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Geometric Origin of Montonen-Olive Duality

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Abstract

We show how $N = 4, D = 4$ duality of Montonen and Olive can be derived for all gauge groups using geometric engineering in the context of type II strings, where it reduces to T-duality. The derivation for the non-simply laced cases involves the use of some well known facts about orbifold conformal theories.

1 Introduction

The aim of this note is to show how the celebrated Montonen-Olive duality [1] for all $N = 4$ gauge theories in $D = 4$ can be derived by geometric engineering in the context of type II strings, where it reduces to T-duality. Even though by now there is a lot of evidence for the Montonen-Olive duality (see e.g. [2]) there is no derivation of this duality. Even with the recent advances in our understanding of dynamics of string theory the derivation of this duality is not yet complete. The aim of this note is to fill this gap. The approach we will follow is in the context of type II compactifications and is quite general and provides a unified approach to all gauge groups. Moreover we gain an understanding of how the field theory duality works by relating it to well understood perturbative symmetries (T-dualities) of strings[1].

[1]For the case of $SU(n)$ gauge group there is another approach suggested in [3] which uses the Hull-Townsend $SL(2,\mathbb{Z})$ duality of type IIB strings [4]. In this approach one considers

2 Montonen-Olive Duality

Let us recall what the Montonen-Olive duality is: We consider $N = 4$ supersymmetric gauge theories in $D = 4$. Consider a gauge group G, with gauge coupling constant g. Then the Montonen-Olive duality suggests that this theory is equivalent to $N = 4$ gauge theory with a dual gauge group $\widehat{G}$ and coupling constant $g' \propto 1/g$ where the electric and magnetic degrees of freedom are exchanged. For the $A_n, D_n, E_{6,7,8}, F_4, G_2$ gauge groups $\widehat{G}$ is again the same gauge group[2]. On the other hand the Montonen-Olive duality exchanges the B_n and C_n gauge groups.

3 Basic Idea of Geometric Engineering of $N = 4$ Theories

We now review the basic idea of geometric engineering of $N = 4$ theories in four dimensions in the context of type IIA strings. We will start with the simpler case of the simply laced groups.

3.1 Simply Laced Cases: A-D-E

Consider type IIA strings propagating on an ALE space of A-D-E type. Then as is well known this gives rise in six dimensions to an $N = 2$ theory with A-D-E gauge group, where the charged gauge particles are obtained by wrapping D2 branes around 2-cycles of the ALE space, and the uncharged (Cartan) gauge fields arise from decomposition of the 3-form gauge potential in terms of the harmonic 2-forms corresponding to the compact cycles of the ALE space. If we compactify on a T^2 from 6 down to 4 dimensions, the gauge coupling in $d = 4$ is proportional to the volume of T^2. The inversion of the coupling constant in the four dimensional theory amounts to the T-duality volume inversion symmetry of type IIA strings on a T^2. This establishes the Montonen-Olive self-duality of A-D-E gauge groups by reducing it to string T-duality[3].

the theory of n parallel $D3$ branes, and uses the fact that the $SL(2,\mathbb{Z})$ symmetry maps $D3$ branes back to itself. It would be interesting to see if this approach can be generalized to all gauge groups.

[2]Here we do not pay attention to global issues and limit ourselves to Lie algebras. However, more precisely the Montonen-Olive duality suggests that the weight lattice of the dual group is dual to the weight lattice of the original group. For example if we start with $SU(n)$ the dual group is $SU(n)/\mathbb{Z}_n$.

[3]The connection between S-duality and T-duality in string theories was first suggested in [5]. In the compact version of the above construction, it was noted in [6] to be a consequence of string-string duality. Note that here we are taking a slightly different view by not utilizing string-string duality, and just using facts about D-branes which were established later, thanks to the observations in [7] and [8]. See in particular [9,10].

3.2 The Non-Simply Laced Cases

To geometrically engineer non-simply laced gauge groups in four dimensions we follow the idea in [11] (see also [12]), by using outer automorphisms of the simply laced groups. We consider an A-D-E ALE space, and compactify on an extra circle and identify translation along $1/n$-th of the circle by a specific outer automorphism $\mathbb{Z}_n$ symmetry acting on an ALE space. In other words we consider the 5-dimensional space

$$M = \frac{\text{ALE} \times S^1}{\mathbb{Z}_n}$$

as the background, where $\mathbb{Z}_n$ acts simultaneously as an outer automorphism of the *ALE* space and an order n translation on the circle. The relevant symmetries for the various non-simply laced groups are:

$$\begin{aligned} (D_{n+1}, \mathbb{Z}_2) &\to B_n \\ (A_{2n-1}, \mathbb{Z}_2) &\to C_n \\ (E_6, \mathbb{Z}_2) &\to F_4 \\ (D_4, \mathbb{Z}_3) &\to G_2, \end{aligned} \tag{3.1}$$

where in the D case the $\mathbb{Z}_2$ exchanges the two end nodes, for the A_{2n-1} and E_6 cases it flips the Dynkin diagram through the middle node and for D_4 it permutes the three outer nodes.

The strategy we will follow is to show that type IIA on

$$M = \frac{\text{ALE} \times S^1(R)}{\mathbb{Z}_n}$$

is equivalent to type IIB strings on

$$\widehat{M} = \frac{\widehat{\text{ALE}} \times S^1(n/R)}{\mathbb{Z}_n},$$

where the $\mathbb{Z}_n$'s are according to 3.1 and the dual $\widehat{\text{ALE}}$ corresponds to the same ALE for E_6 and D_4 but exchanges A_{2n-1} and D_{n+1} ALE spaces. Compactifying further on another circle and using the $R \to 1/R$ symmetry on the other circle converts the theory back from type IIB to type IIA, and we will thus have established Montonen-Olive duality using T-duality, by showing the type IIA string equivalence of

$$\frac{\text{ALE} \times S^1(R)}{\mathbb{Z}_n} \times S^1(R') = \frac{\widehat{\text{ALE}} \times S^1(n/R)}{\mathbb{Z}_n} \times S^1(1/R'). \tag{3.2}$$

4 Some Facts About Orbifold CFT's

In this section we will review some facts about $\mathbb{Z}_n$ orbifold conformal field theories that we will use in the next section.

Let C denote a conformal theory with a $\mathbb{Z}_n$ discrete symmetry. We can consider orbifolding this theory with the $\mathbb{Z}_n$ symmetry. Let us recall some aspects of how this works [13] (see also the review article [14]). There are n twisted sectors, labeled by an integer r mod n, which are the sectors of strings closed up to the $\mathbb{Z}_n$ action. Let us denote the Hilbert space of each sector by C_r. Moreover, we can decompose each sector according to how the $\mathbb{Z}_n$ acts on that sector. Let C_r^s denote the subsector of the r-th twisted sector which transforms according to $\exp[2\pi i s/n]$ where s is also defined mod n. We can associate another $\tilde{\mathbb{Z}}_n$ symmetry by using the grading of the twist sector, i.e. by considering C_r^s to transform as $\exp[2\pi i r/n]$. As is well known, the Hilbert space of the conformal theory $C/\mathbb{Z}_n$ is obtained by considering the Z_n invariant pieces of each sector, i.e.

$$\tilde{C} = \frac{C}{\mathbb{Z}_n} = \sum_r C_r^0.$$

It is easy to see (see [14] for a review) that we can mod $\tilde{C}$ by $\tilde{\mathbb{Z}}_n$ and recover C back, i.e.

$$\frac{\tilde{C}}{\tilde{\mathbb{Z}}_n} = C.$$

In fact the s-th twisted sector of the $\tilde{C}/\tilde{\mathbb{Z}}_n$ can be identified with C_r^s, and projecting to the $\mathbb{Z}_n$ invariant sector means keeping $\sum C_0^s$, which is the definition of the C theory Hilbert space. Thus the two theories C and $\tilde{C}$ are on the same footing: out of the n^2 subsectors C_r^s, exchanging $C \leftrightarrow \tilde{C}$ amounts to exchanging $r \leftrightarrow s$.

Now suppose we have two conformal theories C_1 and C_2 each with a $\mathbb{Z}_n$ symmetry. Consider orbifolding with a single $\mathbb{Z}_n$ which acts on both at the same time, on one as the generator of the original $\mathbb{Z}_n$ and on the other, as the inverse generator. Then it is easy to see that

$$\frac{C_1 \times C_2}{\mathbb{Z}_n} = \frac{\tilde{C}_1 \times \tilde{C}_2}{\tilde{\mathbb{Z}}_n}. \tag{4.1}$$

In fact the Hilbert space for both theories is given by

$$\sum_{r,s} C_{1,r}^s C_{2,-r}^{-s}.$$

Note the symmetrical role r and s play in the above expression, which is a reflection of the equivalence 4.1, as $C \leftrightarrow \tilde{C}$ amounts to $r \leftrightarrow s$. The equivalence 4.1 is what we will use to prove the duality we are after. In fact this is exactly of the form of the equivalence 3.2 that we wish to prove (the second circle plays no major role), where $C_1, \tilde{C}_1$ are to be identified with the ALE theory and its dual and $C_2, \tilde{C}_2$ with the $S^1(R)$ theory and its dual $S^1(n/R)$. The fact that $S^1(R)/\mathbb{Z}_n = S^1(n/R)$ is straight forward. In fact, by definition of the $\mathbb{Z}_n$ action on the circle we have

$$S^1(R)/\mathbb{Z}_n = S^1(R/n).$$

Applying the standard T-duality on this we get $S^1(n/R)$ which is thus the dual theory $\tilde{C}_2$. Note that the $\tilde{\mathbb{Z}}_n$ acting on $\tilde{C}_2$ is a translation of order n on this dual circle. In fact $S^1(n/R)/\tilde{\mathbb{Z}}_n = S^1(1/R)$ which by the standard T-duality is equivalent to $S^1(R)$. So all we are left to do to complete the proof of Montonen-Olive duality is to prove $C_1, \tilde{C}_1$ are dual conformal theories, as predicted by the duality. This we will do in the next section.

5 Proving the Duality

We complete the proof of duality in this section by showing the following dualities of CFT on ALE spaces:

$$A_{2n-1}/\mathbb{Z}_2 = D_{n+1} \qquad D_{n+1}/\mathbb{Z}_2 = A_{2n-1} \qquad (C_n \leftrightarrow B_n)$$

$$E_6/\mathbb{Z}_2 = E_6 \qquad (F_4 \leftrightarrow F_4)$$

$$D_4/\mathbb{Z}_3 = D_4 \qquad (G_2 \leftrightarrow G_2). \tag{5.1}$$

This would identify the $C_1, \tilde{C}_1$ in 4.1 with the appropriate dual needed in 3.1, which completes the proof of Montonen-Olive duality in accordance with 3.2.

In order to do this we need to recall the $N = 2$ superconformal theories associated with string propagation on ALE spaces. It was shown in [15] that this is described by the $N = 2$ Landau-Ginzburg theory with superpotential

$$W = W_{ADE}(x, y, z) - t^{-d},$$

where W_{ADE} denotes the ADE singularity and d is the dual Coxeter number of the corresponding singularity:

$$W_{A_{n-1}} = x^n + y^2 + z^2 \qquad d = n$$

$$W_{D_n} = x^{n-1} + xy^2 + z^2 \qquad d = 2n - 2$$

$$W_{E_6} = x^3 + y^4 + z^2 \qquad d = 12$$

$$W_{E_7} = x^3 + xy^3 + z^2 \qquad d = 18$$

$$W_{E_8} = x^3 + y^5 + z^2 \qquad d = 30.$$

(we also need to mod out by $\exp(2\pi i J_0)$ which is the generator of a $\mathbb{Z}_d$ – this however will not play a major role in the following). As a superconformal theory, the part corresponding to W_{ADE} is equivalent to the corresponding $N = 2$ minimal models, as was shown in [16], and the part corresponding to t^{-d} term is a Kazama-Suzuki model based on the $SL(2)$ group (at level $d + 2$). The symmetries we are interested in modding out act only on the x, y, z variables, thus the latter superconformal theory plays no role. So all we need to show is that the orbifold of minimal $N = 2$ superconformal theories for A_{2n-1},D_{n-1}, E_6 and D_4 behave as expected from 5.1. This fact is actually well known, and can be readily derived since the minimal conformal theories are very well known. Here we shall review it for completeness and present its derivation along the lines suggested in [17].

Consider the A_{2n-1} minimal model:

$$W_{A_{2n-1}} = x^{2n} + y^2 + z^2.$$

The relevant outer automorphism $\mathbb{Z}_2$ acts as

$$x \to -x, \quad y \to -y, \quad z \to z.$$

Thus we introduce the invariant variables

$$\tilde{x} = x^2 \quad \tilde{y} = y/x \quad \tilde{z} = z$$

(which have been chosen to keep the Jacobian of the transformation constant). Then we obtain[4]

$$W_{A_{2n-1}}/\mathbb{Z}_2 = \tilde{x}^n + \tilde{x}\tilde{y}^2 + \tilde{z}^2 = W_{D_{n+1}}.$$

To go in reverse, it is of course true on general grounds discussed above that there is a $\mathbb{Z}_2$ acting on D_{n+1} which gives back A_{2n-1}. However it is not a priori obvious why it should be the one corresponding to the outer automorphism of D_{n+1}. To accomplish this we show directly that the outer automorphism $\mathbb{Z}_2$ leads back to the A_{2n-1} theory. We have

$$W_{D_{n+1}} = x^n + xy^2 + z^2$$

and the outer automorphism $\mathbb{Z}_2$ acts by

$$x \to x, \quad y \to -y, \quad z \to -z.$$

[4]The result of this $\mathbb{Z}_2$ modding out for conformal theories associated to ALE is *not* the one that would be naively expected based on the geometry of *ALE* space for particle theories, where one would end up instead with D_{n+2}.

We introduce the new invariant variables

$$\tilde{x} = x \quad \tilde{y} = y^2 \quad \tilde{z} = z/y,$$

which leads to

$$W_{D_{n+1}}/\mathbb{Z}_2 = \tilde{x}^n + \tilde{y}(\tilde{x} + \tilde{z}^2) = W_{A_{2n-1}}$$

(the last equality follows by shift of variables, or simply by noting that the variation with respect to $\tilde{y}$ sets $\tilde{x} = -\tilde{z}^2$ which leads to $\tilde{z}^{2n}$).

For E_6 we have

$$W_{E_6} = x^3 + y^4 + z^2$$

and the $\mathbb{Z}_2$ outer automorphism is given by

$$x \to x, \quad y \to -y, \quad z \to -z.$$

We introduce the new variables

$$\tilde{x} = x \quad \tilde{y} = y^2 \quad \tilde{z} = z/y,$$

which leads to

$$W_{E_6}/\mathbb{Z}_2 = \tilde{x}^3 + \tilde{y}^2 + \tilde{y}\tilde{z}^2 = \tilde{x}^3 + (\tilde{y} + \frac{1}{2}\tilde{z}^2)^2 - \frac{1}{4}\tilde{z}^4 = W_{E_6}$$

(the last equality follows by shifting $\tilde{y}$ by $-\frac{1}{2}\tilde{z}^2$).

For the D_4 case we have

$$W_{D_4} = x^3 + y^3 + z^2$$

and the $\mathbb{Z}_3$ outer automorphism is given by

$$x \to \omega x, \quad y \to \omega^{-1} y, \quad z \to z; \qquad \omega^3 = 1,$$

and we introduce the invariant variables (again with Jacobian of the transformation being constant)

$$\tilde{x} = x^2/y \quad \tilde{y} = y^2/x \quad \tilde{z} = z$$

which leads to

$$W_{D_4}/\mathbb{Z}_3 = \tilde{x}^2\tilde{y} + \tilde{y}^2\tilde{x} + \tilde{z}^2 = W_{D_4},$$

(where the last equality follows by shifting $\tilde{x}$ and $\tilde{y}$). This completes the proof.

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