Ark. Mat., **61** (2023), 81–97
DOI: 10.4310/ARKIV.2023.v61.n1.a5

A complex-analytic approach to streamline properties of deep-water Stokes waves

Olivia Constantin

Abstract. Using methods from complex analysis we obtain some qualitative results for certain streamline characteristics in a deep-water Stokes flow.

1. Introduction

The study of steady two-dimensional deep-water waves is a classical topic in hydrodynamics, since typically near-surface water flows are analysed in terms of superpositions and interactions of such waves, for which at great depths the water is almost at rest. This interpretation depends on a detailed understanding of the dynamics of these steady water waves, for which no closed-form solution is known. In 1847 Stokes provided a perturbation procedure that yields the first successive approximations to the flow beneath irrotational periodic travelling deep-water waves (termed deep-water "Stokes waves" nowadays), and in recent decades computers were used to obtain explicitly high-order Stokes approximations (see [OS01]). However, while a power series approach proves that the Stokes expansion converges if the wave steepness is very small (see the discussion in [Co11]), it is by now well-established that results obtained by truncation at a certain order fail to be accurate even for waves of moderate steepness (see [Cl07], [Pi20]). A lot of insight into the dynamics of deep-water Stokes waves was gained relying on analytical methods: see [BT03] for existence results and [He06], [NR18], [To96] and [Wa04] for investigations of the flow pattern beneath these waves. In particular, methods from complex analysis (e.g. conformal maps, maximum principles) proved very useful. In this paper we make use of a complex analysis approach to prove convexity properties

Key words and phrases: deep-water wave, irrotational flow, conformal map, convexity.
2010 Mathematics Subject Classification: primary 76B15; secondary 30C20, 30C45.

of some streamline characteristics like the length of a streamline, the area between the surface and a streamline, or the total kinetic energy in regions between two streamlines. For instance, we show that the length of a streamline decreases as the depth increases and that its logarithm is a convex function of the depth.

2. Preliminaries

2.1. Governing equations

For two-dimensional water waves it suffices to investigate the flow characteristics in a cross-section of the flow, oriented towards the direction of wave propagation. We choose Cartesian coordinates (X,Y) with the X-axis pointing in the direction of wave propagation and the Y-axis oriented upwards. Let $Y=\boldsymbol{\eta}(X,t)$ be the free surface and $(U(X,Y,t),\, V(X,Y,t))$ be the velocity field, with the water occupying the region $\{(X,Y)\colon Y<\boldsymbol{\eta}(X,t)\}$ at time t. Under the physically reasonable assumption of a homogeneous inviscid fluid, the governing equations for a flow determined by the balance between the restoring gravity force and the inertia of the system are the equation of mass conservation

$$U_X+V_Y=0\,, \quad Y<\boldsymbol{\eta}(X,t)\,, \tag{1}$$

and Euler's equations

$$\begin{cases} U_t+UU_X+VU_Y=-P_X\,, \\ V_t+UV_X+VV_Y=-P_Y-g\,, \end{cases} \qquad Y<\boldsymbol{\eta}(X,t)\,, \tag{2}$$

where $P(X,Y,t)$ is the pressure and g is the constant gravitational acceleration. The associated boundary conditions are the kinematic boundary condition

$$V=\boldsymbol{\eta}_t+U\boldsymbol{\eta}_X \quad \text{on} \quad Y=\boldsymbol{\eta}(X,t)\,, \tag{3}$$

expressing the fact that the water's free surface is an interface, the dynamic boundary condition

$$P=P_{\text{atm}} \quad \text{on} \quad Y=\boldsymbol{\eta}(X,t)\,, \tag{4}$$

which decouples the motion of the water from that of the air above it, where P_{atm} is the constant atmospheric pressure at sea level, and the assumption that the water is practically at rest at great depths, expressed by the constraint

$$(U,V)\longrightarrow(0,0) \quad \text{as} \quad Y\longrightarrow-\infty \quad \text{uniformly for} \quad (X,t)\in\mathbb{R}^2\,. \tag{5}$$

The absence of underlying currents is ensured by the irrotational character of the flow, requiring

$$U_Y - V_X = 0\,, \quad Y < \boldsymbol{\eta}(X,t)\,. \tag{6}$$

A *Stokes wave* is a travelling wave solution to the governing equations (1)–(6) for which there exists a period $\lambda > 0$ and a wave speed $c > 0$ such that the free surface profile $\boldsymbol{\eta}$, the fluid velocity (U, V) and the pressure P have period λ in the X variable, $\boldsymbol{\eta}$ depends only on $(X - ct)$, while U, V, and P depend only on $(X - ct)$ and Y. We consider Stokes waves whose profile is symmetric. Typically for the investigation of deep-water Stokes waves (see [BT03], [To96]) one requires periodicity and symmetry properties of the velocity field (U, V) and of the pressure P, in addition to the periodicity and symmetry of the wave profile $\boldsymbol{\eta}$. It turns out that in the case of smooth Stokes waves it suffices to require only the periodicity and symmetry of the wave profile, since the periodicity and symmetry properties of the velocity field and the pressure beneath the wave are implied by these, as we shall show below. We would also like to point out that there are symmetric travelling waves with several crests and troughs per minimal period, for which the symmetry is not about the highest crest or about the lowest trough – see [BDT00], [EW15]. We assume that

$$U < c \quad \text{throughout the fluid region} \quad \{(X,Y) \in \mathbb{R}^2 :\ Y \leq \boldsymbol{\eta}(X,t)\}\,, \tag{7}$$

since typically the wave speed is much larger than the horizontal fluid velocity (see [Co11]). The existence of deep-water Stokes waves of small and large amplitude is by now well-established by means of global bifurcation theory (see the discussions in [BT03], [To96]). In particular, it is known that (7) holds for all Stokes waves with exception of the Stokes wave of greatest height, in which case (7) fails precisely at the wave crest, where $U = c$ and the wave profile has a corner – it fails to be differentiable, having lateral tangents at an angle of 120°. In this paper we discuss *smooth Stokes waves*, so that the assumption (7) is justified. We now impose a last condition that will enable us to express our results in terms of the depth of a streamline below the trough: we assume that there is a single crest and trough per wavelength λ, in particular, that the profile is strictly monotonic between successive crests and troughs. The latter condition actually implies the symmetry of the wave profile (see [Ga65], [To98] and [OS01]).

Taking advantage of the (X,t)-dependence of the form $(X - ct)$, passing to the moving frame

$$x = X - ct\,, \quad y = Y\,, \tag{8}$$

and writing

$$U(X,Y,t) = u(X - ct, Y) = u(x,y)\,, \quad V(X,Y,t) = v(X - ct, Y) = v(x,y)\,,$$

$$P(X,Y,t)=p(X-ct,Y)=p(x,y)\,, \quad \boldsymbol{\eta}(X,t)=\eta(X-ct)=\eta(x)\,,$$

we can reformulate the governing equations for Stokes waves as

$$(u-c)u_x+vu_y=-p_x\,, \quad y<\eta(x)\,, \tag{9}$$

$$(u-c)v_x+vv_y=-p_y-g\,, \quad y<\eta(x)\,, \tag{10}$$

$$u_x+v_y=0\,, \quad y<\eta(x)\,, \tag{11}$$

$$u_y=v_x\,, \quad y<\eta(x)\,, \tag{12}$$

$$v=(u-c)\eta_x \quad \text{on} \quad y=\eta(x)\,, \tag{13}$$

$$(u,v)\longrightarrow(0,0) \quad \text{as} \quad y\longrightarrow-\infty\,, \text{ uniformly in } x\in\mathbb{R}, \tag{14}$$

$$p=P_{\text{atm}} \quad \text{on} \quad y=\eta(x)\,, \tag{15}$$

with

$$u-c<0 \quad \text{for all} \quad y\le\eta(x)\,. \tag{16}$$

Regarding regularity, we assume that the free surface $y=\eta(x)$ is the graph of a periodic C^1-function of period λ, and that the velocity field (u,v) is bounded and continuously differentiable throughout the fluid domain (in the moving frame)

$$\mathcal{D}=\{(x,y)\in\mathbb{R}^2:\ y<\eta(x)\}\,,$$

and has a continuous extension to its closure $\overline{\mathcal{D}}=\{(x,y)\in\mathbb{R}^2:\ y\le\eta(x)\}$. Elliptic regularity theory ensures that under these assumptions the curve η is a real-analytic curve and the functions u and v can be extended to smooth functions on a open neighbourhood of $\overline{\mathcal{D}}$, so that the equations can be manipulated at will. Moreover, let us note that structural properties of the system (9)–(15) can be used to reduce the number of unknowns without loss of information. Firstly, the pressure can be eliminated by taking advantage of Bernoulli's law: from (9)–(12) we can see that

$$\frac{(u-c)^2+v^2}{2}+p+gy \tag{17}$$

is constant throughout $\mathcal{D}$.

2.2. Stream function, velocity potential and the hodograph transform

A further reduction is obtained by introducing the *stream function* $\psi(x,y)$, defined up to an additive constant by

$$\psi_x=-v\,, \quad \psi_y=u-c \quad \text{throughout} \quad \mathcal{D}\,. \tag{18}$$

Note that the (11)–(12) ensure that ψ is harmonic throughout $\mathcal{D}$, while the kinematic boundary condition (13) shows that ψ is constant on the free surface $y{=}\eta(x)$. Given that ψ is uniquely defined by (18) up to an additive constant, we may set

$$\psi=0 \quad \text{on} \quad y=\eta(x)\,. \tag{19}$$

Since (16) and (18) yield $\psi_y{<}0$ throughout $\mathcal{D}$, we infer from (19) that $\psi{>}0$ in $\mathcal{D}$.

We now show that the function $x{\mapsto}\psi(x,y)$ inherits the symmetry and periodicity properties of the free surface, that is

$$\psi(x{+}\lambda,y)=\psi(x,y) \quad \text{and} \quad \psi(x,y)=\psi(-x,y) \quad \text{for all} \quad (x,y)\in\mathcal{D}\,.$$

To this end, notice first that the harmonic function $\beta(x,y){=}\psi(x{+}\lambda,y){-}\psi(x,y)$ is bounded throughout $\mathcal{D}$ since v is bounded and the mean-value theorem implies

$$\beta(x,y)=\psi_x(\xi_{x,y},y)\,\lambda=-v(\xi_{x,y},y)\,\lambda\,, \quad (x,y)\in\mathcal{D}\,,$$

for some $\xi_{x,y}{\in}(x,x{+}\lambda)$. Moreover, $\beta{=}0$ on the free boundary $y{=}\eta(x)$ of $\mathcal{D}$. The maximum principle for harmonic functions in unbounded domains (see Corollary 2.3.3 in [Ra95]) yields $\beta{\equiv}0$ on $\mathcal{D}$. With the periodicity of ψ at hand, we may now apply the maximum principle to the harmonic function $\gamma(x,y){=}\psi(x,y){-}\psi(-x,y)$ in the domain

$$\{(x,y)\in\mathbb{R}^2:\ -\lambda/2<x<\lambda/2,\ y<\eta(x)\}\,.$$

Indeed, this function vanishes on the top boundary as well as on the lateral sides of this domain by the symmetry of the free surface $y{=}\eta(x)$, respectively, by the periodicity of ψ, while

$$\lim_{y\to-\infty}[\psi(x,y)-\psi(-x,y)]=0 \quad (\text{uniformly in} \quad x\in[-\lambda/2,\lambda/2])$$

by Lagrange's mean-value theorem combined with (14) and (18). Hence $\gamma{\equiv}0$.

The claim about the periodicity and symmetry properties of the velocity field throughout the fluid domain follows now from (18). The periodicity and symmetry of p are ensured by Bernoulli's law (17).

Without loss of generality, by translating the origin of the Cartesian coordinates, we may assume that

$$\inf_{x\in\mathbb{R}}\eta(x)=\eta(\lambda/2)=0\,, \tag{20}$$

so that the wave trough is located at $x{=}\pm\lambda/2$. We now claim that

$$\psi_{yy}\big(\tfrac{\lambda}{2},y\big)<0 \quad \text{for all} \quad y<0\,. \tag{21}$$

Indeed, consider the domain

$$\Omega_+ = \{(x,y)\in\mathbb{R}^2 : \quad 0\le x\le \tfrac{\lambda}{2}\,,\ y\le \eta(x)\}\,.$$

Since $v(x,y) = -\psi_x(x,y)$ is odd and periodic with period λ in the x-variable throughout $\mathcal{D}$, we have that $v=0$ on the lateral boundaries of Ω_+, while $v\to 0$ as $y\to-\infty$ uniformly in $x\in\big[0,\,\frac{\lambda}{2}\big]$, by (14). On the other hand, from (13), (16) and the fact that $\eta_x(x)\le 0$ for $x\in\big[0,\,\frac{\lambda}{2}\big]$ (as the wave profile in monotone between crest and trough) we obtain

$$v(x,\eta(x))\ge 0\,,\quad x\in\big[0,\,\tfrac{\lambda}{2}\big]\,,$$

that is, $v\ge 0$ on the top boundary of Ω_+. Since v is harmonic by (12) and (11), the maximum principle yields $v>0$ throughout Ω_+, since $v\equiv 0$ in Ω_+ would imply a flat free surface by (13). From Hopf's lemma (see [Co11]) we infer that $v_x\big(\frac{\lambda}{2},y\big)<0$ for all $y<0$ and (21) follows now from (12) and (18).

Let us now define the *velocity potential* $\varphi(x,y)$ as the harmonic conjugate of ψ, defined uniquely up to an additive constant by

$$\varphi_x = \psi_y\,,\quad \varphi_y = -\psi_x\,,\quad (x,y)\in\mathcal{D}\,. \tag{22}$$

From (22), (18), (16) and the fact that $\psi(x,y)$ is even in the x-variable, we deduce that the restriction of φ to the periodicity cell

$$\Omega = \{(x,y)\in\mathcal{D} :\ -\tfrac{\lambda}{2}\le x\le \tfrac{\lambda}{2}\}$$

is constant on the vertical segment $x=\lambda/2$, equal to its minimum value φ_{min}, and constant on $x=-\lambda/2$, equal to its maximum value φ_{max}. For simplicity, we set $\varphi_{min}=0$.

Throughout the paper we identify the complex number $x+\mathrm{i}y$ with the vector $(x,y)\in\mathbb{R}^2$, using the notations interchangeably. In light of (11)–(12), the complex function $F(z)=\varphi(x,y)+\mathrm{i}\,\psi(x,y)$ with $z=x+\mathrm{i}y$, called the *hodograph transform*, is analytic in $\mathcal{D}$. Moreover, we have

$$F'(z) = (u-c)+\mathrm{i}(-v) \tag{23}$$

and relation (14) can be rewritten as

$$\lim_{y\to-\infty}\{\sup_{x\in\mathbb{R}}|F'(x,y)+c|\} = 0\,. \tag{24}$$

Due to the periodic dependence of $\psi(x,y)$ on the x variable, the analytic function $z\mapsto F(z+\lambda)-F(z)$ has vanishing imaginary part throughout $\mathcal{D}$, and it is therefore constant. In order to determine this constant, we let $\Im m(z)\to-\infty$ in

$$F(z+\lambda)-F(z) = \int_z^{z+\lambda} F'(w)\,dw,$$

and, by (24), we find that the right-hand-side above converges to $-c\lambda$. Therefore

$$\varphi(x+\lambda,y)-\varphi(x,y)=F(z+\lambda)-F(z)=-c\lambda, \tag{25}$$

holds for any $z=x+\mathrm{i}y$ in the fluid domain $\mathcal{D}$, so that, by (20), we get

$$\varphi_{max}=\varphi_{max}-\varphi_{min}=\varphi\big(-\tfrac{\lambda}{2},\,\eta\big(-\tfrac{\lambda}{2}\big)\big)-\varphi\big(\tfrac{\lambda}{2},\,\eta\big(\tfrac{\lambda}{2}\big)\big)=\varphi\big(-\tfrac{\lambda}{2},\,0\big)-\varphi\big(\tfrac{\lambda}{2},\,0\big)=c\lambda\,.$$

Notice that F is a biholomorphic map between the interior of the periodicity cell $\{(x,y)\colon\, x\in(-\frac{\lambda}{2},\,\frac{\lambda}{2})\,,\ y<\eta(x)\}$ and the open half strip $\{(q,p)\colon\, 0<q<c\lambda,\ p>0\}$; see Figure 1. As already mentioned, relation (16) ensures that the function η has to be real-analytic, and ψ admits an extension to a harmonic function on an open neighbourhood of $\overline{\mathcal{D}}$ (see [To96] for a proof based on Lewy's theorem [Le52] and [Co11] for an alternative proof relying on regularity for elliptic boundary-value problems).

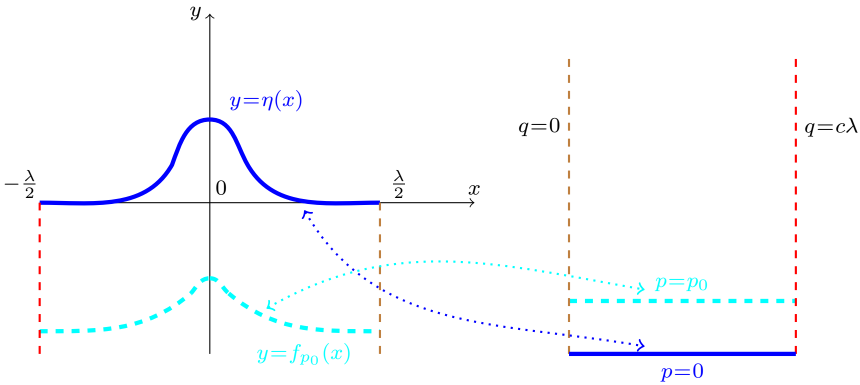

Figure 1. The orientation-reversing conformal hodograph transform $(x,y)\mapsto(\varphi,\psi)$ maps the fluid domain $\mathcal{D}$ into the upper half-plane $[p>0]$, flattening each streamline: the image of the curve $y=f_{p_0}(x)$ is the horizontal line $p=p_0$.

The hodograph transform F induces the following orientation-reversing conformal change of variables

$$\begin{cases} q=\varphi(x,y), \\ p=\psi(x,y)\,, \end{cases} \tag{26}$$

which is a global diffeomorphism from the closure $\overline{\mathcal{D}}$ of the fluid domain $\mathcal{D}$ to the closure of the upper half-plane

$$\mathcal{R}=\{(q,p):\ q\in\mathbb{R}\,,\ p>0\}\,.$$

Furthermore, it is easy to see that

$$\lim_{y\to-\infty}\psi(x,y)=\infty \quad \text{uniformly in} \quad x\in\mathbb{R}\,. \tag{27}$$

Indeed, we infer from (14) and (16) that

$$\delta=-\sup_{(x,y)\in\mathcal{D}}\{u(x,y)-c\}>0\,.$$

The uniform growth of $\psi(x,y)$ as $y\to-\infty$ follows by integration of (18), since

$$\psi(x,0)-\psi(x,y)\le\delta y\,,\quad x\in\mathbb{R}\,,\quad y\le 0\,,$$

and $x\mapsto\psi(x,0)$ is bounded, being periodic. Our claim (27) is thus proven.

Note that the conformal bijection $q+ip\mapsto x+iy$ from $\mathcal{R}$ to the fluid domain $\mathcal{D}$, obtained as the inverse of the change of variables (26),

$$\begin{cases} x=x(q,p), \\ y=y(q,p)\,, \end{cases} \tag{28}$$

has the "periodicity" as well as the asymptotic properties

$$\begin{cases} x(q+c\lambda,p)=x(q,p)-\lambda \quad \text{for} \quad (q,p)\in\mathcal{R}\,, \\ y(q+c\lambda,p)=y(q,p) \quad \text{for} \quad (q,p)\in\mathcal{R}\,, \\ \lim_{p\to\infty} y(q,p)=-\infty \quad \text{uniformly in} \quad q\in\mathbb{R}\,, \end{cases} \tag{29}$$

where the first two relations above are mere reformulations of (25). The asymptotic property in (29) can be shown analogously to (27). Indeed, by (16) we have

$$\delta_0=-\inf_{(x,y)\in\mathcal{D}}\{u(x,y)-c\}>0\,,$$

and integrating (18) we deduce that

$$\psi(x,y)-\psi(x,0)\le-\delta_0 y\,,\quad x\in\mathbb{R}\,,\quad y\le 0\,,$$

which implies

$$\begin{aligned}\delta_0\, y(q,p) &\le \max_{x\in\mathbb{R}}\{\psi(x,0)\}-\psi(x(q,p),\, y(q,p)) \\ &= \max_{x\in\mathbb{R}}\{\psi(x,0)\}-p \longrightarrow -\infty \quad \text{as } p\longrightarrow\infty.\end{aligned}$$

As a consequence of (16), (18) and the implicit function theorem, for every $x\in\mathbb{R}$ there exists a unique $f_p(x)\in(-\infty,\eta(x)]$ such that $\psi(x,f_p(x))=p$ and the streamlines (i.e. the level sets of ψ) are smooth curves. We have

$$f_p'(x)=-\frac{\psi_x(x,f_p(x))}{\psi_y(x,f_p(x))}=\frac{\varphi_y(x,f_p(x))}{\varphi_x(x,f_p(x))}=\frac{v(x,f_p(x))}{u(x,f_p(x))-c}\,. \tag{30}$$

Recall that the free surface $y=\eta(x)$ is the streamline $\psi=0$. One can easily show that each streamline inherits the periodicity, symmetry and monotonicity properties of the free surface. While no explicit expression for the stream function is available, note that linearization yields $\psi(x,y)\approx a\,\mathrm{e}^{ky}\cos(kx)-cy$ for $y\leq 0$ and $0<a<c/k$, where $k=\frac{2\pi}{\lambda}$ (see [Cl07]). However, as pointed out in the introduction, perturbation methods for deep-water Stokes waves can be misleading.

3. Kinetic energy between two streamlines

In this section we investigate convexity properties of the total kinetic energy within the region

$$\mathcal{R}_p=\{(x,y)\in\mathbb{R}^2:\ -\tfrac{\lambda}{2}\leq x\leq\tfrac{\lambda}{2}\,,\ f_p(x)\leq y\leq\eta(x)\}$$

between the free surface $y=\eta(x)$ and an arbitrary streamline $y=f_p(x)$, more precisely, of the quantity

$$\mathbb{E}(p)=\iint_{\mathcal{R}_p}(u^2+v^2)\,\mathrm{d}A,\quad p\geq 0.$$

We start by recalling some basic facts that are needed in our further considerations. A function $f\colon(0,\infty)\to\mathbb{R}$ is convex of $\log r$ if

$$f(r)\leq\alpha f(r_1)+(1-\alpha)f(r_2)$$

holds whenever $\log r=\alpha\log r_1+(1-\alpha)\log r_2$ with $\alpha\in(0,1)$ and $r_1,\,r_2\in(0,\infty)$. Equivalently, $f\colon(0,\infty)\to\mathbb{R}$ is a convex/concave function of $\log r$ if and only if $g(y)=f(\mathrm{e}^y)$ is convex/concave on $\mathbb{R}$.

Remark 3.1. Let N and β be real functions on intervals such that the composition $M:=N\circ\beta$ is well-defined. It is straightforward to check that, if β is concave and increasing and if N is convex and decreasing (respectively, concave and increasing), then $M:=N\circ\beta$ is convex and decreasing (respectively, concave and increasing).

We denote $\mathbb{D}:=\{z\in\mathbb{C}\colon\ |z|<1\}$ and for $r\in(0,1)$ we consider the annuli

$$A(r,1)=\{z\in\mathbb{C}:\ r<|z|<1\}\,.$$

Remark 3.2. For any square-integrable analytic function $f\colon\mathbb{D}\to\mathbb{C}$, the function

$$r\longmapsto\mathcal{A}_f(r)=\iint_{A(r,1)}|f(z)|^2\,\mathrm{d}A(z)\,,\quad r\in(0,1)\,,$$

is concave of $\log r$. Indeed, from

$$\mathcal{A}_f(r) = \int_r^1 \int_0^{2\pi} |f(\rho e^{i\theta})|^2 \cdot \rho \, d\theta \, d\rho$$

we get

$$\frac{d}{dr} \mathcal{A}_f(r) = -r \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = -r \, M_2(f,r) \,, \quad r \in (0,1) \,,$$

where

$$M_2(f,r) = \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \,, \quad r \in [0,1) \,,$$

if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$. Clearly $r \mapsto M_2(f,r)$ is increasing on $[0,1)$. Therefore, setting $g(y) = \mathcal{A}_f(e^y)$, we get

$$g'(y) = \mathcal{A}_f'(e^y) \cdot e^y = -e^{2y} M_2(f, e^y) \,,$$

which is a decreasing function and we conclude by the considerations from the beginning of this section. □

Theorem 3.3. *The total kinetic energy $\mathbb{E}(p)$ between the free surface and the streamline $\psi = p$ is a concave function of $p > 0$, or, equivalently, a concave function of the streamline depth below the wave trough, given by $d = d(p) = -f_p(\frac{\lambda}{2})$.*

Proof. The hodograph transform $F = \varphi + i\psi$ maps $\mathcal{R}_p$ bijectively to the rectangle $[0, c\lambda] \times [0, p]$ and the periodicity cell

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : \ -\tfrac{\lambda}{2} \le x \le \tfrac{\lambda}{2} \,, \ y \le \eta(x)\}$$

onto the half-strip $\mathcal{S} = [0, c\lambda] \times [0, \infty)$; see Figure 1.

Due to (23), the change of variables $z \mapsto F(z)$ yields

$$\begin{aligned}
\mathbb{E}(p) &= \iint_{\mathcal{R}_p} (u^2 + v^2) \, dA = \iint_{\mathcal{R}_p} \frac{|F'(z) + c|^2}{|F'(z)|^2} \cdot |F'(z)|^2 \, dA(z) \\
&= \int_0^p \int_0^{c\lambda} \left| \frac{F' \circ F^{-1} + c}{F' \circ F^{-1}} (q, p_1) \right|^2 dA(q, p_1) \\
&= \int_0^p \int_0^{c\lambda} \left| 1 + \frac{c}{F' \circ F^{-1}(q, p_1)} \right|^2 dA(q, p_1) \\
&= \int_0^p \int_0^{c\lambda} |G(q, p_1)|^2 \, dA(q, p_1) \,, \qquad (31)
\end{aligned}$$

where

$$G := 1 + c\,(F^{-1})' = 1 + \frac{c}{F' \circ F^{-1}}$$

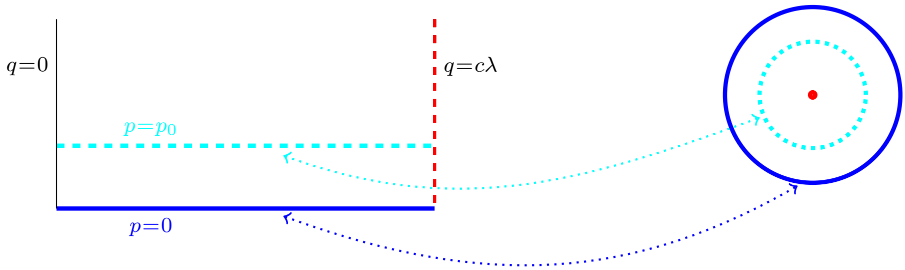

Figure 2. The conformal bijection $z\mapsto e^{ikz}$ with $k=\frac{2\pi}{c\lambda}$ maps the half-strip $\mathcal{S}\setminus\{(c\lambda,p)\colon p\geq 0\}$ onto the punctured closed unit disk $\overline{\mathbb{D}}\setminus\{0\}$.

is analytic in the interior of $\mathcal{S}$ and continuous on its closure.

We aim to make a change of variables in order to turn the integral $\mathbb{E}(p)$ into an integral over an annulus. The transformation $z\mapsto e^{ikz}$ with $k=\frac{2\pi}{c\lambda}$ maps the half-strip $\mathcal{S}\setminus\{(c\lambda,p)\colon p\geq 0\}$ bijectively onto the punctured closed unit disk $\overline{\mathbb{D}}\setminus\{0\}$ (see Figure 2). Its inverse is given by $\alpha(w)=\frac{1}{ik}\operatorname{Log} w$, where

$$\operatorname{Log} w = \log\,|w| + i\operatorname{Arg} w\,,\quad 0\leq \operatorname{Arg} w < 2\pi\,,$$

is the branch of the complex logarithm that 'jumps' over the positive x-axis. We define the map

$$H(w)=G(\alpha(w))=\frac{F'\circ F^{-1}(\alpha(w))+c}{F'\circ F^{-1}(\alpha(w))}\,.$$

The periodicity of F' in the x-variable (expressed as $F'(x+\lambda,y)=F'(x,y)$) together with the relation (see (29))

$$F^{-1}(q+c\lambda,p)=F^{-1}(q,p)+(\lambda,0)\,,\quad q\in\mathbb{R}\,,\quad p\geq 0\,, \tag{32}$$

cancel the 'jump' of Log over the positive x-axis and thus H extends to a continuous map on $\overline{\mathbb{D}}\setminus\{0\}$. Morera's theorem now ensures the analyticity of H in $\mathbb{D}\setminus\{0\}$. Since $F'+c=u+iv$ and, by (29),

$$\lim_{p\to\infty}\Im m\Big(F^{-1}(q,p)\Big)=-\infty\quad\text{uniformly in}\quad q\,,$$

the decay of u and v for $y\to-\infty$ given by (14) yields $\lim_{w\to 0}H(w)=0$. Hence H has a removable singularity at $w=0$ and $H_1(w):=\frac{H(w)}{w}$ extends analytically to $\mathbb{D}$. Substituting $z=\alpha(w)$ (with $|\alpha'(w)|=\frac{1}{k|w|}$) in (31), we get

$$\begin{aligned}\mathbb{E}(p)&=\int_0^p\int_0^{c\lambda}|G(z)|^2\,dA(z)=\frac{1}{k^2}\iint_{A(r(p),1)}|H_1(w)|^2\,dA(w)\\&=\mathcal{A}_{H_1}(r(p))\,,\end{aligned}$$

where $r(p)=\mathrm{e}^{-kp}$. It follows from Remark 3.2 that $r\mapsto\mathcal{A}_{H_1}(r)$ is a concave function of $\log r$, which at its turn implies that $p\mapsto\mathbb{E}(p)$ is a concave function, proving the first assertion.

To prove the second assertion, note that the depth $d=d(p)$ of the streamline $\psi=p$ below the wave trough satisfies

$$p=p(d)=\psi\big(\tfrac{\lambda}{2},\,-d\big)\,.$$

In view of (16), (18) and (21) we obtain

$$p'(d)=-\psi_y\big(\tfrac{\lambda}{2},\,-d\big)>0\,, \tag{33}$$

$$p''(d)=\psi_{yy}\big(\tfrac{\lambda}{2},\,-d\big)<0\,. \tag{34}$$

Hence $d\mapsto p(d)$ is increasing and concave and, since $\mathbb{E}(p)$ is clearly increasing, we may apply Remark 3.1 to infer that $d\mapsto\mathbb{E}(p(d))$ is concave and increasing. □

Remark 3.4. The total kinetic energy *in the moving frame* within the region $\mathcal{R}_p$ between the free surface $y=\eta(x)$ and the streamline $y=f_p(x)$, defined by

$$\mathbb{E}_m(p)=\iint_{\mathcal{R}_p}\big((u-c)^2+v^2\big)\,\mathrm{d}A,$$

is the linear function $\mathbb{E}_m(p)=c\lambda\cdot p$. Indeed, using the hodograph transform, we obtain

$$\mathbb{E}_m(p)=\iint_{\mathcal{R}_p}|F'(z)|^2\,\mathrm{d}A(z)=\int_0^p\int_0^{c\lambda}1\,\mathrm{d}A=c\lambda\cdot p\,. \qquad \square$$

4. Area between two streamlines

Consider the region between a fixed streamline $\psi=p_0$ and the streamline $\psi=p$ with $p\geq p_0$, given by

$$S_{p_0,p}=\{(x,y)\in\mathbb{R}^2:\ -\tfrac{\lambda}{2}\leq x\leq\tfrac{\lambda}{2}\,,\ f_p(x)\leq y\leq f_{p_0}(x)\}.$$

and denote by $\mathbb{A}_{p_0}(p)$ its (Lebesgue) area.

Theorem 4.1. *$\mathbb{A}_{p_0}$ is a concave function of p, or, equivalently, a concave function of the streamline depth below the wave trough, given by $d=d(p)=-f_p(\frac{\lambda}{2})$.*

Proof. By an analogous approach to the one in the proof of Theorem 3.3, we perform a change of variables using the hodograph transform $F=(u-c)+\mathrm{i}v$ followed by $\alpha(w)=-\frac{\mathrm{i}}{k}\operatorname{Log} w$ (with $|\alpha'(w)|=\frac{1}{k|w|}$) to obtain

$$\begin{aligned}\mathbb{A}_{p_0}(p) &= \iint_{S_{p_0,p}} 1\,\mathrm{d}A = \int_0^{c\lambda}\int_{p_0}^{p} |(F^{-1})'(z)|^2\,\mathrm{d}A(z)\\ &= \frac{1}{k^2}\iint_{A(r,r_0)} \left|\frac{(F^{-1})'(\alpha(w))}{w}\right|^2 \mathrm{d}A(w)\,,\end{aligned} \tag{35}$$

where $r=\mathrm{e}^{-kp}$ and $r_0=\mathrm{e}^{-kp_0}$. Just as in the proof of Theorem 3.3 we find that

$$\widetilde{H}(w) := (F^{-1})'(\alpha(w)) = \frac{1}{F'\Big(F^{-1}\Big(-\frac{\mathrm{i}}{k}\operatorname{Log} w\Big)\Big)}$$

extends to an analytic function on $\mathbb{D}\setminus\{0\}$. The decay of $F'+c$ for $y\to-\infty$ (see relation (24)) yields

$$\lim_{w\to 0}\widetilde{H}(w) = -\frac{1}{c}\,,$$

which shows that $\widetilde{H}$ extends to an analytic function on $\mathbb{D}$ with $\widetilde{H}(0)=-\frac{1}{c}$. We may therefore write $\widetilde{H}(w)=\sum_{n=0}^{\infty} a_n w^n$ for $w\in\mathbb{D}$, and thus

$$\frac{\widetilde{H}(w)}{w} = -\frac{1}{c}\,\frac{1}{w}+\sum_{n=0}^{\infty} a_{n+1}w^n = -\frac{1}{c}\,\frac{1}{w}+\frac{\widetilde{H}(w)-\widetilde{H}(0)}{w}\,.$$

We now return to (35), where we use polar coordinates and the uniform convergence of the above power series on circles of radius $\rho\in(0,1)$ to get

$$\begin{aligned}\mathcal{A}_{p_0}(p) &= \frac{1}{k^2}\iint_{A(r,r_0)}\left|\frac{\widetilde{H}(w)}{w}\right|^2 \mathrm{d}A(w)\\ &= \frac{1}{k^2}\int_r^{r_0}\frac{1}{\rho^2}\int_0^{2\pi}|\widetilde{H}(\rho\mathrm{e}^{\mathrm{i}t})|^2\mathrm{d}t\cdot\rho\,\mathrm{d}\rho\\ &= \frac{1}{k^2}\int_r^{r_0}\frac{1}{\rho}\cdot 2\pi\sum_{n=0}^{\infty}|a_n|^2\rho^{2n}\,\mathrm{d}\rho\\ &= \frac{2\pi}{k^2}\Big(\int_r^{r_0}\frac{|a_0|^2}{\rho}\,\mathrm{d}\rho+\int_r^{r_0}\sum_{n=0}^{\infty}|a_{n+1}|^2\rho^{2n+1}\,\mathrm{d}\rho\Big)\\ &= \frac{1}{k^2}\Big(\frac{2\pi}{c^2}\,\log\frac{r_0}{r}+\iint_{A(r,r_0)}\left|\frac{\widetilde{H}(w)-\widetilde{H}(0)}{w}\right|^2\mathrm{d}A(w)\Big)\,.\end{aligned}$$

By Remark 3.2

$$r\longmapsto \iint_{A(r,r_0)} \Big|\frac{\widetilde{H}(w)-\widetilde{H}(0)}{w}\Big|^2\,\mathrm{d}A(w)$$
$$=\iint_{A(r,1)} \Big|\frac{\widetilde{H}(w)-\widetilde{H}(0)}{w}\Big|^2\,\mathrm{d}A(w)-\iint_{A(r_0,1)} \Big|\frac{\widetilde{H}(w)-\widetilde{H}(0)}{w}\Big|^2\,\mathrm{d}A(w)$$

is, for fixed r_0, a concave function of $\log r$ and hence also a concave function of $p=-\frac{1}{k}\log r$. Since $\log\frac{r_0}{r}=k(p-p_0)$ is a linear function of p, it follows that $p\mapsto \mathcal{A}_{p_0}(p)$ is concave. The second assertion follows from this by Remark 3.1, in view of (33)–(34). □

5. The length of a streamline

In this section we show that the length of a streamline decreases as the depth increases, and its logarithm is a convex function of the depth.

Theorem 5.1. *Let $\mathcal{L}(p)$ be the length of the streamline*

$$\gamma_p(x)=(x,f_p(x))\,,\quad -\tfrac{\lambda}{2}\le x\le\tfrac{\lambda}{2}\,,$$

given by $\mathcal{L}(p)=\int_{\gamma_p}1\,\mathrm{d}s$, where s represents the arclength parameter. The function $p\mapsto\log\mathcal{L}(p)$ is convex, while $p\mapsto\mathcal{L}(p)$ is decreasing. Equivalently, the streamline's length is decreasing and its logarithm is a convex function of the streamline's depth below the wave trough, given by $d=d(p)=-f_p(\frac{\lambda}{2})$.

Proof. As in Theorems 3.3 and 4.1, we first use the hodograph transform $F=\varphi+\mathrm{i}\,\psi$. More precisely, its restriction to each given streamline provides a change of variables which enables us to turn the integrals along streamlines into integrals over a line segments, which we may subsequently turn into integrals over concentric circles, and it is at this point that Hardy's convexity theorem (see [Du70]) comes into play. We have

$$F'=(u-c)+\mathrm{i}(-v)=\varphi_x-\mathrm{i}\varphi_y\,.$$

We perform a change of variables in the integral expression for $\mathcal{L}$ by setting

$$q=\tilde{q}_p(x)=\varphi(x,f_p(x))\,.$$

Then, by (30),

$$\tilde{q}_p'(x)=\varphi_x(x,f_p(x))+\varphi_y(x,f_p(x))\cdot f_p'(x)=\frac{\varphi_x^2(x,f_p(x))+\varphi_y^2(x,f_p(x))}{\varphi_x(x,f_p(x))}$$
$$=\frac{|F'(x,f_p(x))|^2}{\varphi_x(x,f_p(x))}<0\,,\quad -\tfrac{\lambda}{2}\le x\le\tfrac{\lambda}{2}\,,\tag{36}$$

and

$$\tilde{q}_p(-\tfrac{\lambda}{2}) = \varphi(-\tfrac{\lambda}{2}, f_p(-\tfrac{\lambda}{2})) = c\lambda\,, \quad \tilde{q}_p(\tfrac{\lambda}{2}) = \varphi(\tfrac{\lambda}{2}, f_p(\tfrac{\lambda}{2})) = 0\,. \tag{37}$$

It now follows that

$$\begin{aligned}
\mathcal{L}(p) &= \int_{-\lambda/2}^{\lambda/2} \sqrt{1+[f_p'(x)]^2}\,\mathrm{d}x = \int_{-\lambda/2}^{\lambda/2} |F'(x, f_p(x))| \cdot \frac{1}{-\varphi_x(x, f_p(x))}\,\mathrm{d}x \\
&= -\int_{-\lambda/2}^{\lambda/2} \frac{1}{|F'(x, f_p(x))|}\,\tilde{q}_p'(x)\,\mathrm{d}x = \int_0^{c\lambda} \frac{1}{\big|F'\big(\tilde{q}_p^{-1}(q),\ f_p(\tilde{q}_p^{-1}(q))\big)\big|}\,\mathrm{d}q \\
&= \int_0^{c\lambda} \frac{1}{|F'(F^{-1}(q,p))|}\,\mathrm{d}q = \int_0^{c\lambda} |(F^{-1})'(q,p))|\,\mathrm{d}q\,,
\end{aligned}$$

where the next to last step above follows from

$$\begin{aligned}
F\big(\tilde{q}_p^{-1}(q),\ f_p(\tilde{q}_p^{-1}(q))\big) &= \Big(\varphi\big(\tilde{q}_p^{-1}(q),\ f_p(\tilde{q}_p^{-1}(q))\big),\ \psi\big(\tilde{q}_p^{-1}(q),\ f_p(\tilde{q}_p^{-1}(q))\big)\Big) \\
&= \Big(\tilde{q}_p(\tilde{q}_p^{-1}(q)), p\Big) = (q,p)\,.
\end{aligned} \tag{38}$$

We now pass from the half strip $[0, c\lambda) \times [0, \infty)$ to the unit disc, where we may apply Hardy's theorem. As shown in the proof of Theorem 3.3, the function $\widetilde{H}(w) = (F^{-1})'(-\frac{i}{k}\ \mathrm{Log}\ w)$ extends to an analytic function in the unit disc. We then have

$$\begin{aligned}
M_2(\widetilde{H}, r) &:= \int_0^{2\pi} |\widetilde{H}(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{d}\theta \\
&= \int_0^{2\pi} \Big|(F^{-1})'\Big(-\tfrac{\mathrm{i}}{k}\log r + \tfrac{\theta}{k}\Big)\Big|^2 \mathrm{d}\theta \\
&= k\int_0^{c\lambda} \Big|(F^{-1})'\Big(q - \tfrac{\mathrm{i}}{k}\log r\Big)\Big|^2 \mathrm{d}q = k\,\mathcal{L}(p)\,,
\end{aligned} \tag{39}$$

where $p = -\frac{\log r}{k}$ and the next to last step above follows by the change of variables $q = \frac{\theta}{k}$. Hardy's convexity theorem (see [Du70]) now ensures that the map $r \mapsto \log M_2(\widetilde{H}, r)$ is a convex function of $\log r$ for $r \in (0,1)$, i.e. for any $r, r_1, r_2 \in (0,1)$ and $\alpha \in [0,1]$ such that $\log r = \alpha \log r_1 + (1-\alpha)\log r_2$, we have

$$M_2(\widetilde{H}, r) \le [M_2(\widetilde{H}, r_1)]^{\alpha} \cdot [M_2(\widetilde{H}, r_2)]^{1-\alpha}. \tag{40}$$

Since $p = -\frac{\log r}{k}$, we deduce from (39) that $p \mapsto \log L(p)$ is a convex function.

As pointed out in Remark 3.2, the map $r \mapsto M_2(\widetilde{H}, r)$ is increasing, and it hence follows from (39) that $p \mapsto \mathcal{L}(p)$ is decreasing.

The two statements regarding the properties of the streamline length as a function of the depth below the wave trough follow from those with respect to the p-variable in view of Remark 3.1 and (33)–(34). □

Acknowledgement. The author is very grateful for helpful comments from the referee, which brought considerable improvement to the paper.

References

[BDT00] Buffoni, B., Dancer, E. N. and Toland, J. F., The sub-harmonic bifurcation of Stokes waves, *Arch. Ration. Mech. Anal.* **152** (2000), 241–270. MR1764946

[BT03] Buffoni, B. and Toland, J. F., *Analytic theory of global bifurcation*, Princeton University Press, Princeton, 2003. MR1956130

[Cl07] Clamond, D., On the Lagrangian description of steady surface gravity waves, *J. Fluid Mech.* **589** (2007), 433–454. MR2360967

[Co11] Constantin, A., *Nonlinear water waves with applications to wave-current interactions and tsunamis*, CBMS-NSF Reg. Conf. Series Appl. Math. **81**, SIAM, Philadelphia, 2011. MR2867413

[Du70] Duren, P. L., *Theory of H^p spaces*, Academic Press, New York, 1970. MR0268655

[EW15] Ehrnström, M. and Wahlén, E., Trimodal steady water waves, *Arch. Ration. Mech. Anal.* **216** (2015), 449–471. MR3317807

[Ga65] Garabedian, P. R., Surface waves of finite depth, *J. Anal. Math.* **14** (1965), 161–169. MR0184511

[He06] Henry, D., The trajectories of particles in deep-water Stokes waves, *IMRN*, 23405 (2006). MR2272104

[Le52] Lewy, H., A note on harmonic functions and a hydrodynamic application, *Proc. Am. Math. Soc.* **3** (1952), 111–113. MR0049399

[NR18] Nachbin, A. and Ribeiro-Junior, R., Capturing the flow beneath water waves, *Philos. Trans. Roy. Soc. A* **376**, no. 2111, 20170098, 17 pp. (2018). MR3744206

[Pi20] Pizzo, N., Theory of deep-water surface gravity waves derived from a Lagrangian, *J. Fluid Mech.* **896** (2020). A7. MR4108882

[OS01] Okamoto, H. and Shōji, M., *The mathematical theory of permanent progressive water-waves*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. MR1869386

[Ra95] Ransford, T., *Potential theory in the complex plane*, Cambridge University Press, Cambridge, 1995. MR1334766

[To96] Toland, J. F., Stokes waves, *Topol. Methods Nonlinear Anal.* **7** (2016), 1–48. MR1422004

[To98] Toland, J. F., On the symmetry theory for Stokes waves of finite and infinite depth, in *Trends in applications of mathematics to mechanics*, *Nice, 1998*, pp. 207–217, Chapman & Hall/CRC, Boca Raton, 2000. MR1734884

[Wa04] Wahlén, E., Uniqueness for autonomous planar differential equations and the Lagrangian formulation of water flows with vorticity, *J. Nonlinear Math. Phys.* **11** (2004), 549–555. MR2098545

Olivia Constantin
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
AT-1090 Vienna
Austria
olivia.constantin@univie.ac.at

Received May 10, 2022
in revised form August 30, 2022