \in K$ and $t \geq 0$ one has

$$\log(e+t) \leq \log\left((e+|x|)(e+t)\right) \leq c \log(e+t), \tag{1.3}$$

for a constant c that only depends on K. This yields in particular

$$\int_K \Psi(x, |f(x)|)dx \sim \int_K \Psi_0(|f(x)|)dx = \int_0^\infty |\{x \in K : |f(x)| > s\}| \, d\Psi_0(s), \tag{1.4}$$

where the last equality follows by integration by parts. Note in particular that for every $\alpha_0 > 0$ it follows that

$$\int_K \Psi(x, |f(x)|)dx \lesssim_K \Psi_0(\alpha_0)\,|K| + \int_{\alpha_0}^\infty \left|\{x \in \mathbb{R}^d : |f(x)| > s\}\right| d\Psi_0(s). \tag{1.5}$$

We also fix a non-negative function $\phi \in C^\infty(\mathbb{R}^d)$, which is supported in the unit ball of $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \phi(y)dy = 1$ and $\phi(x) = c_d$ for all $|x| \leq 1/2$, where c_d is a constant depending on the dimension d. Given an $\varepsilon > 0$, we employ the notation $\phi_\varepsilon(x) := \varepsilon^{-d}\phi(\varepsilon^{-1}x)$, $x \in \mathbb{R}^d$.

Definition ($H^{\log}$, see [4], [35]) Let ϕ be as above. If f is a tempered distribution on $\mathbb{R}^d$, consider its maximal function

$$M_\phi(f)(x) := \sup_{\varepsilon > 0} |(f * \phi_\varepsilon)(x)|, \quad x \in \mathbb{R}^d.$$

The Hardy space $H^{\log}(\mathbb{R}^d)$ is defined to be the space of tempered distributions f on $\mathbb{R}^d$ such that $M_\phi(f) \in L_\Psi(\mathbb{R}^d)$, that is, $M_\phi(f)$ satisfies

$$\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x))dx < \infty.$$

Let us record some elementary lemmas about $H^{\log}$.

Lemma 1. *Consider the function* $g : [0,\infty) \times [0,\infty) \to [0,\infty)$ *given by*

$$g(s,t) := \frac{1}{\log(e+t) + \log(e+s)}, \quad (s,t) \in [0,\infty) \times [0,\infty).$$

Then one has

$$\Psi(x,t) \sim \int_0^t g(|x|, \tau)d\tau \tag{1.6}$$

for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$, *where the implied constants do not depend on* x, t.

This assertion is a version of the general estimate

$$\Psi(x,t)\sim\int_0^t \frac{\Psi(x,\tau)}{\tau}d\tau=\int_0^1 \frac{\Psi(x,t\tau)}{\tau}d\tau.$$

For Orlicz functions, the upper bound is a consequence of the lower type p assumption, while the lower bound is obtained by using the upper type 1 condition. The case of a general Musielak-Orlicz function follows from the fact that the constants that appear satisfy uniform bounds in x.

Remark 2. The inequalities (1.6) yield

$$\Psi_0(t)=\frac{t}{\log(e+t)}\sim\frac{t}{\log(e+t)+1}\sim\int_e^{t+e}\frac{1}{\log s}ds,$$

which implies that for $y>e$

$$e+\int_e^y \frac{1}{\log s}ds\gtrsim\frac{y}{\log y}. \tag{1.7}$$

Lemma 3. *Let $x_0\in\mathbb{R}^d$ be fixed and for any locally integrable function f define $\tau_{x_0}f(x):=f(x+x_0)$, $x\in\mathbb{R}^d$. Then $f\in L_\Psi(\mathbb{R}^d)$ if, and only if, $\tau_{x_0}f\in L_\Psi(\mathbb{R}^d)$.*

Proof. Note that it suffices to prove that for any $x_0\in\mathbb{R}^d$ and $f\in L_\Psi(\mathbb{R}^d)$ one also has that $\tau_{x_0}f\in L_\Psi(\mathbb{R}^d)$. Without loss of generality, we can assume that f is non-negative. Note that for fixed x_0, we have that for all $x\in\mathbb{R}^d$

$$\log(e+|x-x_0|)\sim_{x_0}\log(e+|x|).$$

A change of variables and the previous estimate yield

$$\int_{\mathbb{R}^d}\frac{\tau_{x_0}f(x)}{\log(e+|x|)+\log(e+\tau_{x_0}f(x))}dx\sim_{x_0}\int_{\mathbb{R}^d}\frac{f(x)}{\log(e+|x|)+\log(e+f(x))}dx,$$

which proves the stated result. $\square$

Since the maximal operator M_ϕ commutes with translations, the following statement is a direct consequence of the previous lemma.

Lemma 4. *Let $x_0\in\mathbb{R}^d$ be fixed and for $u\in S(\mathbb{R}^d)$ define $\langle\tau_{x_0}f,u\rangle:=\langle f,\tau_{-x_0}u\rangle$. Then $f\in H^{\log}(\mathbb{R}^d)$ if, and only if, $\tau_{x_0}f\in H^{\log}(\mathbb{R}^d)$.*

We shall also use the fact that functions in $H^{\log}(\mathbb{R}^d)$ have mean zero; see Lemma 1.4 in [3]. One can actually establish the following more general fact.

Lemma 5. *If $f\in H^{\log}(\mathbb{R}^d)$ is a compactly supported distribution, then $\hat{f}(0)=0$.*

Proof. Let f be a compactly supported distribution in $H^{\log}(\mathbb{R}^d)$. In light of Lemma 4, we may assume, without loss of generality, that f is supported in a closed ball B_r centred at 0 with radius $r>0$, i.e. $\operatorname{supp}(f)\subseteq B_r:=\{x\in\mathbb{R}^d:|x|\le r\}$.

Recall that $\hat{f}(0):=\langle f,1\rangle$ and let $\chi\in C_c^\infty(\mathbb{R}^d)$ be supported inside B_{2r} and be equal to 1 on the support of f.

To prove the lemma, take an $x\in\mathbb{R}^d$ with $|x|>2r$ and observe that, by the definition of ϕ_ε, if we take $\varepsilon=4|x|$ we have that

$$|f*\phi_\varepsilon(x)|=\frac{1}{\varepsilon^d}\big|\langle f,\chi\phi(\varepsilon^{-1}(\cdot-x))\rangle\big|\gtrsim\frac{1}{|x|^d}|\langle f,\chi\rangle|$$

as we then have $\phi(\varepsilon^{-1}(x-y))=c_d$ for $y\in B_r$. Notice that $|\langle f,1-\chi\rangle|=0$. Therefore, for all $|x|>2r$ and $\varepsilon=4|x|$, we have

$$M_\phi(f)(x)\gtrsim\frac{1}{|x|^d}|\langle f,1\rangle|$$

and so, we deduce from Lemma 1 that for $|x|$ large enough

$$\Psi(x,M_\phi(f)(x))\gtrsim\frac{1}{|x|^d\log(e+|x|)}|\hat{f}(0)|.$$

Hence, if $\hat{f}(0)\neq0$, then the function $\Psi(x,M_\phi(f)(x))$ does not belong to $L^1(\mathbb{R}^d)$, which is a contradiction. □

The reason for defining $H^{\log}(\mathbb{R}^d)$ comes from the study of products of functions in the real Hardy space $H^1(\mathbb{R}^d)$ and its dual space $BMO(\mathbb{R}^d)$. To be more specific, following earlier work by Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister in [3], it was shown by Bonami, Grellier, and Ky [4] that the product fg, in the sense of distributions, of a function $f\in H^1(\mathbb{R}^d)$ and a function g of bounded mean oscillation in $\mathbb{R}^d$ can in fact be represented as a sum of a continuous bilinear mapping into $L^1(\mathbb{R}^d)$ and a continuous bilinear mapping into $H^{\log}(\mathbb{R}^d)$. Following [4], for a function g of bounded mean oscillation in $\mathbb{R}^d$, we set

$$\|g\|_{BMO^+(\mathbb{R}^d)}:=\sup_{\substack{Q\subset\mathbb{R}:\\ Q\text{ cube}}}\frac{1}{|Q|}\int_Q|g(x)-\langle g\rangle_Q|dx+\Big|\int_{[0,1)^d}g(x)dx\Big|,$$

where $\langle g\rangle_Q:=|Q|^{-1}\int_Q g(x)dx$. The aforementioned result of Bonami, Grellier, and Ky can be stated as follows.

Theorem 6. ([4]) *There exist two bilinear operators S,T and a constant $C_d>0$ such that*

$$\|S(f,g)\|_{L^1(\mathbb{R}^d)}\le C_d\|f\|_{H^1(\mathbb{R}^d)}\|g\|_{BMO^+(\mathbb{R}^d)}$$

and

$$\|T(f,g)\|_{H^{\log}(\mathbb{R}^d)} \leq C_d \|f\|_{H^1(\mathbb{R}^d)} \|g\|_{BMO^+(\mathbb{R}^d)}$$

with

$$f \cdot g = S(f,g) + T(f,g)$$

in the sense of distributions.

The operators S and T in the statement of Theorem 6 are not unique and they are given in [4] in terms of paraproducts that are constructed by using continuous wavelets. See also [2], [17], and [35] for further developments. For an introduction to the theory of wavelets, we refer the reader to Y. Meyer's book [21]. As mentioned on p. 231 in [4], the space $H^{\log}(\mathbb{R}^d)$ in Theorem 6 is in a certain sense optimal. See also Section 4 in [5] regarding sharpness in the one-dimensional case.

Having seen why $H^{\log}(\mathbb{R}^d)$ is worthy of study, we wish to further elucidate its structure. Our paper consists of three parts.

Part I: Sections 2 and 3

In the first part of this paper we present analogues of the aforementioned theorems of Zygmund and Stein for $H^{\log}(\mathbb{R}^d)$. Such results can be derived from more general results previously obtained in the setting of Orlicz spaces; see, for instance, [6], [14]. (We are grateful that these facts were pointed out to us in connection with an earlier note on this subject.) We give a self-contained account here, including a discussion of sharpness, and indicate some minor modifications needed to obtain results in the Musielak-Orlicz setting.

For an $H^{\log}$ version of Stein's theorem, we need to identify the correct analogue of $L\log L$ in this context, which turns out to be $L\log\log L$: given a measurable subset B of $\mathbb{R}^d$, $L\log\log L(B)$ denotes the class of all locally integrable functions f with $\operatorname{supp}(f) \subseteq B$ and

$$\int_B |f(x)| \log^+ \log^+ |f(x)| dx < \infty.$$

Here is our version of Stein's lemma for L_Ψ.

Theorem 7. *Let f be a measurable function supported in a closed ball $B \subsetneq \mathbb{R}^d$. Then $M(f) \in L_\Psi(B)$ if, and only if, $f \in L\log\log L(B)$.*

Our proof in fact leads to a more general version of Theorem 7. We discuss this, and give a proof of Theorem 7 in Section 2.

Next is the analogue of Zygmund's result for $H^{\log}(\mathbb{R}^d)$.

Theorem 8. *Let B denote the closed unit ball in $\mathbb{R}^d$. If f is a measurable function satisfying $f \in L\log\log L(B)$ and $\int_B f(y)dy=0$, then $f \in H^{\log}(\mathbb{R}^d)$.*

We remark that, by Lemma 5 above, the mean-zero condition in the hypothesis is in fact necessary in order to place a compactly supported function in $H^{\log}(\mathbb{R}^d)$.

Part II: Sections 4, 5 and 6

In the second part of this paper we show that one can simplify the argument in [4] that establishes Theorem 6 by reducing matters to appropriate dyadic counterparts. To be more specific, in Section 4 we introduce a dyadic version of $H^{\log}$ in the periodic setting and then, by establishing a characterisation of dyadic $H^{\log}$ in terms of atomic decompositions, we show that $H^{\log}$ coincides with an intersection of two translates of dyadic $H^{\log}$, a result of independent interest; see Section 5. In Section 6, we show that, in view of the aforementioned result of Section 5, one can obtain a simplified proof of Theorem 6 in the periodic setting in which only dyadic paraproducts are involved.

Part III: Section 7

In Section 7, we establish a version of a classical inequality of G. H. Hardy and J. E. Littlewood [11] that gives a description of the order of magnitude of Fourier coefficients of distributions in $H^{\log}(\mathbb{T})$.

2. Proof of the Stein-type Theorem for L_Ψ and further extensions

Proof of Theorem 7. Assume first that $f \in L\log\log L(B)$. The main observation is that, by (1.4), locally the space L_Ψ essentially coincides with the Orlicz space L_{Ψ_0} and so, one can employ the arguments of Stein [26]. In view of this observation, we remark that the fact that $f \in L\log\log L(B)$ implies $M(f) \in L_{\Psi_0}(B)$ is well-known; see, for instance, [6, p.242], [14, Sections 4 and 7]. We shall also include the proof of this implication here for the convenience of the reader.

The distribution function of Mf satisfies for all $\alpha > 0$ (see e.g. §5.2 (a) in Chapter I in [27])

$$|\{x \in \mathbb{R}^d : M(f)(x) > \alpha\}| \leq \frac{C_d}{\alpha} \int_{\{|f|>\alpha/2\}} |f(x)|dx \tag{2.1}$$

and also by using a Calderón-Zygmund decomposition; see (6) in [26],

$$|\{x \in \mathbb{R}^d : M(f)(x) > \alpha\}| \geq \frac{1}{2^d c\alpha} \int_{\{|f|>c\alpha\}} |f(x)|dx, \tag{2.2}$$

where c is a constant depending on the dimension.

Inequality (1.7), integration by parts, and (2.1) imply that

$$\begin{aligned}\int_B \Psi_0(M(f)(x))dx &\lesssim_B 1+\int_{B\cap\{M(f)>e\}}\left(\int_e^{M(f)(x)}\frac{1}{\log\alpha}d\alpha\right)dx\\ &=1+\int_e^\infty \frac{1}{\log\alpha}\cdot|\{x\in B: M(f)(x)>\alpha\}|d\alpha\\ &\lesssim_B 1+\int_B |f(x)|\left(\int_e^{2|f(x)|}\frac{1}{\alpha\log\alpha}d\alpha\right)dx\lesssim 1+\int_B |f(x)|\log^+\log^+|f(x)|dx,\end{aligned}$$

which yields that $M(f)\in L_{\Psi_0}(B)$.

To prove the reverse implication, by the translation invariance of L_{Ψ_0}, and since the maximal operator commutes with translations and dilations, we can assume, without loss of generality, that B is the unit ball centred at the origin.

Assume that for some f supported in B with $f\in L^1(B)$ we have $M(f)\in L_{\Psi_0}(B)$. Our task is to show that $f\in L\log\log L(B)$. In order to accomplish this, we shall make use of the fact that there exists a $\rho>2$, depending only on $\|f\|_{L^1(B)}$ and B, such that we also have $M(f)\in L_\Psi(\rho B)$ and moreover, for every $\alpha\geq e^e$,

$$|\{x\in\rho B: M(f)(x)>c_1\alpha\}|\geq\frac{c_2}{\alpha}\int_{B\cap\{|f|>\alpha\}}|f(x)|dx, \tag{2.3}$$

where c_1, c_2 are positive constants that can be taken to be independent of f and α. Indeed, arguing as in the proof of [26, Lemma 1], note that for every $r>2$

$$M(f)(x)\lesssim\frac{1}{(r-1)^d|B|}\|f\|_{L^1(B)}\quad \text{for all } x\in\mathbb{R}^d\setminus rB. \tag{2.4}$$

Hence, if we choose $\rho>2$ to be large enough, then $M(f)(x)<e^e\leq\alpha$ for all $x\in\mathbb{R}^d\setminus\rho B$ and so, (2.3) follows from (2.2).

Furthermore, one can check that $M(f)\in L_\Psi(\rho B)$. Indeed, as in [26], it follows from the definition of M and the fact that $\text{supp}(f)\subseteq B$ that there exists a constant $c_0>0$, depending only on the dimension, such that for every $x\in 2B\setminus B$ one has

$$M(f)(x)\leq c_0 M(f)\left(\frac{x}{|x|^2}\right) \tag{2.5}$$

and so, $M(f)\in L_\Psi(2B)$. To show that (2.5) implies that $M(f)\in L_\Psi(B)$, observe first that the function Ψ_0 is increasing on $[0,+\infty)$, and satisfies (1.2).

Next, a change to polar coordinates, followed by another a change of variables and elementary estimates yield

$$\int_{2B\setminus B}\Psi_0(Mf(x))dx\lesssim\int_1^2 s^{d-1}\int_{S^{d-1}}\Psi_0(Mf(\theta/s))d\sigma(\theta)ds$$

$$\sim \int_{\frac{1}{2}}^{1} t^{-1-d} \int_{S^{d-1}} \Psi_0(Mf(t\theta))d\sigma(\theta)dt$$
$$\sim \int_{\frac{1}{2}}^{1} t^{d-1} \int_{S^{d-1}} \Psi_0(Mf(t\theta))d\sigma(\theta)dt \lesssim \int_B \Psi_0(Mf(x))dx.$$

Moreover, we deduce from (2.4) that $M(f)$ belongs to $L_\Psi(\rho B\setminus 2B)$ and it thus follows that $M(f)\in L_\Psi(\rho B)$, as desired.

By the same reasoning as in the proof of sufficiency and by Fubini's theorem,

$$\begin{aligned}\int_{\rho B} \Psi_0(M(f)(x))dx &\gtrsim \int_{\rho B\cap\{M(f)>e^e\}} \frac{M(f)(x)}{\log(M(f)(x))}dx\\ &\gtrsim \int_{\rho B\cap\{M(f)>e^e\}} \left(\int_{e^e}^{M(f)(x)} \frac{1}{\log\alpha}d\alpha\right)dx\\ &\gtrsim \int_{e^e}^{\infty} \frac{1}{\log\alpha}|\{x\in\rho B: M(f)(x)>c_2\alpha\}|d\alpha.\end{aligned}$$

By using (2.3), we now get

$$\begin{aligned}\infty > \int_{\rho B} \Psi_0(M(f)(x))dx &\gtrsim \int_B |f(x)|\left(\int_{e^e}^{|f(x)|} \frac{1}{\alpha\log\alpha}d\alpha\right)dx\\ &\gtrsim 1+\int_B |f(x)|\log^+\log^+|f(x)|dx\end{aligned}$$

and this completes the proof of Theorem 7. □

The following example illustrates the previous theorem. Let B_0 denote the closed unit ball in $\mathbb{R}^d$, and given a small $\delta\in(0,e^{-e})$, set $f:=\delta^{-d}\chi_{\{|x|<\delta\}}$. Then $M(f)(x)\sim|x|^{-d}$ for all $|x|>2\delta$ and one checks that

$$\int_{B_0} |f(x)|\log^+\log^+|f(x)|dx \sim \log(\log(\delta^{-1})) \sim \int_{B_0} \Psi(x, M(f)(x))dx. \tag{2.6}$$

2.1. Further generalisations

Assume that $\Psi:\mathbb{R}^d\times[0,\infty)$ is a Musielak-Orlicz growth function satisfying the following properties:

(1) If K is a compact set in $\mathbb{R}^d$, then there exist $x_1, x_2\in K$ and a constant $C_K>0$ such that

$$C_K^{-1} < \Psi(x_1,t) \le \Psi(x,t) \le \Psi(x_2,t) < C_K$$

for every $x\in K$ and for all $t>0$.

(2) If we write $\Psi(x,t)=\Psi_x(t)=\int_0^t \psi_x(s)ds$, then for every α_0, β_0 with $0<\alpha_0<\beta_0$ one has

$$\int_{\alpha_0}^{\beta_0} \frac{\psi_x(s)}{s} ds < \infty$$

for every $x\in\mathbb{R}^d$.

By carefully examining the proof of Theorem 7, one obtains the following result.

Theorem 9. *Let* $\Psi(x,t)=\int_0^t \psi_x(s)ds$, $(x,t)\in\mathbb{R}^d\times[0,\infty)$, *be as above.*

Fix a compact ball $B\subset\mathbb{R}^d$ *and let* f *be such that* $\operatorname{supp}(f)\subseteq B$. *Then,* $M(f)\in L_\Psi(B)$ *if, and only if, for every* $\alpha_0>0$

$$\int_{\{|f|>\alpha_0\}} |f(x)| \left(\int_{\alpha_0}^{|f(x)|} \frac{\psi_x(s)}{s} ds \right) dx < \infty.$$

3. Proof of the Zygmund-type Theorem for $H^{\log}(\mathbb{R}^d)$

Proof of Theorem 8. Let B denote the closed unit ball in $\mathbb{R}^d$. Fix a function f with $\operatorname{supp}(f)\subseteq B$, $\int_B f(y)dy=0$ and $f\in L\log\log L(B)$. First of all, observe that

$$M_\phi(f)(x) \lesssim M(f)(x) \quad \text{for all } x\in\mathbb{R}^d,$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of f; see e.g. Theorem 2 on pp. 62–63 in [27]. We thus deduce from Lemma 1 that

$$\Psi(x, M_\phi(f)(x)) \lesssim \Psi(x, M(f)(x)) \quad \text{for all } x\in\mathbb{R}^d$$

and hence, by using Theorem 7, we obtain

$$\int_{2B} \Psi(x, M_\phi(f)(x))dx \lesssim 1 + \int_{2B} |f(x)| \log^+ \log^+ |f(x)| dx, \tag{3.1}$$

where $2B:=\{x\in\mathbb{R}^d : |x|\le 2\}$.

To estimate the integral of $\Psi(x, M_\phi(f)(x))$ for $x\in\mathbb{R}^d\setminus 2B$, we shall make use of the cancellation of f. To be more specific, observe that if $|x|>2$ then for every $\varepsilon<|x|/2$, one has that

$$f*\phi_\varepsilon(x) = \frac{1}{\varepsilon^d} \int_B f(y)\phi\Big(\frac{x-y}{\varepsilon}\Big) dy = 0$$

since $|x-y|/\varepsilon>1$ whenever $y\in B$. Therefore, we may restrict ourselves to $\varepsilon\ge|x|/2$ when $|x|>2$. Hence, for $\varepsilon\ge|x|/2$, by exploiting the cancellation of f and using a Lipschitz estimate on ϕ_ε, we obtain

$$|f*\phi_\varepsilon(x)| = \frac{1}{\varepsilon^d} \left| \int_B f(y)\phi\Big(\frac{x-y}{\varepsilon}\Big)dy \right| = \frac{1}{\varepsilon^d} \left| \int_B f(y)\Big[\phi\Big(\frac{x-y}{\varepsilon}\Big) - \phi\Big(\frac{x}{\varepsilon}\Big)\Big]dy \right|$$

$$\lesssim_\phi \frac{1}{\varepsilon^{d+1}} \int_B |y\cdot f(y)|dy \lesssim \frac{1}{|x|^{d+1}} \left[1+\int_B |f(y)|\log^+\log^+|f(y)|dy\right].$$

We thus deduce that, for every $x\in\mathbb{R}^d\setminus 2B$,

$$|M_\phi(f)(x)| \lesssim \frac{1}{|x|^{d+1}} \left[1+\int_B |f(y)|\log^+\log^+|f(y)|dy\right]$$

and so,

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus 2B} \Psi(x, M_\phi(x))dx \\
&\lesssim \left[1+\int_B |f(y)|\log^+\log^+|f(y)|dy\right] \int_{\mathbb{R}^d\setminus 2B} \frac{1}{|x|^{d+1}\log(e+|x|)}dx \\
&\lesssim 1+\int_B |f(y)|\log^+\log^+|f(y)|dy,
\end{aligned}$$

as desired. Therefore, Theorem 8 is now established by using the last estimate combined with (3.1). □

3.1. A partial converse

As in the classical setting of the real Hardy space H^1, see [26] as well as §5.3 in Chapter III in [28], Theorem 8 has a partial converse. To be more precise, let B and B' be open balls such that $\overline{B'}\subset B$ and suppose that an integrable function f is positive on B and belongs to $H^{\log}(\mathbb{R}^d)$. Then $f\in L\log\log L(B')$.

Indeed, to see this, we may assume without loss of generality that B and B' are centred at the origin, i.e. $B=\{x\in\mathbb{R}^d:|x|<s\}$ and $B'=\{x\in\mathbb{R}^d:|x|<t\}$ for $t<s$. In addition, as $f\in L^1(\mathbb{R}^d)$, we may assume that $M(f)(0)<\infty$.

We claim that there exists a constant $\alpha_{d,s,t}>0$, depending only on the dimension d and on the radii s and t of the balls B and B' respectively, such that

$$M(\eta f)(x) \le \alpha_{d,s,t}\,[M_\phi(f)(x)+M(f)(0)] \quad \text{for all } x\in\overline{B'} \tag{3.2}$$

for any continuous function η with $\chi_{B'}\le\eta\le\chi_B$.

To prove (3.2), fix an η as above and an $x\in\overline{B'}$. Note that

$$M(\eta f)(x) = \frac{2^d}{c_d\omega_d}\sup_{r>0}\frac{1}{r^d}\left|\int_{|x-y|<r}\eta(y)f(y)\phi\left(\frac{x-y}{r}\right)dy\right|,$$

as f is non-negative on the support of η and $\phi(y)=c_d$ for $|y|\leq 1/2$. Here ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. Hence,

$$M(\eta f)(x)\leq \frac{2^d}{c_d\omega_d}\left[M_\phi(f)(x)+A_{\eta,s,t}(f)(x)\right], \tag{3.3}$$

where

$$A_{\eta,s,t}(f)(x):=\sup_{r>0}\frac{1}{r^d}\left|\int_{|x-y|<r}[\eta(y)-1]\,f(y)\phi\left(\frac{x-y}{r}\right)dy\right|.$$

Observe that

$$A_{\eta,s,t}(f)(x)\leq \sup_{r>0}\frac{1}{r^d}\int_{|y|<\frac{rs}{s-t}}|\eta(y)-1||f(y)|\left|\phi\left(\frac{x-y}{r}\right)\right|dy,$$

as $|x|\leq t$ and so, for $|y|\geq s$ one has $r>|x-y|\geq|y|-|x|\geq|y|(s-t)/s$. Hence,

$$A_{\eta,s,t}(f)(x)\leq 2c_d\sup_{r>0}\frac{1}{r^d}\int_{|y|<\frac{rs}{s-t}}|f(y)|dy\leq 2c_d\omega_d\left(\frac{s}{s-t}\right)^d M(f)(0).$$

The last estimate combined with (3.3) implies (3.2). We thus have

$$\begin{aligned}\infty&>\int_{\mathbb{R}^d}\Psi(x,M_\phi(f)(x))dx+\int_{B'}\Psi(x,M(f)(0))dx\\ &\geq\int_{B'}\Psi(x,M_\phi(f)(x)+M(f)(0))dx\geq\int_{B'}\Psi(x,\alpha_{d,s,t}^{-1}M(\eta f)(x))dx.\end{aligned}$$

Hence, by Theorem 7, one gets $\alpha_{d,s,t}^{-1}\eta f\in L\log\log L(B')$ and so, as $\eta\equiv 1$ on B', we deduce that $f\in L\log\log L(B')$.

3.2. A variant of Theorem 7 on $\mathbb{T}$

There is a periodic version of Theorem 7, namely $M(f)\in L_{\Psi_0}(\mathbb{T})$ if, and only if, $f\in L\log\log L(\mathbb{T})$. Combining this with Lemma 1, one obtains the following result. Here, we adopt the convention that $\mathbb{T}\cong\mathbb{R}/\mathbb{Z}$.

Before we proceed, let us recall that, following [3], $H^{\log}(\mathbb{T})$ is defined as the class of all $f\in\mathcal{D}'$ whose non-tangential maximal function f^* satisfies

$$\int_{[0,1)}\Psi_0(|f^*(\theta)|)d\theta<\infty,$$

where Ψ_0 is as above. Here, $\mathcal{D}'$ denotes the class of all distributions on $\mathbb{T}$. For $f\in H^{\log}(\mathbb{T})$, one sets

$$\|f\|_{H^{\log}(\mathbb{T})}:=\inf\left\{\lambda>0:\int_{[0,1)}\Psi_0(\lambda^{-1}|f^*(\theta)|)d\theta\leq 1\right\}.$$

In what follows, for $f\in\mathcal{D}'$ and $n\in\mathbb{Z}$, we write $\hat{f}(n):=\langle f,e_n\rangle$ where $e_n(\theta):=e^{i2\pi n\theta}$, $\theta\in[0,1)$.

Remark 10. Recall that for $p>0$, the real Hardy space $H^p(\mathbb{T})$ is defined as the class of all $f\in\mathcal{D}'$ such that $f^*\in L^p(\mathbb{T})$.

It is well-known that elements in $H^p(\mathbb{T})$ are functions for $p\geq 1$ and moreover, $H^p(\mathbb{T})\equiv L^p(\mathbb{T})$ for $p>1$. Clearly,

$$H^1(\mathbb{T})\subset H^{\log}(\mathbb{T})\subset H^p(\mathbb{T}) \quad \text{for all } 0<p<1.$$

Proposition 11. *If* $f\in L\log\log L(\mathbb{T})$, *then* $f\in H^{\log}(\mathbb{T})$.

Moreover, arguing as in Section 3 and using the necessity in (an appropriate periodic version of) Theorem 7 as well as Proposition 11 and Lemma 1, one can show that if f is a non-negative integrable function in $H^{\log}(\mathbb{T})$, then $f\in L\log\log L(\mathbb{T})$.

Proposition 12. *One has*

$$\{f\in L\log\log L(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}=\{f\in H^{\log}(\mathbb{T})\cap L^1(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}.$$

Proof. Note that Proposition 11 implies that

$$\{f\in L\log\log L(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}\subseteq \{f\in H^{\log}(\mathbb{T})\cap L^1(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}. \tag{3.4}$$

To prove the reverse inclusion, take a non-negative function f in $H^{\log}(\mathbb{T})\cap L^1(\mathbb{T})$ and notice that it follows from the work of Stein [26] that

$$|\{\theta\in[0,1): M(f)(\theta)>c_1\alpha\}|\geq \frac{c_2}{\alpha}\int_{\{|f|>\alpha\}}|f(\theta)|d\theta, \tag{3.5}$$

where $c_1,c_2>0$ are absolute constants. Hence, by arguing as in the proof of Theorem 7, it follows from (3.5) (noting that the periodic case is easier as one does not need to consider the contribution away from the support of f) that

$$\int_{[0,1)}\Psi_0(M(f))(\theta)d\theta\gtrsim 1+\int_{[0,1)}|f(x)|\log^+\log^+|f(\theta)|d\theta. \tag{3.6}$$

Since $f\geq 0$ a.e. on $\mathbb{T}$, arguing as on p. 308 in [26], one has

$$f^*(\theta)\geq \sup_{0\leq r<1}|(P_r*f)(\theta)|\gtrsim M(f)(\theta) \quad \text{for a.e. } \theta\in\mathbb{T}, \tag{3.7}$$

where P_r denotes the Poisson kernel in the periodic setting. Hence, by using (3.6), (3.7), and Lemma 1, we deduce that $f\in L\log\log L(\mathbb{T})$ and so,

$$\{f\in H^{\log}(\mathbb{T})\cap L^1(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}\subseteq \{f\in L\log\log L(\mathbb{T}): f\geq 0 \text{ a.e. on } \mathbb{T}\}. \tag{3.8}$$

The desired fact is a consequence of (3.4) and (3.8). □

4. Dyadic $H^{\log}$ on $\mathbb{T}$

In this section, we introduce a dyadic variant of $H^{\log}(\mathbb{T})$ and in the next section we shall prove that it admits a characterisation in terms of atomic decompositions.

4.1. Definition of dyadic $H^{\log}$ on $\mathbb{T}$

Let $\mathcal{I}$ be a given system of dyadic arcs in $\mathbb{T}$. In particular, we assume that $\mathcal{I}$ is of the form

$$\mathcal{I}^{\tau} := \left\{ I_{k,m,\tau} : k \in \mathbb{N}_0, m = 0, ..., 2^k - 1 \right\}$$

for some $\tau \in [0,1)$, where

$$I_{k,m,\tau} := \begin{cases} [2^{-k}m+\tau, 2^{-k}(m+1)+\tau) & \text{if } 2^{-k}(m+1)+\tau \leq 1, \\ [2^{-k}m+\tau, 1) \cup [0, 2^{-k}(m+1)+\tau-1) & \text{if } 2^{-k}m+\tau < 1 < 2^{-k}(m+1)+\tau, \\ [2^{-k}m+\tau-1, 2^{-k}(m+1)+\tau-1) & \text{if } 2^{-k}m+\tau \geq 1. \end{cases}$$

Note that for $\tau=0$, $\mathcal{I}^0$ is the usual dyadic system; for $k \in \mathbb{N}_0$ and $m=0, ..., 2^k-1$, one has $I_{k,m,0} = [2^{-k}m, 2^{-k}(m+1))$.

If $I \in \mathcal{I}$, then h_I denotes the cancellative Haar function associated to I, that is,

$$h_I := |I|^{-1/2} \left(\chi_{I_-} - \chi_{I_+} \right),$$

where I_- and I_+ denote the left and right halves of I, respectively.

Here, we shall adopt the following convention: if $\mathbf{f} = \{f_I\}_{I \in \mathcal{I}} \cup \{f_0\}$ is a collection of complex numbers, we consider the associated sequence of functions $\{f_N\}_{N \in \mathbb{N}}$ given by

$$f_N(\theta) := f_0 + \sum_{\substack{I \in \mathcal{I}: \\ |I| \geq 2^{-N}}} f_I h_I(\theta), \quad \theta \in \mathbb{T}.$$

As usual, the dyadic square function $S_{\mathcal{I}}[f_N]$ of f_N is given by

$$S_{\mathcal{I}}[f_N](\theta) := |f_0| + \Big(\sum_{\substack{I \in \mathcal{I}: \\ |I| \geq 2^{-N}}} |f_I|^2 \frac{\chi_I(\theta)}{|I|} \Big)^{1/2}, \quad \theta \in \mathbb{T}.$$

One then defines the dyadic square function $S_{\mathcal{I}}[\mathbf{f}]$ of $\mathbf{f}$ as the pointwise limit

$$S_{\mathcal{I}}[\mathbf{f}] := \lim_{N \to \infty} S_{\mathcal{I}}[f_N] = |f_0| + \Big(\sum_{I \in \mathcal{I}} |f_I|^2 \frac{\chi_I}{|I|} \Big)^{1/2}.$$

Definition We define $h_{\mathcal{I}}^{\log}(\mathbb{T})$ as the class of all collections $\mathbf{f}=\{f_I\}_{I\in\mathcal{I}}\cup\{f_0\}$ of complex numbers satisfying

$$\int_{[0,1)} \Psi_0\big(S_{\mathcal{I}}[\mathbf{f}](\theta)\big)\,d\theta<\infty.$$

If $\mathbf{f}\in h_{\mathcal{I}}^{\log}(\mathbb{T})$, we set

$$\|\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}:=\inf\left\{\lambda>0: \int_{[0,1)} \Psi_0\big(\lambda^{-1}S_{\mathcal{I}}[\mathbf{f}](\theta)\big)\,d\theta\leq 1\right\}.$$

4.1.1. Some remarks

It can easily be seen that there exists an absolute constant $C_0>1$ such that for all $\mathbf{f},\mathbf{g}\in h_{\mathcal{I}}^{\log}(\mathbb{T})$ and $\mu\in\mathbb{C}$ one has

$$\|\mathbf{f}+\mathbf{g}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}\leq C_0(\|\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T}))}+\|\mathbf{g}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})})$$

and

$$\|\mu\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}=|\mu|\|\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}.$$

Moreover, $\mathbf{f}=\mathbf{0}$ if, and only if, $\|\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}=0$. In particular, $\|\cdot\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}$ is a quasi-norm on the linear space $h_{\mathcal{I}}^{\log}(\mathbb{T})$ and one can show that $(h_{\mathcal{I}}^{\log}(\mathbb{T}),\|\cdot\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})})$ is complete.

Let $F_{\mathcal{I}}(\mathbb{T})$ denote the class of all functions $f\in L^1(\mathbb{T})$ such that the collection $\{\langle f,h_I\rangle\}_{I\in\mathcal{I}}\cup\{\hat{f}(0)\}$ consists of finitely many non-zero terms. Note that if $f\in F_{\mathcal{I}}(\mathbb{T})$ one can write $f=\hat{f}(0)+\sum_{I\in\mathcal{I}}\langle f,h_I\rangle h_I$ and moreover, by identifying functions in $F_{\mathcal{I}}(\mathbb{T})$ with the corresponding collections of their Haar coefficients, we may regard $F_{\mathcal{I}}(\mathbb{T})$ as a dense subspace of $h_{\mathcal{I}}^{\log}(\mathbb{T})$.

4.2. $h_{\mathcal{I}}^{\log}(\mathbb{T})$ and $H_{\mathcal{I}}^{\log}(\mathbb{T})$

Our goal in this section is to show that every collection in $h_{\mathcal{I}}^{\log}(\mathbb{T})$ can be regarded in a 'canonical' way as an element of $\mathcal{D}'$. More specifically, we shall prove that if $\mathbf{f}\in h_{\mathcal{I}}^{\log}(\mathbb{T})$, then the corresponding sequence of functions $\{f_N\}_{N\in\mathbb{N}}$ in $F_{\mathcal{I}}(\mathbb{T})$ converges in the sense of distributions to some $f\in\mathcal{D}'$. If ι denotes the associated map from $h^{\log}(\mathbb{T})$ to $\mathcal{D}'$, then one defines $H_{\mathcal{I}}^{\log}(\mathbb{T}):=\iota\big[h_{\mathcal{I}}^{\log}(\mathbb{T})\big]$ and $\|f\|_{H_{\mathcal{I}}^{\log}(\mathbb{T})}:=\|\mathbf{f}\|_{h_{\mathcal{I}}^{\log}(\mathbb{T})}$ for $\mathbf{f}\in h_{\mathcal{I}}^{\log}(\mathbb{T})$ with $f=\iota(\mathbf{f})$.

To this end, let $\mathbf{f}=\{f_I\}_{I\in\mathcal{I}}\cup\{f_0\}$ be a given collection in $h_{\mathcal{I}}^{\log}(\mathbb{T})$. It can easily be seen that for every $p\in(1/2,1)$ one has

$$\Psi_0(t)\geq(1-p)t^p \quad \text{for all } t\geq 1. \tag{4.1}$$

Hence, by using (4.1), we deduce that

$$\|S_{\mathcal{I}}[\mathbf{f}]\|_{L^p(\mathbb{T})} \leq D(p,\mathbf{f}), \tag{4.2}$$

where $D(p,\mathbf{f})$ is a (finite) positive constant given by

$$D(p,\mathbf{f}) := 1+(1-p)^{-1/p}\Big(\int_{[0,1)} \Psi_0(S_{\mathcal{I}}[\mathbf{f}](\theta))d\theta\Big)^{1/p}. \tag{4.3}$$

Since

$$S_{\mathcal{I}}[\mathbf{f}] \geq \Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^2 \frac{\chi_I}{|I|}\Bigg)^{1/2} \quad \text{for all } k\in\mathbb{N}_0,$$

we deduce from (4.2) that

$$D(p,\mathbf{f}) \geq \Bigg(\int_{[0,1)} \Big(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^2 \frac{\chi_I(\theta)}{|I|}\Big)^{p/2} d\theta\Bigg)^{1/p} \quad \text{for all } k\in\mathbb{N}_0.$$

We thus have

$$D(p,\mathbf{f}) \geq \Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p |I|^{1-p/2}\Bigg)^{1/p} \quad \text{for all } k\in\mathbb{N}_0, \tag{4.4}$$

where we used the fact that the arcs $I\in\mathcal{I}$ with $|I|=2^{-k}$ are mutually disjoint. We shall combine (4.4) with the following standard estimate.

Lemma 13. *There exists an absolute constant $C_0>0$ such that*

$$|\langle \phi, h_I\rangle| \leq C_0 \|\phi'\|_{L^\infty(\mathbb{T})} |I|^{3/2}$$

for all $\phi\in C^1(\mathbb{T})$ and for all $I\in\mathcal{I}$.

Proof. We may assume without loss of generality that $I\subset\mathbb{T}$ can be regarded as an interval in $[0,1)$. By using a change of variables and the mean value theorem, we have

$$\begin{aligned}|\langle \phi, h_I\rangle| &= |I|^{-1/2}\Big|\int_{I_-}\phi(\theta)d\theta - \int_{I_+}\phi(\theta)d\theta\Big|\\ &= |I|^{-1/2}\Big|\int_{I_-}[\phi(\theta)-\phi(\theta+|I|/2)]d\theta\Big| \lesssim \|\phi'\|_{L^\infty(\mathbb{T})}|I|^{3/2},\end{aligned}$$

as desired. □

As mentioned above, we shall prove that the sequence of functions $\{f_N\}_{N\in\mathbb{N}}$ associated to $\mathbf{f}$ converges in the sense of distributions. Towards this aim, by using Lemma 13, for $N,M\in\mathbb{N}$ with $N>M$, we have

$$|\langle f_N - f_M, \phi\rangle| = \Bigg|\sum_{\substack{I\in\mathcal{I}:\\ 2^{-M}>|I|\geq 2^{-N}}} f_I\langle h_I,\phi\rangle\Bigg| \leq \sum_{k=M+1}^{N}\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I||\langle\phi,h_I\rangle|$$
$$\lesssim \|\phi'\|_{L^\infty(\mathbb{T})}\sum_{k=M+1}^{N}\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I||I|^{3/2}.$$

Fix a $p\in(1/2,1)$ and note that the previous estimate implies that

$$|\langle f_N - f_M, \phi\rangle| \lesssim \|\phi'\|_{L^\infty(\mathbb{T})}\sum_{k=M+1}^{N}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p|I|^{3p/2}\Bigg)^{1/p}. \tag{4.5}$$

Since

$$\sum_{k=M+1}^{N}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p|I|^{3p/2}\Bigg)^{1/p} = \sum_{k=M+1}^{N}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p|I|^{1-p/2}|I|^{2p-1}\Bigg)^{1/p}$$
$$= \sum_{k=M+1}^{N} 2^{-(2-1/p)k}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p|I|^{1-p/2}\Bigg)^{1/p},$$

it follows from (4.4) that

$$\sum_{k=M+1}^{N}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|=2^{-k}}} |f_I|^p|I|^{3p/2}\Bigg)^{1/p} \leq \Bigg(\sum_{k=M+1}^{N} 2^{-(2-1/p)k}\Bigg)D(p,\mathbf{f}). \tag{4.6}$$

Therefore, (4.5) and (4.6) imply that $\{f_N\}_{N\in\mathbb{N}}$ is Cauchy in $\mathcal{D}'$ and so, it converges to some $f\in\mathcal{D}'$. A completely analogous argument shows that

$$\sup_{N\in\mathbb{N}}|\langle f_N,\phi\rangle| \leq |f_0|\|\phi\|_{L^\infty(\mathbb{T})} + c_p D(p,\mathbf{f})\|\phi'\|_{L^\infty(\mathbb{T})} \quad \text{for all } \phi\in C^1(\mathbb{T}),$$

where $c_p>0$ is a constant that depends only on p. Hence,

$$|\langle f,\phi\rangle| \leq |f_0|\|\phi\|_{L^\infty(\mathbb{T})} + c_p D(p,\mathbf{f})\|\phi'\|_{L^\infty(\mathbb{T})} \quad \text{for all } \phi\in C^1(\mathbb{T}). \tag{4.7}$$

5. Atomic decomposition of $H_{\mathcal{I}}^{\log}(\mathbb{T})$

5.1. Atomic decomposition of $H^{\log}(\mathbb{T})$

It follows from the work of B. Viviani [31], see also [17], [35], that $H^{\log}(\mathbb{T})$ admits a characterisation in terms of atomic decompositions. To state this characterisation of $H^{\log}(\mathbb{T})$, we need the following definitions.

Definition Let $I \subset \mathbb{T}$ be an arc. A measurable function a_I on $\mathbb{T}$ is said to be an $H^{\log}(\mathbb{T})$-atom associated to I whenever

- $\mathrm{supp}(a_I) \subseteq I$,
- $\int_I a_I(\theta) d\theta = 0$, and
- $\|a_I\|_{L^\infty(\mathbb{T})} \leq \|\chi_I\|^{-1}_{L^{\log}(\mathbb{T})}$.

Definition The atomic Hardy-Orlicz space $H^{\log}_{\mathrm{at}}(\mathbb{T})$ is defined as the space of all $f \in \mathcal{D}'$ that can be written as

$$f - \hat{f}(0) = \sum_{k \in \mathbb{N}} b_{I_k} \quad \text{in } \mathcal{D}',$$

where, for $k \in \mathbb{N}$, b_{I_k} is multiple of an atom in $H^{\log}(\mathbb{T})$ associated to some arc I_k and

$$\sum_{k \in \mathbb{N}} |I_k| \Psi_0(\|b_{I_k}\|_{L^\infty(\mathbb{T})}) < \infty.$$

If $\{b_{I_k}\}_{k \in \mathbb{N}}$ is as above, let

$$\Lambda_\infty\left(f, \{b_{I_k}\}_{k \in \mathbb{N}}\right) := \inf\left\{\lambda > 0 : \Psi_0(\lambda^{-1}|\hat{f}(0)|) + \sum_{k \in \mathbb{N}} |I_k| \Psi_0(\lambda^{-1}\|b_{I_k}\|_{L^\infty(\mathbb{T})}) \leq 1\right\}$$

and define

$$\|f\|_{H^{\log}_{\mathrm{at}}(\mathbb{T})} := \inf\left\{\Lambda_\infty\left(f, \{b_{I_k}\}_{k \in \mathbb{N}}\right) : f - \hat{f}(0) = \sum_{k \in \mathbb{N}} b_{I_k} \text{ with } \{b_{I_k}\}_{k \in \mathbb{N}} \text{ being as above}\right\}.$$

By arguing as in [17], [31] one can show that

$$H^{\log}_{\mathrm{at}}(\mathbb{T}) \cong H^{\log}(\mathbb{T}). \tag{5.1}$$

5.2. Atomic decomposition of $H^{\log}_{\mathcal{I}}(\mathbb{T})$

Definition Let I be an arc in $\mathcal{I}$. A measurable function a_I on $\mathbb{T}$ is said to be an $H^{\log}_{\mathcal{I}}$-atom associated to $I\in\mathcal{I}$ whenever

- $\operatorname{supp}(a_I)\subset I$,
- $\int_I a_I(\theta)d\theta=0$, and
- $\|a_I\|_{L^\infty(\mathbb{T})}\le\|\chi_I\|^{-1}_{L^{\log}(\mathbb{T})}$.

In analogy with the non-dyadic case, define $H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})$ to be the class of all $f\in\mathcal{D}'$ for which there exists a sequence $\{\beta_{I_k}\}_{k\in\mathbb{N}}$ of scalar multiples of $H^{\log}_{\mathcal{I}}$-atoms such that

$$f-\hat{f}(0)=\sum_{k\in\mathbb{N}}\beta_{I_k}\quad\text{in } \mathcal{D}'\quad\text{and}\quad\sum_{k\in\mathbb{N}}|I_k|\Psi_0\big(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})}\big)<\infty.$$

For $f\in H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})$, we set

$$\|f\|_{H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})}:=\inf\left\{\Lambda_\infty\big(f,\{\beta_{I_k}\}_{k\in\mathbb{N}}\big):f-\hat{f}(0)=\sum_{k\in\mathbb{N}}\beta_{I_k}\text{ in } \mathcal{D}'\right\}.$$

We shall prove that $H^{\log}_{\mathcal{I}}(\mathbb{T})$ is contained in $H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})$. To this end, we shall show first that every function $f\in F_{\mathcal{I}}(\mathbb{T})$ admits a decomposition in terms of $H^{\log}_{\mathcal{I}}$-atoms.

Proposition 14. *For every $f\in F_{\mathcal{I}}(\mathbb{T})$ there exists a finite collection of multiples of $H^{\log}_{\mathcal{I}}$-atoms $\{\beta_{I_k}\}_{k=1}^N$ such that*:

- $f-\hat{f}(0)=\sum_{k=1}^N\beta_{I_k}$ *and*
- $\Lambda_\infty(f,\{\beta_{I_k}\}_{k=1}^N)\le C_0\|f\|_{H^{\log}_{\mathcal{I}}(\mathbb{T})}$,

where $C_0>0$ is an absolute constant that is independent of f.

Proof. The proof is a variant of the non-dyadic case presented in Chapter 1 of [35].

Fix an $f\in F_{\mathcal{I}}(\mathbb{T})$. Without loss of generality, we may assume that $\hat{f}(0)=0$. Let $M_{\mathcal{I}}[f]$ denote the dyadic maximal function of f, namely

$$M_{\mathcal{I}}[f](\theta):=\sup_{N\in\mathbb{N}_0}|\mathbb{E}_{\mathcal{I},N}[f](\theta)|,$$

where

$$\mathbb{E}_{\mathcal{I},N}[f]:=\sum_{\substack{I\in\mathcal{I}:\\|I|=2^{-N}}}\langle f\rangle_I\chi_I.$$

Notice that, as $f\in F_{\mathcal{I}}(\mathbb{T})$, there exists an $N_0\in\mathbb{N}$ such that

$$\mathbb{E}_{\mathcal{I},N}[f]\equiv f\quad\text{for all } N\ge N_0. \tag{5.2}$$

For $\lambda>0$, by arguing as on p. 3 in [12], one can show that the set

$$\Omega_\lambda := \{\theta \in \mathbb{T} : M_{\mathcal{I}}[f](\theta) > \lambda\}$$

can be written as a finite union of mutually disjoint arcs in $\mathcal{I}$. We may thus write $\Omega_\lambda = \bigcup_k I(\lambda,k)$, where the union is finite and the arcs $\{I(\lambda,k)\}_k$ are mutually disjoint and in $\mathcal{I}$. We then define

$$g_\lambda(\theta) := \begin{cases} f(\theta) & \text{if } x \in \mathbb{T}\setminus\Omega_\lambda, \\ \langle f\rangle_I & \text{if } \theta \in I(\lambda,k) \end{cases}$$

and $b_\lambda := f - g_\lambda = \sum_k b_{\lambda,k}$, where $b_{\lambda,k} := \chi_{I(\lambda,k)} b_\lambda$. Using (5.2), one checks that

$$\|g_\lambda\|_{L^\infty(\mathbb{T})} \le \lambda. \tag{5.3}$$

Note that for $N\in\mathbb{N}$ large enough, one has $\Omega_{2^n} = \varnothing$ for all $n \ge N$ and so, $g_{2^n} \equiv f$ for all $n \ge N$. We thus have

$$f = \sum_{n=0}^{N-1} (g_{2^{n+1}} - g_{2^n}) = \sum_{n=0}^{N-1} (b_{2^n} - b_{2^{n+1}}).$$

We write $\Omega_{2^n} = \bigcup_k I(2^n,k)$ and define $\beta_{n,k} := (b_{2^n} - b_{2^{n+1}})\chi_{I(2^n,k)}$. It can easily be seen that $\int_{I(2^n,k)} \beta_{n,k}(x')dx' = 0$. Moreover, it follows from (5.3) that

$$\|\beta_{n,k}\|_{L^\infty(\mathbb{T})} \le \|g_{2^n}\|_{L^\infty(\mathbb{T})} + \|g_{2^{n+1}}\|_{L^\infty(\mathbb{T})} \le 3\cdot 2^n.$$

Hence, $\beta_{n,k}$ are multiples of $H^{\log}_{\mathcal{I}}$-atoms and moreover,

$$\begin{aligned} \sum_{n=0}^{N-1}\sum_k |I(2^n,k)|\Psi_0(\|\beta_{n,k}\|_{L^\infty(\mathbb{T})}/\lambda) &\lesssim \sum_{n=0}^{N-1}\sum_k |I(2^n,k)|\Psi_0(2^n/\lambda) \\ &= \sum_{n=0}^{N-1}\Psi_0(2^n/\lambda)\sum_k |I(2^n,k)| = \sum_{n=0}^{N-1}\Psi_0(2^n/\lambda)\sum_{l\in\mathbb{N}_0} |\{2^{n+l} < M_{\mathcal{I}}[f] \le 2^{n+l+1}\}|. \end{aligned}$$

Note that there exists a $c_0>0$ such that $\Psi_0(2^{-l}t) \le 2^{-c_0 l}\Psi_0(t)$ for all $t\ge 0$ and $l\in\mathbb{N}_0$. Hence,

$$\begin{aligned} &\sum_{n=0}^{N-1}\Psi_0(2^n/\lambda)\sum_{l\in\mathbb{N}_0} |\{2^{n+l} < M_{\mathcal{I}}[f] \le 2^{n+l+1}\}| \\ &= \sum_{l\in\mathbb{N}_0}\sum_{n=0}^{N-1}\Psi_0(2^n/\lambda)|\{2^{n+l} < M_{\mathcal{I}}[f] \le 2^{n+l+1}\}| \end{aligned}$$

$$\leq \sum_{l\in\mathbb{N}_0}\sum_{m\geq l}\Psi_0(2^{m-l}/\lambda)|\{2^m < M_{\mathcal{I}}[f]\leq 2^{m+1}\}|$$
$$\leq \sum_{l\in\mathbb{N}_0} 2^{-c_0 l}\sum_{m\geq l}\Psi_0(2^{m}/\lambda)|\{2^m < M_{\mathcal{I}}[f]\leq 2^{m+1}\}| \lesssim \int_{[0,1)}\Psi_0(\lambda^{-1}M_{\mathcal{I}}[f](\theta))d\theta.$$

To complete the proof of the proposition, note that there exists an absolute constant $C>0$ such that

$$\int_{[0,1)}\Psi_0(\lambda^{-1}M_{\mathcal{I}}[f](\theta))d\theta \leq C\int_{[0,1)}\Psi_0(\lambda^{-1}S_{\mathcal{I}}[f](\theta))d\theta,$$

which follows from, e.g., an appropriate periodic variant of [8, Corollary 3.1]. See also [7]. □

By arguing as on p. 109 in [28], the density of $F_{\mathcal{I}}(\mathbb{T})$ in $H^{\log}_{\mathcal{I}}(\mathbb{T})$ and Proposition 14 imply the following result.

Proposition 15. *One has* $H^{\log}_{\mathcal{I}}(\mathbb{T})\subseteq H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})$.

5.3. Proof of the reverse inclusion

The main result of this section is that the converse of Proposition 15 also holds.

Proposition 16. *One has* $H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})\subseteq H^{\log}_{\mathcal{I}}(\mathbb{T})$.

In order to prove Proposition 16, we shall establish first the following dyadic variant of [35, Lemma 1.3.5].

Lemma 17. *Let $I\in\mathcal{I}$ be a given arc. For every L^∞-function β_I supported in I one has*

$$\int_I \Psi_0(S_{\mathcal{I}}[\beta_I](\theta))d\theta \lesssim |I|\Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})}).$$

Proof. Let β_I be as in the statement of the lemma. We may assume that $\|\beta_I\|_{L^\infty(\mathbb{T})}>0$. We argue as in the proof of [35, Lemma 1.3.5]. More specifically, since Ψ_0 is increasing and $\Psi_0(ct)\leq c\Psi_0(t)$ for all $c\geq 1$ and $t\geq 0$, we have

$$\begin{aligned}\int_I \Psi_0(S_{\mathcal{I}}[\beta_I](\theta))d\theta &= \int_I \Psi_0\Big(\frac{S_{\mathcal{I}}[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\|\beta_I\|_{L^\infty(\mathbb{T})}\Big)d\theta\\ &\leq \int_I \Psi_0\Big(\Big(1+\frac{S_{\mathcal{I}}[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\Big)\|\beta_I\|_{L^\infty(\mathbb{T})}\Big)d\theta\\ &\leq \Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})})\int_I \Big(1+\frac{S_{\mathcal{I}}[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\Big)d\theta\end{aligned}$$

$$= \Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})}) \Big(|I| + \|\beta_I\|^{-1}_{L^\infty(\mathbb{T})} \int_I S_{\mathcal{I}}[\beta_I](\theta) d\theta \Big).$$

To complete the proof of the lemma, observe that by using the Cauchy-Schwarz inequality and the fact that $S_{\mathcal{I}}$ is an isometry on $L^2(\mathbb{T})$, one has

$$\int_I S_{\mathcal{I}}[\beta_I](\theta) d\theta \le |I|^{1/2} \|S_{\mathcal{I}}[\beta_I]\|_{L^2(\mathbb{T})} = |I|^{1/2} \|\beta_I\|_{L^2(\mathbb{T})} \le |I| \|\beta_I\|_{L^\infty(\mathbb{T})},$$

as desired. □

5.4. Proof of Proposition 16

Let f be a given distribution in $H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})$. Without loss of generality, we may assume that $\hat{f}(0)=0$.

By definition, there exists a sequence of multiples of $H^{\log}_{\mathcal{I}}$-atoms $\{\beta_{I_k}\}_{k\in\mathbb{N}}$ such that

$$f = \lim_{N\to\infty} \sum_{k=1}^{N} \beta_{I_k} \text{ in } \mathcal{D}' \quad \text{and} \quad \sum_{k=1}^{\infty} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})}) < \infty.$$

For $N\in\mathbb{N}$, we set $b_N := \sum_{k=1}^{N} \beta_{I_k}$. Note that since $b_N \in L^\infty(\mathbb{T})$ one has $b_N = \sum_{I\in\mathcal{I}} \langle b_N, h_I\rangle h_I$ a.e. on $\mathbb{T}$ and in $L^2(\mathbb{T})$.

In what follows, we shall use several times the fact that

$$\Psi_0\left(\sum_{j=1}^{L} t_j\right) \le \sum_{j=1}^{L} \Psi_0(t_j) \tag{5.4}$$

for any finite collection of non-negative numbers $\{t_j\}_{j=1}^{L}$; see the proof of [35, Lemma 1.1.6 (i)].

Lemma 18. *Let $I\in\mathcal{I}$ be given. If $\{b_N\}_{N\in\mathbb{N}}$ is as above, then the sequence of complex numbers $\{\langle b_N, h_I\rangle\}_{N\in\mathbb{N}}$ converges.*

Proof. It follows from Lemma 17 and (5.4) that for $N > M$ one has

$$\begin{aligned}\int_{[0,1)} \Psi_0(S_{\mathcal{I}}[b_N - b_M](\theta)) d\theta &\le \sum_{k=M+1}^{N} \int_{[0,1)} \Psi_0(S_{\mathcal{I}}[\beta_{I_k}](\theta)) d\theta \\ &\lesssim \sum_{k=M+1}^{N} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})})\end{aligned}$$

and so,

$$\lim_{M,N\to\infty}\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[b_N-b_M](\theta))d\theta=0. \tag{5.5}$$

Observe that, since Ψ_0 is increasing, one has

$$\begin{aligned}\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[b_N-b_M](\theta))d\theta &\geq \int_{[0,1)}\Psi_0(|\langle b_N,h_I\rangle-\langle b_M,h_I\rangle||I|^{-1/2}\chi_I)d\theta\\ &=|I|\Psi_0(|\langle b_N,h_I\rangle-\langle b_M,h_I\rangle||I|^{-1/2})\end{aligned}$$

for all $I\in\mathcal{I}$. Since Ψ_0 is continuous and $\Psi_0(t)=0$ if, and only if, $t=0$, we deduce from the previous inequality and (5.5) that

$$\lim_{M,N\to\infty}|\langle b_N,h_I\rangle-\langle b_M,h_I\rangle|=0.$$

Hence, the sequence $\{\langle b_N,h_I\rangle\}_{N\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ and so, it converges. $\square$

In view of Lemma 18, we may define $\mathbf{b}:=\{b_I\}_{I\in\mathcal{I}}$ with $b_I:=\lim_{N\to\infty}\langle b_N,h_I\rangle$. We claim that $\mathbf{b}\in h^{\log}_{\mathcal{I}}(\mathbb{T})$ with

$$\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[\mathbf{b}](\theta))d\theta\lesssim\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})}). \tag{5.6}$$

Indeed, by Lemma 17 and the definition of $\{b_N\}_{N\in\mathbb{N}}$, one has

$$\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[b_N](\theta))d\theta\lesssim\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})})\quad\text{for all } N\in\mathbb{N}. \tag{5.7}$$

Fix an $M\in\mathbb{N}$ and note that, by combining (5.7) with Fatou's lemma, one gets

$$\begin{aligned}&\int_{[0,1)}\liminf_{N\to\infty}\Psi_0\left(\left\{\sum_{\substack{I\in\mathcal{I}:\\|I|\geq 2^{-M}}}|\langle b_N,h_I\rangle|^2|I|^{-1}\chi_I(\theta)\right\}^{1/2}\right)d\theta\\ &\leq\liminf_{N\to\infty}\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[b_N](\theta))d\theta\lesssim\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})}).\end{aligned}$$

Since Ψ_0 is continuous, we deduce that

$$\int_{[0,1)}\Psi_0\left(\left\{\sum_{\substack{I\in\mathcal{I}:\\|I|\geq 2^{-M}}}|b_I|^2|I|^{-1}\chi_I(\theta)\right\}^{1/2}\right)d\theta\lesssim\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})}) \tag{5.8}$$

for all $M\in\mathbb{N}$. Hence, (5.6) is obtained by using (5.8), the monotone convergence theorem, and the continuity of Ψ_0.

If we now define the sequence of functions $\{B_M\}_{M\in\mathbb{N}}$ with finite wavelet expansions given by

$$B_M(\theta) := \sum_{\substack{I\in\mathcal{I}:\\ |I|\geq 2^{-M}}} b_I h_I(\theta) \quad (\theta\in\mathbb{T}),$$

then, as explained in Section 4.2, B_M converges in $\mathcal{D}'$ to some $b\in\mathcal{D}'$. It thus suffices to prove that $b\equiv f$. To this end, it is enough to show that, in view of the definition of $\{b_N\}_{N\in\mathbb{N}}$, one has

$$b = \lim_{N\to\infty} b_N \text{ in } \mathcal{D}'. \tag{5.9}$$

For $M, N\in\mathbb{N}$, consider the function $\delta_{M,N}$ given by

$$\delta_{M,N}(\theta) := B_M(\theta) - (b_N)_M(\theta) \quad (\theta\in\mathbb{T}),$$

where $(b_N)_M$ is the 'truncation' of the Haar series representation of b_N allowing Haar projections corresponding to arcs $I\in\mathcal{I}$ with $|I|\geq 2^{-M}$, that is,

$$(b_N)_M(\theta) := \sum_{\substack{I\in\mathcal{I}:\\ |I|\geq 2^{-M}}} \langle b_N, h_I\rangle h_I(\theta) \quad (\theta\in\mathbb{T}).$$

We claim that for any fixed $\phi\in C^\infty(\mathbb{T})$ one has

$$\lim_{M\to\infty} \langle \delta_{M,N}, \phi\rangle = \langle b - b_N, \phi\rangle \quad \text{uniformly in } N\in\mathbb{N}. \tag{5.10}$$

Indeed, fix a $\phi\in C^\infty(\mathbb{T})$ and write

$$|\langle \delta_{M,N} - (b-b_N), \phi\rangle| \leq |\langle B_M - b, \phi\rangle| + |\langle (b_N)_M - b_N, \phi\rangle|. \tag{5.11}$$

Let $p\in(1/2,1)$ be fixed. By arguing as in Section 4.2, one deduces that there exists an absolute constant $c_0>0$ such that

$$|\langle B_M - b, \phi\rangle| \leq c_0 \|\phi'\|_{L^\infty(\mathbb{T})} D(p,\mathbf{b}) \sum_{k=M+1}^{\infty} 2^{-(2-1/p)k}, \tag{5.12}$$

where $D(p,\mathbf{b})$ is as in Section 4.2 and is finite, in view of (5.6). Similarly, one has

$$|\langle (b_N)_M - b_N, \phi\rangle| \leq c_0 \|\phi'\|_{L^\infty(\mathbb{T})} D(p,\mathbf{b}_N) \sum_{k=M+1}^{\infty} 2^{-(2-1/p)k}, \tag{5.13}$$

where

$$D(p,\mathbf{b}_N)=1+(1-p)^{-1/p}\left(\int_{[0,1)}\Psi_0(S_{\mathcal{I}}[\mathbf{b}_N](\theta))d\theta\right)^{1/p}.$$

By using Lemma 17 and the definition of b_N, one deduces that

$$\begin{aligned}D(p,\mathbf{b}_N)&\leq 1+(1-p)^{-1/p}\left(\sum_{k=1}^{N}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})})\right)^{1/p}\\&\leq 1+(1-p)^{-1/p}\left(\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})})\right)^{1/p}\end{aligned}$$

that is, $D(p,\mathbf{b}_N)$ is bounded by a finite constant that is independent of $N\in\mathbb{N}$. Hence, (5.13) implies that

$$|\langle (b_N)_M-b_N,\phi\rangle|\leq c_p\|\phi'\|_{L^\infty(\mathbb{T})}\left[1+\left(\sum_{k=1}^{\infty}|I_k|\Psi_0(\|\beta_{I_k}\|_{L^\infty(\mathbb{T})})\right)^{1/p}\right]\sum_{k=M+1}^{\infty}2^{-(2-1/p)k}, \tag{5.14}$$

where $c_p>0$ is a constant depending only on p. Therefore, by combining (5.12) with (5.14), we deduce that (5.10) holds.

Moreover, by arguing again as in Section 4.2, one shows that for any fixed $p\in(1/2,1)$ there exists a constant $c_p'>0$, depending only on p, such that

$$|\langle \delta_{M,N},\phi\rangle|\leq c_p'\|\phi'\|_{L^\infty(\mathbb{T})}\left[\int_{[0,1)}\Psi_0\left(\left\{\sum_{\substack{I\in\mathcal{I}:\\|I|\geq 2^{-M}}}|b_I-\langle b_N,h_I\rangle|^2|I|^{-1}\chi_I(\theta)\right\}^{1/2}\right)d\theta\right]^{1/p} \tag{5.15}$$

for all $M,N\in\mathbb{N}$ and $\phi\in C^\infty(\mathbb{T})$. We shall prove that the right-hand side of (5.15) tends to 0 as $N\to\infty$ for all $M\in\mathbb{N}$ and $\phi\in C^\infty(\mathbb{T})$. To this end, note that for any $L\in\mathbb{N}$ and for every collection $\{t_l\}_{l=1}^L$ of non-negative numbers, one has

$$\Psi_0\left[\left(\sum_{j=1}^{L}t_j\right)^{1/2}\right]\leq\sum_{j=1}^{L}\Psi_0(t_j^{1/2}). \tag{5.16}$$

Indeed, (5.16) is obtained by combining (5.4) with $\left(\sum_{j=1}^L t_j\right)^{1/2}\leq\sum_{j=1}^L t_j^{1/2}$. Observe that by using (5.16) one has

$$\int_{[0,1)}\Psi_0\left(\left\{\sum_{\substack{I\in\mathcal{I}:\\|I|\geq 2^{-M}}}|b_I-\langle b_N,h_I\rangle|^2|I|^{-1}\chi_I(\theta)\right\}^{1/2}\right)d\theta\leq$$

$$\sum_{\substack{I\in\mathcal{I}:\\ |I|\geq 2^{-M}}} |I|\Psi_0\big(|b_I-\langle b_N,h_I\rangle||I|^{-1/2}\big)$$

and hence, (5.15) implies that

$$|\langle\delta_{M,N},\phi\rangle|\leq c_p'\|\phi'\|_{L^\infty(\mathbb{T})}\Bigg(\sum_{\substack{I\in\mathcal{I}:\\ |I|\geq 2^{-M}}} |I|\Psi_0\big(|b_I-\langle b_N,h_I\rangle||I|^{-1/2}\big)\Bigg)^{1/p}. \tag{5.17}$$

Since the sum on the right-hand side of (5.17) is finite and Ψ_0 is continuous, it follows from the definition of $\mathbf{b}$ that

$$\lim_{N\to\infty}\langle\delta_{M,N},\phi\rangle=0 \quad \text{for all } M\in\mathbb{N}. \tag{5.18}$$

Therefore, by combining (5.10) and (5.18), it follows that

$$\lim_{N\to\infty}\langle b-b_N,\phi\rangle=\lim_{N\to\infty}\lim_{M\to\infty}\langle\delta_{M,N},\phi\rangle=\lim_{M\to\infty}\lim_{N\to\infty}\langle\delta_{M,N},\phi\rangle=0$$

for all $\phi\in C^\infty(\mathbb{T})$. Hence, (5.9) holds and so, the proof of Proposition 16 is complete.

5.5. Concluding remarks

By combining Propositions 15 and 16 one obtains the following theorem.

Theorem 19. *One has* $H^{\log}_{\mathrm{at},\mathcal{I}}(\mathbb{T})\cong H^{\log}_{\mathcal{I}}(\mathbb{T})$.

By using [20, Proposition 2.1] and the atomic decomposition of $H^{\log}(\mathbb{T})$; see (5.1), one shows that $H^{\log}_{\mathrm{at},\mathcal{I}^0}(\mathbb{T})+H^{\log}_{\mathrm{at},\mathcal{I}^{1/3}}(\mathbb{T})\cong H^{\log}(\mathbb{T})$. We thus deduce from Theorem 19 the following variant of T. Mei's theorem [20] for $H^{\log}(\mathbb{T})$.

Theorem 20. *One has*

$$H^{\log}_{\mathcal{I}^0}(\mathbb{T})+H^{\log}_{\mathcal{I}^{1/3}}(\mathbb{T})\cong H^{\log}(\mathbb{T}).$$

Let $\mathcal{I}$ be a given system of dyadic arcs in $\mathbb{T}$. For $p\in(0,\infty)$, define the dyadic Hardy space $h^p_{\mathcal{I}}(\mathbb{T})$ as the class of all collections of complex numbers $\mathbf{f}=\{f_I\}_{I\in\mathcal{I}}\cup\{f_0\}$ such that $S_{\mathcal{I}}[\mathbf{f}]\in L^p(\mathbb{T})$.

By arguing as in Sections 4 and 5, one can show that $h^p_{\mathcal{I}}(\mathbb{T})$ can be identified with a dyadic H^p space $H^p_{\mathcal{I}}(\mathbb{T})$ of distributions on $\mathbb{T}$ and moreover, the following extension of T. Mei's theorem [20] holds.

Theorem 21. *One has*

$$H^p_{\mathcal{I}^0}(\mathbb{T})+H^p_{\mathcal{I}^{1/3}}(\mathbb{T})\cong H^p(\mathbb{T})$$

for all $p\in(1/2,1]$.

We remark that dyadic Hardy spaces for $p<1$ have also been considered in [33] and [23], but the definitions there are different than ours.

6. Proof of the Bonami-Grellier-Ky theorem in the periodic setting

For a function $b\in L^1(\mathbb{T})$, we set

$$\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}:=\Big(\sup_{I\in\mathcal{I}}\frac{1}{|I|}\int_I|b(\theta)-\langle b\rangle_I|^2d\theta\Big)^{1/2}+\Big|\int_{[0,1)}b(\theta)d\theta\Big|.$$

We then define $BMO^+_{\mathcal{I}}(\mathbb{T})$ as the class of all functions $b\in L^1(\mathbb{T})$ such that $\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}<\infty$. One defines $BMO^+(\mathbb{T})$ similarly.

Recall the following standard consequence of the John-Nirenberg type result in the dyadic case.

Lemma 22. *There exists an absolute constant* $C_0>0$ *such that for every function* $b\in BMO^+_{\mathcal{I}}(\mathbb{T})$ *one has*

$$\|b\|_{\exp L(\mathbb{T})}\le C_0\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})},$$

where $\|b\|_{\exp L(\mathbb{T})}:=\inf\{\lambda>0:\int_{[0,1)}\Phi(|b(x)|/\lambda)dx\le 1\}$ *and* $\Phi(t):=e^t-t-1$, $t\ge 0$.

The following variant of [4, Proposition 2.1] is obtained by combining Lemma 22 with [4, Lemma 2.1].

Proposition 23. *For all functions* f, b *such that* $f\in L^1(\mathbb{T})$ *and* $b\in BMO^+_{\mathcal{I}}(\mathbb{T})$, *one has*

$$\|f\cdot b\|_{L^{\log}(\mathbb{T})}\lesssim\|f\|_{L^1(\mathbb{T})}\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}.$$

In this section, we present the following dyadic version of [4, Theorem 1.1].

Theorem 24. *There exist two bilinear operators* S *and* T *on the product space* $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ *such that*

$$f\cdot b=S_{\mathcal{I}}(f,b)+T_{\mathcal{I}}(f,b)\quad\text{in }\mathcal{D}'$$

with $S_{\mathcal{I}}:H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})\to L^1(\mathbb{T})$ *and* $T_{\mathcal{I}}:H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})\to H^{\log}_{\mathcal{I}}(\mathbb{T})$.

The proof of Theorem 24 that we present here is a variant of the corresponding one given by Bonami, Grellier, and Ky in [4] (that establishes [4, Theorem 1.1]). To be more specific, let $f\in H^1_{\mathcal{I}}(\mathbb{T})$ be a function with finite wavelet expansion. If b is a function in $\mathrm{BMO}^+_{\mathcal{I}}(\mathbb{T})$ that also has a finite wavelet expansion, then we may write

$$f\cdot b=\Pi_1^{\mathcal{I}}(f,b)+\Pi_2^{\mathcal{I}}(f,b)+\Pi_3^{\mathcal{I}}(f,b),$$

where

$$\Pi_1^{\mathcal{I}}(f,b)(\theta):=\sum_{\substack{I,J\in\mathcal{I}:\\ J\supsetneq I}} f_I g_J h_I(\theta)h_J(\theta),$$

$$\Pi_2^{\mathcal{I}}(f,b)(\theta):=\sum_{\substack{I,J\in\mathcal{I}:\\ I\supsetneq J}} f_I g_J h_I(\theta)h_J(\theta),$$

and

$$\Pi_3^{\mathcal{I}}(f,b)(\theta):=\sum_{\substack{I,J\in\mathcal{I}:\\ I=J}} f_I g_J h_I(\theta)h_J(\theta).$$

We shall prove that:

- $\Pi_1^{\mathcal{I}}$ can be extended as a bounded bilinear operator from $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $H^{\log}_{\mathcal{I}}(\mathbb{T})$,
- $\Pi_2^{\mathcal{I}}$ can be extended as a bounded bilinear operator from $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $H^1_{\mathcal{I}}(\mathbb{T})$, and
- $\Pi_3^{\mathcal{I}}$ can be extended as a bounded bilinear operator from $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $L^1(\mathbb{T})$.

One can thus conclude that Theorem 24 holds by taking $S_{\mathcal{I}}:=\Pi_1^{\mathcal{I}}$ and $T_{\mathcal{I}}:=\Pi_2^{\mathcal{I}}+\Pi_3^{\mathcal{I}}$.

Proposition 25. *The bilinear operator $\Pi_3^{\mathcal{I}}$ extends into a bounded bilinear operator from $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $L^1(\mathbb{T})$.*

Proof. It is well-known that $H^1_{\mathcal{I}}(\mathbb{T})$ admits a characterisation in terms of atoms. More specifically, recall that a measurable function a is said to be an L^2-atom in $H^1_{\mathcal{I}}(\mathbb{T})$ if it is either the constant function or there exists an $\Omega\in\mathcal{I}$ such that $\mathrm{supp}(a)\subseteq\Omega$, $\int_{[0,1)}a(\theta)d\theta=0$, and $\|a\|_{L^2(\mathbb{T})}\leq|\Omega|^{-1/2}$. Then, $f\in H^1_{\mathcal{I}}(\mathbb{T})$ if, and only if, there exist a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of non-negative scalars and a sequence $\{a_k\}_{k\in\mathbb{N}}$ of atoms such that

$$f=\sum_{k\in\mathbb{N}}\lambda_k a_k$$

in the $H^1_{\mathcal{I}}$-norm and moreover, one has

$$\|f\|_{H^1_{\mathcal{I}}(\mathbb{T})}\sim\inf\left\{\sum_{k\in\mathbb{N}}|\lambda_k|:f=\sum_{k\in\mathbb{N}}\lambda_k a_k\right\}.$$

Hence, to show that $\Pi_3^{\mathcal{I}}$ maps $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $L^1(\mathbb{T})$, it is enough to prove that there exists a constant $C>0$ such that

$$\|\Pi_3^{\mathcal{I}}(a,b)\|_{L^1(\mathbb{T})}\le C\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \tag{6.1}$$

for all non-constant $H^1_{\mathcal{I}}(\mathbb{T})$-atoms a and every $b\in BMO^+_{\mathcal{I}}(\mathbb{T})$ with $\int_{[0,1)} b(\theta)d\theta=0$ such that a and b have finite wavelet expansions. To this end, assume that a and b have finite wavelet expansions and a is associated to some $\Omega\in\mathcal{I}$. If we write $a=\sum_{I\subseteq\Omega} a_I h_I$ and $b=\sum_I b_I h_I$, then

$$\Pi_3^{\mathcal{I}}(a,b)(\theta)=\sum_{\substack{I,J\in\mathcal{I}:\\ I=J}} a_I b_J h_I(\theta)h_J(\theta)=\sum_{I\subseteq\Omega} a_I b_I \frac{\chi_I(\theta)}{|I|}$$

and hence, by using the Cauchy-Schwarz inequality, one gets the pointwise estimate

$$|\Pi_3^{\mathcal{I}}(a,b)(\theta)|\le S_{\mathcal{I}}[a](\theta)\cdot S_{\mathcal{I}}[P_\Omega b](\theta),$$

where $P_\Omega b(\theta):=\sum_{I\subseteq\Omega} b_I h_I(\theta)$. We thus have by using the Cauchy-Schwarz inequality and the L^2-boundedness of $S_{\mathcal{I}}$,

$$\|\Pi_3^{\mathcal{I}}(a,b)\|_{L^1(\mathbb{T})}\le \|a\|_{L^2(\mathbb{T})}\|P_\Omega b\|_{L^2(\mathbb{T})}. \tag{6.2}$$

Since $\|a\|_{L^2(\mathbb{T})}\le|\Omega|^{-1/2}$ and $\|P_\Omega b\|_{L^2(\mathbb{T})}\le|\Omega|^{1/2}\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}$, (6.1) follows from (6.2). □

Proposition 26. *The bilinear operator* $\Pi_2^{\mathcal{I}}$ *extends into a bounded bilinear operator from* $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ *to* $H^1_{\mathcal{I}}(\mathbb{T})$.

Proof. As in the proof of the previous proposition, it suffices to prove that there exists an absolute constant $C>0$ such that

$$\|\Pi_2^{\mathcal{I}}(a,b)\|_{H^1_{\mathcal{I}}(\mathbb{T})}\le C\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \tag{6.3}$$

for each non-constant $H^1_{\mathcal{I}}(\mathbb{T})$-atom a and for each $b\in BMO^+_{\mathcal{I}}(\mathbb{T})$ with $\int_{[0,1)} b(\theta)d\theta=0$ such that a and b have finite wavelet expansions. Towards this aim, write $a=\sum_{I\subseteq\Omega} a_I h_I$ for some $\Omega\in\mathcal{I}$ and notice that

$$\begin{aligned}\Pi_2^{\mathcal{I}}(a,b)(\theta)=\sum_{\substack{I,J\in\mathcal{I}:\\ \Omega\supseteq I\supsetneq J}} a_I b_J h_I(\theta)h_J(\theta)&=\sum_{\substack{I,J\in\mathcal{I}:\\ \Omega\supseteq I\supsetneq J}} a_I (P_\Omega b)_J h_I(\theta)h_J(\theta)\\ &=\sum_{\substack{I,J\in\mathcal{I}:\\ \Omega\supseteq I\supsetneq J}} a_I (P_\Omega b)_J h_I(c_J)h_J(\theta),\end{aligned}$$

where c_J denotes the centre of J. Moreover, observe that since

$$\langle a\rangle_J = |J|^{-1}\int_J a(\theta)d\theta = \sum_{\substack{I\in\mathcal{I}:\\ I\subseteq\Omega}} a_I|J|^{-1}\int_J h_I(\theta)d\theta = \sum_{\substack{I\in\mathcal{I}:\\ J\subsetneq I\subseteq\Omega}} a_I h_I(c_J),$$

one may rewrite $\Pi_2^{\mathcal{I}}(a,b)$ as

$$\Pi_2^{\mathcal{I}}(a,b)(\theta) = \sum_{J\in\mathcal{I}} \langle a\rangle_J (P_\Omega b)_J h_J(\theta).$$

Hence,

$$S_{\mathcal{I}}[\Pi_2^{\mathcal{I}}(a,b)](\theta) = \Big(\sum_{J\in\mathcal{I}} |\langle a\rangle_J|^2 |(P_\Omega b)_J|^2 \frac{\chi_J(\theta)}{|J|}\Big)^{1/2} \leq M(a)(\theta)\cdot S_{\mathcal{I}}[P_\Omega b](\theta),$$

where M denotes the Hardy-Littlewood maximal operator acting on functions defined over $\mathbb{T}$. Hence, by using the Cauchy-Schwarz inequality and the L^2-boundedness of M and $S_{\mathcal{I}}$, one gets

$$\|\Pi_2^{\mathcal{I}}(a,b)\|_{H^1_{\mathcal{I}}(\mathbb{T})} = \|S_{\mathcal{I}}\big(\Pi_2^{\mathcal{I}}(a,b)\big)\|_{L^1(\mathbb{T})} \lesssim \|a\|_{L^2(\mathbb{T})}\|P_\Omega b\|_{L^2(\mathbb{T})}.$$

As in the proof of the previous lemma, note that one has $\|a\|_{L^2(\mathbb{T})}\leq|\Omega|^{-1/2}$ and $\|P_\Omega b\|_{L^2(\mathbb{T})}\leq|\Omega|^{1/2}\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}$ and so, (6.3) follows from the last estimate. □

It follows from Propositions 25 and 26 that if we define

$$T(f,b)(\theta) := \Pi_2^{\mathcal{I}}(f,b)(\theta) + \Pi_3^{\mathcal{I}}(f,b)(\theta),$$

then T is a bilinear operator that maps $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $L^1(\mathbb{T})$. Therefore, to complete the proof of Theorem 24, it remains to handle $\Pi_1^{\mathcal{I}}$.

Proposition 27. *The bilinear operator $\Pi_1^{\mathcal{I}}$ extends into a bounded bilinear operator from $H^1_{\mathcal{I}}(\mathbb{T})\times BMO^+_{\mathcal{I}}(\mathbb{T})$ to $H^{\log}_{\mathcal{I}}(\mathbb{T})$.*

Proof. Fix an $f\in H^1_{\mathcal{I}}(\mathbb{T})$ and $b\in BMO^+_{\mathcal{I}}(\mathbb{T})$ with finite wavelet expansions and moreover, assume that $\int_{[0,1)} b(\theta)d\theta=0$.

First of all, arguing as above, one may write

$$\Pi_1^{\mathcal{I}}(f,b)(\theta) = \sum_{I\in\mathcal{I}} f_I\langle b\rangle_I h_I(\theta).$$

Let a be a non-constant $H^1_{\mathcal{I}}(\mathbb{T})$-atom with finite wavelet expansion that is supported in some $\Omega\in\mathcal{I}$ so that $\|a\|_{L^2(\mathbb{T})}\leq|\Omega|^{-1/2}$. We claim that

$$\Pi_1^{\mathcal{I}}(a,b)(\theta) = \Pi_1^{\mathcal{I}}(a,P_\Omega b)(\theta) + \langle b\rangle_\Omega\cdot a(\theta). \tag{6.4}$$

Indeed, to see this, write

$$\Pi_1^{\mathcal{I}}(a,b)(\theta)=\sum_{I\in\mathcal{I}} a_I\langle b\rangle_I h_I(\theta)=\Pi_1^{\mathcal{I}}(a,P_\Omega b)(\theta)+\sum_{I\in\mathcal{I}} a_I\langle b-P_\Omega b\rangle_I h_I(\theta)$$

and observe that

$$\begin{aligned}\sum_{I\in\mathcal{I}} a_I\langle b-P_\Omega b\rangle_I h_I(\theta)&=\sum_{I\in\mathcal{I}} a_I\Big(|I|^{-1}\int_I \sum_{\substack{J\in\mathcal{I}:\\ J\supsetneq\Omega}} b_J h_J(\theta')d\theta'\Big)h_I(\theta)\\ &=\sum_{I\in\mathcal{I}} a_I\Big(\sum_{\substack{I\in\mathcal{I}:\\ J\supsetneq\Omega}} b_J h_J(c_\Omega)\Big)h_I(\theta)=\Big(\sum_{\substack{J\in\mathcal{I}:\\ J\supsetneq\Omega}} b_J h_J(c_\Omega)\Big)a(\theta)=\langle b\rangle_\Omega\cdot a(\theta),\end{aligned}$$

where c_Ω denotes the centre of Ω. Hence, the proof of (6.4) is complete.

We may assume that $f=\sum_{k=1}^N \lambda_k a_k$, where each atom a_k has a finite wavelet expansion. By using (6.4), one may write

$$\Pi_1^{\mathcal{I}}(f,b)=\beta_1+\beta_2,$$

where

$$\beta_1(\theta):=\sum_{k=1}^N \lambda_k\Pi_1^{\mathcal{I}}(a_k,P_{\Omega_k}b)(\theta)\quad\text{and}\quad\beta_2(\theta):=\sum_{k=1}^N \lambda_k\langle b\rangle_{\Omega_k}a_k(\theta).$$

For the first term, we have

$$\begin{aligned}\|\Pi_1^{\mathcal{I}}(a_k,P_{\Omega_k}b)\|_{H^1_{\mathcal{I}}(\mathbb{T})}&=\|S_{\mathcal{I}}[\Pi_1^{\mathcal{I}}(a_k,P_{\Omega_k}b)]\|_{L^1(\mathbb{T})}\\ &=\Big\|\Big(\sum_{\substack{I\in\mathcal{I}\\ I\subseteq\Omega_k}}|\langle P_\Omega\rangle_I|^2|(a_k)_I|^2\frac{\chi_I}{|I|}\Big)^{1/2}\Big\|_{L^1(\mathbb{T})}\le\|M(P_{\Omega_k}b)S_{\mathcal{I}}[a_k]\|_{L^1(\mathbb{T})}\\ &\le\|M(P_{\Omega_k}b)\|_{L^2(\mathbb{T})}\|S_{\mathcal{I}}[a_k]\|_{L^2(\mathbb{T})}\lesssim\|P_{\Omega_k}b\|_{L^2(\mathbb{T})}\|a_k\|_{L^2(\mathbb{T})}\\ &\lesssim|\Omega_k|^{1/2}\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}|\Omega_k|^{-1/2}=\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}\end{aligned}$$

for all $k=1,\dots,N$. Hence,

$$\|\beta_1\|_{H^1_{\mathcal{I}}(\mathbb{T})}\le\sum_{k=1}^N|\lambda_k|\|\Pi_1^{\mathcal{I}}(a_k,P_{\Omega_k}b)\|_{H^1_{\mathcal{I}}(\mathbb{T})}\le C\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}\sum_{k=1}^N|\lambda_k|$$

and so, one deduces that

$$\|\beta_1\|_{H^1_{\mathcal{I}}(\mathbb{T})}\lesssim\|f\|_{H^1_{\mathcal{I}}(\mathbb{T})}\|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})}.$$

It remains to treat β_2. The goal is to prove that

$$\|S_{\mathcal{I}}[\beta_2]\|_{L^{\log}(\mathbb{T})} \lesssim \|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \sum_{k=1}^{N} |\lambda_k|, \tag{6.5}$$

where the implied constant is independent of b, f (and N). To this end, observe that

$$\begin{aligned}
S_{\mathcal{I}}[\beta_2](\theta) &\le \sum_{k=1}^{N} |\lambda_k| |\langle b\rangle_{\Omega_k}| S_{\mathcal{I}}[a_k](\theta) \\
&\le \sum_{k=1}^{N} |\lambda_k| |b(\theta) - \langle b\rangle_{\Omega_k}| S_{\mathcal{I}}[a_k](\theta) + |b(\theta)| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k](\theta) \\
&= \sum_{k=1}^{N} |\lambda_k| |P_{\Omega_k} b(s)| S_{\mathcal{I}}[a_k](\theta) + |b(\theta)| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k](\theta).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|S_{\mathcal{I}}[\beta_2]\|_{L^{\log}(\mathbb{T})} &\lesssim \Big\| \sum_{k=1}^{N} |\lambda_k| |P_{\Omega_k} b| S_{\mathcal{I}}[a_k] \Big\|_{L^{\log}(\mathbb{T})} + \Big\| |b| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k] \Big\|_{L^{\log}(\mathbb{T})} \\
&\lesssim \Big\| \sum_{k=1}^{N} |\lambda_k| |P_{\Omega_k} b| S_{\mathcal{I}}[a_k] \Big\|_{L^{1}(\mathbb{T})} + \Big\| |b| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k] \Big\|_{L^{\log}(\mathbb{T})}.
\end{aligned}$$

By arguing as above, it can easily be seen that

$$\Big\| \sum_{k=1}^{N} |\lambda_k| |P_{\Omega_k} b| S_{\mathcal{I}}[a_k] \Big\|_{L^{1}(\mathbb{T})} \lesssim \|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \sum_{k=1}^{N} |\lambda_k|.$$

Therefore, the proof of (6.5) is reduced to showing that

$$\Big\| |b| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k] \Big\|_{L^{\log}(\mathbb{T})} \lesssim \|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \sum_{k=1}^{N} |\lambda_k|. \tag{6.6}$$

To this end, note that by using Proposition 23 one gets

$$\Big\| |b| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k] \Big\|_{L^{\log}(\mathbb{T})} \lesssim \|b\|_{BMO^+_{\mathcal{I}}(\mathbb{T})} \Big\| \sum_{k=1}^{N} |\lambda_k| S_{\mathcal{I}}[a_k] \Big\|_{L^{1}(\mathbb{T})}$$

and since for each $H^1_{\mathcal{I}}$-atom one has $\|S_{\mathcal{I}}[a_k]\|_{L^1(\mathbb{T})} \le 1$, (6.6) follows from the last estimate. This completes the proof of (6.5). □

6.1. Passing from dyadic to non-dyadic decompositions

In this subsection, it is explained how the following theorem can be deduced from the corresponding dyadic case that we studied in the previous section.

Theorem 28. *There exist bounded bilinear operators* $\mathbf{\Pi}_1 : H^1(\mathbb{T}) \times BMO^+(\mathbb{T}) \to H^{\log}(\mathbb{T})$, $\mathbf{\Pi}_2 : H^1(\mathbb{T}) \times BMO^+(\mathbb{T}) \to H^1(\mathbb{T})$, $\mathbf{\Pi}_3 : H^1(\mathbb{T}) \times BMO^+(\mathbb{T}) \to L^1(\mathbb{T})$ *such that for all* $f \in H^1(\mathbb{T})$ *and* $b \in BMO^+(\mathbb{T})$ *one has*

$$f \cdot b = \mathbf{\Pi}_1(f,b) + \mathbf{\Pi}_2(f,b) + \mathbf{\Pi}_3(f,b) \quad \text{in } \mathcal{D}'. \tag{6.7}$$

Note that if $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$, and $\mathbf{\Pi}_3$ are as in Theorem 28, then a periodic version of Theorem 6 is obtained by taking $S := \mathbf{\Pi}_2 + \mathbf{\Pi}_3$ and $T := \mathbf{\Pi}_1$.

6.1.1. Proof of Theorem 28

We shall construct bilinear operators that are initially defined on the product space $H^1_{\mathrm{fin}}(\mathbb{T}) \times BMO^+(\mathbb{T})$. Here $H^1_{\mathrm{fin}}(\mathbb{T})$ denotes the dense subspace of $H^1(\mathbb{T})$ that consists of functions in $L^1(\mathbb{T})$ that can be written as finite linear combinations of L^2-atoms in $H^1(\mathbb{T})$. For $f \in H^1_{\mathrm{fin}}(\mathbb{T})$, if one defines

$$\|f\|_{H^1_{\mathrm{fin}}(\mathbb{T})} := \inf\left\{ \sum_{k=1}^{N} |\lambda_k| : \ f = \sum_{k=1}^{N} \lambda_k a_k, \lambda_n \in \mathbb{C}, \ a_k \text{ is an } L^2\text{-atom in } H^1(\mathbb{T}), \ k = 1, \ldots, N \right\},$$

then

$$\|f\|_{H^1_{\mathrm{fin}}(\mathbb{T})} \sim \|f\|_{H^1(\mathbb{T})}; \tag{6.8}$$

see [19, Theorem 3.1 (i)] for the corresponding Euclidean case. For $\tau \in [0,1)$, $H^1_{\mathrm{fin},\mathcal{I}^\tau}(\mathbb{T})$ is defined in an analogous way.

Remark 29. Let $\tau \in [0,1)$ be given. Since any L^2-atom in $H^1(\mathbb{T})$ can be regarded as a multiple of an L^2-atom in $H^1_{\mathcal{I}^\tau}(\mathbb{T})$, one has

$$H^1_{\mathrm{fin}}(\mathbb{T}) \subset L^2(\mathbb{T}) \subset H^1_{\mathcal{I}^\tau}(\mathbb{T}). \tag{6.9}$$

In addition, one has

$$BMO^+(\mathbb{T}) \subset BMO^+_{\mathcal{I}^\tau}(\mathbb{T}) \subset L^2(\mathbb{T}). \tag{6.10}$$

To simplify the notation, the operators considered in the previous section will be denoted by Π_j^τ (instead of $\Pi_j^{\mathcal{I}^\tau}$), $j\in\{1,2,3\}$.

Observe that, for $j\in\{1,2,3\}$, in view of Remark 29, Π_j^τ is well-defined on $H^1_{\rm fin}(\mathbb{T})\times BMO^+(\mathbb{T})$. In addition, it is well-known that there exists an absolute constant $C>0$ such that for all $\tau\in[0,1)$ and $u,v\in L^2(\mathbb{T})$ one has

$$\|\Pi_j^\tau(u,v)\|_{H^1_{\mathcal{I}^\tau}(\mathbb{T})}\le C\|u\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{T})},\quad j\in\{1,2\}, \tag{6.11}$$

and

$$\|\Pi_3^\tau(u,v)\|_{L^1(\mathbb{T})}\le C\|u\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{T})}. \tag{6.12}$$

For $f\in H^1_{\rm fin}(\mathbb{T})$ and $b\in BMO^+(\mathbb{T})$, we define

$$\mathbf{\Pi}_j(f,b):=\int_{[0,1)}\Pi_j^\tau(f,b)d\tau,\quad j\in\{1,2,3\}.$$

It follows from the definition of $\mathbf{\Pi}_j$, $j\in\{1,2,3\}$, that (6.7) holds on $H^1_{\rm fin}(\mathbb{T})\times BMO^+(\mathbb{T})$.

We shall prove that there exists an absolute constant $C>0$ such that for all $f\in H^1_{\rm fin}(\mathbb{T})$ and $b\in BMO^+(\mathbb{T})$ one has

$$\|\mathbf{\Pi}_1(f,b)\|_{H^{\log}(\mathbb{T})}\le C\|f\|_{H^1(\mathbb{T})}\|b\|_{BMO^+(\mathbb{T})}, \tag{6.13}$$

$$\|\mathbf{\Pi}_2(f,b)\|_{H^1(\mathbb{T})}\le C\|f\|_{H^1(\mathbb{T})}\|b\|_{BMO^+(\mathbb{T})}, \tag{6.14}$$

$$\|\mathbf{\Pi}_3(f,b)\|_{L^1(\mathbb{T})}\le C\|f\|_{H^1(\mathbb{T})}\|b\|_{BMO^+(\mathbb{T})}. \tag{6.15}$$

Fix $f\in H^1_{\rm fin}(\mathbb{T})$ and $b\in BMO^+(\mathbb{T})$. Since

$$\|\mathbf{\Pi}_2(f,b)\|_{H^1(\mathbb{T})}\le\int_{[0,1)}\|\Pi_2^\tau(f,b)\|_{H^1(\mathbb{T})}d\tau,$$

(6.14) follows directly from the well-known fact

$$\|u\|_{H^1}\lesssim\|u\|_{H^1_{\mathcal{I}^\tau}(\mathbb{T})},\quad \tau\in[0,1),\ u\in H^1_{\mathcal{I}^\tau}(\mathbb{T}), \tag{6.16}$$

Proposition 26, and the following theorem of B. Davis [9, Theorem 3.1] (see also [30, (0.6)]):

$$\int_{[0,1)}\|u\|_{H^1_{\mathcal{I}^\tau}(\mathbb{T})}d\tau\lesssim\|u\|_{H^1(\mathbb{T})}. \tag{6.17}$$

One shows (6.15) in an analogous way; as

$$\|\mathbf{\Pi}_3(f,b)\|_{L^1(\mathbb{T})}\le\int_{[0,1)}\|\Pi_3^\tau(f,b)\|_{L^1(\mathbb{T})}d\tau,$$

the desired estimate is a consequence of Proposition 25, (6.16), and (6.17).

The case of $\mathbf{\Pi}_1$ is more complicated, because of the lack of convexity of the unit ball in $H^{\log}(\mathbb{T})$.

To start, let us assume that f is an L^2-atom in $H^1(\mathbb{T})$ supported on some arc $I\subset\mathbb{T}$. For $\tau\in\mathbb{T}$, let $J_\tau(I)$ denote the minimal arc in $\mathcal{I}^\tau$ containing I.

We say that I fits into $\mathcal{I}^\tau$ with constant c, if $|J_\tau(I)|\leq c|I|$. For $r\in\mathbb{N}_0$, let

$$\mathbb{T}_r(I):=\{\tau\in\mathbb{T}: I \text{ fits into } \mathcal{I}^\tau \text{ with constant } 2^r\}.$$

Lemma 30. *One has* $|\mathbb{T}\backslash\mathbb{T}_r(I)|<2^{-r+1}$.

Proof. Let $I\subset\mathbb{T}$ be an arc, and let $N\in\mathbb{Z}$ be such that $2^N\leq|I|<2^{N+1}$. We can assume without loss of generality that $I=\{e^{2\pi it}:0\leq t\leq|I|\}$.

Let

$$\mathcal{I}_r^\tau:=\{J\in\mathcal{I}^\tau:|J|=2^{r+N}\}.$$

We write

$$A_r(I):=\{\tau\in\mathbb{T}: \text{ there exists } J\in\mathcal{I}_r^\tau \text{ with } I\subseteq J\}.$$

Clearly $\mathcal{I}_r^\tau=\mathcal{I}_r^{\tau+2^{r+N}}$ and $A_r(I)\subseteq\mathbb{T}_r(I)$. Noting that $\tau\in A_r(I)$ for $|I|<\tau<2^{r+N}$, it follows that

$$|\mathbb{T}_r(I)|\geq|A_r(I)|\geq\frac{2^{r+N}-|I|}{2^{r+N}}$$

and hence

$$|\mathbb{T}\backslash\mathbb{T}_r(I)|\leq\frac{|I|}{2^{r+N}}<\frac{1}{2^{r-1}}. \quad \square$$

Note that by the definition of $J_\tau(I)$, $\Pi_1^\tau(f,b)=\Pi_1^\tau(f,\chi_{J_\tau(I)}b)$ and hence, $\Pi_1^\tau(f,b)=\Pi_1^\tau(f,\chi_{J_\tau(I)}(b-\langle b\rangle_{J_\tau(I)}))+\Pi_1^\tau(f,\chi_{J_\tau(I)}\langle b\rangle_{J_\tau}(I))$. We may thus write

$$\mathbf{\Pi}_1(f,b)=g_1+g_2,$$

where

$$g_1:=\int_{[0,1)}\Pi_1^\tau(f,\chi_{J_\tau(I)}(b-\langle b\rangle_{J_\tau(I)}))d\tau$$

and

$$g_2:=\int_{[0,1)}\Pi_1^\tau(f,\chi_{J_\tau(I)}\langle b\rangle_{J_\tau(I)})d\tau.$$

By the definition of $J_\tau(I)$, the Haar expansion of f relative to the dyadic grid $\mathcal{I}^\tau$ contains only terms for $J\in\mathcal{I}^\tau$ with $J\subseteq J_\tau(I)$. Hence,

$$\Pi_1^\tau(f,\chi_{J_\tau(I)})=f \text{ for } \tau\in[0,1) \tag{6.18}$$

and so,

$$g_2 = f \int_{[0,1)} \langle b \rangle_{J_\tau(I)} d\tau.$$

Fix a $\tau_0 \in \mathbb{T}_2(I)$ and write J_0 for $J_{\tau_0}(I)$. Note that f is a multiple of an atom associated to J_0 with $\|f\|_{H^1_{\mathcal{I}^{\tau_0}}(\mathbb{T})} \le 2$. Then

$$S_{\mathcal{I}^{\tau_0}}[g_2] = \left| \int_{[0,1)} \langle b \rangle_{J_\tau(I)} d\tau \right| S_{\mathcal{I}^{\tau_0}}[f].$$

Recalling that $|J_0| \le 2|I|$ with $I \subseteq J_0$ and $\operatorname{supp}(S_{\mathcal{I}^{\tau_0}}[f]) \subset J_0$, we obtain the pointwise inequality

$$S_{\mathcal{I}^{\tau_0}}[g_2] \le 2 S_{\mathcal{I}^{\tau_0}}[f]\, M(b).$$

Hence, by using a periodic version of [1, Theorem 4.2] and Proposition 23, we deduce that $S_{\mathcal{I}^{\tau_0}}[g_2] \in L^{\log}(\mathbb{T})$. So, $g_2 \in H^{\log}_{\mathcal{I}^{\tau_0}}(\mathbb{T}) \subset H^{\log}(\mathbb{T})$.

To handle g_1, note that since

$$\bigcup_{r\in\mathbb{N}} (\mathbb{T}_r(I) \setminus \mathbb{T}_{r-1}(I)) = \mathbb{T},$$

by using (6.16), (6.11) for $j=1$, and Lemma 30, we have

$$\begin{aligned}
\|g_1\|_{H^1(\mathbb{T})} &\lesssim \sum_{r=1}^{\infty} \int_{\mathbb{T}_r(I)\setminus\mathbb{T}_{r-1}(I)} \|\Pi_1^\tau(f, \chi_{J_\tau(I)}(b - \langle b \rangle_{J_\tau(I)})\|_{H^1_{\mathcal{I}^{\tau_0}}(\mathbb{T})} d\tau \\
&\lesssim \sum_{r=1}^{\infty} \int_{\mathbb{T}_r(I)\setminus\mathbb{T}_{r-1}(I)} \|f\|_{L^2(\mathbb{T})} \|\chi_{J_\tau(I)}(b - \langle b \rangle_{J_\tau(I)})\|_{L^2(\mathbb{T})} d\tau \\
&\le \|b\|_{BMO^+(\mathbb{T})} |I|^{-1/2} \sum_{r=1}^{\infty} \int_{\mathbb{T}_r(I)\setminus\mathbb{T}_{r-1}(I)} |J_\tau(I)|^{1/2} d\tau \\
&\le \|b\|_{BMO^+(\mathbb{T})} |I|^{-1/2} \sum_{r=1}^{\infty} 2^{-(r-2)} 2^{r/2} |I|^{1/2} \\
&\lesssim \|b\|_{BMO^+(\mathbb{T})}.
\end{aligned}$$

We now move to the special case where $f \in H^1_{\mathrm{fin}, \mathcal{I}^{\tau_0}}(\mathbb{T})$ for some $\tau_0 \in [0,1)$, that is, one may write

$$f = \sum_{k=1}^{N} \lambda_k a_{I_k},$$

where $\{\lambda_k\}_{k=1}^N$ is a finite collection of complex numbers and $\{a_{I_k}\}_{k=1}^N$ is a finite collection of L^2-atoms in $H^1_{\mathcal{I}^{\tau_0}}$ such that each a_{I_k} is supported in some $I_k \in \mathcal{I}^{\tau_0}$.

With a corresponding decomposition and the notation $J_\tau(I_k)$ for the minimal arc in $\mathcal{I}^\tau$ containing I_k, we obtain

$$\mathbf{\Pi}_1(f,b)=\tilde{g}_1+\tilde{g}_2,$$

where

$$\tilde{g}_1:=\sum_{k=1}^N \lambda_k \int_{[0,1)} \Pi_1^\tau(a_{I_k},\chi_{J_\tau(I_k)}(b-\langle b\rangle_{J_\tau(I_k)}))d\tau$$

and

$$\tilde{g}_2:=\sum_{k=1}^N \lambda_k \int_{[0,1)} \Pi_1^\tau(a_{I_k},\chi_{J_\tau(I_k)}\langle b\rangle_{J_\tau(I_k)})d\tau.$$

The desired H^1-estimate for $\tilde{g}_1$ follows directly from the corresponding estimate for single atoms. We are left to estimate $\tilde{g}_2$. By (6.18),

$$\tilde{g}_2=\sum_{k=1}^N \lambda_k \int_{[0,1)} \Pi_1^\tau(a_{I_k},\chi_{J_\tau(I_k)}\langle b\rangle_{J_\tau(I_k)})d\tau=\sum_{k=1}^N \lambda_k \; a_{I_k}\int_{[0,1)}\langle b\rangle_{J_\tau(I_k)}d\tau$$

and, using that $S_{\mathcal{I}^{\tau_0}}[a_{I_k}]$ is supported on I_k for each $k\in\{1,\dots,N\}$, we obtain

$$S_{\mathcal{I}^{\tau_0}}[\tilde{g}_2]\le\sum_{k=1}^N|\lambda_k|S_{\mathcal{I}^{\tau_0}}[a_{I_k}]\,M(b).$$

Hence, a periodic version of [1, Theorem 4.2] combined with Proposition 23 yields

$$\|\tilde{g}_2\|_{H^{\log}_{\mathcal{I}^{\tau_0}}(\mathbb{T})}\lesssim\|b\|_{BMO^+(\mathbb{T})}\sum_{k=1}^N|\lambda_k|.$$

We have thus shown that

$$\|\mathbf{\Pi}_1(f,b)\|_{H^{\log}(\mathbb{T})}\lesssim\|f\|_{H^1_{\mathrm{fin},\mathcal{I}^{\tau_0}}(\mathbb{T})}\|b\|_{BMO^+(\mathbb{T})} \tag{6.19}$$

for all $f\in H^1_{\mathrm{fin},\mathcal{I}^{\tau_0}}(\mathbb{T})$ and $b\in BMO^+(\mathbb{T})$.

For general $f\in H^1_{\mathrm{fin}}(\mathbb{T})$, following the proof of [20, Corollary 2.4], write

$$f=f_1+f_2$$

with $f_1\in H^1_{\mathrm{fin},\mathcal{I}^0}(\mathbb{T})$, $f_2\in H^1_{\mathrm{fin},\mathcal{I}^{1/3}}(\mathbb{T})$, and

$$\|f_1\|_{H^1_{\mathrm{fin},\mathcal{I}^0}(\mathbb{T})}+\|f_2\|_{H^1_{\mathrm{fin},\mathcal{I}^{1/3}}(\mathbb{T})}\sim\|f\|_{H^1_{\mathrm{fin}}(\mathbb{T})}. \tag{6.20}$$

Hence, by using (5.1), Theorem 19, (6.19), (6.20), and (6.8), we obtain

$$\|\mathbf{\Pi}_1(f,b)\|_{H^{\log}(\mathbb{T})}\lesssim\|\mathbf{\Pi}_1(f_1,b)\|_{H^{\log}(\mathbb{T})}+\|\mathbf{\Pi}_1(f_2,b)\|_{H^{\log}(\mathbb{T})}$$

$$
\begin{aligned}
&\sim \|\mathbf{\Pi}_1(f_1,b)\|_{H^{\log}_{\text{at}}(\mathbb{T})}+\|\mathbf{\Pi}_1(f_2,b)\|_{H^{\log}_{\text{at}}(\mathbb{T})}\\
&\lesssim \|\mathbf{\Pi}_1(f_1,b)\|_{H^{\log}_{\text{at},\mathcal{I}^0}(\mathbb{T})}+\|\mathbf{\Pi}_1(f_2,b)\|_{H^{\log}_{\text{at},\mathcal{I}^{1/3}}(\mathbb{T})}\\
&\sim \|\mathbf{\Pi}_1(f_1,b)\|_{H^{\log}_{\mathcal{I}^0}(\mathbb{T})}+\|\mathbf{\Pi}_1(f_2,b)\|_{H^{\log}_{\mathcal{I}^{1/3}}(\mathbb{T})}\\
&\lesssim \left(\|f_1\|_{H^1_{\mathcal{I}^0}(\mathbb{T})}+\|f_2\|_{H^1_{\mathcal{I}^{1/3}}(\mathbb{T})}\right)\|b\|_{BMO^+(\mathbb{T})}\\
&\lesssim \|f\|_{H^1(\mathbb{T})}\|b\|_{BMO^+(\mathbb{T})},
\end{aligned}
$$

as desired. The general case follows now from a standard approximation argument.

7. A variant of an inequality of Hardy and Littlewood for $H^{\log}(\mathbb{T})$

A classical result due to Hardy and Littlewood asserts that for every $p\in(0,1]$ there exists a constant $C_p>0$ such that

$$
\left(\sum_{n=1}^{\infty}\frac{|f_n|^p}{|n|^{2-p}}\right)^{1/p}\leq C_p\left(\sup_{0\leq r<1}\int_{[0,1)}|F(re^{i2\pi\theta})|^p d\theta\right)^{1/p} \tag{7.1}
$$

for all analytic functions $F(z)=\sum_{n=0}^{\infty}f_n z^n$ in the Hardy space $H^p(\mathbb{D})$ on the unit disc $\mathbb{D}$; [11, Theorem 16]. It follows from (7.1) that for every $p\in(0,1]$ there exists a constant $B_p>0$

$$
\left(\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{|\hat{f}(n)|^p}{|n|^{2-p}}\right)^{1/p}\leq B_p\,\|f\|_{H^p(\mathbb{T})}\,. \tag{7.2}
$$

Since $H^{\log}(\mathbb{T})\subset H^p(\mathbb{T})$, $p\in(0,1)$ (see Remark 10), one deduces from (7.2) that $\{|n|^{p-2}\hat{f}(n)|^p\}_{n\in\mathbb{Z}\setminus\{0\}}$ is summable for any $f\in H^{\log}(\mathbb{T})$ for all $p\in(0,1)$. The next theorem establishes a more accurate description of the behaviour of the Fourier coefficients of distributions in $H^{\log}(\mathbb{T})$.

Theorem 31. *There exists a constant $C>0$ such that*

$$
\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\Psi_0(|n\hat{f}(n)|)}{n^2}\leq C\int_{[0,1)}\Psi_0(f^*(\theta))d\theta.
$$

Proof. We shall prove that there exists an absolute constant $C_0>0$ such that

$$
\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\Psi_0(|n\widehat{a_I}(n)|)}{n^2}\leq C_0|I|\Psi_0(\|a_I\|_{L^\infty(\mathbb{T})}) \tag{7.3}
$$

for any L^∞-function a_I supported in some arc I in $\mathbb{T}$ with $\int_I a_I(\theta)d\theta=0$.

To this end, we fix such a function a_I (and an arc I) and write

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\Psi_0(|n\widehat{a_I}(n)|)}{n^2} = A+B,$$

where

$$A := \sum_{1\leq|n|\leq|I|^{-1}} \frac{\Psi_0(|n\widehat{a_I}(n)|)}{n^2} \quad \text{and} \quad B := \sum_{|n|>|I|^{-1}} \frac{\Psi_0(|n\widehat{a_I}(n)|)}{n^2}.$$

We shall prove that

$$A \lesssim |I|\Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big) \tag{7.4}$$

and

$$B \lesssim |I|\Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big). \tag{7.5}$$

To prove (7.4), by using the cancellation of a_I and the fact that $|e^{-i2\pi nx}-e^{-i2\pi ny}|\leq 2\pi|n||x-y|\leq 2\pi|n||I|$ for all $n\in\mathbb{Z}$ and $x,y\in I$, one has

$$|\widehat{a_I}(n)| \leq 2\pi|n||I|^2\|a_I\|_{L^\infty(\mathbb{T})} \quad \text{for all } n\in\mathbb{Z}. \tag{7.6}$$

Since Ψ_0 is increasing and of lower type $2/3$ i.e. there exists an absolute constant $A_0>0$ such that $\Psi_0(st)\leq A_0 s^{2/3}\Psi_0(t)$ for all $t>0$ and $s\in(0,1)$; see [35, Example 1.1.5 (i)], it follows from (7.6) that

$$\begin{aligned} A \leq \sum_{1\leq|n|\leq|I|^{-1}} \frac{\Psi_0\big(2\pi n^2|I|^2\|a_I\|_{L^\infty(\mathbb{T})}\big)}{n^2} &\lesssim \sum_{1\leq n\leq|I|^{-1}} (|I|^2n^2)^{2/3}\frac{\Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big)}{n^2} \\ &= |I|^{4/3}\Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big) \sum_{1\leq n\leq|I|^{-1}} n^{-2/3} \\ &\lesssim |I|\Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big). \end{aligned}$$

Hence, (7.4) holds. To establish (7.5), note that by using Hölder's inequality for $p=4$ and $p'=4/3$ and Parseval's identity, one obtains $B\leq B_1\cdot B_2$, where

$$B_1 := \|a_I\|^{1/2}_{L^2(\mathbb{T})} \leq |I|^{1/4}\|a_I\|^{1/2}_{L^\infty(\mathbb{T})} \quad \text{and} \quad B_2 := \Bigg(\sum_{|n|>|I|^{-1}} \frac{\widetilde{\Psi}_0\big(|n\widehat{a_I}(n)|\big)}{n^2}\Bigg)^{3/4}$$

with $\widetilde{\Psi}_0(t) := t^{2/3}\cdot[\log(e+t)]^{-4/3}$, $t\geq 0$. Since $|\widehat{a_I}(n)|\leq|I|\|a_I\|_{L^\infty(\mathbb{T})}$ and $\widetilde{\Psi}_0$ is increasing on $[0,\infty)$, we have

$$B_2 \leq \Bigg(\sum_{|n|>|I|^{-1}} \frac{\widetilde{\Psi}_0\big(|n||I|\|a_I\|_{L^\infty(\mathbb{T})}\big)}{n^2}\Bigg)^{3/4}$$

$$\leq |I|^{1/2} \frac{\|a_I\|_{L^\infty(\mathbb{T})}^{1/2}}{\log\big(e+\|a_I\|_{L^\infty(\mathbb{T})}\big)} \Bigg(\sum_{|n|>|I|^{-1}} n^{-4/3} \Bigg)^{3/4}$$
$$\lesssim |I|^{3/4} \frac{\|a_I\|_{L^\infty(\mathbb{T})}^{1/2}}{\log\big(e+\|a_I\|_{L^\infty(\mathbb{T})}\big)}$$

and so, (7.5) holds as

$$B \leq B_1 \cdot B_2 \lesssim |I| \Psi_0\big(\|a_I\|_{L^\infty(\mathbb{T})}\big).$$

Therefore, in view of (7.4) and (7.5), (7.3) holds.

To complete the proof of the theorem, take an $f \in H^{\log}(\mathbb{T})$ and note that there exists a sequence $\{b_{I_k}\}_{k\in\mathbb{N}}$ of multiples of atoms in $H^{\log}(\mathbb{T})$, supported in arcs I_k, such that

$$f - \hat{f}(0) = \sum_{k\in\mathbb{N}} b_{I_k} \quad \text{in } \mathcal{D}'$$

and

$$\sum_{k\in\mathbb{N}} |I_k| \Psi_0\big(\|b_{I_k}\|_{L^\infty(\mathbb{T})}\big) \leq M_0 \int_{[0,1)} \Psi_0(f^*(\theta))d\theta,$$

where $M_0>0$ is an absolute constant. Hence, by using (5.4) and (7.3) we get

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\Psi_0(|n\hat{f}(n)|)}{n^2} \leq \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\Psi_0(|n\widehat{b_{I_k}}(n)|)}{n^2}$$
$$\lesssim \sum_{k\in\mathbb{N}} |I_k| \Psi_0\big(\|b_{I_k}\|_{L^\infty(\mathbb{T})}\big) \lesssim \int_{[0,1)} \Psi_0(f^*(\theta))d\theta$$

and this completes the proof of our theorem. □

Observe that if $\{a_n\}_{n\in\mathbb{Z}}$ is a collection of complex numbers with at most polynomial growth at infinity, then

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|a_n|}{(|n|+1)\log(e+|n|)} \lesssim \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\Psi_0(|na_n|)}{n^2}.$$

We thus deduce from Theorem 31 that $\{[(|n|+1)\log(e+|n|)]^{-1}\}_{n\in\mathbb{Z}}$ is a Fourier multiplier from $H^{\log}(\mathbb{T})$ to $\ell^1(\mathbb{Z})$, that is, the following result holds true.

Corollary 32. *For any $f \in H^{\log}(\mathbb{T})$ one has*

$$\sum_{n\in\mathbb{Z}} \frac{|\hat{f}(n)|}{(|n|+1)\log(e+|n|)} \leq C\|f\|_{H^{\log}(\mathbb{T})}.$$

Theorem 31 can be used to exhibit distributions in $H^p(\mathbb{T})\setminus H^{\log}(\mathbb{T})$ for $p\in(0,1)$. For instance, it follows from Theorem 31 that the Dirac distribution δ_0 does not belong to $H^{\log}(\mathbb{T})$.

Furthermore, Theorem 31 is sharp in the following sense: if $\widetilde{\Psi}:[0,\infty)\to[0,\infty)$ is any increasing function with $\lim_{t\to\infty}\widetilde{\Psi}(t)/\Psi_0(t)=\infty$, then there is no constant $C>0$ such that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\widetilde{\Psi}(|n\hat{f}(n)|)}{n^2} \le C\int_{[0,1)} \Psi_0(f^*(\theta))d\theta \tag{7.7}$$

for all $f\in H^{\log}(\mathbb{T})$. Indeed, take a function $\widetilde{\Psi}$ as above and suppose that (7.7) holds true. Let N be a large positive integer that will eventually be sent to infinity. Consider the function

$$a_N(\theta) := N2^N e^{i2\pi 2^N\theta}\chi_{[0,2^{-N})}(\theta), \quad \theta\in[0,1).$$

One can easily check that

$$\|a_N\|_{H^{\log}(\mathbb{T})} \lesssim 1, \tag{7.8}$$

where the implied constant is independent of N.

Consider the interval $I_N:=[2^{N-2},2^{N-1})$ and observe that there exists an absolute constant $c_0>0$ such that for every natural number n in I_N one has

$$|\widehat{a_N}(n)| = N2^N \frac{\left|e^{-i2\pi(2^{-N}n-1)}-1\right|}{2\pi|n-2^N|} = N2^N \frac{\left|\sin\left[\pi\left(n2^{-N}-1\right)\right]\right|}{\pi(2^N-n)} \ge c_0 N,$$

where we used the identity $|e^{is}-e^{it}|=2|\sin[(s-t)/2]|$ for $s=-2\pi(2^{-N}n-1)$ and $t=0$ as well as the fact that for $n\in I_N$ one has $2^N-n\sim 2^N$ and $|\sin[\pi(2^{-N}n-1)]|\sim 1$.

Hence, (7.7) and (7.8) imply that

$$\begin{aligned} 1 &\gtrsim \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\widetilde{\Psi}(|n\widehat{a_N}(n)|)}{n^2} \ge \sum_{n=2^{N-2}}^{2^{N-1}} \frac{\widetilde{\Psi}(|n\widehat{a_N}(n)|)}{n^2} \ge \widetilde{\Psi}(c_0 2^{N-2}N) \sum_{n=2^{N-2}}^{2^{N-1}} n^{-2} \\ &\sim \frac{\widetilde{\Psi}(c_0 2^{N-2}N)}{2^N} = \frac{\Psi_0(c_0 2^{N-2}N)}{2^N}\cdot\frac{\widetilde{\Psi}(c_0 2^{N-2}N)}{\Psi_0(c_0 2^{N-2}N)} \sim \frac{\widetilde{\Psi}(c_0 2^{N-2}N)}{\Psi_0(c_0 2^{N-2}N)}, \end{aligned}$$

which yields a contradiction by taking $N\in\mathbb{N}$ 'large enough'.

Remark 33. As a consequence of sharpness of Theorem 31 discussed above, one deduces that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\hat{f}(n)|}{|n|\left[\log\left(e+|n\hat{f}(n)|\right)\right]^s} \lesssim \int_{[0,1)} \Psi_0(f^*(\theta))d\theta$$

is false when $0<s<1$.

It follows from Theorem 31 and [3, (8.3)] that there exists a constant $C>0$ such that

$$\sum_{n=1}^{\infty} \frac{\Psi_0\big(n|f_n|\big)}{n^2} \leq C \sup_{0\leq r<1} \int_{[0,1)} \Psi_0\big(|F(re^{i2\pi\theta})|\big)d\theta \tag{7.9}$$

for all analytic functions $F(z)=\sum_{n=0}^{\infty} f_n z^n$ in the unit disc $\mathbb{D}$ for which the quantity on the right-hand side of (7.9) is finite.

We remark that variants of (7.1) and (7.2) for certain classes of Hardy-Orlicz spaces have been obtained in [24] and [34] (see also [15], [25], [32]), which do not include the case of $H^{\log}(\mathbb{T})$ treated above. Moreover, our methods are completely different from those in the aforementioned references.

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