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Removability of product sets for Sobolev functions in the plane

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Abstract. We study conditions on closed sets $C, F \subset \mathbb{R}$ making the product $C \times F$ removable or non-removable for $W^{1,p}$. The main results show that the Hausdorff-dimension of the smaller dimensional component C determines a critical exponent above which the product is removable for some positive measure sets F, but below which the product is not removable for another collection of positive measure totally disconnected sets F. Moreover, if the set C is Ahlfors-regular, the above removability holds for any totally disconnected F.

1. Introduction

In this paper we study the Sobolev-removability of closed subsets of the Euclidean plane. The Sobolev space $W^{1,p}(\Omega)$, for $1 \leq p \leq \infty$ and a domain $\Omega \subset \mathbb{R}^2$, consists of $f \in L^p(\Omega)$ for which the weak first order partial derivatives $\partial_i f$ are also in $L^p(\Omega)$. A subset $E \subset \mathbb{R}^2$ of Lebesgue measure zero is called removable for $W^{1,p}$, or simply p-removable, for $1 \leq p \leq \infty$, if $W^{1,p}(\mathbb{R}^2 \setminus E) = W^{1,p}(\mathbb{R}^2)$ as sets. Since E has Lebesgue measure zero, E is removable for $W^{1,p}$ if and only if every $u \in W^{1,p}(\mathbb{R}^2 \setminus E)$ has an L^p-representative that is absolutely continuous on almost every line-segment parallel to the coordinate axis.

Let us make some observations on p-removable sets. By Hölder's inequality, p-removable sets are also q-removable for every $q > p$. In particular, each p-removable set is ∞-removable. Hence, the complement of a p-removable set is always quasi-convex meaning that any two points in the complement can be joined by a curve in the complement whose length is comparable to the distance between the points. Since the sets E we consider have Lebesgue measure zero, the quasi-convexity of the

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complement implies that closed p-removable sets are actually metrically removable, see [11, Proposition 3.3].

Ahlfors and Beurling [1] studied removable sets for analytic functions with finite Dirichlet integral (see also the work of Carleson [4]). This class of sets coincides with planar 2-removable sets. Consequently, a lot of work was done on removable sets for quasiconformal maps that are globally homeomorphisms, see for instance [3], [7], [9], [10], [14], [15], [26] and [27]. Let us point out that Sobolev-removability has also been considered for globally continuous functions; see for example [21] and references therein. The version of Sobolev-removability we consider here can be characterized via condenser capacities or extremal distances [1], [2], [8], [23], [25], [27] and [29]. However, these conditions are not easy to check. Because of this, Koskela [17] and Wu [28] considered Sobolev-removability in terms of different kinds of porosities that are easier to verify. Removability of porous sets for weighted Sobolev spaces [6] and (weighted) Orlicz-Sobolev spaces [12], [13] has also been studied. Generalizations of the removability results in the spirit of Ahlfors and Beurling have been done for weighted Sobolev spaces, see for example [5].

The sets E whose p-removability we consider here are of the form $E=C\times F$ where $C,F\subset\mathbb{R}$ are closed. If C or F contains an interval of positive length, it is easy to see that the set E is not $W^{1,p}$-removable for any $1\leq p\leq\infty$. Therefore, we may assume that both C and F are totally disconnected. Now, on one hand, if both C and F have zero Lebesgue measure, the set E is automatically $W^{1,p}$-removable since almost every line segment parallel to a coordinate axis has empty intersection with E. On the other hand, if C and F both have positive Lebesgue measure, the set E has positive Lebesgue measure, hence cannot be removable. We have reduced our study to the following.

Problem 1.1. Let $1\leq p\leq\infty$ and $C,F\subset\mathbb{R}$ be totally disconnected closed subsets with C having zero Lebesgue measure and F positive Lebesgue measure. Under what conditions on C and F is the set $C\times F$ removable for $W^{1,p}$?

Examples of p-removable and non-removable product sets of the type considered in Problem 1.1 have appeared in [17], [28, Example 2], and [19, Lemma 4.4]. In the case $p=1$, the answer to Problem 1.1 is easy: Every set $C\times F$ with F positive Lebesgue measure and $C\neq\varnothing$ is non-removable for $W^{1,1}$ by the isoperimetric inequality. For $1<p<2$ partial answers were given by Koskela [17]. In this paper we generalize his results to give partial answers to Problem 1.1 for the range $2<p<\infty$.

Koskela considered the case $C=\{0\}$ and observed in [17, Theorem 2.2] that $\{0\}\times F$ is not p-removable for $1\leq p\leq 2$, when $\mathcal{H}^1(F)>0$ and $F=[0,1]\setminus\bigcup_{i=1}^\infty I_i$ with I_i pairwise disjoint open intervals with $\sum_{i=1}^\infty |I_i|^{2-p}<\infty$. The generalization of this result is done in Theorem 1.2 below. In the other direction, Koskela proved in [17, Theorem 2.3] that $\{0\}\times F$ is p-removable for $1<p<2$, if for almost every $x\in F$ there exist a sequence of numbers $r_i\searrow 0$ and a constant c so that $(x-r_ix,+r_i)\setminus F$ contains an interval of length $cr_i^{1/(2-p)}$. We generalize this result in Theorem 1.3.

In [17] and [28] different porosity parameters of sets determined the p-removability. In our results the porosity type conditions have only a secondary role and the main parameter is the Hausdorff dimension of the set C. Some ideas of the proofs we present here were present in [19, Lemma 4.4], where also the dimension of C was seen to affect the p-removability.

Our main results (Theorem 1.2 and Theorem 1.3) connect the Hausdorff dimension of C with the p-removability of $C\times F$ in the following way. They roughly say that $C\times F$ is not p-removable for some F, but is q-removable for other F when

$$p<\frac{2-\dim(C)}{1-\dim(C)}<q.$$

Thus, the dimension of C is sharp for the transition between non-removable and removable examples. However, we emphasize that in our results the positive measure set F needs to be thick (Theorem 1.2) or thin (Theorem 1.3) enough.

The threshold $(2-\dim(C))/(1-\dim(C))$ has been observed already earlier in similar contexts. For instance in [18, Theorem 3.1] in connection to the validity of $(1,p)$-Poincaré inequality in the complement of a Cantor diamond construction. This is very much related to the question of removability since by Koskela [17], the removability of a measure zero set is equivalent to the validity of a Poincaré inequality for the complement.

Theorem 1.2. *Let $2\le p<\infty$ and $s>\frac{p-2}{p-1}$. Then for any closed subset $C\subset\mathbb{R}$ with $\mathcal{H}^s(C)>0$ and any set F of the form $F=[0,1]\setminus\bigcup_{j=1}^\infty I_j$, where I_j are open intervals satisfying*

$$\sum_{j=1}^{\infty}|I_j|^{1-(1-s)(p-1)}<\infty, \tag{1}$$

and $\mathcal{H}^1(F)>0$, the set $C\times F$ is not p-removable. Moreover, the set $C\times F$ is not removable even for $W^{1,p}$-functions that are continuous on the whole plane.

We do not know what are the sets C in Theorem 1.2 for which $C\times F$ is not p-removable for every closed set $F\subset\mathbb{R}$ of positive Lebesgue measure. On one hand, if C is a singleton, the p-removability depends on F by the results of Koskela [17], as we already discussed above. On the other hand, in Section 4 we show that if C is Ahlfors s-regular with $0<s<1$, then the p-removability is independent of F.

Theorem 1.3. *Let* $2 \le p < \infty$ *and* $s < \frac{p-2}{p-1}$. *Suppose that* $C \subset \mathbb{R}$ *is a closed set with* $\mathcal{H}^s(C) < \infty$ *and that* $F \subset \mathbb{R}$ *is a closed set for which at* $\mathcal{H}^1$*-almost every point* $y \in F$ *there exists* $r_y > 0$ *and* $c_y > 0$ *so that for any* $0 < r < r_y$ *we have*

$$\mathcal{H}^1([y-r, y+r] \setminus F) \ge c_y r^{(1-s)(p-1)}. \tag{2}$$

Then the set $C \times F$ *is p-removable.*

Notice that $(1-s)(p-1) > 1$ in Theorem 1.3 with the choices of p and s. Thus, there exist closed sets $F \subset \mathbb{R}$ of positive Lebesgue measure that satisfy (2) at every point $y \in F$. One might wonder why in Theorem 1.3 we require (2) for all small scales r instead of a sequence of scales as in [17, Theorem A]. One reason for our stricter requirement is that in our proof we argue using a sequence of dyadic scales. Even if this could be avoided, the fact that we assume the Hausdorff measure of C to be finite would force us to work on many scales at once. Replacing the Hausdorff measure assumption by box counting dimension assumption might yield the analogous result with a weaker requirement on F.

The proof of Theorem 1.2 is inspired by the proof of [19, Lemma 4.4] by the second named author together with Koskela and Zhang. In [19, Lemma 4.4], the non-removability was proven for a more regular set C, while the removability was done via a curve condition to which we return in Section 4. The proof of Theorem 1.2 is done in Section 2 while Theorem 1.3 is proven in Section 3. In the final Section 4 we study the relations between p-removability, curve conditions, and Ahlfors regularity and lower porosity of C. In particular, we show that for Ahlfors-regular C the p-removability of $C \times F$ is independent of F. The non-removability of $C \times F$ for Ahlfors-regular C might still depend on F.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. The proof is similar to the proof of [19, Lemma 4.4], where a standard Cantor staircase function was extended by hand from horizontal lines passing through F to the whole set $\mathbb{R}^2 \setminus (C \times F)$. This was possible because of the regularity of the Cantor set C that was used. In the proof of Theorem 1.2 we give a more general construction of a suitable Cantor staircase function via Frostman's Lemma, and an extension of the staircase function via averages.

Up to taking a subset of C, we can assume that $0 < \mathcal{H}^s(C) < \infty$ and that C is compact (say, $C \subset [0,1]$). For $R > 1$, we will construct a function $u \in W^{1,p}((-R,R)^2 \setminus (C \times F))$ which is not absolutely continuous on any segment $(-R,R) \times \{y\}$ for $y \in F$. It follows that u cannot be in $W^{1,p}((-R,R)^2)$.

By Frostman's Lemma (see for instance [20, Theorem 8.8]), there exists a Borel probability measure μ concentrated on C satisfying

$$\mu(B(x,r)) \le c_F r^s \tag{3}$$

for some constant $c_F > 0$.

We define on $[0,1]$ the non-decreasing function $f(x)=\mu([0,x])$ and we extend it to the interval $(-R,R)$ by letting $f=0$ on $(-R,0)$ and $f=1$ on $(1,R)$. Observe that f is not absolutely continuous, since it is constant $\mathcal{H}^1$-a.e. on $[0,1]$, but $f(1)-f(0)=1$. However, since $s>0$, the function f is Hölder-continuous.

We extend the function f to $(-R,R)\times[0,+\infty)$ by letting

$$v(x,y) = \frac{1}{2y}\int_{x-y}^{x+y} f(t)\,dt. \tag{4}$$

The function v is also Hölder-continuous by the Hölder-continuity of f.

Lemma 2.1. *The extension v defined in (4) is differentiable on $(-R,R)\times(0,+\infty)$, and*

$$\nabla v(x,y) = \frac{1}{2y}\left(f(x+y)-f(x-y), f(x+y)+f(x-y)-2v(x,y)\right).$$

In particular,

$$|\nabla v(x,y)| \le \frac{f(x+y)-f(x-y)}{\sqrt{2}y} = \frac{\mu((x-y,x+y))}{\sqrt{2}y}.$$

Proof. By the Leibniz integral rule we have

$$\frac{dv}{dx}(x,y) = \frac{1}{2y}\frac{d}{dx}\int_{x-y}^{x+y} f(t)\,dt = \frac{f(x+y)-f(x-y)}{2y}$$

and

$$\begin{aligned}\frac{dv}{dy}(x,y) &= \frac{-1}{2y^2}\int_{x-y}^{x+y} f(t)\,dt + \frac{1}{2y}\frac{d}{dy}\int_{x-y}^{x+y} f(t)\,dt \\ &= \frac{1}{2y}\left(-2v(x,y)+f(x+y)+f(x-y)\right).\end{aligned}$$

The final estimate comes from the fact that f is non-decreasing, which implies

$$v(x,y) = \frac{1}{2y}\int_{x-y}^{x+y} f(t)\,dt \ge f(x-y). \quad \square$$

It will be useful to estimate the integral of $|\nabla v|$ on a rectangle $(-R,R)\times(0,r)$. For $0<y<r$, let $\{(x_i-r_i,x_i+r_i)\}_i$ be a finite cover of C with disjoint intervals of radii $r_i<y$. Then we have

$$\begin{aligned}\int_{-R}^{R}\mu((x-y,x+y))\,dx &= \int_{-R}^{R}\sum_i \mu((x-y,x+y)\cap(x_i-r_i,x_i+r_i))\,dx\\ &\leq \sum_i \int_{x_i-2y}^{x_i+2y} \mu((x-y,x+y)\cap(x_i-r_i,x_i+r_i))\,dx\\ &\leq 4y\sum_i \mu((x_i-r_i,x_i+r_i))=4y,\end{aligned}$$

where we used that μ is a probability measure on C. Combining this with Lemma 2.1 and (3) yields

$$\begin{aligned}\int_0^r\int_{-R}^{R}|\nabla v|^p\,dx\,dy &\leq 2^{-\frac{p}{2}}\int_0^r\int_{-R}^{R}\frac{\mu((x-y,x+y))^p}{y^p}\,dx\,dy\\ &\leq 2^{-\frac{p}{2}}c_F\int_0^r\frac{y^{s(p-1)}}{y^p}\int_{-R}^{R}\mu((x-y,x+y))\,dx\,dy\\ &\leq 2^{2-\frac{p}{2}}c_F\int_0^r y^{(s-1)(p-1)}\,dy.\end{aligned}\tag{5}$$

We now define the function u as $u(x,y)=v(x,\operatorname{dist}(y,F))$. As the composition of a Lipschitz-continuous mapping $(x,y)\mapsto(x,\operatorname{dist}(y,F))$ and a Hölder-continuous function v, the function u is also Hölder-continuous. Observe that $u(x,y)=f(x)$ for every $y\in F$, so u is not absolutely continuous on every segment $(-R,R)\times\{y\}$, $y\in F$.

Since by hypothesis $(s-1)(p-1)>-1$, making use of (5), for each interval I_j in the complement of F we have

$$\int_{I_j}\int_{-R}^{R}|\nabla u|^p\,dx\,dy=2\int_0^{|I_j|/2}\int_{-R}^{R}|\nabla v|^p\,dx\,dy\leq c(p,s)c_F\,|I_j|^{1-(1-s)(p-1)},$$

where $c(p,s)=2^{1+\frac{p}{2}-s(p-1)}$. By summing over j and using (1) we obtain that

$$u\in W^{1,p}((-R,R)^2\setminus(C\times F)),$$

as wanted.

3. Proof of Theorem 1.3

Let $u \in W^{1,p}(\mathbb{R}^2 \setminus E)$. We aim at showing that $u \in W^{1,p}(\mathbb{R}^2)$, which holds exactly when u has an L^p-representative that is ACL in $\mathbb{R}^2$. Without changing the notation, let u be the continuous ACL representative of u in $\mathbb{R}^2 \setminus E$. Since $\mathcal{H}^1(C)=0$, u is absolutely continuous on almost every vertical line-segment in $\mathbb{R}^2$. Hence, we only need to verify that u is absolutely continuous on almost every horizontal line-segment.

Let us write $\alpha=(1-s)(p-1)$. Let $y \in F$ be such that there exist $c_y>0$ and $r_y>0$ so that for any $0<r<r_y$ we have

$$\mathcal{H}^1((y-r, y+r) \setminus F) \geq c_y r^\alpha. \tag{6}$$

By assumption, such constants exist for almost every y. Let us abbreviate $f(x)=u(x,y)$. It remains to show that f is absolutely continuous.

Let $0<\delta<r_y$. Recalling that $\mathcal{H}^s(C)<\infty$, we take a collection of open intervals $\{J_i\}_{i=1}^n$ such that $C \subset \bigcup_{i=1}^n J_i$, $|J_i|<\delta$ for all i and

$$\sum_{i=1}^n |J_i|^s \leq 2\mathcal{H}^s(C). \tag{7}$$

Without loss of generality, we may assume that no point in $\mathbb{R}$ is contained in more than two different intervals J_i. Define for every i the open square

$$Q_i = J_i \times \left(y-\frac{1}{2}|J_i|, y+\frac{1}{2}|J_i|\right).$$

Lemma 3.1. *For every i we have the inequality*

$$\left|\inf_{x\in J_i} f(x) - \sup_{x\in J_i} f(x)\right| \leq c(s,p,y)\, |J_i|^{\frac{s}{q}}\, \|\nabla u\|_{L^p(Q_i)}, \tag{8}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $c(s,p,y)>0$ is a constant depending only on s, p, and y.

Assuming for the moment Lemma 3.1, we conclude the proof as follows. By Hölder's inequality, (7), and (8) we obtain

$$\begin{aligned}
\sum_{i=1}^n \left|\inf_{x\in J_i} f(x) - \sup_{x\in J_i} f(x)\right| &\leq \left(\sum_{i=1}^n |J_i|^s\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |J_i|^{-\frac{sp}{q}} \left|\inf_{x\in J_i} f(x) - \sup_{x\in J_i} f(x)\right|^p\right)^{\frac{1}{p}} \\
&\leq (2\mathcal{H}^s(C))^{\frac{1}{q}} \left(\sum_{i=1}^n c(s,p,y)\, \|\nabla u\|^p_{L^p(Q_i)}\right)^{\frac{1}{p}} \\
&\leq c(s,p,y)\,(2\mathcal{H}^s(C))^{\frac{1}{q}}\, \|\nabla u\|_{L^p(\mathbb{R}\times[y-\delta,y+\delta])} \longrightarrow 0
\end{aligned}$$

as $\delta\to 0$. Since f is absolutely continuous outside C, the above shows that f is absolutely continuous on the whole $\mathbb{R}$.

It remains to prove Lemma 3.1.

Proof of Lemma 3.1. Fix $i\in\{1,\dots,n\}$ and let $I\subset J\subset J_i$ be intervals such that $|J|=2\,|I|$. By (6), the open set $K=(y-|I|\,,y+|I|)\setminus F$ satisfies $\mathcal{H}^1(K)\geq c_y\,|I|^{\alpha}$. Define a collection $\gamma_t\colon[0,1]\to Q_i$, $t\in[0,1]$ of curves so that each γ_t is the concatenation of three line-segments γ_t^1, γ_t^2, and γ_t^3 that are defined as follows. Write $J=[a,b]$, $I=[c,d]$ and set $x_1(t)=a+t(b-a)$ and $x_2(t)=c+t(d-c)$. Define

$$y(t)=\inf\left\{\tilde{y}\in[y-|I|,y+|I|]\,:\,\mathcal{H}^1(K\cap(-\infty,\tilde{y}])\geq t\mathcal{H}^1(K)\right\}.$$

Now, γ_t^1 is taken to be the line-segment from $(x_1(t),y)$ to $(x_1(t),y(t))$, γ_t^2 the line-segment from $(x_1(t),y(t))$ to $(x_2(t),y(t))$, and γ_t^3 the line-segment from $(x_2(t),y(t))$ to $(x_2(t),y)$. Notice that the image of γ_t does not intersect E for $\mathcal{H}^1$-almost every $t\in[0,1]$.

By integrating over the curves γ_t we obtain

$$\begin{aligned}\left|\frac{1}{|I|}\int_I f(x)\,dx-\frac{1}{|J|}\int_J f(x)\,dx\right| &= \left|\int_0^1 u(\gamma_t(1))-u(\gamma_t(0))\,dt\right|\\ &\leq \int_0^1 |u(\gamma_t(1))-u(\gamma_t(0))|\,dt\\ &\leq \int_0^1\int_{\gamma_t}|\nabla u(z)|\,ds(z)\,dt\\ &= \sum_{k=1}^{3}\int_0^1\int_{\gamma_t^k}|\nabla u(z)|\,ds(z)\,dt\end{aligned}$$

First we treat the integrals along the vertical lines γ_t^1,γ_t^3. By Hölder's inequality we have

$$\begin{aligned}\int_0^1\int_{\gamma_t^1}|\nabla u(z)|\,ds(z)\,dt &\leq \int_0^1\int_{y-\frac{1}{2}|J_i|}^{y+\frac{1}{2}|J_i|}|\nabla u(x_1(t),z)|\,dz\,dt\\ &= \int_I\int_{y-\frac{1}{2}|J_i|}^{y+\frac{1}{2}|J_i|}|\nabla u(x,z)|\,dz\,dx\\ &\leq (|I|\cdot|J|)^{\frac{1}{q}}\,\|\nabla u\|_{L^p(Q_i)}\\ &= c(p)\delta^{\frac{2-s}{q}}\,|J|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}\,.\end{aligned}$$

A similar computation shows that $\int_0^1\int_{\gamma_t^3}|\nabla u(z)|\,ds(z)\,dt\leq c(p)\delta^{\frac{2-s}{q}}\,|J|^{\frac{s}{q}}$ $\|\nabla u\|_{L^p(Q_i)}$.

To evaluate the integrals along γ_t^2, observe that the map $t\mapsto y(t)$ is piecewise affine (on a countable union of open intervals) with $y'(t)=\frac{1}{\mathcal{H}^1(K)}$ a.e. on $(0,1)$. Thus we have

$$\begin{aligned}\int_0^1\int_{\gamma_t^1}|\nabla u(z)|\,ds(z)\,dt&\le\int_{J\times K}\mathcal{H}^1(K)^{-1}\,|\nabla u(w,z)|\,dw\,dz\\&\le|J|^{\frac{1}{q}}\,\mathcal{H}^1(K)^{\frac{1}{q}-1}\,\|\nabla u\|_{L^p(J\times K)}\\&\le 2^{\frac{\alpha}{p}}c_y^{-\frac{1}{p}}\,|J|^{\frac{1}{q}-\frac{\alpha}{p}}\,\|\nabla u\|_{L^p(Q_i)}=2^{\frac{\alpha}{p}}c_y^{-\frac{1}{p}}\,|J|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}\,,\end{aligned}$$

where we used $\mathcal{H}^1(K)\ge 2^{-\alpha}c_y\,|J|^\alpha$ and the definition of α.

By putting all together we get

$$\left|\frac{1}{|I|}\int_I f(x)\,dx-\frac{1}{|J|}\int_J f(x)\,dx\right|\le c(s,p,y)\,|J|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}\,. \tag{9}$$

Now, let $z_1,z_2\in J_i$ be such that

$$\left|\inf_{x\in J_i}f(x)-\sup_{x\in J_i}f(x)\right|\le 2\,|f(z_1)-f(z_2)|\,.$$

Let $\{I_k\}_{k=1}^\infty$ be subintervals of J_i so that $I_1=J_i$, $z_1\in I_k$ for every $k\in\mathbb{N}$, and $|I_k|=2\,|I_{k+1}|$ for every $k\in\mathbb{N}$. Now, by (9), we have

$$\begin{aligned}\left|f(z_1)-\frac{1}{|J_i|}\int_{J_i}f(x)\,dx\right|&\le\sum_{k=1}^\infty\left|\frac{1}{|I_k|}\int_{I_k}f(x)\,dx-\frac{1}{|I_{k+1}|}\int_{I_{k+1}}f(x)\,dx\right|\\&\le\sum_{k=1}^\infty c(s,p,y)\,|I_k|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}\\&\le c(s,p,y)\,|J_i|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}.\end{aligned}$$

Together with an analogous estimate for z_2, we obtain

$$\begin{aligned}\frac{1}{2}\left|\inf_{x\in J_i}f(x)-\sup_{x\in J_i}f(x)\right|&\le|f(z_1)-f(z_2)|\\&\le\left|f(z_1)-\frac{1}{|J_i|}\int_{J_i}f(x)\,dx\right|+\left|f(z_2)-\frac{1}{|J_i|}\int_{J_i}f(x)\,dx\right|\\&\le 2c(s,p,y)\,|J_i|^{\frac{s}{q}}\,\|\nabla u\|_{L^p(Q_i)}.\quad\square\end{aligned}$$

4. Curve-condition, porosity and Ahlfors-regular sets

In this section we study the case where the set $E=C\times F$ consists of a set F of positive measure and a zero measure set C with more regularity. The most regular case is when C is (Ahlfors) s-regular, that is, if there exists a constant $c_R>0$ so that

$$\frac{1}{c_R}r^s\leq\mathcal{H}^s((x-r,x+r)\cap C)\leq c_R r^s$$

for every $x\in C$ and $0<r<\operatorname{diam}(C)$. The set C in [19, Lemma 4.4] was not exactly s-regular, but almost. A small perturbation to s-regularity was required there to have the nonremovability at the critical exponent.

In [19, Lemma 4.4], the p-removability of $C\times F$ was proven via the following sufficient condition from [16], [24]. Suppose $E\subset\mathbb{R}^2$ is closed set of measure zero and $2\leq p<\infty$. If there exists a constant $c_\Gamma>0$ such that for every $z_1,z_2\in\mathbb{R}^2\setminus E$ there exists a curve $\gamma\subset\mathbb{R}^2\setminus E$ connecting z_1 to z_2 and satisfying

$$\int_\gamma \operatorname{dist}(z,E)^{\frac{1}{1-p}}\,ds(z)\leq c_\Gamma\,|z_1-z_2|^{\frac{p-2}{p-1}}\,, \tag{10}$$

then E is p-removable. If the above holds, we say that $E\subset\mathbb{R}^2$ satisfies the curve condition (10).

By adapting the proof in [19], we get a p-removability result that is independent of the structure of F.

Theorem 4.1. *Let $C\subset\mathbb{R}$ be a closed s-regular set with $0<s<1$, and $F\subset\mathbb{R}$ totally disconnected closed set. Then $C\times F$ is p-removable for every $p>\frac{2-s}{1-s}$.*

A slightly more general result for p-removability via the curve condition (10) than the one stated in Theorem 4.1 is in terms of porosity. Recall that a set $C\subset\mathbb{R}$ is called uniformly lower α-porous, if for every $x\in C$ and $r>0$ there exists $y\in(x-r,x+r)$ so that $(y-\alpha r,y+\alpha r)\cap C=\varnothing$.

Theorem 4.2. *Let $C\subset\mathbb{R}$ be a closed uniformly lower α-porous set and $F\subset\mathbb{R}$ totally disconnected closed set. Then $C\times F$ is p-removable for every $p>\hat{p}$, where $\hat{p}>2$ depends only on the parameter α.*

Proof of Theorems 4.1 and 4.2. Both of the theorems are proven by verifying the condition (10). Towards verifying this condition, let $z_1,z_2\in\mathbb{R}^2\setminus E$. Write these points in coordinates as $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$. Let us abbreviate $r=|z_1-z_2|$. Since F is totally disconnected and E is closed, we may assume that $y_1,y_2\notin F$.

Notice that an s-regular set is uniformly lower porous. Thus, in both cases by porosity of C there exists a point $x\in(x_1-r,x_1+r)$ so that $(x-\alpha r,x+\alpha r)\cap C=\varnothing$.

We now connect z_1 to z_2 by concatenating three line-segments γ_1, γ_2, and γ_3. The curve γ_1 connects (x_1,y_1) to (x,y_1), γ_2 connects (x,y_1) to (x,y_2), and γ_3 connects (x,y_2) to (x_2,y_2). The choice of x now gives

$$\int_{\gamma_2} \operatorname{dist}(z,E)^{\frac{1}{1-p}}\, ds(z) \le \int_{\gamma_2} (\alpha r)^{\frac{1}{1-p}}\, ds(z) = (\alpha r)^{\frac{1}{1-p}}\, |y_1-y_2| \le \alpha^{\frac{1}{1-p}} r^{\frac{p-2}{p-1}}$$

for the vertical part γ_2.

For the horizontal parts γ_1 and γ_3 we first show that the following condition holds for s-regular sets C and for uniformly lower α-porous sets C with some $0<s<1$: there exists a constant $c_s<\infty$ such that for all $0<\delta\le 1$ and every $-\infty<a<b<\infty$, the set $(a,b)\setminus C$ contains at most $c_s\delta^{-s}$ connected components of length more than $\delta\,|b-a|$.

Let us first show this for an s-regular set C. Suppose that $\{I_i\}_{i=1}^n$ are the connected components of $(a,b)\setminus C$ of length more than $\delta\,|b-a|$. For each i let v_i be the left-most point of $\overline{I}_i$. The sets $((v_i-\delta\,|b-a|\,, v_i+\delta\,|b-a|)\cap C)\subset[a-|b-a|\,, b+|b-a|]$ are pairwise disjoint. Thus, by s-regularity (notice that the left-most v_i might not be in C)

$$\frac{n-1}{c_R}\,(\delta\,|b-a|)^s \le \mathcal{H}^s([a-|b-a|\,, b+|b-a|]\cap C) < c_R(2\,|b-a|)^s,$$

which gives the claim for s-regular sets C.

Let us now suppose that C is uniformly lower α-porous, fix δ and denote by $\{I_i\}_{i=1}^n$ the intervals of $(a,b)\setminus C$ of length at least $\delta\,|b-a|$, and by $\{J_i\}_{i=1}^\infty$ the remaining intervals of $(a,b)\setminus C$. Consider the set

$$C' = \left\{ z+t \,:\, z\in C, t\in\left(-\frac{\delta}{2}\,|b-a|\,, \frac{\delta}{2}\,|b-a|\right)\right\}.$$

By a result of Salli [22, Theorem 3.5], we have

$$\mathcal{H}^1(C') \le c(\alpha)\,|b-a|\,\delta^{1-s}, \tag{11}$$

where $s=\frac{\log 2}{\log\left(\frac{2-\alpha}{1-\alpha}\right)}\in(0,1)$ and $c(\alpha)$ is a positive constant depending on α. Observe that $\bigcup_i J_i\subset C'$ and, for every interval I_i, $|I_i\setminus C'|\le|I_i|-\frac{\delta}{2}\,|b-a|$. Thus, using (11), we have

$$|b-a| = \sum_{i=1}^n |I_i| + \sum_{i=1}^\infty |J_i| = \sum_{i=1}^n |I_i\setminus C'| + \mathcal{H}^1(C') \le |b-a| - \frac{1}{2}n\delta\,|b-a| + c(\alpha)\,|b-a|\,\delta^{1-s},$$

yielding $n\le 2c(\alpha)\delta^{-s}$.

Let us then estimate the integral along γ_1. Without loss of generality we may assume that $x_1 < x$. Denote by $\{J_i\}_i$ the collection of open intervals constituting the connected components of $(x_1,x)\setminus C$. Let $k_0 \in \mathbb{Z}$ be so that $2^{-k_0} < |x-x_1| \le 2^{-k_0+1}$. Then

$$\begin{aligned}\int_{\gamma_1} \operatorname{dist}(z,E)^{\frac{1}{1-p}}\,ds(z) &\le \sum_i 2\int_0^{|J_i|} t^{\frac{1}{1-p}}\,dt = 2\frac{p-1}{p-2}\sum_i |J_i|^{\frac{p-2}{p-1}} \\ &\le c(p)\sum_{k=k_0}^{\infty} \#\left\{i:\ 2^{-k-1} < |J_i| \le 2^{-k}\right\} 2^{-k\frac{p-2}{p-1}} \\ &\le c(p)\sum_{k=k_0}^{\infty} c_s 2^{(k-k_0)s} 2^{-k\frac{p-2}{p-1}} \\ &\le c(p)\sum_{k=k_0}^{\infty} c_s 2^{(k-k_0)\left(s-\frac{p-2}{p-1}\right)} |x-x_1|^{\frac{p-2}{p-1}} \\ &\le c(p,s)\,|x-x_1|^{\frac{p-2}{p-1}} \le c(p,s)\,|z_1-z_2|^{\frac{p-2}{p-1}}\end{aligned}$$

as long as $s < \frac{p-2}{p-1}$.

The integral along γ_3 is handled analogously. □

We end this section by showing that the p-removability results that are proven via the curve condition (10) give removability only for porous sets.

Proposition 4.3. *Suppose that $E = C \times F \subset \mathbb{R}^2$ is a compact set satisfying the curve condition (10) and that $F \subset \mathbb{R}$ is a totally disconnected set with positive Lebesgue measure. Then C is uniformly lower α-porous for some $\alpha > 0$.*

Proof. Let $c_\Gamma > 0$ be the constant in (10). Let $y \in F$ be a Lebesgue density-point of F and $\varepsilon := \sqrt{2}c_\Gamma^{1-p}$. Then there exists $r_0 > 0$ such that for all $0 < r < r_0$ we have

$$\mathcal{H}^1((y-r,y+r)\setminus F) < \varepsilon r. \tag{12}$$

Let $x \in C$ and $0 < r < r_0$. Define $\tilde{z}_1 = (x-r/2,y)$ and $\tilde{z}_1 = (x+r/2,y)$, and select points $z_1 \in B(\tilde{z}_1, r/4)\setminus E$ and $z_2 \in B(\tilde{z}_2, r/4)\setminus E$. Let $\gamma \subset \mathbb{R}^2\setminus E$ be a curve connecting z_1 to z_2 and satisfying (10). Define $A := [x-7r/8, x+7r/8]\times[y-r/2, y+r/2]$ and $d := \max\{\operatorname{dist}(z,E):\ z \in \gamma\cap A\}$. Now, by (10)

$$d^{\frac{1}{1-p}}\frac{r}{2} \le \int_{\gamma\cap A} \operatorname{dist}(z,E)^{\frac{1}{1-p}}\,ds(z) \le c_\Gamma\,|z_1-z_2|^{\frac{p-2}{p-1}} \le c_\Gamma(2r)^{\frac{p-2}{p-1}}.$$

Thus,

$$d \ge 2c_\Gamma^{1-p} r = \sqrt{2}\varepsilon r,$$

which together with (12) gives the ε-porosity of C at x at the scale r. From the compactness of C it then follows that C is uniformly lower α-porous for some $\alpha>0$. □

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