

Strong Coupling $N = 2$ Gauge Theory with Arbitrary Gauge Group

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Abstract

A explicit definition of the cycles, on the auxiliary Riemann surface defined by Martinec and Warner for describing pure $N = 2$ gauge theories with arbitrary group, is provided. The strong coupling monodromies around the vanishing cycles are shown to arise from a set of dyons which becomes massless at the singularities. It is shown how the correct weak coupling monodromies are reproduced and how the dyons have charges which are consistent with the spectrum that can be calculated at weak coupling using conventional semi-classical methods. In particular, the magnetic charges are co-root vectors as required by the Dirac-Schwinger-Zwanziger quantization condition.

1 Introduction

A huge leap forward in the understanding of $N = 2$ gauge theories in the coulomb phase was initiated by Seiberg and Witten [1]. Their original papers dealt only with the case of gauge group $SU(2)$. Since then there have been a number of papers setting out the generalization to arbitrary gauge group. In particular, Martinec and Warner [2] develop a general construction of the auxiliary Riemann surface which arise in Seiberg and Witten's approach. (Their work subsumes earlier piece-meal generalizations to certain gauge

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groups, [3–8] although, it is not equivalent to the hyper-elliptic approach put forward in [9, 10] for some groups, an approach that has been shown to be incorrect for other reasons [11].) In the Martinec-Warner approach, the Riemann surface is the spectral curve of an integrable system—the Toda equations based on an associated affine algebra. In principle, this relation to the integrable system allows one to extract the quantities in the low energy effective action of the $N = 2$ supersymmetric gauge theory. More specifically, the cycles on the surfaces that are needed to provide the solution of the low energy effective action are determined by a special Prym subvariety of the Jacobian of the surface. One goal of this paper is to provide an explicit construction of these preferred cycles for any gauge group. The primary motivation is to use these explicit expressions to subject the construction for arbitrary gauge groups to the same rigorous tests as the $SU(2)$ case. In particular, we shall show how the strong coupling monodromies arise from renormalization around massless dyons singularities and how the strong coupling monodromies reproduce all the weak monodromies that can be calculated in perturbation theory. As a by-product, we shall derive the electric and magnetic charges of the dyons that drive the strong coupling dynamics. This allows for another highly non-trivial test of the construction because we can show that these dyons are indeed in the spectrum of the theory at weak coupling where conventional semi-classical methods are available for computing the dyon spectrum. The result of this work is that the physics of the pure $N = 2$ gauge theories with arbitrary gauge groups is now placed on a much firmer footing.

Seiberg and Witten’s approach determines the exact prepotential of the low energy effective action of gauge theories with $N = 2$ supersymmetry. BPS states carry charges $Q = (\mathbf{g}, \mathbf{q})$ and have a mass which is determined by the prepotential

$$M_Q = |Z_Q| = |Q \cdot A|, \quad (1.1)$$

where $A = (\mathbf{a}, \mathbf{a}_D)$ is a function on the moduli space of vacua \mathcal{M} and the symplectic inner product is defined as

$$Q \cdot A = Q \Omega A^T = \mathbf{a}_D \cdot \mathbf{g} - \mathbf{a} \cdot \mathbf{q} \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1.2)$$

The vector \mathbf{g} is the magnetic charge of the BPS state with respect to the unbroken $U(1)^r$ gauge group. It is a topological charge and is consequently quantized such that, with a suitable choice of overall normalization, it is a vector of the co-root lattice Λ_R^\vee of \mathfrak{g} , the (complexified) Lie algebra of the gauge group G . This is the lattice spanned by the simple co-roots α_i^\vee , where we define the dual operation

$$\alpha^\vee = 2\alpha/\alpha^2. \quad (1.3)$$

The vector \mathbf{q} determines the electric charge of the BPS state. It is not quite the true electric charge of the state, with respect to the unbroken $U(1)^r$ gauge group, because of theta-like terms in the low energy effective action [12]. More precisely, it is Noether's charge corresponding to global $U(1)^r$ gauge transformations. Nevertheless, with a slight abuse of language, we shall refer to \mathbf{q} as the electric charge. This charge is also quantized since the abelian group is embedded in the gauge group G . This means that the allowed electric charge vectors \mathbf{q} must lie in the weight lattice of g . However, in a pure gauge model, where all the fields are adjoint-valued, only charges in the root lattice Λ_R are actually realized. To summarize

$$(\mathbf{g}, \mathbf{q}) \in (\Lambda_R^\vee, \Lambda_R). \quad (1.4)$$

Notice that the charges satisfy the generalized Dirac-Schwinger-Zwanziger quantization condition:

$$Q_1 \cdot Q_2 = Q_1 \Omega Q_2^T = \mathbf{g}_1 \cdot \mathbf{q}_2 - \mathbf{g}_2 \cdot \mathbf{q}_1 \in \mathbb{Z}. \quad (1.5)$$

Conventional electro-magnetic duality transformations act on the fields in the low energy effective action. These transformations induce an action on A of the form $A \rightarrow AD$, where D is some matrix acting to the left. The transformation takes the low energy effective action to an identical action in the dual variables provided that D is a symplectic transformation, i.e.

$$D\Omega D^T = \Omega. \quad (1.6)$$

It is easy to see that the BPS mass formula is invariant under this transformation if the charges transform as $Q \rightarrow QD$.

The goal of the analysis is to determine A . This is a multi-valued function on \mathcal{M} since there are non-trivial monodromies along paths which encircle certain co-dimension two subspaces. Physically, the monodromy is caused by a certain BPS state becoming massless on the subspace, causing a logarithmic running of the effective coupling, in a way that we calculate in section 2. A monodromy transformation around a cycle C acts as $A \rightarrow AM$ and they are duality transformations since $M^T \Omega M = \Omega$. This implies that a BPS state Q taken around the cycle will end up as the state QM^{-1} , unless the state passes across a surface on which it decays to other BPS states of the same total charge.

Seiberg and Witten's major insight was that the multi-valued function A is determined solely in terms of a conjectured set of the co-dimension two singular subspaces on which certain BPS states become massless along with their associated monodromies. For $SU(2)$ they surmised that there were two such singularities, but in general it turns out that there are $2r$ singularities corresponding to a set of $2r$ BPS states Q_i^a , $i = 1, \dots, r$ and

$a = 1, 2$. Although, this is only a subset of all the BPS states there is a ‘democracy of dyons’, as first described in the $SU(2)$ case, whereby any dyon in the (semi-classical) spectrum becomes massless at one of the singularities by following a path to the singularity with some appropriate non-trivial monodromy which transforms it to one of the set Q_i^a . The construction proceeds by defining a Riemann surface with a moduli space identified with \mathcal{M} . For the surface, there exists a special meromorphic one-form λ and a set of $2r$ preferred homology one-cycles ν_i^a , such that

$$Z_{Q_i^a} = Q_i^a \cdot A = \oint_{\nu_i^a} \lambda. \quad (1.7)$$

Since the charges Q_i^a are linearly independent these relations are enough to determine the function A and a mapping between charges Q and cycles C_Q . The picture is now clear: a subspace where a dyon Q becomes massless is precisely the subspace on which the cycle C_Q vanishes and the Riemann surface degenerates.

Rather than begin with a conjectured set of charges for the massless BPS states at the singularities and then proceed to construct the Riemann surface and the other data, as Seiberg and Witten did in the $SU(2)$ case, we shall start with the Riemann surface data and show that resulting function A has the correct monodromy properties which agree with calculations performed in dual perturbation theory in the neighbourhood of the singularities and at weak coupling in conventional perturbation theory.

2 The Monodromies in Perturbation Theory

In this section, we calculate the monodromies of the function A . At weak coupling these may be calculated using standard perturbation theory. At strong coupling, it is conjectured that the monodromies arise from paths around singularities on which certain dyons become massless, and consequently these monodromies can also be calculated in perturbation theory, but now in dual variables.

At weak coupling the moduli space of vacua can be parameterized by the classical Higgs VEV Φ , modulo global gauge transformations. These transformations can always be used to conjugate Φ into the Cartan subalgebra $\mathbf{a} \cdot \mathbf{H}$, which defines the complex r -dimensional vector \mathbf{a} . The remaining freedom to perform conjugations in the Weyl group of g , can be used to choose, say, $\text{Re}(\mathbf{a})$ to be in the fundamental Weyl chamber of g , i.e.

$$\alpha_i \cdot \text{Re}(\mathbf{a}) \geq 0, \quad (2.1)$$

where α_i , $i = 1, \dots, r$, are the set of simple roots of g . On the wall of the fundamental Weyl chamber, where $\mathbf{a} \cdot \alpha_i = 0$, points are identified under

σ_i , the Weyl reflection in α_i . At weak coupling we parameterize \mathcal{M} by \mathbf{a} so constrained.

The function \mathbf{a}_D can be calculated as a function of the Higgs VEV, i.e. \mathbf{a} , in perturbation theory. The calculation is standard and to one-loop

$$\mathbf{a}_D = \frac{i}{2\pi} \sum_{\beta} \beta(\beta \cdot \mathbf{a}) \ln \left(\frac{\beta \cdot \mathbf{a}}{\Lambda} \right), \quad (2.2)$$

where Λ is the usual strong coupling scale and the sum is over all the root of g . The perturbative regime is valid as long as $|\mathbf{a} \cdot \alpha_i| \gg \Lambda$, for $i = 1, \dots, r$. It is immediately apparent that \mathbf{a}_D is not single-valued as one follows a cycle around one of the r co-dimension two subspaces $\mathbf{a} \cdot \alpha_i = 0$. Bearing in mind that such a cycle involves an identification by the Weyl reflection in the simple root α_i , on the wall $\text{Re}(\mathbf{a}) \cdot \alpha_i = 0$, one finds that the monodromy, for some choice of orientation, is $A \rightarrow AM_i$, where [5–7]

$$M_i = \begin{pmatrix} \sigma_i & \alpha_i \otimes \alpha_i \\ 0 & \sigma_i \end{pmatrix}. \quad (2.3)$$

Notice that $M\Omega M^T = \Omega$, so that the monodromy transformation is a duality transformation. It is important that non-perturbative corrections to \mathbf{a}_D do not make any additional contributions to weak coupling monodromies. The monodromy transformations generate a representation of the Brieskorn Braid Group.

It will turn out that the weak coupling monodromies M_i are the remnant of a larger set of monodromies that are present at strong coupling. These monodromies arise from following cycles around co-dimension two subspace on which a certain BPS state Q becomes massless, i.e. $Z_Q = 0$. The resulting monodromy M_Q around the singularity can be calculated by performing a duality transformation to dual variables, which are local with respect to the BPS state, and then using perturbation theory in those dual variables. The result was written down in [5]:

$$M_Q = 1 + (\Omega Q^T) Q. \quad (2.4)$$

Notice that $M_Q \Omega M_Q^{-1} = \Omega$ so that the monodromy transformation is a duality transformation. For completeness we will sketch the proof of this relation.

For any given charge Q , there always exists some duality transformation D such that

$$Q' = QD = (\mathbf{0}, \mathbf{q}'), \quad A' = AD. \quad (2.5)$$

So in the transformed frame the state is purely electrically charged. The effective field theory in the vicinity of the subspace on which the state becomes

massless consists of a vector supermultiplet containing a set of photons which are related by the duality transformation D to the photons of the unbroken $U(1)^r$ symmetry, and a light hypermultiplet of electric charge \mathbf{q}' describing the BPS state. These light charged states cause the dual coupling constant to run in an asymptotically infra-red free way. To one-loop perturbation theory in the dual variables one has

$$\mathbf{a}'_D = -\frac{i}{2\pi} \mathbf{q}' (\mathbf{q}' \cdot \mathbf{a}') \ln \left(\frac{\mathbf{q}' \cdot \mathbf{a}'}{\Lambda'} \right). \quad (2.6)$$

This one-loop expression is enough to determine the monodromy around the singularity $Z_Q = \mathbf{a}' \cdot \mathbf{q}' = 0$:

$$A' \rightarrow A' M_{Q'}, \quad M_{Q'} = \begin{pmatrix} 1 & \mathbf{q}' \otimes \mathbf{q}' \\ 0 & 1 \end{pmatrix} = 1 + (\Omega Q'^T) Q'. \quad (2.7)$$

To find the monodromy transformation M_Q we simply have to transform back to the original variables $M_Q = D^{-1} M_{Q'} D$, which given that D is a duality transformation, yields the expression in (2.4). Using (2.4), the action of the monodromy transformation M_Q on the charge \tilde{Q} is

$$\tilde{Q} \rightarrow \tilde{Q} M_Q = \tilde{Q} + (\tilde{Q} \cdot Q) Q. \quad (2.8)$$

3 The Riemann Surface: Simply-Laced

In this section, we explain how the Riemann surface and the associated data λ and the $2r$ preferred cycles ν_i^a are constructed. It is helpful to consider the simply-laced and non-simply-laced cases separately and so in this section we shall be considering the former cases only, the non-simply-laced cases will be considered in a later section.

When the gauge group G has a simply-laced Lie algebra \mathfrak{g} one chooses a representation ρ of \mathfrak{g} and defines a Riemann surface via a characteristic polynomial in two auxiliary variables x and z :

$$\det [\rho(A(z)) - x \cdot \mathbf{1}] = 0, \quad (3.1)$$

where

$$A(z) = \varphi \cdot \mathbf{H} + \sum_{i=1}^r (E_{\alpha_i} + E_{-\alpha_i}) + z E_{-\theta} + \mu z^{-1} E_{\theta}. \quad (3.2)$$

In the above, E_{α_i} is the step generator of \mathfrak{g} associated to the simple root α_i and E_{θ} is the generator associated to the highest root θ . The Cartan generators of \mathfrak{g} are denoted by the $r = \text{rank}(\mathfrak{g})$ dimensional vector \mathbf{H} . The

r -dimensional complex vector φ parametrizes the moduli space of the surface which is identified with \mathcal{M} . The parameter μ is equal to $\Lambda^{2\dim(\rho)}$, where Λ is the familiar scale of strong coupling effects. It turns out that the surface that has been defined is the spectral curve of an integral system. In fact $\mathbf{A}(z)$ is the Lax operator of the $g^{(1)}$ Toda system, with z playing the role of the loop variable [2]. This fascinating observation will not play any central role in the present discussion.

Following [2], it is most convenient to think of the Riemann surface as a foliation over the Riemann sphere for z by extending x to an analytic function of z . The number of leaves of the foliation is then equal to the dimension of the representation ρ . The operator $\mathbf{A}(z)$ is always conjugate to some element of the Cartan subalgebra $\tilde{\varphi}(z) \cdot \mathbf{H}$, however, there remains the freedom to perform conjugations within the Weyl subgroup of the gauge group. This freedom can be fixed by choosing, for example, $\text{Re}(\tilde{\varphi}(z))$ to be in, or on the wall of, the Fundamental Weyl Chamber, i.e.

$$\alpha_i \cdot \text{Re}(\tilde{\varphi}(z)) \geq 0. \quad (3.3)$$

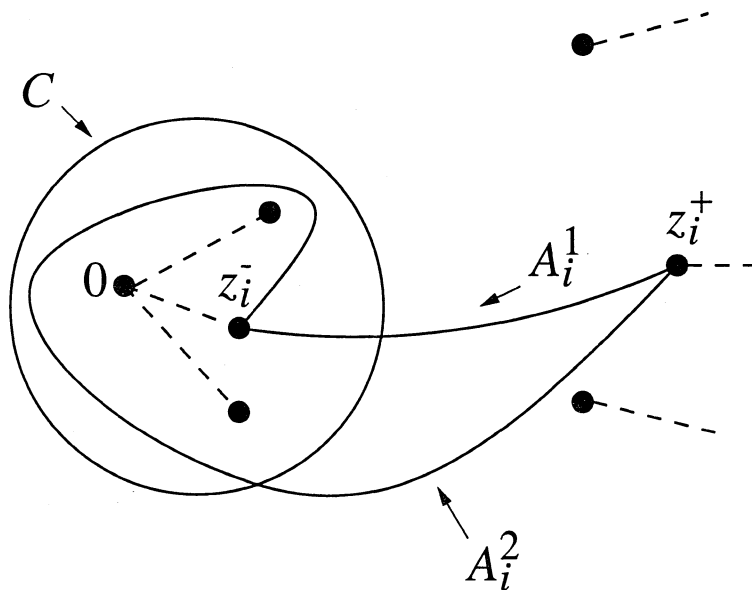
This condition determines a series of cuts on the z -plane.

On the leaf of the foliation associated to the weight vector ω , we have $x = \omega \cdot \tilde{\varphi}(z)$. Two leaves are connected at branch-points whenever two such eigenvalues of $\mathbf{A}(z)$ coincide. Notice that due to our choice (3.3), this can only happen when

$$\alpha_i \cdot \tilde{\varphi}(z) = 0, \quad (3.4)$$

for a simple root α_i . At such a point, the pair of sheets labelled by ω and ω' , for which $\omega' = \sigma_i(\omega)$, are joined. (Here, σ_i is the Weyl reflection in the simple root α_i .) It follows from this, that the foliation splits into disconnected components corresponding to the separate orbits of the weights under the Weyl group of g . Since $\rho(\mathbf{A}(z))^T = \rho(\mathbf{A}(\mu/z))$, it follows that (3.1) is invariant under $z \rightarrow \mu/z$. As a consequence, the branch-points will occur in pairs z_i^\pm , related by $z_i^+ z_i^- = \mu$. To fix the definition, we define z_i^- to be the branch-points that tend to 0 in the weak coupling limit $\mu \rightarrow 0$. There are also branch-points at 0 and ∞ that connect all the sheets corresponding to weights on a particular orbit of the Weyl group. The branch-points are connected by cuts which are specified by our Weyl Group-fixing choice in (3.3).

Notice that the positions of the branch-points are independent of the representation ρ . Clearly this will prove significant when we come to show that the construction is independent of the choice of representation ρ . Notice also, that the lift of the contour, labelled C in Fig. 1, which encircles the origin and the branch-points z_i^- , will return to the same sheet after one circuit.

Figure 1: Definition of contours A_i^a and C

As alluded to in the introduction, the connection between the Riemann surface and the $N = 2$ supersymmetric gauge theories is established through the existence of a special meromorphic differential λ and a preferred set of $2r$ cycles on the surfaces. The differential is simply

$$\lambda = \frac{1}{2\pi i} \frac{x dz}{z}, \quad (3.5)$$

which, due to dependence on x , has a different expression on each sheet.

The vanishing cycles correspond to points in the moduli space where a pair of branch-points z_i^\pm become coincident. Obviously since $z_i^- z_i^+ = \mu$, this occurs at either $z = \pm\sqrt{\mu}$. A given pair of branch-points can come together along an infinite number paths depending on the arbitrary number of circuits around the origin that are made before the points become coincident. In order to specify the solution, it is necessary to choose just two vanishing paths, for each i , one associated to when z_i^\pm come together at $z = \sqrt{\mu}$, and the other at $-\sqrt{\mu}$. This will then provide the $2r$ cycles required for defining \mathbf{a} and \mathbf{a}_D . The $2r$ contours will be denoted by A_i^a , $i = 1, \dots, r$ and $a = 1, 2$. For fixed i , each pair joins z_i^- to z_i^+ along a path which avoids any cut on the same sheet of the foliation as z_i^\pm . There is a certain ambiguity present in the choice of the contours which we will later interpret as a manifestation of the ‘democracy of dyons’ described in the context of $SU(2)$ in [1]. Let us suppose we choose a set of contours A_i^1 with intersections defined by the

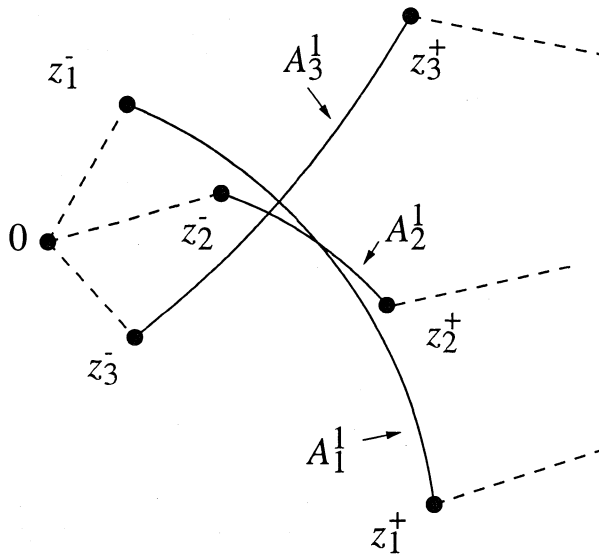


Figure 2: Choice of contours A_i^1 for $g = A_3$

anti-symmetric matrix \mathcal{I}_{ij} . We will fix our definition of orientation by saying that the intersection of two contours A and B , denoted $A \circ B$ is 1 (–1) if B crosses A from right (left). It is always possible to choose, for example, $\mathcal{I}_{ij} = 0$ when ever $\alpha_i \cdot \alpha_j \neq 0$, which fixes the ambiguity up to an overall integer corresponding to the number of times the contours as a whole wind around the origin. The partner contours A_i^2 are then defined as

$$A_i^2 = C + A_i^1, \quad (3.6)$$

i.e. with an extra winding around the origin. This is illustrated in Fig. 1. It is easy to see that the pair A_i^a , for fixed i , correspond to paths where z_i^\pm come together at each of the two singular points $\pm\sqrt{\mu}$. For example, Fig. 2 shows the branch-points and a choice of contours A_i^1 for $g = A_3$. In this case $\mathcal{I}_{12} = \mathcal{I}_{13} = \mathcal{I}_{23} = 1$.

The set of $2r$ cycles ν_i^a that are used to specify the solution in (1.7) are defined as particular lifts of the contours A_i^a to the leaves of the foliation. The weighting given to each contour will turn out to be crucial and we shall find that the correct lift is

$$\nu_i^a = \frac{1}{N_\rho} \sum_{\omega} (\omega \cdot \alpha_i) A_i^a(\omega), \quad (3.7)$$

where $A_i^a(\omega)$ is the lift of the contour A_i^a to the sheet labelled by ω and N_ρ is a normalization factor which will be determined below. Before we

continue it is important to show that the ν_i^a are indeed (closed) one-cycles on the surface. The proof becomes trivial when one notices that contour $A_i^a(\omega)$ joining z_i^- to z_i^+ on the sheet associated to ω is always accompanied by a return contour $A_i^a(\omega')$, having opposite weight in (3.7), on the sheet associated to $\omega' = \sigma_i(\omega)$. Hence the ν_i^a are indeed closed one-cycles. In the following, it is convenient to make this explicit by introducing the closed cycles

$$\hat{A}_i^a(\omega) = A_i^a(\omega) - A_i^a(\sigma_i(\omega)), \quad (3.8)$$

in terms of which

$$\nu_i^a = \frac{1}{2N_\rho} \sum_{\omega} (\omega \cdot \alpha_i) \hat{A}_i^a(\omega). \quad (3.9)$$

The extra factor of $1/2$ is to prevent over-counting due to the fact that $\hat{A}_i(\sigma_i(\omega)) = -\hat{A}_i(\omega)$. It is now a simple matter to show that \mathbf{a} and \mathbf{a}_D are independent of the representation ρ , once the normalization factor N_ρ has been fixed. We first notice that

$$\oint_{\hat{A}_i^a(\omega)} \lambda = \int_{A_i^a} (\omega \cdot \tilde{\varphi}(z) - \omega' \cdot \tilde{\varphi}(z)) \frac{dz}{z} = \frac{\omega \cdot \alpha_i}{\alpha_i^2} \int_{A_i^a} \alpha_i \cdot \tilde{\varphi}(z) \frac{dz}{z} \quad (3.10)$$

This means that $\oint_{\nu_i^a} \lambda$ only depends on the representation ρ through the overall normalization $\sum_{\omega} (\omega \cdot \alpha_i)^2 / N_\rho$. By taking

$$N_\rho \cdot \mathbf{1} = \sum_{\omega} \omega \otimes \omega, \quad (3.11)$$

guarantees that the integrals of λ around the ν_i^a and hence \mathbf{a} and \mathbf{a}_D are independent of ρ . The cycles that we have constructed pick out the special Prym subvariety of the Jacobian of the surface. It is precisely the subvariety associated to the reflection representation of the Weyl group [2]. This is explicit in (3.7), where the Weyl group acts on the contours as $\sigma : A_i(\omega) \mapsto A_i(\sigma(\omega))$.

In order to make connection with the electric and magnetic charges, it is useful to notice that in the weak coupling limit, although the integrals of λ around ν_i^a are logarithmically divergent, the integral around the difference $\nu_i^1 - \nu_i^2$, which is a lift of the cycle C , is well-defined. To see this, notice that in the weak coupling limit $\mu \rightarrow 0$ so all the branch-points $z_i^- \rightarrow 0$. At $\mu = 0$, the function $\tilde{\varphi}(z)$ is analytic at the origin and therefore using (3.10) we find that the integral is equal to $\alpha_i \cdot \tilde{\varphi}_{\mu=0}(0)$. We can therefore identify $\tilde{\varphi}_{\mu=0}(0)$ as the vector which specifies the classical Higgs VEV, that is \mathbf{a} in the weak coupling limit. Hence $\nu_i^1 - \nu_i^2$ is a purely electric cycle corresponding to an electric charge vector α_i . This means that the charges of the dyons satisfy:

$$Q_i^1 - Q_i^2 = (\mathbf{0}, \alpha_i). \quad (3.12)$$

4 The Strong Coupling Monodromies: Simply-Laced

In this section, we will show that the monodromy transformations around the $2r$ vanishing cycles ν_i^a can be identified with the monodromies associated to a set of $2r$ BPS states, as calculated in section 2. Once we have identified these states we can then calculate the weak coupling monodromies and show that they reproduce those calculated in section 2.

In order to calculate the monodromy transformations around a vanishing cycle, we use the Picard-Lefschetz Theorem which allows one to calculate the action of monodromies on the cycles ν_i^a themselves. We can then lift this action to the charges Q_i^a by using (1.7). The Picard-Lefschetz Theorem states that the monodromy associated to a vanishing cycle with a single connected component δ on another cycle is

$$\mathcal{M}_\delta(\zeta) = \zeta + (\zeta \circ \delta)\delta. \quad (4.1)$$

Here, $\zeta \circ \delta$ is the intersection number of the two cycles. Since our vanishing cycles have, in general, more than one connected component we will need a refinement of the Picard-Lefschetz Theorem. If $\delta = \sum_i n_i \delta_i$, a union of disconnected components with certain non-zero coefficients n_i , then

$$\mathcal{M}_\delta(\zeta) = \zeta + \sum_i (\zeta \circ \delta_i) \delta_i. \quad (4.2)$$

Applying this formula allows us to deduce the action of the monodromies $\mathcal{M}_{\nu_i^a}$ associated to the vanishing cycles ν_i^a on the other vanishing cycles. Before we do this, it is helpful to know that

$$\hat{A}_i^a(\omega) \circ \hat{A}_j^b(\omega') = (\epsilon_{ab} + \mathcal{I}_{ij}) \left(\delta_{\omega, \omega'} - \delta_{\sigma_i(\omega), \omega'} - \delta_{\omega, \sigma_j(\omega')} + \delta_{\sigma_i(\omega), \sigma_j(\omega')} \right). \quad (4.3)$$

In the above, \mathcal{I}_{ij} is the intersection form of the A_i^1 contours in the z -plane defined previously and ϵ_{ab} is the two-dimensional anti-symmetric tensor.

Using (4.2) and (4.3), we can now calculate the monodromy transformations around the vanishing cycles:

$$\mathcal{M}_{\nu_i^a} \left(\nu_j^b \right) = \nu_j^b + \frac{1}{2} \sum_{\omega} \left(\nu_j^b \circ \hat{A}_i^a(\omega) \right) \hat{A}_i^a(\omega). \quad (4.4)$$

The extra factor of $1/2$ is to prevent over-counting due to $\hat{A}_i(\sigma_i(\omega)) =$

$-\hat{A}_i(\omega)$. The intersection number above can be written as

$$\begin{aligned}\nu_j^b \circ \hat{A}_i^a(\omega) &= \frac{1}{2N_\rho} \sum_{\omega'} \omega' \cdot \alpha_j \left(\hat{A}_j^b(\omega') \circ \hat{A}_i^a(\omega) \right) \\ &= \frac{1}{N_\rho} (\epsilon_{ba} + \mathcal{I}_{ji}) (\alpha_j \cdot \omega - \alpha_j \cdot \sigma_i(\omega)) \\ &= \frac{1}{N_\rho} (\epsilon_{ba} + \mathcal{I}_{ji}) \frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2} \alpha_i \cdot \omega,\end{aligned}\tag{4.5}$$

where we have used (4.3). Since g is simply-laced, all the roots have the same length which we normalize as $\alpha_i^2 = 2$, in which case (4.4) becomes

$$\mathcal{M}_{\nu_i^a} \left(\nu_j^b \right) = \nu_j^b + (\epsilon_{ba} + \mathcal{I}_{ji}) (\alpha_i \cdot \alpha_j) \nu_i^a.\tag{4.6}$$

This is equivalent, via (1.7), to the following action on the charges Q_i^a :

$$\mathcal{M}_{Q_i^a} \left(Q_j^b \right) = Q_j^b + (\epsilon_{ba} + \mathcal{I}_{ji}) (\alpha_i \cdot \alpha_j) Q_i^a.\tag{4.7}$$

By comparing (4.7) with (2.8) we see that the charges must have the following symplectic inner products:

$$Q_j^b \cdot Q_i^a = (\epsilon_{ba} + \mathcal{I}_{ji}) \alpha_i \cdot \alpha_j.\tag{4.8}$$

The charges are further constrained by (3.12). This determines the charges in the form

$$Q_i^1 = (\alpha_i, p_i \alpha_i), \quad Q_i^2 = (\alpha_i, (p_i + 1) \alpha_i),\tag{4.9}$$

where, for i and j such that $\alpha_i \cdot \alpha_j \neq 0$,

$$p_i - p_j = \mathcal{I}_{ji},\tag{4.10}$$

otherwise $p_i - p_j$ is unconstrained. Since the Dynkin diagrams have no closed loops, these equations always admit a solution for integer p_i , up to an overall integer. The integer p_i is correlated with the choice of the contour A_i^1 in the sense that if we shift it by an additional winding around the origin, i.e. $A_i^1 \rightarrow A_i^1 + C$, then $p_i \rightarrow p_i + 1$. The overall integer reflects the freedom for all the contours to wind around the origin an arbitrary number of times. Recall that it is always possible to choose the A_i^1 contours so that $\mathcal{I}_{ji} = 0$ whenever $\alpha_i \cdot \alpha_j \neq 0$. With this choice we have $p_i = n$, for some integer n , for all i . For the A_3 example in Fig. 2 the charges are

$$Q_1^1 = (\alpha_1, n\alpha_1), \quad Q_2^1 = (\alpha_2, (n+1)\alpha_2), \quad Q_3^1 = (\alpha_3, (n+2)\alpha_3),\tag{4.11}$$

for integer n .

From the above, we deduce that the dyon of charge $(\alpha_i, n\alpha_i)$ has a vanishing cycle that is the lift of $A_i^1 + (n - p_i)C$. This is a manifestation of the democracy of dyons.

5 The Riemann Surface: Non-Simply-Laced

In this section, we explain how to generalize the discussion to the theories whose gauge group has a non-simply-laced Lie algebra. Many of the details are similar to the simply-laced cases.

First of all, we define the dual algebra g^\vee to be the algebra obtained by interchanging long and short roots, i.e. by replacing each root α by its co-root $\alpha^\vee = 2\alpha/\alpha^2$. The construction of the characteristic polynomial for a non-simply-laced algebra g , involves a simply-laced algebra \tilde{g} related to g by $\tilde{g}^{(\tau)} = (g^{(1)})^\vee$. In other words the dual of the affine algebra $g^{(1)}$ is the twisted affinization of the simply-laced algebra \tilde{g} of respect to an outer (diagram) automorphism π with order τ . For the non-simply-laced Lie algebras the associated simply-laced algebras are given by

$$\left(B_r^{(1)}\right)^\vee = A_{2r-1}^{(2)}, \quad \left(C_r^{(1)}\right)^\vee = D_{r+1}^{(2)}, \quad \left(F_4^{(1)}\right)^\vee = E_6^{(2)}, \quad \left(G_2^{(1)}\right)^\vee = D_4^{(3)}. \quad (5.1)$$

Under the outer automorphism, \tilde{g} is decomposed into eigenspaces:

$$\tilde{g} = \bigoplus_{p=0}^{\tau-1} \tilde{g}_p, \quad (5.2)$$

where $\pi(\tilde{g}_p) = \exp(2\pi ip/\tau)\tilde{g}_p$. The invariant subalgebra \tilde{g}_0 is precisely the original non-simply-laced Lie algebra g and the eigenspaces are irreducible highest weight representations of g . The simple roots of g are identified with the π -invariant combinations of the simple-roots of \tilde{g} for each orbit of the simple-roots under π :

$$\alpha_i = \sum_{p=1}^{\tau_i} \tilde{\alpha}_{i_p}, \quad (5.3)$$

where $\tilde{\alpha}_{i_p} = \pi^{p-1}(\tilde{\alpha}_{i_1})$, $p = 1, \dots, \tau_i$, are the simple roots on the i^{th} orbit of the outer automorphism π . The orbit is either of dimension τ , in which case α_i is a long root of g , or one dimensional, in which case α_i is a short root of g . In the latter case, we have by definition $i_p = i_1$. It is convenient to define L and S as the set of i such that α_i is long and short, respectively. We will choose the labelling in each of the long orbits so that $\alpha_{i_p} \cdot \alpha_{j_q} \neq 0$ only for $p = q$.

The Riemann surface for g is constructed as in (3.1) by replacing $g^{(1)}$ by the twisted affine algebra $\tilde{g}^{(\tau)}$. One fixes a representation $\tilde{\rho}$ of \tilde{g} and the $\tilde{r} = \text{rank}(\tilde{g})$ dimensional complex vector φ is constrained to lie in the

r -dimensional subspace invariant under the outer automorphism π .² The generator E_θ is now associated to θ the highest weight of the representation of \tilde{g}_1 of g , rather than the highest root. As in section 3, we shall think of the Riemann surface as a foliation over the z plane and we shall fix the details of the foliation by choosing

$$\tilde{\alpha}_{i_p} \cdot \operatorname{Re}(\tilde{\varphi}(z)) \geq 0. \quad (5.4)$$

There are branch-points connecting two sheets $\tilde{\omega}$ and $\tilde{\omega}' = \sigma_{i_p}(\tilde{\omega})$ of the foliation when $\tilde{\alpha}_{i_p} \cdot \tilde{\varphi}(z) = 0$. These branch-points come in τr pairs $z_i^\pm(p)$, $i = 1, \dots, r$ and $p = 1, \dots, \tau$. To see this, we remark that because $\tilde{\alpha}_{i_p} \cdot \tilde{\varphi}(z)$ is an element of $\tilde{g}^{(\tau)}$ it is invariant under a combination of the action of the outer automorphism π along with $z \rightarrow \exp(-2\pi i/\tau)z$, i.e.

$$\pi(\tilde{\alpha}_{i_p}) \cdot \tilde{\varphi}(z \exp(-2\pi i/\tau)) = \tilde{\alpha}_{i_p} \cdot \tilde{\varphi}(z). \quad (5.5)$$

Therefore the outer automorphism has an action on the branch-points in the z -plane given by

$$z_i^\pm(p+1) = \exp(2\pi i/\tau) z_i^\pm(p). \quad (5.6)$$

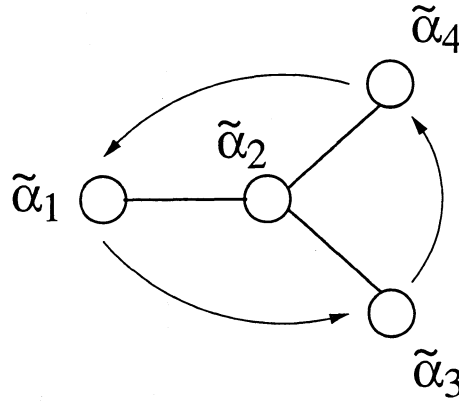
Notice that the branch-points associated to a simple root $\tilde{\alpha}_{i_1}$ which is fixed under π come with a multiplicity of τ . Since $\tilde{\rho}(A(z))^T = \tilde{\rho}(A'(\mu/z))$, where the prime indicates a re-labelling of the simple roots $i_p \leftrightarrow i_{-p}$ (where the label p is understood to be defined modulo τ) one easily deduces the additional relation

$$z_i^+(p) z_i^-(-p) = \mu. \quad (5.7)$$

The preferred set of $2r$ vanishing cycles ν_i^a , $i = 1 \dots, r$ and $a = 1, 2$ are defined as follows. Each of the pairs $z_i^\pm(p)$ can come together at two points $\pm\sqrt{\mu} \exp(2\pi i p/\tau)$ and for each pair we define two contours in the z -plane $A_i^a(p)$, $a = 1, 2$, which vanish when $z_i^\pm(p)$ come together at each of two points above. As for the simply-laced cases, there is a certain ambiguity in defining contours. Firstly, we choose the set of contours such that, under $z \rightarrow z \exp(2\pi i/\tau)$, $A_i^1(p+1)$ is the image of $A_i^1(p)$. Let us denote the intersection number of these contours by $\mathcal{I}_{ij}(p - q \bmod \tau)$. The fact that the intersection form depends on $p - q$ alone is due to its invariance under the outer automorphism. Each contour has a partner $A_i^2(p)$ defined in the following way. If the simple root $\tilde{\alpha}_{i_p}$ is not fixed under π then

$$i \in L : \quad A_i^2(p) = C + A_i^1(p), \quad (5.8)$$

²We will use the single symbol π to describe the various actions of the outer automorphism: (i) on the simple roots (ii) lifted into the Lie algebra (iii) on vectors in Cartan space, since no confusions should arise.

Figure 3: Action of outer automorphism on D_4 giving G_2

where C is the contour that surrounds the origin and all the $z_i^-(p)$, i.e. $A_i^2(p)$ is equal to $A_i^1(p)$ plus an extra winding around the origin. If the simple root $\tilde{\alpha}_{i_p}$ is fixed under π , then $A_i^2(p)$ is the contour which connects $z_i^-(p)$ to $z_i^+(p+1)$ (with p defined modulo τ) in such a way that

$$i \in S: \quad A_i^2(p) = C(p) + A_i^1(p), \quad (5.9)$$

where $C(p)$ is the contour which connects $z_i^+(p)$ to $z_i^+(p+1)$. These contours lift to the foliation in an identical way to those in section 3. We define $A_i^a(\tilde{\omega}; p)$ to be the lift of $A_i^a(p)$ to the sheet labelled by the weight $\tilde{\omega}$. Notice the essential difference between $A_i^2(\tilde{\omega}; p)$, according to whether $i \in L$ or $i \in S$. In the latter case, the $z_+(p)$ are on the same sheets of the foliation, hence it is consistent for a contour to emerge from $z_i^-(p)$ and disappear down $z_i^+(p+1)$. This is not so when $i \in L$.

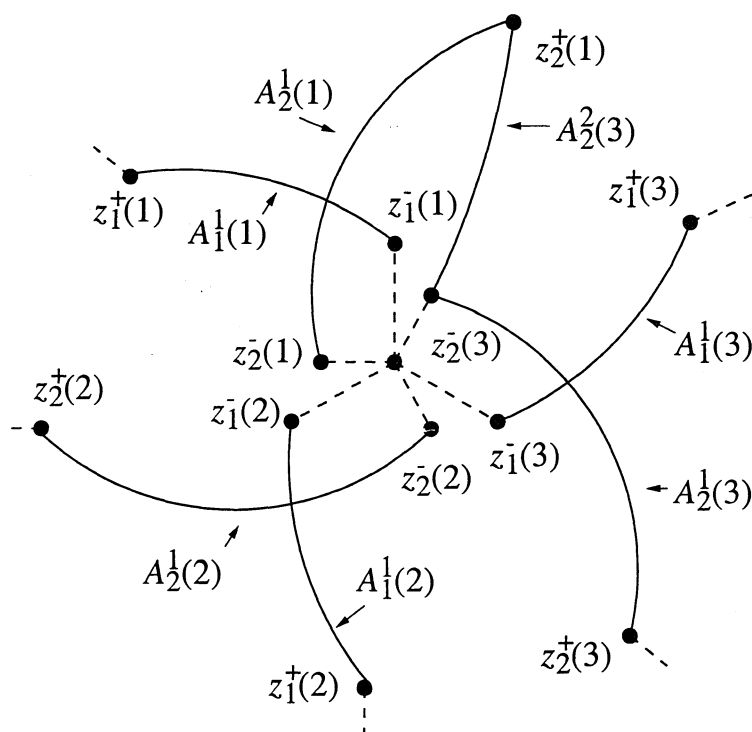
As an example consider the case of G_2 which arises via a third order outer automorphism of D_4 . There are two orbits for the simple roots under π , $i = 1$ containing $\tilde{\alpha}_1$, $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ and $i = 2$ containing $\tilde{\alpha}_2$ only. This is illustrated in Fig. 3.

The positions of the branch-points and a choice for the contours $A_i^1(p)$ (along with only $A_2^2(3)$ for clarity) is illustrated in Fig. 4. With this choice $\mathcal{I}_{12}(0) = -1$ and $\mathcal{I}_{12}(1) = \mathcal{I}_{12}(2) = 0$.

In terms of these cycles the basis set of preferred cycles are

$$\nu_i^a = \frac{1}{N_{\tilde{\rho}}} \sum_{\tilde{\omega}} \sum_{p=1}^{\tau} (\tilde{\omega} \cdot \pi^{p-1}(\tilde{\alpha}_{i_1})) A_i^a(\tilde{\omega}; p). \quad (5.10)$$

where $N_{\tilde{\rho}}$ is a constant, dependent upon $\tilde{\rho}$, given in (3.11). As in the last section, it is straightforward to see that the ν_i^a are indeed a sum of closed

Figure 4: Choice of contours for G_2

cycles on the foliated surface. To see this notice that they can be re-expressed in terms of the closed cycles $\hat{A}_i^a(\tilde{\omega}; p) = A_i^a(\tilde{\omega}; p) - A_i^a(\sigma_{i_p}(\tilde{\omega}); p)$. One can show, in a way completely analogous to section 3, that the integrals of λ around the ν_i^a are independent of the representation $\tilde{\rho}$.

As for the simply-laced groups, it is straightforward to determine the weak coupling limit of the integral around the cycle $\nu_i^1 - \nu_i^2$. One finds that

$$(Q_i^1 - Q_i^2) \cdot A = \alpha_i \cdot \tilde{\varphi}_{\mu=0}(0), \quad (5.11)$$

Hence, as before,

$$Q_i^1 - Q_i^2 = (\mathbf{0}, \alpha_i), \quad (5.12)$$

is purely electrically charged and in the weak coupling limit $\tilde{\varphi}(0)$ is proportional to the classical Higgs VEV.

6 The Strong Coupling Monodromies: Non-Simply-Laced

In this section, we calculate the strong coupling monodromies for the non-simply-laced cases. First of all, let us write the intersection of the closed cycles $\hat{A}_i^a(\tilde{\omega}; p)$ in the form

$$\begin{aligned} \hat{A}_i^a(\tilde{\omega}; p) \circ \hat{A}_j^b(\tilde{\omega}; q') = \\ \mathcal{I}_{ij}^{ab}(p - q) \left(\delta_{\tilde{\omega}, \tilde{\omega}'} - \delta_{\sigma_{i_p}(\tilde{\omega}), \tilde{\omega}'} - \delta_{\tilde{\omega}, \sigma_{j_q}(\tilde{\omega}')} + \delta_{\sigma_{i_p}(\tilde{\omega}), \sigma_{j_q}(\tilde{\omega}')} \right), \end{aligned} \quad (6.1)$$

where the intersection form $\mathcal{I}_{ij}^{ab}(p - q)$, for $a = b = 1$, is equal to $\mathcal{I}_{ij}(p - q)$ defined in the last section.

We now apply the Picard-Lefschetz theorem to find the monodromies around the vanishing cycles as in section 4. One finds

$$\begin{aligned} \mathcal{M}_{\nu_i^a}(\nu_j^b) = \\ \nu_j^b + \frac{1}{2N_{\tilde{\rho}}} \sum_{\tilde{\omega}} \sum_{p, q=1}^{\tau} (\tilde{\alpha}_{j_1} \cdot \pi^{p-q}(\tilde{\alpha}_{i_1})) (\tilde{\alpha}_{i_p} \cdot \tilde{\omega}) \mathcal{I}_{ji}^{ba}(q - p) \hat{A}_i^a(\omega; p). \end{aligned} \quad (6.2)$$

In order to simplify the above expression, we have to consider the various cases that arise. If $i, j \in L$ then $\tilde{\alpha}_{j_q} \cdot \tilde{\alpha}_{i_p} = \tilde{\alpha}_{j_1} \cdot \pi^{p-q}(\tilde{\alpha}_{i_1}) \neq 0$ only when $p = q$, in which case

$$\mathcal{M}_{\nu_i^a}(\nu_j^b) = \nu_j^b + \mathcal{I}_{ji}^{ba}(0) \tilde{\alpha}_{j_1} \cdot \tilde{\alpha}_{i_1} \nu_i^a. \quad (6.3)$$

For the other three possible cases, where at least i or j or both $\in S$, one finds

$$\mathcal{M}_{\nu_i^a}(\nu_j^b) = \nu_j^b + \left(\sum_{p=0}^{\tau-1} \mathcal{I}_{ji}^{ba}(p) \right) 2\tilde{\alpha}_{j_1} \cdot \tilde{\alpha}_{i_1} \nu_i^a. \quad (6.4)$$

We can further simplify the results above and writing them in terms of the intersection form $\mathcal{I}_{ij}(p)$ and in terms of the inner product of the simple roots of g :

$$\mathcal{M}_{\nu_i^a}(\nu_j^b) = \nu_j^b + (\xi_{ba}(\alpha_i^2/\alpha_j^2) + D_{ji}) \frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2} \nu_i^a. \quad (6.5)$$

In the above, D_{ji} is defined as follows:

$$\begin{aligned} j \in L : \quad D_{ji} &= \begin{cases} \mathcal{I}_{ji}(0) & i \in L \\ \frac{1}{\tau} \sum_{p=0}^{\tau-1} \mathcal{I}_{ji}(p) & i \in S \end{cases} \\ j \in S : \quad D_{ji} &= \sum_{p=0}^{\tau-1} \mathcal{I}_{ji}(p). \end{aligned} \quad (6.6)$$

$$(6.7)$$

The matrix $\xi_{ba}(x)$ is defined as

$$\xi_{ba}(x) = \begin{pmatrix} 0 & x \\ -1 & x-1 \end{pmatrix} \quad (6.8)$$

The charges must therefore have symplectic inner products

$$Q_j^b \cdot Q_i^a = (\xi_{ba}(\alpha_i^2/\alpha_j^2) + D_{ji}) \frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2}, \quad (6.9)$$

and be further constrained by (5.12). Notice that the expression on the right-hand-side of (6.9) is antisymmetric under the interchange of i and j , and a and b , as it should be for consistency. The charges are determined in the form

$$Q_i^1 = (\alpha_i^\vee, p_i \alpha_i), \quad Q_i^2 = (\alpha_i^\vee, (p_i + 1) \alpha_i), \quad (6.10)$$

where, for i and j such that $\alpha_i \cdot \alpha_j \neq 0$,

$$\frac{\alpha_i^2}{\alpha_j^2} p_i - p_j = D_{ji}. \quad (6.11)$$

Notice immediately that the magnetic charges are indeed co-roots of g as required by the DSZ quantization condition. The equations (6.11) always admit a solution up to an overall integer ambiguity

$$p_i \rightarrow p_i + \begin{cases} n & i \in L \\ n\tau & i \in S. \end{cases} \quad (6.12)$$

This reflects the usual ambiguity in the number of times that the contours as a whole wind around the origin. For the G_2 example of Fig. 4, the charges are

$$Q_1^1 = (\alpha_1^\vee, n\alpha_1), \quad Q_2^1 = (\alpha_i^\vee, (3n-1)\alpha_2), \quad (6.13)$$

for integer n .

If we re-define $A_i^1(p)$, with $i \in L$, to have an additional winding around the origin this corresponds to a shift $p_i \rightarrow p_i + 1$. Similarly, if we redefine $A_i^1(p)$, $i \in S$, so that it connects $z_i^-(p)$ to $z_i^+(p+1)$, rather than $z_i^+(p)$, then this also corresponds to a shift $p_i \rightarrow p_i + 1$. Hence, a dyon with charge $(\alpha_i^\vee, n\alpha_i)$ is related to a vanishing cycle which is the lift of $A_i^1(p) + (n - p_i)C$, when $i \in L$, and $A_i^1(p) + \sum_{a=0}^{n-p_i-1} C(p+a \bmod \tau)$, when $i \in S$. As a consequence of this, there is a vanishing cycle for each of the dyons of charge $(\alpha_i^\vee, n\alpha_i)$, for any $n \in \mathbb{Z}$. As before, this reflects the democracy of dyons.

7 Discussion

Now that we have shown that the strong coupling monodromies can be explained by dyons, we can subject the Martinec-Warner construction to two additional non-trivial tests.

The first of these involves showing that the correct weak coupling monodromies are produced. A monodromy at weak coupling M_i corresponds to a path for which $z_i^\pm \rightarrow \exp(\pm 2\pi i)z_i^\pm$, for the simply-laced algebras, and $z_i^\pm(p) \rightarrow \exp(\pm 2\pi i\tau_i/\tau)z_i^\pm(p)$, for the non-simply-laced algebras. These transformations can be achieved without leaving the weak coupling regime. Such a path corresponds precisely to the encircling the pair of strong coupling singularities corresponding to the vanishing cycles ν_i^a , $a = 1, 2$. The picture is then a rather simple generalization of the $SU(2)$ case discussed by Seiberg and Witten [1]. Being careful with the order, one finds that

$$M_i = M_{Q_i^2} M_{Q_i^1}. \quad (7.1)$$

It is a simple matter to verify that the right-hand-side is equal to the weak coupling monodromy in (2.3), when computed using (2.4), and the facts that $Q_i^1 = (\alpha_i^\vee, n\alpha_i)$, for some $n \in \mathbb{Z}$, and $Q_i^1 - Q_i^2 = (0, \alpha_i)$.

The second non-trivial test is to compare the spectrum of dyons which are responsible for the strong coupling singularities with those present at weak coupling. There is no absolute guarantee that dyons present at weak coupling will survive without decay all the way to the strong coupling singularities, although it can be shown that they do in $SU(2)$ [1]. Nevertheless, the match is perfect because the weak coupling spectrum consists of dyons of charge $(\alpha_i^\vee, n\alpha_i)$, for $n \in \mathbb{Z}$, precisely the ones responsible for the strong coupling singularities. The weak coupling spectrum also consists of dyons whose magnetic charges are non-simple co-roots. The spectrum of such dyons is complicated by the fact the allowed electric charges vary in different cells of moduli space separated by surfaces on which these dyons decay [12, 13]. However, such dyons are related by products of weak coupling monodromies to those dyons whose magnetic charge is a simple root. Hence there is a true democracy, since for the dyons whose magnetic charge is a non-simply root, one can first follow a path which undoes the semi-classical monodromy and then proceed to a singularity.

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