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On Deformed $\mathcal{W}$-algebras and Quantum Affine Algebras

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Abstract

We discuss some aspects of the deformed $\mathcal{W}$-algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$. In particular, we derive an explicit formula for the Kac determinant, and discuss the center when t^2 is a primitive k-th root of unity. The relation of the structure of $\mathcal{W}_{q,t}[\mathfrak{g}]$ to the representation ring of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, as discovered recently by Frenkel and Reshetikhin, is further elucidated in some examples.

1 Introduction

In recent years there has been a considerable interest in understanding the role of infinite dimensional quantum algebras in the theory of off-critical integrable models of statistical mechanics. In particular, it has become clear that the algebraic framework of those theories closely parallels that of their critical counterpart – the conformal field theories.

In this context the algebras of particular interest are the so-called deformed $\mathcal{W}$-algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$ associated to arbitrary simple finite dimensional Lie algebras $\mathfrak{g}$. They can be considered to be deformations of the $\mathcal{W}$-algebras

$\mathcal{W}[\mathfrak{g}]$ that arise in conformal field theory.[1] In analogy with the undeformed case, the algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$ are most conveniently defined as the centralizer of a set of screening operators $S_i^{\pm}$ in a deformed Heisenberg algebra $\mathcal{H}_{q,t}[\mathfrak{g}]$.

The simplest example, the deformed Virasoro algebra $\mathrm{Vir}_{q,t} \cong \mathcal{W}_{q,t}[\mathfrak{sl}_2]$, was introduced in [2,3] and further studied in [4–10]. In [6,11] it has been argued that this deformed Virasoro algebra plays the role of the dynamical symmetry algebra in the Andrews-Baxter-Forrester RSOS models. The higher rank generalizations, $\mathcal{W}_{q,t}[\mathfrak{sl}_N]$, were introduced in [12–17]. The deformed $\mathcal{W}$-algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$, for arbitrary simple finite dimensional Lie algebras $\mathfrak{g}$, were introduced recently by Frenkel and Reshetikhin [16] and further studied in, e.g., [18].

In this paper we further investigate the structure of $\mathcal{W}_{q,t}[\mathfrak{g}]$. In particular, we derive explicit expressions for elements in the centralizer of the screening operators for $\mathfrak{g} = \mathfrak{sl}_2$, i.e., generators of $\mathcal{W}_{q,t}[\mathfrak{sl}_2]$, and formulate an algorithm to obtain generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ by 'pasting' of the various $\mathfrak{sl}_2$ directions. We illustrate this procedure in the case of the rank 2 simple Lie algebras. This explicit construction further elucidates the connection of $\mathcal{W}_{q,t}[\mathfrak{g}]$ to the Grothendieck ring $\mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ of finite dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ as discovered in [16].

The main ideas and some of the formulae of this part of the paper are already, at least implicitly, contained in [16]. The main results of this paper are a formula for the determinant of the contravariant bilinear form ('Shapovalov form') on the Verma modules over $\mathcal{W}_{q,t}[\mathfrak{g}]$ (Theorem 4.4) and an explicit construction of the center of $\mathcal{W}_{q,t}[\mathfrak{g}]$ when one of the deformation parameters t^2 is a primitive k-th root of unity (Theorem 5.1). These two results generalize our results for $\mathfrak{g} = \mathfrak{sl}_2$ in [10].

The paper is organized as follows: In Section 2 we first recall, following [16], the definition of the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}[\mathfrak{g}]$ and formulate a conjecture for the structure of its generators $T_i(z)$. This conjecture (Assumption 2.4) has been verified in a number of cases, including $\mathfrak{g} = \mathfrak{sl}_N$ and all rank 2 simple Lie algebras. It is a slight refinement of Conjecture 1 in [16] and will be the starting point for the analysis in this paper. We also prove some simple corollaries of this conjecture which will be needed in the discussion of the main results of this paper. In Section 3 we provide further evidence for the conjectured structure of the $\mathcal{W}_{q,t}[\mathfrak{g}]$ generators by explicitly working out the centralizer of the screening operators in the case of $\mathfrak{g} = \mathfrak{sl}_2$. We also formulate an algorithm to obtain generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ by 'pasting' of the various $\mathfrak{sl}_2$ directions, and further elucidate the connection of those results to the representation theory of quantum affine algebras. In Section 4 we prove a formula for the Kac determinant of $\mathcal{W}_{q,t}[\mathfrak{g}]$ and in Section 5 we discuss the center of $\mathcal{W}_{q,t}[\mathfrak{g}]$ for t^2 a primitive k-th root of unity. In an

[1] For a review of $\mathcal{W}$-algebras $\mathcal{W}[\mathfrak{g}]$ see, e.g., [1] and the references therein.

appendix we illustrate several of the issues raised above in the case of the deformed algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$ where $\mathfrak{g}$ is a simple finite dimensional Lie algebra of rank 2. In particular, we give explicit formulae for all the generators $T_i(z)$ and their commutation relations.

2 Deformed $\mathcal{W}$-Algebras

In this section we review the construction of the deformed algebra $\mathcal{W}_{q,t}[\mathfrak{g}]$, where $\mathfrak{g}$ is an arbitrary simple Lie algebra of rank ℓ, and $q, t \in \mathbb{C}$ are deformation parameters. The first part of this section closely follows [16], which the reader should consult for further details.

Definition 2.1 ([16]). The deformed Heisenberg algebra $\mathcal{H}_{q,t}[\mathfrak{g}]$ is the associative algebra with the (root type) generators $a_i[n]$, $i = 1, \ldots, \ell$, $n \in \mathbb{Z}$, satisfying the relations

$$[a_i[m], a_j[n]] = \frac{1}{m}(q^m - q^{-m})(t^m - t^{-m})B_{ij}(q^m, t^m)\delta_{m+n,0}, \tag{2.1}$$

where $B(q,t) = D(q,t)C(q,t)$ and[2]

$$C_{ij}(q,t) = (q^{r_i}t^{-1} + q^{-r_i}t)\delta_{ij} - [I_{ij}]_q, \tag{2.2}$$

$$D_{ij}(q,t) = [r_i]_q\delta_{ij}, \tag{2.3}$$

are, respectively, deformations of the Cartan matrix $C = (C_{ij})$ and the diagonal matrix $D = \mathrm{diag}(r_1, \ldots, r_n)$ of $\mathfrak{g}$.

We recall that $I_{ij} = 2\delta_{ij} - C_{ij}$ is the incidence matrix of $\mathfrak{g}$, while the relatively prime integers r_i are given in terms of lengths of canonically normalized simple roots by $r_i = r^\vee(\alpha_i, \alpha_i)/2$, where $r^\vee$ is the dual tier number of $\mathfrak{g}$, i.e., the maximal number of edges connecting two vertices of the Dynkin diagram. In particular, when $\mathfrak{g}$ is simply laced, we have $r^\vee = 1$, $r_i = 1$, and

$$B_{ij}(q,t) = C_{ij}(q,t) = (qt^{-1} + q^{-1}t)\delta_{ij} - I_{ij}. \tag{2.4}$$

The set of fundamental weight type generators, $y_i[n]$, $i = 1, \ldots, \ell$, $n \in \mathbb{Z}$, is defined by

$$a_i[m] = \sum_j C_{ji}(q^m, t^m)y_j[m]. \tag{2.5}$$

[2]We use the standard q-notation, $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

They satisfy

$$[a_i[m], y_j[n]] = \frac{1}{m}(q^{r_i m} - q^{-r_i m})(t^m - t^{-m})\delta_{ij}\delta_{m+n,0}\,, \tag{2.6}$$

and

$$[y_i[m], y_j[n]] = \frac{1}{m}(q^m - q^{-m})(t^m - t^{-m})M_{ij}(q^m, t^m)\delta_{m+n,0}\,, \tag{2.7}$$

where $M(q,t) = D(q,t)C(q,t)^{-1}$. An explicit formula for the matrices $M(q,t)$ for Lie algebras of classical type can be found in Appendix C of [16].

In the following we will use the generating series

$$A_i(z) = t^{2(\rho^\vee,\alpha_i)}q^{-2r^\vee(\rho,\alpha_i)+2a_i[0]} :\exp\left(\sum_{m\neq 0} a_i[m]z^{-m}\right):, \tag{2.8}$$

and

$$Y_i(z) = t^{2(\rho^\vee,\omega_i)}q^{-2r^\vee(\rho,\omega_i)+2y_i[0]} :\exp\left(\sum_{m\neq 0} y_i[m]z^{-m}\right):, \tag{2.9}$$

where ω_i are the fundamental weights and ρ is the Weyl vector of $\mathfrak{g}$, i.e., $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ and $(\rho, \alpha_i^\vee) = 1$.

A Fock module, $F(\mu)$, of $\mathcal{H}_{q,t}[\mathfrak{g}]$, where μ is a weight of $\mathfrak{g}$, is freely generated by $a_i[m]$, $m < 0$, from the vacuum state $|\mu\rangle$, which satisfies $a_i[0]|\mu\rangle = (\mu, \alpha_i)|\mu\rangle$ and $a_i[m]|\mu\rangle = 0$, $m > 0$. It decomposes as $F(\mu) = \coprod_{n\geq 0} F(\mu)_{(n)}$ under the action of, d, the derivation of $\mathcal{H}_{q,t}[\mathfrak{g}]$ defined by

$$[d, a_i[m]] = -m a_i[m]\,, \quad m \in \mathbb{Z}\,. \tag{2.10}$$

There are two canonical bases of $F(\mu)_{(n)}$ consisting of vectors

$$o[-\boldsymbol{\lambda}]|\mu\rangle \equiv o_1[-\lambda_1^{(1)}]\ldots o_1[-\lambda_{l_1}^{(1)}]\ldots o_\ell[-\lambda_1^{(\ell)}]\ldots o_\ell[-\lambda_{l_\ell}^{(\ell)}]|\mu\rangle\,, \tag{2.11}$$

where $o_i[m] = a_i[m]$ and $o_i[m] = y_i[m]$, respectively, and $\boldsymbol{\lambda} \vdash n$ runs over all multi-partitions of n, i.e. $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ with $|\boldsymbol{\lambda}| = \sum_i |\lambda^{(i)}| = n$.

The formal power series generated by terms of the form

$$:\partial_z^{n_1} Y_{i_1}(zq^{a_1}t^{b_1})^{\epsilon_1}\ldots \partial_z^{n_l} Y_{i_l}(zq^{a_l}t^{b_l})^{\epsilon_l}:,$$

where $\epsilon_i = \pm 1$, $n_i \geq 0$ and $(a_i, b_i) \in L \subset \mathbb{Z}\times\mathbb{Z}$, together with the Fock module $F(0)$, form a deformed chiral algebra in the sense of [19]. It will be denoted by $\mathbf{H}_{q,t}[\mathfrak{g}]$.

To construct maps between Fock modules we extend $\mathcal{H}_{q,t}[\mathfrak{g}]$ to $\mathcal{H}'_{q,t}[\mathfrak{g}]$ by introducing operators Q_i, $i = 1, \dots, \ell$, such that e^{Q_i} are shift operators satisfying

$$[a_i[m], e^{Q_j}] = B_{ij}\beta\delta_{m,0}e^{Q_j}, \tag{2.12}$$

or, equivalently,

$$[y_i[m], e^{Q_j}] = r_i\beta\delta_{ij}\delta_{m,0}e^{Q_j}, \tag{2.13}$$

where β is given by $t = q^\beta$ and $B_{ij} = B_{ij}(1,1)$. Now, the screening currents, $S_i^{\pm}(z)$, are defined as the generating series

$$S_i^+(z) = e^{-Q_i/r_i} z^{-s_i^+[0]} :\exp\left(\sum_{m\neq 0} s_i^+[m]z^{-m}\right):, \tag{2.14}$$

$$S_i^-(z) = e^{Q_i/\beta} z^{s_i^-[0]} :\exp\left(-\sum_{m\neq 0} s_i^-[m]z^{-m}\right):, \tag{2.15}$$

where

$$s_i^+[m] = \frac{a_i[m]}{q^{mr_i} - q^{-mr_i}}, \quad m \neq 0, \qquad s_i^+[0] = \frac{a_i[0]}{r_i}, \tag{2.16}$$

$$s_i^-[m] = \frac{a_i[m]}{t^m - t^{-m}}, \quad m \neq 0, \qquad s_i^-[0] = \frac{a_i[0]}{\beta}. \tag{2.17}$$

The screening operators $S_i^+ : F(0) \to F(-\beta\alpha_i^\vee)$ and $S_i^- : F(0) \to F(r^\vee\alpha_i)$ are defined by $S_i^{\pm} = S_i^{\pm}[1]$, where $S_i^{\pm}(z) = \sum_{m\in\mathbb{Z}} S_i^{\pm}[m]z^{-m}$.

Definition 2.2 ([16]). The deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}[\mathfrak{g}]$ is the associative algebra, topologically generated by the Fourier coefficients of the maximal deformed chiral subalgebra of $\mathbf{H}_{q,t}[\mathfrak{g}]$, which commutes with the screening operators $S_i^{\pm}$, $i = 1, \dots, \ell$.

An explicit construction of the generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ has been carried out completely in the case $\mathfrak{g} = \mathfrak{sl}_N$ [12, 13] and, partially, when $\mathfrak{g}$ is a Lie algebra of classical type [16], where, in particular, the generator $T_1(z)$ was constructed for all classical Lie algebras $\mathfrak{g}$. Additional results are known for the classical limit $t \to 1$, i.e. for the Poisson algebra $\mathcal{W}_{q,1}[\mathfrak{g}]$ (see, in particular, [2, 16, 18]). In Appendix A we give a complete result when $\mathfrak{g}$ is one of the rank 2 simple Lie algebras.

To elucidate the structure of those generators, let us first consider the case $\mathfrak{g} = \mathfrak{sl}_N$.

Theorem 2.3 ([3, 12, 13, 16]). *The algebra $\mathcal{W}_{q,t}[\mathfrak{sl}_N]$ is generated by the Fourier modes of fields*

$$T_i(z) = \sum_{1\leq j_1<\ldots<j_i\leq N} :\Lambda_{j_1}(zp^{-i+1})\Lambda_{j_2}(zp^{-i+3})\ldots\Lambda_{j_i}(zp^{i-1}):, \qquad i=1,\ldots,N-1, \tag{2.18}$$

where $\Lambda_i(z)$ are defined recursively by

$$\begin{aligned} \Lambda_1(z) &= Y_1(z), \\ \Lambda_i(z) &= :\Lambda_{i-1}(z)A_{i-1}(zp^{-i+1})^{-1}:, \qquad i=2,\ldots,N, \end{aligned} \tag{2.19}$$

and $p=qt^{-1}$. The $\Lambda_i(z)$ satisfy

$$:\Lambda_1(z)\Lambda_2(zp^2)\ldots\Lambda_N(zp^{2(N-1)}): = 1. \tag{2.20}$$

In the remaining cases the structure of the generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ can be motivated by the explicit examples, various limiting cases (such as the conformal limit $q\to 1$, $\beta=\text{const}$) in which the algebra is known, and, most interestingly, a natural interpretation of formulae like (2.18) as characters of finite dimensional irreducible representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, where $\widehat{\mathfrak{g}}$ is the affine Lie algebra corresponding to $\mathfrak{g}$. A conjecture was first formulated in [16]; in order to prove the main results of this paper we will need a slightly sharper version of that conjecture. Henceforth, we will assume that the following holds:

Assumption 2.4. *The algebra $\mathcal{W}_{q,t}[\mathfrak{g}]$ is generated by the Fourier modes $T_i[m]$ of the fields $T_i(z)=\sum_{m\in\mathbb{Z}}T_i[m]z^{-m}$, $i=1,\ldots,\ell$,*

$$T_i(z) = \sum_{\lambda\in P(V(\omega_i))}\sum_{k_\lambda=1}^{\operatorname{mult}\lambda} c_\lambda^{\omega_i,(k_\lambda)}(q,t)\,Y_\lambda^{\omega_i,(k_\lambda)}(z), \tag{2.21}$$

where λ runs over the weights $P(V(\omega_i))$ of the finite dimensional irreducible representation $V(\omega_i)$ of $U_q(\widehat{\mathfrak{g}})$ with highest weight ω_i, and each $Y_\lambda^{\omega_i,(k_\lambda)}(z)$, with $\lambda=\omega_i-\sum\alpha_{i_j}$, is of the form

$$Y_\lambda^{\omega_i,(k_\lambda)}(z) = :Y_i(z)A_{i_1}(zq^{a_1}t^{b_1})^{-1}\ldots A_{i_k}(zq^{a_k}t^{b_k})^{-1}:, \tag{2.22}$$

for some choice of integers $a_i,b_i\in\mathbb{Z}$. Furthermore, we assume that (2.21) is obtained by the algorithm of pasting $\mathfrak{sl}_2$ directions (see Section 3). In particular, $T_i(z)$ is uniquely determined by its highest weight component $Y_{\omega_i}^{\omega_i}(z)=Y_i(z)$. We normalize $T_i(z)$ by choosing $c_{\omega_i}^{\omega_i}(q,t)=1$.

Remark. For $V(\omega_1)$ the set of weights is the same as for the irreducible $\mathfrak{g}$ module of highest weight ω_1. However, in general, this set is bigger, except for $\mathfrak{sl}_N$ (see, e.g., [20] for background material on quantum affine algebras).

Again, we emphasize that Assumption 2.4 holds in all known cases including, in particular, $\mathfrak{g} = \mathfrak{sl}_N$ [12, 13, 16] and all rank 2 simple Lie algebras $\mathfrak{g}$ (Appendix A). In Section 3 we will generalize it to arbitrary finite dimensional irreducible $U_q(\widehat{\mathfrak{g}})$ modules V.

Theorem 2.5. *The coefficients* $c_\lambda^{\omega_i,(k_\lambda)}(q,t)$ *defined by* (2.21) *satisfy*

$$c_\lambda^{\omega_i,(k_\lambda)}(q^{-1},t^{-1}) \;=\; c_\lambda^{\omega_i,(k_\lambda)}(q,t)\,, \tag{2.23}$$

and

$$\sum_{k_\lambda=1}^{\mathrm{mult}\,\lambda} c_{w\lambda}^{\omega_i,(k_\lambda)}(q,t) \;=\; \sum_{k_\lambda=1}^{\mathrm{mult}\,\lambda} c_\lambda^{\omega_i,(k_\lambda)}(q,t)\,, \qquad w \in W\,. \tag{2.24}$$

In particular, if $\mathrm{mult}\,\lambda = 1$, *then the coefficients* $c_\lambda^{\omega_i}(q,t)$ *are invariant with respect to the action of the Weyl group* W *of* $\mathfrak{g}$. *In addition, for simply-laced* $\mathfrak{g}$, *we have*

$$c_\lambda^{\omega_i,(k_\lambda)}(t,q) \;=\; c_\lambda^{\omega_i,(k_\lambda)}(q,t)\,. \tag{2.25}$$

Since the proof requires a more detailed analysis of the structure of $T_i(z)$, we defer it to Section 3.

Note that the matrices $C_{ij}(q,t)$, $B_{ij}(q,t)$ and $M_{ij}(q,t)$ are all invariant under $(q,t) \to (q^{-1},t^{-1})$. This implies that $\mathcal{H}_{q,t}[\mathfrak{g}] \cong \mathcal{H}_{q^{-1},t^{-1}}[\mathfrak{g}]$. In fact, let us combine this transformation with a $\mathbb{Z}_2$-automorphism of $\mathcal{H}_{q,t}[\mathfrak{g}]$ and define

$$\begin{aligned} \vartheta(q) \;&=\; q^{-1}\,, \qquad \vartheta(t) \;=\; t^{-1}\,, \\ \vartheta(q^{a_i[0]}) \;&=\; q^{-a_i[0]}\,, \qquad \vartheta(a_i[m]) \;=\; -\,a_i[m]\,, \quad m \neq 0\,, \end{aligned} \tag{2.26}$$

then

$$\vartheta(A_i(z)) \;=\; A_i(z)^{-1}\,, \qquad \vartheta(Y_i(z)) \;=\; Y_i(z)^{-1}\,, \tag{2.27}$$

while the screening currents (2.14) and (2.15) are invariant under ϑ provided we define

$$\vartheta(Q_i) \;=\; Q_i\,. \tag{2.28}$$

The invariance of the screening operators implies that $\mathcal{W}_{q,t}[\mathfrak{g}]$ is invariant under ϑ. Indeed, in Section 3 we prove

Theorem 2.6 ([16]). *Under the transformation* ϑ,

$$\vartheta(T_i(z)) = T_{i^*}(zq^{r^\vee h^\vee}t^{-h}), \tag{2.29}$$

where $T_{i^*}(z)$ *is the generator corresponding to the weight* $\omega_i^* = -w_0\omega_i$, *conjugate to* ω_i.

The fields $T_i(z)$ satisfy the exchange relations [12, 16]

$$T_i(z)T_j(w) = S_{T_i,T_j}(\frac{w}{z})T_j(w)T_i(z), \tag{2.30}$$

where $S_{T_i,T_j}(x) = f_{ij}(x)^{-1}f_{ij}(1/x)$ and $f_{ij}(x)$ is defined by

$$Y_i(z)Y_j(w) = f_{ij}(\frac{w}{z})^{-1} :Y_i(z)Y_j(w): . \tag{2.31}$$

A straightforward calculation using (2.7) yields

$$f_{ij}(x) = \exp\left(- \sum_{m>0}(q^m - q^{-m})(t^m - t^{-m})M_{ij}(q^m, t^m)\frac{x^m}{m}\right). \tag{2.32}$$

Finally, the products of operators in (2.32) are understood in the sense of analytic continuation. Using standard techniques one can derive from (2.32) the corresponding commutation relations for the modes $T_i[m]$ (see [12, 13] for the case of $\mathfrak{sl}_N$ and Appendix A for the rank 2 simple Lie algebras).

The Verma module, $M(h)$, of $\mathcal{W}_{q,t}[\mathfrak{g}]$ is defined as usual [12, 13]. It is generated from the highest weight state, $|h\rangle$, satisfying $T_i[0]|h\rangle = h_i|h\rangle$ and $T_i[m]|h\rangle = 0$, $m > 0$, and decomposes under the action of d as $M(h) = \coprod_{n\geq 0} M(h)_{(n)}$. A basis of $M(h)_{(n)}$ consists of vectors

$$T[-\boldsymbol{\lambda}]|\mu\rangle \equiv T_1[-\lambda_1^{(1)}]\ldots T_1[-\lambda_{l_1}^{(1)}]\ldots T_\ell[-\lambda_1^{(\ell)}]\ldots T_\ell[-\lambda_{l_\ell}^{(\ell)}]|\mu\rangle, \tag{2.33}$$

indexed by multi-partitions $\boldsymbol{\lambda}$.

The realization of $T_i(z)$ in terms of $y_i(z)$ defines a homomorphism $\imath : M(h(\mu)) \to F(\mu)$, uniquely determined by $\imath(|h(\mu)\rangle) = |\mu\rangle$. The highest weight, $h(\mu)$, of the Verma module can be found by evaluating $T_i[0]$ on the highest weight vector of $F(\mu)$, $T_i[0]|\mu\rangle = h_i(\mu)|\mu\rangle$. As a consequence of Assumption 2.4 we have

Corollary 2.7. *The eigenvalues* $h_i(\mu)$ *are given by*

$$h_i(\mu) = \sum_{\lambda\in P(V(\omega_i))} \left(\sum_{k_\lambda=1}^{\mathrm{mult}\,\lambda} c_\lambda^{\omega_i,(k_\lambda)}(q,t)\right) q^{-2r^\vee(\rho,\lambda)}t^{2(\rho^\vee,\lambda)}q^{2(\mu,\lambda)}, \tag{2.34}$$

where the sum runs over the weights of the irreducible module $V(\omega_i)$ *of* $U_q(\widehat{\mathfrak{g}})$.

It follows from Theorem 2.5 that the $h_i(\mu)$ are invariant under

$$\mu \ \rightarrow\ w*\mu \ \equiv\ w(\mu - r^\vee\rho + \beta\rho^\vee) + (r^\vee\rho - \beta\rho^\vee)\,, \tag{2.35}$$

for each w in the Weyl group W of $\mathfrak{g}$. In fact,

Lemma 2.8. *For generic $q,t \in \mathbb{C}$, we have $h_i(\mu) = h_i(\mu')$ for all $i = 1,\ldots,\ell$, if and only if there exists a $w \in W$ such that $\mu' = w*\mu$.*

For future use, also note that

$$h_i(\bar\mu) \ =\ h_{i^*}(\mu)\,, \tag{2.36}$$

where

$$\bar\mu \ \equiv\ -\mu + 2r^\vee\rho - 2\beta\rho^\vee\,. \tag{2.37}$$

3 Explicit Generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$

In this section we outline an algorithm to explicitly compute elements in the commutant of the screening operators $S_i^\pm$, i.e., generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$. First we analyze the case $\mathfrak{g} = \mathfrak{sl}_2$ in detail and then indicate how to obtain the result for arbitrary $\mathfrak{g}$ by pasting together the various $\mathfrak{sl}_2$ directions. In Appendix A we illustrate this procedure in the case of the fundamental generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$, for all rank 2 simple Lie algebras $\mathfrak{g}$. The main ideas of this section, as well as some of the explicit formulae, are already contained in [16,19] – our main motivation is to make these ideas more explicit in order to provide further support for Assumption 2.4, and to prove Theorems 2.5 and 2.6 which will play a crucial role in our analysis of the Kac determinant for $\mathcal{W}_{q,t}[\mathfrak{g}]$.

From the commutation relations (2.6) and (2.13) it follows[3]

$$\begin{aligned}
Y_i(z)S_i^+(w) &= t^{-2}\left(\frac{1-t\frac{w}{z}}{1-t^{-1}\frac{w}{z}}\right) :Y_i(z)S_i^+(w):\,,\\
Y_i(z)S_i^-(w) &= q^{2r_i}\left(\frac{1-q^{-r_i}\frac{w}{z}}{1-q^{r_i}\frac{w}{z}}\right) :Y_i(z)S_i^+(w):\,,\\
S_i^+(w)Y_i(z) &= \left(\frac{1-t^{-1}\frac{z}{w}}{1-t\frac{z}{w}}\right) :Y_i(z)S_i^+(w):\,,\\
S_i^-(w)Y_i(z) &= \left(\frac{1-q^{r_i}\frac{z}{w}}{1-q^{-r_i}\frac{z}{w}}\right) :Y_i(z)S_i^-(w):\,,
\end{aligned} \tag{3.1}$$

[3]The normal ordering includes the prescription to put the $y_i[0]$ to the right of the Q_j.

while, for $i \neq j$,

$$\begin{aligned} Y_i(z)S_j^{\pm}(w) &= :Y_i(z)S_j^{\pm}(w):, \\ S_j^{\pm}(w)Y_i(z) &= :Y_i(z)S_j^{\pm}(w): . \end{aligned} \tag{3.2}$$

Furthermore, we have the difference relations [16]

$$\begin{aligned} S_i^+(z) &= q^{2r_i}t^{-2} :A_i(zq^{r_i})S_i^+(zq^{2r_i}):, \\ S_i^-(z) &= q^{2r_i}t^{-2} :A_i(zt^{-1})S_i^-(zt^{-2}): . \end{aligned} \tag{3.3}$$

Our aim is to find certain combinations of vertex operators such that, by using the difference relations (3.3), the commutator with the screening currents $S_i^+(w)$ and $S_i^-(w)$ can be written as total q^{2r_i}- and t^2-differences, respectively. We recall the definition of a total a-difference $\mathcal{D}_a$,

$$\mathcal{D}_a \cdot f(w) = \frac{f(w) - f(wa)}{w(1-a)} . \tag{3.4}$$

This then obviously implies that those combinations are in the commutant of $S_i^{\pm} = S_i^{\pm}[1]$.

Now consider $\mathfrak{g} = \mathfrak{sl}_2$, and a vertex operator of the form (we write $Y(z) = Y_1(z)$ and $A(z) = A_1(z)$ in the case of $\mathfrak{sl}_2$)

$$\Psi(z) = :Y(z_1)\dots Y(z_m): . \tag{3.5}$$

For our purposes it will be sufficient to consider the case

$$z_i = zq^{2a_i}t^{2b_i} , \qquad i = 1, \dots , m , \tag{3.6}$$

for some $(\mathbf{a}, \mathbf{b}) = ((a_1, b_1), \dots , (a_m, b_m)) \in \mathbb{R}^m \times \mathbb{R}^m$. For convenience we define

$$\xi_{ij} = \sqrt{\frac{z_i}{z_j}} = q^{a_i - a_j}t^{b_i - b_j} = \xi_{ji}^{-1} . \tag{3.7}$$

We first consider the case when the z_i are in generic position, i.e., $\xi_{ij} \notin q^{\mathbb{Z}}t^{\mathbb{Z}}$ for all pairs (i, j), and take the non-generic limit of the resulting expressions at a later stage. Let us introduce

$$Y^{(+)}(z) = Y(z) , \qquad Y^{(-)}(z) = :Y(z)A(zq^{-1}t)^{-1}: = Y(zq^{-2}t^2)^{-1} , \tag{3.8}$$

and consider the 2^m vertex operators

$$\Psi_{(\epsilon_1 \dots \epsilon_m)}(z) = :Y^{(\epsilon_1)}(z_1) \dots Y^{(\epsilon_m)}(z_m):, \tag{3.9}$$

in particular $\Psi(z) = \Psi_{(++\ldots+)}(z)$. A tedious, but straightforward, calculation yields

$$
\begin{aligned}
[\Psi_{(\epsilon_1\ldots\epsilon_m)}(z), S^-(w)] &= q^{m_+-m_-}(q-q^{-1}) \\
&\cdot \sum_i \mathrm{sgn}(\epsilon_i)\left(\prod_{j\neq i} \eta_{ij}^{(\epsilon_i\epsilon_j)}(q,t)\right)\delta(x_i^{(\epsilon_i)}q) :\Psi_{(\epsilon_1\ldots\epsilon_m)}(z)S^-(w):,
\end{aligned} \tag{3.10}
$$

where

$$
M_\pm = \{i \mid \epsilon_i = \pm\}, \qquad m_\pm = |M_\pm|, \tag{3.11}
$$

$$
x_i^{(+)} = x_i = \frac{w}{z_i}, \qquad x_i^{(-)} = x_i t^{-2}, \tag{3.12}
$$

and

$$
\eta_{ij}^{(\epsilon_i,\epsilon_j)}(q,t) \equiv \eta^{(\epsilon_i,\epsilon_j)}(q,t;\xi_{ij}), \tag{3.13}
$$

with

$$
\begin{aligned}
\eta^{(++)}(q,t;\xi) &= \frac{q\xi^{-1}-q^{-1}\xi}{\xi^{-1}-\xi}, \\
\eta^{(-+)}(q,t;\xi) &= \frac{qt^{-1}\xi^{-1}-q^{-1}t\xi}{t^{-1}\xi^{-1}-t\xi}, \\
\eta^{(+-)}(q,t;\xi) &= \frac{qt^{-1}\xi-q^{-1}t\xi^{-1}}{t^{-1}\xi-t\xi^{-1}}, \\
\eta^{(--)}(q,t;\xi) &= \frac{q\xi-q^{-1}\xi^{-1}}{\xi-\xi^{-1}}.
\end{aligned} \tag{3.14}
$$

The coefficients $\eta^{(\epsilon,\epsilon')}(q,t;\xi)$ have the following easily verifiable properties

$$
\begin{aligned}
\eta^{(\epsilon,\epsilon')}(q,t;\xi^{-1}) &= \eta^{(\bar\epsilon,\bar\epsilon')}(q,t;\xi), \\
\eta^{(\epsilon,\epsilon')}(q^{-1},t^{-1};\xi) &= \eta^{(\bar\epsilon,\bar\epsilon')}(q,t;\xi), \\
\eta^{(\epsilon,\epsilon')}(q,t;(qt^{-1})^{\epsilon'}\xi) &= \eta^{(\epsilon,\bar\epsilon')}(q,t;\xi)^{-1},
\end{aligned} \tag{3.15}
$$

where $\bar\epsilon = \mp$ for $\epsilon = \pm$.

Theorem 3.1. *For parameters* $z_i = zq^{2a_i}t^{2b_i}$, $(\mathbf{a},\mathbf{b}) = ((a_1,b_1),\ldots,((a_m,b_m)) \in \mathbb{R}^m \times \mathbb{R}^m$, *in generic position (i.e.,* $\xi_{ij} \notin q^{\mathbb{Z}}t^{\mathbb{Z}}$*), the vertex operators*

$$
T_{(\mathbf{a},\mathbf{b})}(z) = \sum_{\epsilon_i} \gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t)\,\Psi_{(\epsilon_1\ldots\epsilon_m)}(z), \tag{3.16}
$$

where

$$\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t) = \prod_{\substack{i\in M_- \\ j\in M_+}} \left(\frac{\eta_{ij}^{(++)}(q,t)}{\eta_{ij}^{(-+)}(q,t)} \right), \tag{3.17}$$

are in the commutant of $S^{\pm}$. The coefficients $\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t)$ satisfy the following properties

$$\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,1) = 1, \tag{3.18}$$

$$\gamma_{(\epsilon_1\ldots\epsilon_m)}(q^{-1},t^{-1}) = \gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t), \tag{3.19}$$

$$\gamma_{(\epsilon_1\ldots\epsilon_m)}(t,q) = \gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t), \tag{3.20}$$

$$\sum_{\substack{\epsilon_i \\ |M_-|=n}} \gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t) = \sum_{\substack{\epsilon_i \\ |M_-|=m-n}} \gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t). \tag{3.21}$$

Furthermore,

$$\vartheta(T_{(\mathbf{a},\mathbf{b})}(z)) = T_{(-\mathbf{a},-\mathbf{b})}(zq^2t^{-2}). \tag{3.22}$$

Proof. Consider the expression (3.16), where we normalize $\gamma_{(+\ldots+)} = 1$. Using (3.3) it is clear that the $\delta(x_i q)$-term in

$$[\Psi_{(\epsilon_1\ldots\epsilon_{i-1}+\epsilon_{i+1}\ldots\epsilon_m)}(z), S^-(w)]$$

is going to combine with the $\delta(x_i q t^{-2})$-term in

$$[\Psi_{(\epsilon_1\ldots\epsilon_{i-1}-\epsilon_{i+1}\ldots\epsilon_m)}(z), S^-(w)]$$

into a total t^2-difference, i.e., that we have

$$[T_{(\mathbf{a},\mathbf{b})}(z), S^-(w)] = \mathcal{D}_{t^2} \cdot R(z,w), \tag{3.23}$$

for some $R(z,w)$, provided we can choose the $\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t)$ in (3.16) such that

$$\gamma_{(\epsilon_1\ldots\epsilon_{i-1}-\epsilon_{i+1}\ldots\epsilon_m)}(q,t) = \prod_{j\neq i} \left(\frac{\eta_{ij}^{(+\epsilon_j)}(q,t)}{\eta_{ij}^{(-\epsilon_j)}(q,t)} \right) \gamma_{(\epsilon_1\ldots\epsilon_{i-1}+\epsilon_{i+1}\ldots\epsilon_m)}(q,t), \tag{3.24}$$

for all choices of i and $\epsilon_j\,, j \neq i$. The solution of (3.24) is given by (3.17).[4] The analysis for the other screening current $S^+(w)$ proceeds similarly. The properties (3.18) – (3.20) trivially follow from (3.14) and (3.15). To prove (3.21), consider both sides as a (bounded) meromorphic function of one variable, say z_1. Such a function is uniquely determined by the residues at its poles and its value at infinity. Using the third relation in (3.15) it is easy to check that the residues of both sides indeed agree. Finally, (3.22) follows straightforwardly from (3.19). □

For parameters z_i in non-generic position, some of the coefficients $\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t)$ in (3.17) may be vanishing or singular. Four different situations might occur:

(i) There exists a pair (i,j) such that $\xi_{ij} = q$. In this case $\eta_{ij}^{(++)}(q,t) = 0$ such that
$$\gamma_{(\epsilon_1\ldots\epsilon_{i-1}-\epsilon_{i+1}\ldots\epsilon_{j-1}+\epsilon_{j+1}\ldots\epsilon_m)} = 0\,,$$
and the number of terms in (3.16) is effectively reduced by 1/4.

(ii) There exists a pair (i,j) such that $\xi_{ij} = t$. This case is similar to case (i) due to the symmetry $(q,t) \to (t,q)$.

(iii) There exists a pair (i,j) such that $\xi_{ij} = 1$. In this case $\eta_{ij}^{(++)}(q,t)$ and $\eta_{ji}^{(++)}(q,t)$, and thus both
$$\gamma_{(\epsilon_1\ldots\epsilon_{i-1}-\epsilon_{i+1}\ldots\epsilon_{j-1}+\epsilon_{j+1}\ldots\epsilon_m)}$$
and
$$\gamma_{(\epsilon_1\ldots\epsilon_{i-1}+\epsilon_{i+1}\ldots\epsilon_{j-1}-\epsilon_{j+1}\ldots\epsilon_m)}$$
are singular. However, the residue at this singularity vanishes, so the expression (3.16) makes perfect sense provided we interpret the right hand side as the limit of the generic expression in which we let $\xi_{ij} \to 1$. Note that, by doing so, the $\Psi_{(\epsilon_1\ldots\epsilon_m)}(z)$ terms are no longer of the form (3.9), but contain derivative terms.

(iv) There exists a pair (i,j) such that $\xi_{ij} = qt^{-1}$. In this case $\eta_{ij}^{(-+)}(q,t) = 0$ and the coefficient
$$\gamma_{(\epsilon_1\ldots\epsilon_{i-1}-\epsilon_{i+1}\ldots\epsilon_{j-1}+\epsilon_{j+1}\ldots\epsilon_m)}$$
becomes singular. This is a genuine singularity. By renormalizing the expression for $T_{(\mathbf{a},\mathbf{b})}(z)$ we obtain an element in the commutant of $S^\pm$

[4]Note that it is a non-trivial fact that the system (3.24) has a solution at all. Indeed, we have $m2^{m-1}$ equations for $2^m - 1$ unknowns. The dependence of the equations is however guaranteed by (3.15).

with terms which are precisely the complement of the terms remaining in case (i). Note that, in this case, $:Y^{(-)}(z_i)Y^{(+)}(z_j):= 1$. Thus, the generator obtained this way corresponds effectively to an expression (3.16) with $m \to m-2$.

Of course, various combinations of the above cases can occur simultaneously. A particularly important example is when the z_i line up in a single q-string, i.e.,

$$z_i \;=\; zq^{2a+2(i-1)}t^{2b}\,, \qquad i=1,\ldots,m\,, \tag{3.25}$$

for some $a,b \in \mathbb{R}$. One easily verifies that the only nonvanishing coefficients $\gamma_{(\epsilon_1\ldots\epsilon_m)}(q,t)$ in (3.16) are

$$\gamma_{m,n}(q,t) \;\equiv\; \gamma_{(\underbrace{-\ldots-}_{n}\underbrace{+\ldots+}_{m-n})}(q,t)\,, \tag{3.26}$$

and, therefore, the corresponding element $T_{(\mathbf{a},\mathbf{b})}(z) \equiv T_m(z)$ in the commutant has $m+1$ terms. Explicitly, $\gamma_{m,0}(q,t) = 1$ and, for $n = 1,\ldots,m$,

$$\gamma_{m,n}(q,t) \;=\; \prod_{k=1}^{n} \frac{(q^k t^{-1} - q^{-k}t)(q^{m-k+1} - q^{-(m-k+1)})}{(q^k - q^{-k})(q^{m-k+1}t^{-1} - q^{-(m-k+1)}t)}\,. \tag{3.27}$$

From Theorem 3.1 it follows that the coefficients $\gamma_{m,n}(q,t)$ have the following properties

$$\begin{aligned} \gamma_{m,n}(q,1) &= 1\,, \\ \gamma_{m,n}(q^{-1},t^{-1}) &= \gamma_{m,n}(q,t)\,, \\ \gamma_{m,n}(t,q) &= \gamma_{m,n}(q,t)\,, \\ \gamma_{m,m-n}(q,t) &= \gamma_{m,n}(q,t)\,, \end{aligned} \tag{3.28}$$

which can easily be verified explicitly from (3.27).

The foregoing construction of the operators $T_{(\mathbf{a},\mathbf{b})}(z)$ is intimately related to the structure of the representation ring of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. Before formulating a more precise conjecture, let us discuss the extension of the above results to arbitrary simple $\mathfrak{g}$. Suppose we start with the operator

$$Y_{(\mathbf{a},\mathbf{b})}(z) \;\equiv\; :\prod_{i=1}^{\ell}\prod_{j_i=1}^{m_i} Y_i(z_{j_i}^{(i)}):\,, \tag{3.29}$$

where

$$z_j^{(i)} \;=\; zq^{2r_i a_j^{(i)}} t^{2b_j^{(i)}}\,, \tag{3.30}$$

with $(\mathbf{a}, \mathbf{b}) = ((a_1^{(1)}, b_1^{(1)}), \ldots, (a_{m_\ell}^{(\ell)}, b_{m_\ell}^{(\ell)}))$, and try to complete it to an operator in the commutant of all $S_i^{\pm}$, i.e., an element of $\mathcal{W}_{q,t}[\mathfrak{g}]$. In each $\mathfrak{sl}_2$ direction i, we may apply the results of Theorem 3.1, with the replacement $q \to q^{r_i}$, and the final result is obtained by pasting together all the $\mathfrak{sl}_2$ directions. Clearly, for this algorithm to work, certain consistency conditions at the intersections of the various $\mathfrak{sl}_2$ directions must be satisfied. To illustrate the general procedure we have summarized the construction of the generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ for the rank 2 simple Lie algebras $\mathfrak{g}$ in Appendix A.

Note that the above algorithm bears close resemblance to the construction of the irreducible finite dimensional representation $L(\Lambda)$ of $\mathfrak{g}$ of highest weight $\Lambda = \sum_i m_i \omega_i$. In fact, the construction would be exactly identical if, at all $\mathfrak{sl}_2$ highest weights in direction i, the operators $Y_i(z)$ would line up in q^{r_i}-strings. This is not the case in general, though, as the completion of $Y_{(\mathbf{a},\mathbf{b})}(z)$ to an element in the commutant of $S_i^{\pm}$ in general has more terms than the dimension of $L(\Lambda)$. In fact, the conjecture (cf. [16]) is that the number of terms and their weights are the same as the dimension and the weights of an irreducible finite dimensional representation V of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, which decomposes under $U_q(\mathfrak{g})$ as $V \cong L(\Lambda) \oplus \ldots$, where the dots stands for 'smaller' representations. An example of this, which is worked out in detail in Appendix A.3, is the generator corresponding to the **14** of $U_q(G_2)$, to which an additional singlet term has to be added, in accordance with the minimal affinization of the **14**. This conjecture has been verified in all cases where the generators $T_V(z)$ are explicitly known. Obviously, when V is one of the fundamental representations it coincides with Assumption 2.4.

The main point here is that the algorithm should hold for other than the fundamental representations as well [16,21,22]. This leads to a generalization of Assumption 2.4, which will be formulated after we recall some basic facts from the representation theory of quantum affine algebras (see, e.g., [20] and references therein).

Let $\mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ denote the ring of finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$. It is well-known that there is a 1–1 correspondence between the irreducibles $V \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ and monic polynomials $P_{i,V}(u)$, $i = 1, \ldots, \ell$ [23]. Let $\{u_{j_i}^{(i)} \,|\, j_i = 1, \ldots, m_i\}$ be the roots of $P_{i,V}(u)$.

Conjecture 3.2. *Let $V \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ be irreducible. We have a map $V \mapsto T_V(z) \in \mathcal{W}_{q,t}[\mathfrak{g}]$ given by*[5]

$$T_V(z) = \sum_{\lambda \in P(V)} \sum_{k_\lambda = 1}^{\mathrm{mult}\,\lambda} c_\lambda^{V,(k_\lambda)}(q,t)\, Y_\lambda^{V,(k_\lambda)}(z)\,, \tag{3.31}$$

[5] Of course, this map is not unique, e.g., it can be twisted by an automorphism of $U_q(\widehat{\mathfrak{g}})$.

where λ runs over the weights $P(V)$ of V, and is uniquely determined by the above $\mathfrak{sl}_2$-pasting algorithm from the highest weight component

$$Y_V(z) \equiv Y_\Lambda^V(z) = :\prod_{i=1}^{\ell}\Big(\prod_{j_i=1}^{m_i} Y_i(zu_{j_i}^{(i)})\Big):, \tag{3.32}$$

with highest weight $\Lambda = \sum_i m_i\omega_i$.

Remark. In most cases the $Y_\lambda^{V,(k_\lambda)}(z)$, with $\lambda = \Lambda - \sum_j \alpha_{i_j}$, will be of the form

$$:Y_V(z)A_{i_1}(zq^{a_1}t^{b_1})^{-1}\dots A_{i_k}(zq^{a_k}t^{b_k})^{-1}: . \tag{3.33}$$

However, it can happen that in some $\mathfrak{sl}_2$ direction there exists a pair of arguments (z,w) such that $w/z = 1$ (case (iii) above), in which case there will be derivative terms. This, for example, happens in the case of the $\mathcal{W}_{q,t}[G_2]$ generator with highest weight component $Y_1(zq^{-1})Y_1(zq)$, i.e., the minimal $U_q(\widehat{G_2})$ affinization of the $U_q(G_2)$ irrep. $L(2\omega_1) \cong \mathbf{27}$ which is a $\mathbf{27} \oplus \mathbf{7}$.

Remark. It is known that there is a map $V \mapsto t_V(z) = \sum_{m\in\mathbb{Z}} t_V[m]z^{-m}$ from $\mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ to generating series of central elements in $U_q(\widehat{\mathfrak{g}})_{k=-\mathrm{h}^\vee}$ [24, 25], which at $q=1$ reduces to the character of V. These q-characters satisfy the following natural properties

(I) $t_{V\oplus W}(z) = t_V(z) + t_W(z)$, for all $V, W \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$,

(II) $t_{V\otimes W}(z) = t_V(z)t_W(z)$, for all $V, W \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$,

(III) $t_{V(a)}(z) = t_V(za)$, for all $V, W \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$, $a \in \mathbb{C}^*$, and where $V(a)$ is the image of V under the twist automorphism.

Moreover, it has been shown, at least for $\mathfrak{g} = \mathfrak{sl}_N$ [2], that the evaluation of the image of $t_V(z)$ under the free field realization $U_q(\widehat{\mathfrak{g}}) \to \mathcal{H}_{q,1}(\mathfrak{g})$ coincides with the Bethe Ansatz formula for the eigenvalues of the transfer matrix corresponding to the finite dimensional representation $V \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$. Conjecture 3.2, which is a slight extension of the conjectures in [16,21], is a t-deformation of this classical result in the sense that to each $V \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ one can associate a field $T_V(z)$ in $\mathcal{W}_{q,t}[\mathfrak{g}]$ such that $T_V(z) \to t_V(z)$ for $t \to 1$.

Comparing our explicit calculations in the case of $\mathfrak{sl}_2$ above with the representation theory of $U_q(\widehat{\mathfrak{sl}_2})$ [26] shows that Conjecture 3.2 holds for $\mathfrak{sl}_2$. The $\mathfrak{sl}_2$ results indicate that the tensor product structure of $\mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ is also reflected in $\mathcal{W}_{q,t}[\mathfrak{g}]$ through a quantization of property (II) above (properties (I) and (III) continue to hold for the quantization $T_V(z)$). Specifically, since $\mathcal{W}_{q,t}[\mathfrak{g}]$ has been defined as a deformed chiral algebra, by the very axioms of this algebra [19] the residues at the poles of the (meromorphically continued) composition of two generators are again elements of $\mathcal{W}_{q,t}[\mathfrak{g}]$. We expect that the composition $T_V(z)T_W(w)$, $V, W \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$, in particular contains poles corresponding to *all* subquotients of $V \otimes W$. More precisely[6]

Conjecture 3.3. *Let $V, W \in \mathrm{Rep}\,(U_q(\widehat{\mathfrak{g}}))$ and $T_V(z)$ and $T_W(z)$ be the corresponding elements of $\mathcal{W}_{q,t}[\mathfrak{g}]$. For each subquotient U of $V \otimes W$ there exists a meromorphic function $\eta_U^{V\otimes W}(x)$ and a choice of $a, b, a', b' \in \mathbb{Z}$ such that*

$$T_U(zq^{a'}t^{b'}) \;=\; \lim_{w\to zq^a t^b} \eta_U^{VW}(\frac{w}{z}) T_V(z) T_W(w) \,. \tag{3.34}$$

For $\mathfrak{sl}_2$ the validity of this conjecture, which is also implicit in [19], follows from the observation that (cf. [10], (2.37)–(2.40))

$$\begin{aligned} f(\frac{z_2}{z_1}) Y^{(+)}(z_1) Y^{(-)}(z_2) &= \frac{\eta_{21}^{(++)}(q,t)}{\eta_{21}^{(-+)}(q,t)} :Y^{(+)}(z_1)Y^{(-)}(z_2):\,, \\ f(\frac{z_2}{z_1}) Y^{(-)}(z_1) Y^{(+)}(z_2) &= \frac{\eta_{12}^{(++)}(q,t)}{\eta_{12}^{(-+)}(q,t)} :Y^{(-)}(z_1)Y^{(+)}(z_2):\,, \end{aligned} \tag{3.35}$$

and comparison to the explicit tensor product structure of $U_q(\widehat{\mathfrak{sl}_2})$ [26]. In Appendix A we will see examples for the simple Lie algebras $\mathfrak{g}$ of rank 2.

Remark. For V and W in generic position the $U_q(\widehat{\mathfrak{g}})$ module $V \otimes W$ will be irreducible. In that case we can simply take $a = b = 0$, $\xi = 1$ and

$$\eta_{V\otimes W}^{VW}(x) \;=\; f_{VW}(x)\,, \tag{3.36}$$

where $f_{VW}(x)$ is determined by

$$Y_V(z)Y_W(w) \;=\; f_{VW}(\frac{w}{z})^{-1} :Y_V(z)Y_W(w):\;. \tag{3.37}$$

For $V \otimes W$ reducible, and $U \subset V \otimes W$ the subquotient with highest weight given by the highest weight of $V \otimes W$, the choice (3.36) suffices as well. The other subquotients can be projected out by using the singularity structure of $(\eta_{ij}^{(-+)})^{-1}$ in the case of qt^{-1}-strings (see case (iv) in the $\mathfrak{sl}_2$ case).

The structure of the commutant for $\mathfrak{sl}_2$ suggests (see, in particular, (3.9)) the definition of a 'deformed Weyl group action' as follows. Define, for each $i = 1, \ldots, \ell$, a transformation $T_i : \mathcal{H}_{q,t}[\mathfrak{g}] \to \mathcal{H}_{q,t}[\mathfrak{g}]$ by

$$T_i\, y_j[m] \;=\; y_j[m] - \delta_{ij}\, p_i^m\, a_i[m]\,, \tag{3.38}$$

[6] See, also [22].

or, equivalently,

$$T_i\, a_j[m] \;=\; a_j[m] - C_{ij}(q^m, t^m)\, p_i^m\, a_i[m]\,, \tag{3.39}$$

where $p_i \equiv q^{r_i} t^{-1}$. Let us denote by $T_{q,t}[\mathfrak{g}]$ the algebra generated by the T_i, $i = 1, \ldots, \ell$. It is easy to check that $T_{q,t}[\mathfrak{g}]$ acts by automorphisms of $\mathcal{H}_{q,t}[\mathfrak{g}]$. The T_i's satisfy various relations, the simplest of which are

$$(T_i - 1)(T_i + p_i^{-2d}) \;=\; 0\,, \tag{3.40}$$

or, equivalently,

$$T_i^{-1} \;=\; p_i^{2d} T_i + (1 - p_i^{2d})\,, \tag{3.41}$$

where d is defined in (2.10). In addition,

$$\begin{aligned} T_i T_j \;&=\; T_j T_i\,, \qquad \text{if} \quad C_{ij}C_{ji} \;=\; 0\,, \\ T_i T_j T_i \;&=\; T_j T_i T_j\,, \qquad \text{if} \quad C_{ij}C_{ji} \;=\; 1\,. \end{aligned} \tag{3.42}$$

Note that, for $\mathfrak{sl}_2$, we have

$$\Psi_{(-\ldots-)}(z) \;=\; T\Psi_{(+\ldots+)}(z)\,. \tag{3.43}$$

Thus, the algebra $T_{q,t}[\mathfrak{g}]$ can be used to construct 'Weyl orbits' of terms in the expressions for the commutant.

In particular, let $w_0 = r_{i_1} \ldots r_{i_n}$, $(n = |\Delta_+|)$, be a reduced expression for the longest Weyl group element w_0. Define

$$T_{w_0} \;=\; T_{i_1} \ldots T_{i_n}\,, \tag{3.44}$$

then, as one can verify on a case by case basis,[7]

$$T_{w_0} Y_i(z) \;=\; Y_{i^*}(zq^{-r^\vee h^\vee} t^{h})^{-1}\,. \tag{3.45}$$

Finally, let us return to the proofs of Theorems 2.5 and 2.6 in Section 2. Under Assumption 2.4, Theorem 2.5 follows immediately from (3.18) – (3.21), while Theorem 2.6 follows from (3.45) by using, in particular, the assumption that $T_i(z)$ is uniquely determined by its highest component. Moreover, both theorems generalize to $\mathcal{W}_{q,t}[\mathfrak{g}]$ elements with more general highest weight component (3.29). The generalization of Theorem 2.5 is self-evident, while the generalization of Theorem 2.6 is (cf. (3.22) for $\mathfrak{g} = \mathfrak{sl}_2$)

$$\vartheta(T_{(\mathbf{a},\mathbf{b})}(z)) \;=\; T_{(-\mathbf{a}^*,-\mathbf{b}^*)}(zq^{r^\vee h^\vee} t^{-h})\,, \tag{3.46}$$

where

$$(a_j^{*(i)}, b_j^{*(i)}) \;\equiv\; (a_j^{(i^*)}, b_j^{(i^*)})\,. \tag{3.47}$$

[7]Albeit not completely obvious, the result is independent of the choice of reduced expression for w_0.

4 Involutions, Contravariant Forms and the Kac Determinant

The algebra $\mathcal{H}'_{q,t}[\mathfrak{g}]$ has a natural two-parameter family of anti-involutions $\omega_{a,b}$, $a, b \in \mathbb{C}$, defined by

$$\begin{aligned} \omega_{a,b}(a_i[0]) &= -a_i[0] + 2r^\vee(\rho,\alpha_i) - 2\beta(\rho^\vee,\alpha_i)\,, \\ \omega_{a,b}(a_i[m]) &= -(q^a t^b)^m a_i[-m]\,, \qquad m \neq 0\,, \end{aligned} \tag{4.1}$$

and

$$\omega_{a,b}(Q_i) = Q_i\,, \tag{4.2}$$

or, equivalently,

$$\omega_{a,b}(A_i(z)) = A_i(\frac{q^a t^b}{z})^{-1}\,, \qquad \omega_{a,b}(Y_i(z)) = Y_i(\frac{q^a t^b}{z})^{-1}\,. \tag{4.3}$$

Also

$$\omega_{a,b}(S_i^\pm(z)) = \frac{1}{z^2} S_i^\pm(\frac{q^a t^b}{z})\,. \tag{4.4}$$

In particular, the screening operators $S_i^\pm$ are invariant under $\omega_{a,b}$, up to a multiplicative factor, and therefore $\omega_{a,b}$ is well-defined on $\mathcal{W}_{q,t}[\mathfrak{g}]$.

Lemma 4.1. *The action of $\omega_{a,b}$ on the generators $T_i(z)$ of $\mathcal{W}_{q,t}[\mathfrak{g}]$ is given by*

$$\omega_{a,b}(T_i(z)) = T_{i^*}(\frac{q^{a+r^\vee \mathrm{h}^\vee} t^{b-\mathrm{h}}}{z})\,. \tag{4.5}$$

Proof. Note that the action of $\omega_{a,b}$ on $T_i(z)$ in (2.21) is obtained by composing the transformation ϑ of (2.26) with the transformation $z \to q^a t^b/z$. The lemma follows by using Theorem 2.6 and then applying the transformation $z \to q^a t^b/z$. □

In the following we will set $a = -r^\vee \mathrm{h}^\vee$ and $b = \mathrm{h}$, and denote the corresponding anti-involution by ω. Obviously, we then have $\omega(T_i(z)) = T_{i^*}(1/z)$.

The anti-involution ω induces a unique contravariant bilinear form $\langle -|-\rangle_F$ on $F(\bar\mu) \times F(\mu)$ such that $\langle \bar\mu|\mu\rangle_F = 1$, where $\bar\mu = -\mu + 2r^\vee\rho - 2\beta\rho^\vee$ (cf. (2.37)). We will denote by $g_{\lambda\lambda'} = \langle\bar\mu|\omega(y[-\lambda])y[-\lambda']|\mu\rangle_F$ the matrix elements of this form in the basis (2.11). Similarly, ω induces a unique contravariant bilinear form $\langle -|-\rangle$ on $M(h) \times M(h)$ with matrix elements $G_{\lambda\lambda'} = \langle h|\omega(T[-\lambda])T[-\lambda']|h\rangle$. Clearly, both $g_{\lambda\lambda'}$ and $G_{\lambda\lambda'}$ vanish unless $|\lambda| = |\lambda'|$. The map $\imath : M(h(\mu)) \to F(\mu)$ is an isometry for generic values of $(q, t; q^\mu)$.

Lemma 4.2. *Let $g^{(n)}(q,t) = \det(g_{\lambda\lambda'})_{|\lambda|=|\lambda'|=n}$ be the determinant at level n. Then*

$$g^{(n)}(q,t) = C_n \prod_{\substack{r,s\geq 1 \\ rs\leq n}} \left((q^{r^\vee \mathrm{h}^\vee} t^{-\mathrm{h}})^{\ell r} \left(\det M(q^r,t^r) \right) (q^r - q^{-r})^\ell (t^r - t^{-r})^\ell \right)^{p_\ell(n-rs)}, \tag{4.6}$$

where $p_\ell(m)$ is the number of multi-partitions of m and C_n is a constant independent of q and t.

We give an explicit formula for $\det C(q,t)$ and $\det D(q,t)$ for all simple Lie algebras $\mathfrak{g}$, from which $\det M(q,t)$ follows.

A_ℓ. $\det C = (p^{\ell+1} - p^{-\ell-1})/(p - p^{-1})$

B_ℓ. $\det C = q^{2\ell-1}t^{-\ell} + q^{-2\ell+1}t^\ell$, $\quad \det D = (q+q^{-1})^{\ell-1}$

C_ℓ. $\det C = q^{\ell+1}t^{-\ell} + q^{-\ell-1}t^\ell$, $\quad \det D = q + q^{-1}$

D_ℓ. $\det C = (p+p^{-1})(p^{\ell-1} + p^{-\ell+1})$

E_6. $\det C = p^6 + p^4 - 1 + p^{-4} + p^{-6}$

E_7. $\det C = p^7 + p^5 - p - p^{-1} + p^{-5} + p^{-7}$

E_8. $\det C = p^8 + p^6 - p^2 - p^{-2} + p^{-6} + p^{-8}$

F_4. $\det C = q^6t^{-4} - 1 + q^{-6}t^4$, $\quad \det D = (q+q^{-1})^2$

G_2. $\det C = q^4t^{-2} - 1 + q^{-4}t^2$, $\quad \det D = q^2 + 1 + q^{-2}$

For $\mathfrak{g}$ simply-laced the expressions above follow from the lemma below, the expressions for B_ℓ and C_ℓ were given in [16]. The remaining cases were computed by brute force.

Lemma 4.3. *The eigenvalues $\lambda_i(q,t)$ of $C_{ij}(q,t)$ for $\mathfrak{g}$ simply-laced are given by*

$$\lambda_i(q,t) = (p+p^{-1}) + 2\cos\left(\frac{\pi e_i}{\mathrm{h}}\right), \tag{4.7}$$

where $p = qt^{-1}$, and e_i, $i = 1, \ldots, \ell$, are the exponents of $\mathfrak{g}$.

Let $\alpha \in \Delta$, $\mu \in \mathfrak{h}^*$, and $r, s \in \mathbb{N}$. Define

$$\begin{aligned} G_\alpha^{(r,s)}(q,t;q^\mu) &= q^{-r^\vee(\rho,\alpha)+\frac{1}{2}r^\vee s(\alpha,\alpha)} t^{(\rho^\vee,\alpha)-r} q^{(\mu,\alpha)} - \\ & \qquad q^{r^\vee(\rho,\alpha)-\frac{1}{2}r^\vee s(\alpha,\alpha)} t^{-(\rho^\vee,\alpha)+r} q^{-(\mu,\alpha)}, \end{aligned} \tag{4.8}$$

then, under the action of W (see (2.35)), we have

$$G_\alpha^{(r,s)}(q,t;q^{w*\mu}) = G_{w^{-1}\alpha}^{(r,s)}(q,t;q^\mu) \tag{4.9}$$

In addition,

$$G_\alpha^{(r,s)}(q,t;q^{\bar{\mu}}) = G_{-\alpha}^{(r,s)}(q,t;q^\mu), \tag{4.10}$$

where

$$\bar{\mu} = -\mu + 2r^\vee\rho - 2\beta\rho^\vee . \tag{4.11}$$

We define the matrix of $\imath$ in the standard bases (2.33) and (2.11) by

$$\imath\left(T[-\lambda]|h(\mu)\rangle\right) = \sum_{\lambda'} \Pi_{\lambda\lambda'}(q,t;q^\mu)\, y[-\lambda']|\mu\rangle, \tag{4.12}$$

and denote by $\Pi^{(n)}(q,t;q^\mu) \equiv \det(\Pi_{\lambda\lambda'})_{|\lambda|=|\lambda'|=n}$ the determinant of this matrix at level n.

Theorem 4.4. *Given Assumption 2.4, the Kac determinant of $\mathcal{W}_{q,t}[\mathfrak{g}]$ at level n is given by*

$$G^{(n)}(q,t;q^\mu) = \Pi^{(n)}(q,t;q^{\bar{\mu}})\; g^{(n)}(q,t)\; \Pi^{(n)}(q,t;q^\mu), \tag{4.13}$$

where

$$\Pi^{(n)}(q,t;q^\mu) = C_n \prod_{\substack{r,s\geq 1\\ rs\leq n}} \left((q^{r^\vee h^\vee} t^{-h})^{-\frac{\ell r}{2}} \prod_{\alpha\in\Delta_+} G_\alpha^{(r,s)}(q,t;q^\mu) \right)^{p_\ell(n-rs)}, \tag{4.14}$$

and $g^{(n)}(q,t)$ is given in (4.6). That is, using (4.10),

$$\begin{aligned} G^{(n)}(q,t;q^\mu) &= C_n \prod_{\substack{r,s\geq 1\\ rs\leq n}} \Bigg(\left(\prod_{\alpha\in\Delta} G_\alpha^{(r,s)}(q,t;q^\mu) \right) \\ & \qquad \cdot \left(\det M(q^r,t^r) \right) (q^r - q^{-r})^\ell (t^r - t^{-r})^\ell \Bigg)^{p_\ell(n-rs)} . \end{aligned} \tag{4.15}$$

Proof. The proof is a generalization of the proof in [10] for $\mathfrak{g} = \mathfrak{sl}_2$ to which we refer for more details. First we observe that the determinant $G^{(n)}(q,t;q^\mu)$ can be factorized as in (4.13) by using the norm preserving homomorphism $\imath : M(h(\mu)) \to F(\mu)$. The Fock space determinant $g^{(n)}(q,t)$ was computed in Lemma 4.2, so it remains to compute a sufficient number of vanishing lines of $\Pi^{(n)}(q,t;q^\mu)$ (note that $\Pi^{(n)}(q,t;q^\mu)$ is a Laurent polynomial in q^μ). The construction of a set of vanishing lines of $\Pi^{(n)}(q,t;q^\mu)$ proceeds as follows. For every weight μ of the form $\mu = \beta\mu^{(+)} - r^\vee\mu^{(-)}$ with $\mu^{(+)} \in P_+^\vee$ and $\mu^{(-)} \in P_+$ and $i = 1,\ldots,\ell$, we can construct a $\mathcal{W}_{q,t}[\mathfrak{g}]$ singular vector in $F(\bar\mu)$ (where $\bar\mu$ is given by (4.11)) at level $(\mu^{(+)}+\rho^\vee,\alpha_i)(\mu^{(-)}+\rho,\alpha_i^\vee)$. Explicitly, this singular vector is the image of the highest weight vector $|\bar\mu + r\beta\alpha_i^\vee\rangle$ under the map

$$\oint \prod_{j=1}^{r} \left(dz_j S_+^i(z_j)\right) \ : \ F(\bar\mu + r\beta\alpha_i^\vee) \ \to \ F(\bar\mu)\,, \tag{4.16}$$

where

$$r \ = \ (\mu^{(+)}+\rho^\vee,\alpha_i)\,, \qquad s \ = \ (\mu^{(-)}+\rho,\alpha_i^\vee)\,. \tag{4.17}$$

Note that, with the definition (4.17),

$$(\mu - r^\vee\rho + \beta\rho^\vee,\alpha_i) \ = \ r\beta - \tfrac{1}{2}r^\vee s(\alpha_i,\alpha_i)\,. \tag{4.18}$$

For $\mathcal{W}_{q,t}[\mathfrak{sl}_N]$ this construction was carried out in [13], where it was also shown that in this case the result could be expressed in terms of Macdonald polynomials. The construction for general $\mathfrak{g}$ is a straightforward generalization (cf. [1] in the conformal case). Due to the non-degenerate pairing between $F(\mu)$ and $F(\bar\mu)$ (for generic values of $(q,t;q^\mu)$) there must exist a vector in the cokernel of the map $\imath : M(h(\mu)) \to F(\mu)$ at level rs, i.e., by using (4.18), we conclude that $\Pi^{(n)}(q,t;q^\mu)$ has vanishing lines $G_{\alpha_i}^{(r,s)}(q,t;q^\mu)$ for all $i = 1,\ldots,\ell$, and $rs \geq n$. Using the Weyl group invariance (4.9), one then proves that $\Pi^{(n)}(q,t;q^\mu)$ is given by (4.14) up to a Laurent polynomial C_n in q, t and q^μ. To prove that C_n is actually a constant it suffices to compute the leading order term in $\Pi^{(n)}(q,t;q^\mu)$ (the (partial) ordering is given by $q^\mu \succeq q^{\mu'}$ iff $\mu - \mu' \in \mathbb{Z}_{\geq 0} \cdot \Delta_+$). We find (cf. [10] for more details)

$$\begin{aligned}
&\prod_{\lambda \vdash n} \prod_{i=1}^{\ell} \left(q^{-2r^\vee(\rho,\omega_i)} t^{2(\rho^\vee,\omega_i)} q^{2(\mu,\omega_i)} \right)^{\mathrm{length}(\lambda^{(i)})} \\
&\quad = \prod_{\substack{r,s\geq 1 \\ rs \leq n}} \prod_{\alpha\in\Delta_+} \left(q^{-r^\vee(\rho,\alpha)} t^{(\rho^\vee,\alpha)} q^{(\mu,\alpha)} \right)^{p_\ell(n-rs)} \\
&\quad = \prod_{\substack{r,s\geq 1 \\ rs \leq n}} \left((q^{r^\vee \mathrm{h}^\vee} t^{-\mathrm{h}})^{-\frac{\ell r}{2}} \right. \\
&\qquad \left. \cdot \prod_{\alpha\in\Delta_+} \left(q^{-r^\vee(\rho,\alpha)+\frac{1}{2}r^\vee s(\alpha,\alpha)} t^{(\rho^\vee,\alpha)-r} q^{(\mu,\alpha)} \right) \right)^{p_\ell(n-rs)} ,
\end{aligned} \tag{4.19}$$

where we have used

$$\sum_i \omega_i = \tfrac{1}{2} \sum_{\alpha\in\Delta_+} \alpha , \tag{4.20}$$

and

$$\ell \mathrm{h} = 2|\Delta_+| , \qquad \ell \mathrm{h}^\vee = \sum_{\alpha\in\Delta_+} (\alpha,\alpha) . \tag{4.21}$$

This concludes the proof of the theorem. □

Remark. It follows from Theorem 2.6 that the determinant of the matrix

$$\Pi'^{(n)}_{\lambda\lambda'}(q,t;q^\mu) = (q^{r^\vee \mathrm{h}^\vee} t^{-\mathrm{h}})^{\frac{n}{2}} \, \Pi^{(n)}_{\lambda\lambda'}(q,t;q^\mu) , \tag{4.22}$$

satisfies (cf. (3.19) in [10])

$$\Pi'^{(n)}(q^{-1},t^{-1};q^{-\mu}) = \Pi'^{(n)}(q,t;q^\mu) . \tag{4.23}$$

This can indeed be verified from the explicit expression (4.14) of this determinant.

5 The Center of $\mathcal{W}_{q,t}[\mathfrak{g}]$ at Roots of Unity

In this section we consider the limit of $\mathcal{W}_{q,t}[\mathfrak{g}]$ when t^2 is a primitive k-th root of unity, $t^2 \to \sqrt[k]{1}$, and q^2 is generic, i.e., we have $t^{2k} = 1$ and $t^{2j} \neq 1$ for all $j = 1, \ldots, k-1$, while $q^{2j} \neq 1$ for all $j \neq 0$. We will restrict our discussion mainly to the simplest case $\mathfrak{g} = \mathfrak{sl}_N$. Note that, because of the duality $\mathcal{W}_{q,t}[\mathfrak{g}] \simeq \mathcal{W}_{t,q}[\mathfrak{g}]$, which holds when $\mathfrak{g}$ is simply laced, it does not matter with respect to which deformation parameter the limit is taken. For $\mathfrak{g}$ non-simply laced the situation is different as the deformation is not symmetric under the interchange of q and t. (In fact, the duality above is then replaced by a more complicated relation [16].) It follows from the definitions in Section 2 that the dependence of $\mathcal{W}_{q,t}[\mathfrak{g}]$ on t, unlike that on q, is in some sense universal for all $\mathfrak{g}$. This allows an extension of the construction to the general case, which is then verified on examples for rank 2 algebras using the explicit realizations from Appendix A.

The definition of $\mathcal{W}_{q,t}[\mathfrak{g}]$, in the limit $t^2 \to \sqrt[k]{1}$, presents a subtlety in that the oscillators (2.17) and thus the screening currents (2.15) are not well defined in this limit. Note, however, that the divergent factor in the definition of the oscillators (2.17) cancels out in the commutators between the screening operators S_i^- and the fields in $\mathbf{H}_{q,t}[\mathfrak{g}]$. Thus the problem of computing $\mathcal{W}_{q,t}[\mathfrak{g}]$ as a commutant defined through those equations is well posed, also in the limit, and will simply lead to an algebra $\mathcal{W}_{q,t}[\mathfrak{g}]$ with generators $T_i(z)$, as in Assumption 2.4, specialized from the generic case to $t^2 = \sqrt[k]{1}$.

It is an obvious observation that, as follows from (2.1) and (2.7), in the limit $t^2 \to \sqrt[k]{1}$ (or $q^2 \to \sqrt[k]{1}$) the algebra $\mathcal{H}_{q,t}[\mathfrak{g}]$ has a large center generated by the oscillators $a_i[m]$ (equivalently, $y_i[m]$), with $m = 0 \bmod k$. In turn, this implies that there should be a corresponding center of $\mathcal{W}_{q,t}[\mathfrak{g}]$ and our goal is to verify that by constructing this center explicitly in terms of the generators $T_i(z)$.

The existence of this center may also be inferred from the formula for the Kac determinant of $\mathcal{W}_{q,t}[\mathfrak{g}]$ in (4.15). Namely, the determinant, $G^{(n)}$, contains a factor $\prod_r (q^r - q^{-r})(t^r - t^{-r})$, and thus vanishes for either t^2 or q^2 a root of unity and n sufficiently large. Thus, for those values of the deformation parameters, the Verma module should have additional singular vectors that are independent of the highest weight. Indeed, it follows from the nonvanishing of (4.14) for a generic q and μ, that the Verma module, $M(h(\mu))$, is isomorphic with the Fock module, $F(\mu)$. Obviously, the latter has infinitely many singular vectors corresponding to the center of $\mathcal{H}_{q,t}[\mathfrak{g}]$, which in turn give rise to singular vectors in the Verma module independent of the highest weight.

To make this discussion more explicit let us now consider the case $\mathfrak{g} = \mathfrak{sl}_N$. The generators of $\mathcal{W}_{q,t}[\mathfrak{g}]$ are given in Theorem 2.3. Note that if we recast (2.18) as in (2.21), all the coefficients $c_\lambda^{\omega_i}(q,t)$ are equal to one and

$$Y_\lambda^{\omega_i}(z) \;=\; :\Lambda_{l_1}(zp^{-i+1})\Lambda_{l_2}(zp^{-i+3})\ldots\Lambda_{l_i}(zp^{i-1}): \,. \tag{5.1}$$

The (nondegenerate) weight $\lambda \in P(V(\omega_i))$ in (5.1) corresponds to the sequence $(l_1, \ldots, l_i)$, $l_1 < \ldots < l_i$, such that $\lambda = \sum_j \epsilon_{l_j}$ where $\epsilon_i \,, i = 1, \ldots, N$, is an overcomplete basis in terms of which the simple roots of $\mathfrak{sl}_N$ are given as

$\alpha_i = \epsilon_i - \epsilon_{i+1}$. In particular, $(1, 2, \ldots, i)$ is the sequence for ω_i, in accordance with

$$Y_{\omega_i}^{\omega_i}(z) \;=\; Y_i(z) \;=\; :\Lambda_1(zp^{-i+1})\Lambda_2(zp^{-i+3})\ldots\Lambda_i(zp^{i-1}): . \tag{5.2}$$

For $\lambda \in P(V(\omega_i))$ and $\lambda' \in P(V(\omega_j))$, define $f^{ij}_{\lambda\lambda'}(x)$ by

$$Y_\lambda^{\omega_i}(z)Y_{\lambda'}^{\omega_j}(w) \;=\; f^{ij}_{\lambda\lambda'}(\frac{w}{z})^{-1} :Y_\lambda^{\omega_i}(z)Y_{\lambda'}^{\omega_j}(w): . \tag{5.3}$$

Setting $\lambda = \lambda' = \omega_i$, we obtain $f^{ii}_{\omega_i\omega_i}(x) = f_{ii}(x)$, which is given in (2.32). Then an arbitrary $f^{ij}_{\lambda\lambda'}(x)$ can be computed using [12]:

$$\Lambda_l(z)\Lambda_{l'}(w) \;=\; s_{l,l'}(\frac{w}{z})f_{11}(\frac{w}{z})^{-1} :\Lambda_l(z)\Lambda_{l'}(w):, \tag{5.4}$$

where

$$s_{l,l'}(x) \;=\; \begin{cases} s(x) & \text{for} \quad l < l' \\ 1 & \text{for} \quad l = l' \\ s(xp^2) & \text{for} \quad l > l' \end{cases} \tag{5.5}$$

and

$$s(x) \;=\; \frac{(q - q^{-1}x)(t^{-1} - tx)}{(p - p^{-1}x)(1 - x)} . \tag{5.6}$$

For our purposes it will suffice to consider only weights in the same representation, i.e., we take $\lambda, \lambda' \in P(V(\omega_i))$ corresponding to sequences $\lambda = (l_1, \ldots, l_i)$ and $\lambda' = (l'_1, \ldots, l'_i)$, respectively.

An immediate consequence of (5.5) is that $f^{ii}_{\lambda\lambda}(x)$ does not depend on a particular choice of the weight,

$$f^{ii}_{\lambda\lambda}(x) \;=\; f^{ii}_{\omega_i\omega_i}(x) \;=\; f_{ii}(x) . \tag{5.7}$$

For $\lambda \neq \lambda'$, we can use (5.5) to compute the additional factor that arises from the points at which $(l_1, \ldots, l_i)$ and $(l'_1, \ldots, l'_i)$ are different. Hence we write

$$f^{ii}_{\lambda\lambda'}(x)^{-1} \;=\; s_{\lambda\lambda'}(x)f_{ii}(x)^{-1} , \tag{5.8}$$

where

$$s_{\lambda\lambda'}(x) \;\equiv\; s_{l_1\ldots l_i, l'_1\ldots l'_i}(x) \;=\; \prod_{a=1}^{i} s_{l_a, l'_1\ldots l'_i}(x) , \tag{5.9}$$

and $s_{l_a,l'_1\ldots l'_i}(x)$ is the contribution due to the ordering of l_a with respect to $l'_1,\ldots,l'_i$. We find (cf. [12])

$$s_{l_a,l'_1\ldots l'_i}(x) \;=\; \begin{cases} 1 & \text{for} \quad l_a = l'_b \\ s(xp^{2m}) & \text{for} \quad l'_{a+m-1} < l_a < l'_{a+m} \end{cases} \tag{5.10}$$

where the (in)equality is satisfied for some $1 \leq b \leq m$ or $-a \leq m \leq i-a+1$.

The center can be constructed in terms of the generators $T_i(z)$ by generalizing the corresponding result for $\mathfrak{sl}_2$ in [10].

Theorem 5.1. *For $t^2 = \sqrt[k]{1}$ and q generic, define*

$$\Psi_i^{(k)}(z) \;=\; \lim_{z_m \to zt^{2(k-m)}} \Big(\prod_{m<n} f_{ii}(\frac{z_n}{z_m})\Big) T_i(z_1)\ldots T_i(z_k)\,. \tag{5.11}$$

Then we have

$$\Psi_i^{(k)}(z) \;=\; \sum_{\lambda\in P(V(\omega_i))} :Y_\lambda^{\omega_i}(zt^{2(k-1)})Y_\lambda^{\omega_i}(zt^{2(k-2)})\ldots Y_\lambda^{\omega_i}(z): \,. \tag{5.12}$$

and $\Psi_i^{(k)}(z)$ is a well defined series of central elements of $\mathcal{W}_{q,t}[\mathfrak{sl}_N]$.

Proof. After expanding (5.11) using (2.18), (5.1) and (5.3) we obtain a sum of terms of the form

$$\Big(\prod_{m<n} s_{\lambda_m\lambda_n}(\frac{z_n}{z_m})\Big) :Y_{\lambda_1}^{\omega_i}(z_1)\ldots Y_{\lambda_k}^{\omega_i}(z_k): \,. \tag{5.13}$$

It follows from (5.6) that for a generic q, and thus p, none of the factors $s_{\lambda_m\lambda_n}(z_n/z_m)$ develop a pole in the limit $z_n/z_m \to t^{2(m-n)}$. (Note that we have $|m-n| < k$.)

Consider the weights $\lambda_m = (l_{m,1},\ldots,l_{m,i})$ and $\lambda_{m+1} = (l_{m+1,1},\ldots,l_{m+1,i})$. Using (5.10) it is easy to see that $s_{\lambda_m\lambda_{m+1}}(t^{-2})$ has a vanishing factor of $s(t^{-2})$ unless $l_{m,a} \geq l_{m+1,a}$ for all $a = 1,\ldots,i$. Next consider λ_1 and λ_k. By the previous argument we may assume $\lambda_{1,a} \geq \lambda_{2,a} \geq \ldots \geq \lambda_{k,a}$. Suppose $\lambda_{1,i} > \lambda_{k,i}$. This results in a factor $s(p^2t^{2k-2})$, which vanishes for $t^2 = \sqrt[k]{1}$. Thus we must have $l_{1,i} = l_{2,i} = \ldots = l_{k,i}$. Proceeding by induction we then find that the only nonvanishing terms in (5.13) arise for $\lambda_1 = \ldots = \lambda_k$, which proves (5.12).

It follows from the explicit expressions (2.8), (2.9) and (2.19), that for $t^2 = \sqrt[k]{1}$ all terms in the sum on the right hand side of (5.12) have an expansion in terms of the oscillators $a_i[nk]$, $n \in \mathbb{Z}$. Thus, at least formally, (5.12) is in the center of $\mathcal{W}_{q,t}[\mathfrak{g}]$. $\square$

If one tries to expand the right hand side of (5.11) in terms of the modes $T_i[n]$, the resulting series is divergent. It is however well defined when acting on the vacuum of a Verma module. We then obtain a series of singular vectors that are manifestly independent of the highest weight. We refer the reader to [10] for explicit formulae for the low lying singular vectors in the $\mathfrak{sl}_2$ case.

Here let us consider as an example the case of $\mathfrak{sl}_3$ with $k = 2$, i.e., $t^2 = -1$. For a generic q and h we find the following pairs of singular vectors at levels $2m$, $m \geq 1$:

$$\begin{aligned}
\Psi_1^{(2)}[-2m]|h\rangle &= \Big(\sum_{\lambda \vdash 2m} m_\lambda(1,t^2) f_{11}(R_{12}) T_1[-\lambda_1] T_1[-\lambda_2] \\
&\qquad \cdot -2(-1)^m p^{2m} [4m-3]_p T_2[-2m] \Big) |h\rangle \,, \\
\Psi_2^{(2)}[-2m]|h\rangle &= \Big(\sum_{\lambda \vdash 2m} m_\lambda(1,t^2) f_{11}(R_{12}) T_2[-\lambda_1] T_2[-\lambda_2] \\
&\qquad \cdot -2(-1)^m p^{-2m} [4m-3]_p T_1[-2m] \Big) |h\rangle \,,
\end{aligned} \tag{5.14}$$

where the sum runs over partitions $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2 \geq 0$ of $2m$, R_{12} is the raising operator acting by

$$R_{12} T_i[m] T_i[n] = T_i[m-1] T_i[n+1] \,, \tag{5.15}$$

and m_λ is the monomial symmetric function [27]. We can make these formulae even more explicit for $m = 1$ where we find

$$\begin{aligned}
\Psi_1^{(2)}[-2]|h\rangle &= -\left(T_1[-1]T_1[-1] + \frac{2}{[3]_p} T_1[-2]T_1[0] - 2p^2 T_2[-2] \right) |h\rangle \,, \\
\Psi_2^{(2)}[-2]|h\rangle &= -\left(T_2[-1]T_2[-1] + \frac{2}{[3]_p} T_2[-2]T_2[0] - \frac{2}{p^2} T_1[-2] \right) |h\rangle \,.
\end{aligned} \tag{5.16}$$

The above discussion has a simple generalization to arbitrary $\mathcal{W}_{q,t}[\mathfrak{g}]$.

Conjecture 5.2. *For $t^2 = \sqrt[k]{1}$ and q generic, define $\Psi_i^{(k)}(z)$ as in (5.11). Then*

$$\Psi_i^{(k)}(z) = \sum_{\lambda \in P(V(\omega_i))} \sum_{j_\lambda = 1}^{\mathrm{mult}\,\lambda} :Y_\lambda^{\omega_i,(j_\lambda)}(zt^{2k-2}) Y_\lambda^{\omega_i,(j_\lambda)}(zt^{2k-4}) \ldots Y_\lambda^{\omega_i,(j_\lambda)}(z): \,, \tag{5.17}$$

is a series of central elements in $\mathcal{W}_{q,t}[\mathfrak{g}]$.

The last assertion in the conjecture is obvious, provided we prove the expansion (5.17). We have verified Conjecture 5.2 in all cases where the generators $T_i(z)$ are known explicitly.

As remarked before, for simply laced $\mathfrak{g}$, the center for q^2 a primitive k-th root of unity (and t generic) follows from Conjecture 5.2 by using the duality invariance $(q,t) \to (t,q)$. For non-simply laced, the situation $q^2 = \sqrt[k]{1}$ is more complicated. In particular, the generating series of central elements will in general not be homogeneous of fixed order in the generators $T_i(z)$, due to a different rescaling $q \to q^{r_i}$ in the various $\mathfrak{sl}_2$ directions. We will leave this case for further investigation.

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Appendix: Examples – the $\mathcal{W}_{q,t}[\mathfrak{g}]$ Algebras of Rank 2

In this appendix we illustrate some of the ideas of the paper in the case of the deformed $\mathcal{W}$-algebras $\mathcal{W}_{q,t}[\mathfrak{g}]$ corresponding to the rank 2 simple Lie algebras A_2, B_2 and G_2. We provide explicit expressions for the generators and their relations and illuminate the connection to the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

To simplify the notation, let us define

$$\frac{\langle a_1, \dots , a_r\rangle}{\langle b_1, \dots , b_s\rangle} = \frac{(a_1 - a_1^{-1}) \dots (a_r - a_r^{-1})}{(b_1 - b_1^{-1}) \dots (b_s - b_s^{-1})} . \tag{A.1}$$

Also, we recall that the function $f^{ij}_{\lambda\lambda'}(x)$ is defined by (cf. (5.3))

$$Y^{\omega_i}_{\lambda}(z) Y^{\omega_j}_{\lambda'}(w) = f^{ij}_{\lambda\lambda'}(\frac{w}{z})^{-1} :Y^{\omega_i}_{\lambda}(z) Y^{\omega_j}_{\lambda'}(w): . \tag{A.2}$$

A.1 $\mathcal{W}_{q,t}[\mathfrak{sl}_3]$

The case $\mathfrak{g} = A_2 = \mathfrak{sl}_3$ has been discussed in detail in [12,13]. For completeness we give a brief review.

We adopt the following conventions. In terms of an overcomplete basis of $\mathbb{R}^2$ given by vectors $\{\epsilon_1, \epsilon_2, \epsilon_3\}$, satisfying

$$\epsilon_i \cdot \epsilon_j \;=\; \delta_{ij} - \tfrac{1}{3}\,, \qquad \epsilon_1 + \epsilon_2 + \epsilon_3 \;=\; 0\,, \tag{A.3}$$

the simple roots and fundamental weights of $\mathfrak{sl}_3$ are written as

$$\begin{aligned} \alpha_1 &= \epsilon_1 - \epsilon_2\,, & \alpha_2 &= \epsilon_2 - \epsilon_3\,, \\ \omega_1 &= \epsilon_1\,, & \omega_2 &= \epsilon_1 + \epsilon_2 \;=\; -\,\epsilon_3\,. \end{aligned} \tag{A.4}$$

We have $(r_1, r_2) = (1,1)$, and $r^\vee = 1$, $\mathrm{h} = \mathrm{h}^\vee = 3$. The weights of the irreducible $\mathfrak{sl}_3$ representations $L(\omega_1) = \mathbf{3}$ and $L(\omega_2) = \mathbf{3}^*$ are given by $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\{-\epsilon_1, -\epsilon_2, -\epsilon_3\}$, respectively.

The deformed Cartan matrix is given by

$$C_{ij}(q,t) \;=\; \begin{pmatrix} qt^{-1} + q^{-1}t & -1 \\ -1 & qt^{-1} + q^{-1}t \end{pmatrix}. \tag{A.5}$$

In particular,

$$\begin{aligned} A_1(z) &= \;:Y_1(zqt^{-1})Y_1(zq^{-1}t)Y_2(z)^{-1}:\,, \\ A_1(z) &= \;:Y_1(z)^{-1}Y_2(zqt^{-1})Y_2(zq^{-1}t):\;. \end{aligned} \tag{A.6}$$

The generators of $\mathcal{W}_{q,t}[\mathfrak{sl}_3]$ follow by applying Theorem 3.1 in the various $\mathfrak{sl}_2$ directions. One finds (see also (2.19))

$$T_i(z) \;=\; \sum_{\lambda \in P(V(\omega_i))} c_\lambda^{\omega_i}(q,t)\, Y_\lambda^{\omega_i}(z)\,, \tag{A.7}$$

where

$$\begin{aligned} \Lambda_1(z) &\equiv Y_{\epsilon_1}^{\omega_1}(z) = \;:Y_1(z):\,, \\ \Lambda_2(z) &\equiv Y_{\epsilon_2}^{\omega_1}(z) = \;:Y_{\epsilon_1}^{\omega_1}(z)A_1(zq^{-1}t)^{-1}: \;=\; :Y_1(zq^{-2}t^2)^{-1}Y_2(zq^{-1}t):\,, \\ \Lambda_3(z) &\equiv Y_{\epsilon_3}^{\omega_1}(z) = \;:Y_{\epsilon_2}^{\omega_1}(z)A_2(zq^{-2}t^2)^{-1}: \;=\; :Y_2(zq^{-3}t^3)^{-1}:\,, \end{aligned} \tag{A.8}$$

$$\begin{aligned} Y_{-\epsilon_3}^{\omega_2}(z) &= \;:Y_2(z):\,, \\ Y_{-\epsilon_2}^{\omega_2}(z) &= \;:Y_{-\epsilon_3}^{\omega_2}(z)A_2(zq^{-1}t)^{-1}: \;=\; :Y_1(zq^{-1}t)Y_2(zq^{-2}t^2)^{-1}:\,, \\ Y_{-\epsilon_1}^{\omega_2}(z) &= \;:Y_{-\epsilon_2}^{\omega_2}(z)A_1(zq^{-2}t^2)^{-1}: \;=\; :Y_1(zq^{-3}t^3)^{-1}:\;. \end{aligned} \tag{A.9}$$

and all $c_\lambda^{\omega_i}(q,t) = 1$, in agreement with (2.24).

Note that we can write (cf. (2.18))

$$\begin{aligned} Y^{\omega_2}_{-\epsilon_3}(z) &= :\Lambda_1(zq^{-1}t)\Lambda_2(zqt^{-1}):, \\ Y^{\omega_2}_{-\epsilon_2}(z) &= :\Lambda_1(zq^{-1}t)\Lambda_3(zqt^{-1}):, \\ Y^{\omega_2}_{-\epsilon_1}(z) &= :\Lambda_2(zq^{-1}t)\Lambda_3(zqt^{-1}): . \end{aligned} \tag{A.10}$$

In fact, it is not hard to see that,

$$T_2(zqt^{-1}) = \lim_{w\to zq^2t^{-2}} f^{11}_{\epsilon_1\epsilon_2}(\frac{w}{z})\, T_1(z)T_1(w)\,, \tag{A.11}$$

which illustrates Conjecture 3.3. Similarly, one can verify, e.g.,

$$\begin{aligned} T_{V(2\omega_1)}(zq) &= \lim_{w\to zq^2} f^{11}_{\epsilon_1\epsilon_1}(\frac{w}{z})\, T_1(z)T_1(w)\,, \\ T_{V(\omega_1+\omega_2)}(z) &= T_1(z)T_2(zq)\,, \end{aligned} \tag{A.12}$$

where $T_{V(2\omega_1)}(z)$ and $T_{V(\omega_1+\omega_2)}(z)$ are the $\mathcal{W}_{q,t}[\mathfrak{sl}_3]$ generators corresponding to the $V(2\omega_1) = \mathbf{6}$ and $V(\omega_1+\omega_2) = \mathbf{8}$ of $U_q(\widehat{\mathfrak{sl}_3})$, respectively.

For the commutation relations one finds

$$\begin{aligned} &f_{11}(\frac{w}{z})T_1(z)T_1(w) - f_{11}(\frac{z}{w})T_1(w)T_1(z) = \\ &\qquad \frac{\langle q,t^{-1}\rangle}{\langle qt^{-1}\rangle}\Big(\delta(q^2t^{-2}\frac{w}{z})T_2(wqt^{-1}) - \delta(q^{-2}t^2\frac{w}{z})T_2(zqt^{-1})\Big)\,, \\ &f_{12}(\frac{w}{z})T_1(z)T_2(w) - f_{12}(\frac{z}{w})T_2(w)T_1(z) = \\ &\qquad \frac{\langle q,t^{-1}\rangle}{\langle qt^{-1}\rangle}\Big(\delta(q^3t^{-3}\frac{w}{z}) - \delta(q^{-3}t^3\frac{w}{z})\Big)\,, \\ &f_{22}(\frac{w}{z})T_2(z)T_2(w) - f_{22}(\frac{z}{w})T_2(w)T_2(z) = \\ &\qquad \frac{\langle q,t^{-1}\rangle}{\langle qt^{-1}\rangle}\Big(\delta(q^2t^{-2}\frac{w}{z})T_1(wqt^{-1}) - \delta(q^{-2}t^2\frac{w}{z})T_1(zqt^{-1})\Big)\,, \end{aligned} \tag{A.13}$$

where $f_{ij}(x)$ is defined in (2.31). Note that $f_{11}(x) = f_{22}(x)$.

A.2 $\mathcal{W}_{q,t}[B_2]$

In this appendix we compute explicit expressions for the generators and relations of the deformed $\mathcal{W}_{q,t}[B_2]$ algebra. The classical limit $t \to 1$, i.e., $\mathcal{W}_{q,1}[B_2]$, has been discussed already in [2, 16].

We adopt the following conventions. In terms of an orthonormal basis $\{\epsilon_1, \epsilon_2\}$ of $\mathbb{R}^2$, the simple roots and fundamental weights of B_2 are given by

$$\begin{aligned} \alpha_1 &= \epsilon_1 - \epsilon_2\,, & \alpha_2 &= \epsilon_2\,, \\ \omega_1 &= \epsilon_1\,, & \omega_2 &= \tfrac{1}{2}(\epsilon_1 + \epsilon_2)\,. \end{aligned} \tag{A.14}$$

We have $(r_1, r_2) = (2,1)$ and $r^\vee = 2$, $\mathrm{h} = 4$, $\mathrm{h}^\vee = 3$. The weights of the B_2 irreducible representations $L(\omega_1) = \mathbf{5}$ and $L(\omega_2) = \mathbf{4}$ are given by $\{\pm\epsilon_1, \pm\epsilon_2, 0\}$ and $\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2)\}$, respectively.

The deformed Cartan matix is given by

$$C_{ij}(q,t) = \begin{pmatrix} q^2t^{-1} + q^{-2}t & -1 \\ -(q+q^{-1}) & qt^{-1} + q^{-1}t \end{pmatrix}. \tag{A.15}$$

In particular,

$$\begin{aligned} A_1(z) &= :Y_1(zq^2t^{-1})Y_1(zq^{-2}t)Y_2(zq^{-1})^{-1}Y_2(zq)^{-1}:\,, \\ A_2(z) &= :Y_1(z)^{-1}Y_2(zqt^{-1})Y_1(zq^{-1}t):\,. \end{aligned} \tag{A.16}$$

Using the $\mathfrak{sl}_2$ pasting procedure outlined in Section 3 we find the following generators of $\mathcal{W}_{q,t}[B_2]$:[8]

$$T_i(z) = \sum_{\lambda \in P(V(\omega_i))} c_\lambda^{\omega_i}(q,t)\; Y_\lambda^{\omega_i}(z)\,, \tag{A.17}$$

where

$$\begin{aligned} \Lambda_1(z) \equiv Y_{\epsilon_1}^{\omega_1}(z) &= :Y_1(z):\,, \\ \Lambda_2(z) \equiv Y_{\epsilon_2}^{\omega_1}(z) &= :Y_{\epsilon_1}^{\omega_1}(z)A_1(zq^{-2}t)^{-1}: \\ &= :Y_1(zq^{-4}t^2)^{-1}Y_2(zq^{-3}t)Y_2(zq^{-1}t):\,, \\ \Lambda_0(z) \equiv Y_0^{\omega_1}(z) &= :Y_{\epsilon_2}^{\omega_1}(z)A_2(zq^{-4}t^2)^{-1}: \\ &= :Y_2(zq^{-5}t^3)^{-1}Y_2(zq^{-1}t):\,, \\ \Lambda_{\bar{2}}(z) \equiv Y_{-\epsilon_2}^{\omega_1}(z) &= :Y_0^{\omega_1}(z)A_2(zq^{-2}t^2)^{-1}: \\ &= :Y_1(zq^{-2}t^2)Y_2(zq^{-5}t^3)^{-1}Y_2(zq^{-3}t^3)^{-1}:\,, \\ \Lambda_{\bar{1}}(z) \equiv Y_{-\epsilon_1}^{\omega_1}(z) &= :Y_{-\epsilon_2}^{\omega_1}(z)A_1(zq^{-4}t^4)^{-1}: \;=\; :Y_1(zq^{-6}t^4)^{-1}:\,, \end{aligned} \tag{A.18}$$

[8]In fact, $T_2(z)$ of $\mathcal{W}_{q,t}[B_2]$ coincides with $T_1(z)$ of $\mathcal{W}_{q,t}[C_2]$, so that those generators can be read-off from the results in [16].

$$\begin{aligned}
Y^{\omega_2}_{\frac{1}{2}(\epsilon_1+\epsilon_2)}(z) &= :Y_2(z):, \\
Y^{\omega_2}_{\frac{1}{2}(\epsilon_1-\epsilon_2)}(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1+\epsilon_2)}(z)A_2(zq^{-1}t)^{-1}: \\
&= :Y_1(zq^{-1}t)Y_2(zq^{-2}t^2)^{-1}:, \\
Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1-\epsilon_2)}(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1-\epsilon_2)}(z)A_1(zq^{-3}t^2)^{-1}: \\
&= :Y_1(zq^{-5}t^3)^{-1}Y_2(zq^{-4}t^2):, \\
Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1+\epsilon_2)}(z) &= :Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1-\epsilon_2)}(z)A_2(zq^{-5}t^3)^{-1}: \\
&= :Y_2(zq^{-6}t^4)^{-1}:,
\end{aligned} \tag{A.19}$$

and the coefficients $c^{\omega_i}_\lambda(q,t)$ are given by

$$c^{\omega_1}_0(q,t) \;=\; \gamma_{2,1}(q,t) \;=\; \frac{\langle q^2, qt^{-1}\rangle}{\langle q, q^2t^{-1}\rangle}, \qquad c^{\omega_1}_\lambda(q,t) \;=\; 1, \quad \lambda \neq 0. \tag{A.20}$$

$$c^{\omega_2}_\lambda(q,t) \;=\; 1, \qquad \forall\lambda. \tag{A.21}$$

The construction of $T_1(z)$ illustrates an important feature that does not occur in the $\mathcal{W}_{q,t}[\mathfrak{sl}_N]$ case, namely the occurance of an $\mathfrak{sl}_2$ string with 3 terms (the one built on $Y^{\omega_1}_{\epsilon_2}(z)$), and consequently the nontrivial coefficient $c^{\omega_1}_0(q,t) = \gamma_{2,1}(q,t)$. The generators above are in perfect agreement with Assumption 2.4 as well as Theorems 2.5 and 2.6.

Note that we can write

$$\begin{aligned}
\Lambda_1(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1+\epsilon_2)}(zq^{-1}t)Y^{\omega_2}_{\frac{1}{2}(\epsilon_1-\epsilon_2)}(zqt^{-1}):, \\
\Lambda_2(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1+\epsilon_2)}(zq^{-1}t)Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1-\epsilon_2)}(zqt^{-1}):, \\
\Lambda_0(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1+\epsilon_2)}(zq^{-1}t)Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1+\epsilon_2)}(zqt^{-1}):, \\
\Lambda_{\bar{2}}(z) &= :Y^{\omega_2}_{\frac{1}{2}(\epsilon_1-\epsilon_2)}(zq^{-1}t)Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1+\epsilon_2)}(zqt^{-1}):, \\
\Lambda_{\bar{1}}(z) &= :Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1-\epsilon_2)}(zq^{-1}t)Y^{\omega_2}_{-\frac{1}{2}(\epsilon_1+\epsilon_2)}(zqt^{-1}): .
\end{aligned} \tag{A.22}$$

In fact, we have

$$T_1(zqt^{-1}) \;=\; \lim_{w\to zq^2t^{-2}} f^{11}_{\frac{1}{2}(\epsilon_1+\epsilon_2),\frac{1}{2}(\epsilon_1-\epsilon_2)}(\frac{w}{z})T_2(z)T_2(w)\,, \tag{A.23}$$

once more illustrating Conjecture 3.3.

For the commutation relations one finds

$$
\begin{aligned}
& f_{11}(\frac{w}{z})T_1(z)T_1(w) - f_{11}(\frac{z}{w})T_1(w)T_1(z) \;= \\
& \quad \frac{\langle q^2, t^{-1}\rangle}{\langle q^2t^{-1}\rangle}\left(\delta(q^4t^{-2}\frac{w}{z})T_{V(2\omega_2)}(wq^2t^{-1}) - \delta(q^{-4}t^2\frac{w}{z})T_{V(2\omega_2)}(zq^2t^{-1})\right) \\
& \qquad + \frac{\langle q^2, t^{-1}, q^3t^{-1}, qt^{-2}\rangle}{\langle qt^{-1}, q^2t^{-1}, q^3t^{-2}\rangle}\left(\delta(q^6t^{-4}\frac{w}{z}) - \delta(q^{-6}t^4\frac{w}{z})\right),
\end{aligned}
$$

$$
\begin{aligned}
& f_{12}(\frac{w}{z})T_1(z)T_2(w) - f_{12}(\frac{z}{w})T_2(w)T_1(z) \;= \\
& \qquad \frac{\langle q^2, t^{-1}\rangle}{\langle q^2t^{-1}\rangle}\left(\delta(q^5t^{-3}\frac{w}{z})T_2(zq^{-1}t) - \delta(q^{-5}t^3\frac{w}{z})T_2(zqt^{-1})\right),
\end{aligned}
$$

$$
\begin{aligned}
& f_{22}(\frac{w}{z})T_2(z)T_2(w) - f_{22}(\frac{z}{w})T_2(w)T_2(z) \;= \\
& \qquad \frac{\langle q, t^{-1}\rangle}{\langle qt^{-1}\rangle}\left(\delta(q^2t^{-2}\frac{w}{z})T_1(wqt^{-1}) - \delta(q^{-2}t^2\frac{w}{z})T_1(zqt^{-1})\right) \\
& \qquad + \frac{\langle q, t^{-1}, q^3t^{-1}, q^2t^{-2}\rangle}{\langle qt^{-1}, q^2t^{-1}, q^3t^{-2}\rangle}\left(\delta(q^6t^{-4}\frac{w}{z}) - \delta(q^{-6}t^4\frac{w}{z})\right),
\end{aligned}
\tag{A.24}
$$

where

$$
\begin{aligned}
T_{V(2\omega_2)}(z) &= \lim_{w\to zq^2} f_{22}(\frac{w}{z})T_2(zq^{-1})T_2(wq^{-1}) \\
&= :Y_2(zq^{-1})Y_2(zq): + \ldots,
\end{aligned}
\tag{A.25}
$$

is the $\mathcal{W}_{q,t}[B_2]$ generator corresponding to the irreducible $U_q(\widehat{B_2})$ representation that decomposes under $U_q(B_2)$ as a $\mathbf{10} \oplus \mathbf{1}$.

A.3 $\mathcal{W}_{q,t}[G_2]$

In this appendix we illustrate our algorithm to compute explicit expressions for the generators and relations of the algebra $\mathcal{W}_{q,t}[G_2]$. For $t = 1$, i.e.

$\mathcal{W}_{q,1}[G_2]$, the generators and (part of) the Poisson algebra structure were already discussed in [18].

We adopt the following conventions. The simple roots are normalized as

$$(\alpha_1,\alpha_1) \;=\; \tfrac{2}{3}\,, \qquad (\alpha_2,\alpha_2) \;=\; 2\,, \qquad (\alpha_1,\alpha_2) \;=\; -1\,, \tag{A.26}$$

such that $r_1 = 1$, $r_2 = 3$ and $r^\vee = 3$. The Coxeter and dual Coxeter numbers are $\mathrm{h} = 6$, $\mathrm{h}^\vee = 4$, while the deformed Cartan matrix is given by

$$C_{ij}(q,t) \;=\; \begin{pmatrix} qt^{-1}+q^{-1}t & -(q^2+1+q^{-2}) \\ -1 & q^3t^{-1}+q^{-3}t \end{pmatrix}\,, \tag{A.27}$$

such that, in particular,

$$\begin{aligned} A_1(z) &= \;:Y_1(zqt^{-1})Y_1(zq^{-1}t)Y_2(z)^{-1}:\,, \\ A_2(z) &= \;:Y_1(zq^2)^{-1}Y_1(z)^{-1}Y_1(zq^{-2})^{-1}Y_2(zq^3t^{-1})Y_2(zq^{-3}t):\;. \end{aligned} \tag{A.28}$$

The two fundamental representations of G_2 are $L(\omega_1) = \mathbf{7}$ and $L(\omega_2) = \mathbf{14}$. While the representation $L(\omega_1)$ can be affinized to a finite dimensional $U_q(\widehat{G_2})$ module $V(\omega_1)$, the minimal affinization $V(\omega_2)$ of $L(\omega_2)$ involves the addition of a singlet [23], i.e., as a representation of $U_q(G_2)$ this $V(\omega_2)$ decomposes as $\mathbf{14} \oplus \mathbf{1}$.

The corresponding $\mathcal{W}_{q,t}[G_2]$ generators are given by

$$T_i(z) \;=\; \sum_{\lambda\in P(V(\omega_i))} c_\lambda^{\omega_i}(q,t)\, Y_\lambda^{\omega_i}(z)\,, \tag{A.29}$$

where

$$
\begin{aligned}
\Lambda_1(z) &\equiv Y^{\omega_1}_{2\alpha_1+\alpha_2}(z) &&= \,:\!Y_1(z)\!:\,, \\
\Lambda_2(z) &\equiv Y^{\omega_1}_{\alpha_1+\alpha_2}(z) &&= \,:\!\Lambda_1(z)A_1(zq^{-1}t)^{-1}\!: \\
& &&= \,:\!Y_1(zq^{-2}t^2)^{-1}Y_2(zq^{-1}t)\!:\,, \\
\Lambda_3(z) &\equiv Y^{\omega_1}_{\alpha_1}(z) &&= \,:\!\Lambda_2(z)A_2(zq^{-4}t^2)^{-1}\!: \\
& &&= \,:\!Y_1(zq^{-6}t^2)Y_1(zq^{-4}t^2)Y_2(zq^{-7}t^3)^{-1}\!:\,, \\
\Lambda_4(z) &\equiv Y^{\omega_1}_{0}(z) &&= \,:\!\Lambda_3(z)A_1(zq^{-7}t^3)^{-1}\!: \\
& &&= \,:\!Y_1(zq^{-8}t^4)^{-1}Y_1(zq^{-4}t^2)\!:\,, \\
\Lambda_5(z) &\equiv Y^{\omega_1}_{-\alpha_1}(z) &&= \,:\!\Lambda_4(z)A_1(zq^{-5}t^3)^{-1}\!: \\
& &&= \,:\!Y_1(zq^{-8}t^4)^{-1}Y_1(zq^{-6}t^4)^{-1}Y_2(zq^{-5}t^3)\!:\,, \\
\Lambda_6(z) &\equiv Y^{\omega_1}_{-(\alpha_1+\alpha_2)}(z) &&= \,:\!\Lambda_5(z)A_2(zq^{-8}t^4)^{-1}\!: \\
& &&= \,:\!Y_1(zq^{-10}t^4)Y_2(zq^{-11}t^5)^{-1}\!:\,, \\
\Lambda_7(z) &\equiv Y^{\omega_1}_{-(2\alpha_1+\alpha_2)}(z) &&= \,:\!\Lambda_6(z)A_1(zq^{-11}t^5)^{-1}\!: \;=\; :\!Y_1(zq^{-12}t^6)^{-1}\!:\,,
\end{aligned}
\tag{A.30}
$$

while

$$
\begin{aligned}
Y^{\omega_2}_{3\alpha_1+2\alpha_2}(z) &= \,:\!Y_2(z)\!:\,, \\
Y^{\omega_2}_{3\alpha_1+\alpha_2}(z) &= \,:\!Y^{\omega_2}_{3\alpha_1+2\alpha_2}(z)A_2(zq^{-3}t)^{-1}\!: \\
&= \,:\!Y_1(zq^{-5}t)Y_1(zq^{-3}t)Y_1(zq^{-1}t)Y_2(zq^{-6}t^2)^{-1}\!:\,, \\
Y^{\omega_2}_{2\alpha_1+\alpha_2}(z) &= \,:\!Y^{\omega_2}_{3\alpha_1+\alpha_2}(z)A_1(zq^{-6}t^2)^{-1}\!: \\
&= \,:\!Y_1(zq^{-3}t)Y_1(zq^{-1}t)Y_1(zq^{-7}t^3)^{-1}\!:\,, \\
Y^{\omega_2}_{\alpha_1+\alpha_2}(z) &= \,:\!Y^{\omega_2}_{2\alpha_1+\alpha_2}(z)A_1(zq^{-4}t^2)^{-1}\!: \\
&= \,:\!Y_1(zq^{-1}t)Y_1(zq^{-7}t^3)^{-1}Y_1(zq^{-5}t^3)^{-1}Y_2(zq^{-4}t^2)\!:\,, \\
Y^{\omega_2}_{\alpha_1}(z) &= \,:\!Y^{\omega_2}_{\alpha_1+\alpha_2}(z)A_2(zq^{-7}t^3)^{-1}\!: \\
&= \,:\!Y_1(zq^{-9}t^3)Y_1(zq^{-1}t)Y_2(zq^{-10}t^4)^{-1}\!:\,,
\end{aligned}
$$

$$\begin{aligned}
Y^{\omega_2}_{\alpha_2}(z) &= :Y^{\omega_2}_{\alpha_1+\alpha_2}(z)A_1(zq^{-2}t^2)^{-1}: \\
&= :Y_1(zq^{-7}t^3)^{-1}Y_1(zq^{-5}t^3)^{-1} \\
&\qquad \cdot Y_1(zq^{-3}t^3)^{-1}Y_2(zq^{-4}t^2)Y_2(zq^{-2}t^2):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{0}(z) &= :Y^{\omega_2}_{\alpha_1}(z)A_1(zq^{-10}t^4)^{-1}: \\
&= :Y_1(zq^{-11}t^5)^{-1}Y_1(zq^{-1}t):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2\prime}_{0}(z) &= :Y^{\omega_2}_{\alpha_1}(z)A_1(zq^{-2}t^2)^{-1}: \;=\; :Y^{\omega_2}_{\alpha_2}(z)A_2(zq^{-7}t^3)^{-1}: \\
&= :Y_1(zq^{-9}t^3)Y_1(zq^{-3}t^3)^{-1}Y_2(zq^{-10}t^4)^{-1}Y_2(zq^{-2}t^2):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2\prime\prime}_{0}(z) &= :Y^{\omega_2}_{\alpha_2}(z)A_2(zq^{-5}t^3)^{-1}: \\
&= :Y_2(zq^{-8}t^4)^{-1}Y_2(zq^{-4}t^2):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-\alpha_2}(z) &= :Y^{\omega_2\prime}_{0}(z)A_2(zq^{-5}t^3)^{-1}: \\
&= :Y_1(zq^{-9}t^3)Y_1(zq^{-7}t^3)Y_1(zq^{-5}t^3) \\
&\qquad \cdot Y_2(zq^{-10}t^4)^{-1}Y_2(zq^{-8}t^4)^{-1}:,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-\alpha_1}(z) &= :Y^{\omega_2\prime}_{0}(z)A_1(zq^{-10}t^4)^{-1}: \\
&= :Y_1(zq^{-11}t^5)^{-1}Y_1(zq^{-3}y^3)^{-1}Y_2(zq^{-2}t^2):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-\alpha_1-\alpha_2}(z) &= :Y^{\omega_2}_{-\alpha_1}(z)A_2(zq^{-5}t^3)^{-1}: \;=\; :Y^{\omega_2}_{-\alpha_2}(z)A_1(zq^{-10}t^4)^{-1}: \\
&= :Y_1(zq^{-11}t^5)^{-1}Y_1(zq^{-7}t^3)Y_1(zq^{-5}t^3)Y_2(zq^{-8}t^4)^{-1}:,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-2\alpha_1-\alpha_2}(z) &= :Y^{\omega_2}_{-\alpha_1-\alpha_2}(z)A_1(zq^{-8}t^4)^{-1}: \\
&= :Y_1(zq^{-11}t^5)^{-1}Y_1(zq^{-9}t^5)^{-1}Y_1(zq^{-5}t^3):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-3\alpha_1-\alpha_2}(z) &= :Y^{\omega_2}_{-2\alpha_1-\alpha_2}(z)A_1(zq^{-6}t^4)^{-1}: \\
&= :Y_1(zq^{-11}t^5)^{-1}Y_1(zq^{-9}t^5)^{-1} \\
&\qquad \cdot Y_1(zq^{-7}t^5)^{-1}Y_2(zq^{-6}t^4):,
\end{aligned}$$

$$\begin{aligned}
Y^{\omega_2}_{-3\alpha_1-2\alpha_2}(z) &= :Y^{\omega_2}_{-3\alpha_1-2\alpha_2}(z)A_2(zq^{-9}t^5)^{-1}: \\
&= :Y_2(zq^{-12}t^6)^{-1}:,
\end{aligned} \tag{A.31}$$

and the coefficients $c_\lambda^{\omega_i}(q,t)$ are given by

$$\begin{aligned} c^{\omega_1}_{2\alpha_1+\alpha_2}(q,t) &= c^{\omega_1}_{\alpha_1+\alpha_2}(q,t) = c^{\omega_1}_{\alpha_1}(q,t) = c^{\omega_1}_{-\alpha_1}(q,t) \\ &= c^{\omega_1}_{-(\alpha_1+\alpha_2)}(q,t) = c^{\omega_1}_{-(2\alpha_1+\alpha_2)}(q,t) = 1\,, \\ c^{\omega_1}_0(q,t) &= \gamma_{2,1}(q,t) = \frac{\langle q^2, qt^{-1}\rangle}{\langle q, q^2t^{-1}\rangle}\,. \end{aligned} \tag{A.32}$$

and

$$\begin{aligned} c^{\omega_2}_{3\alpha_1+2\alpha_2}(q,t) &= c^{\omega_2}_{3\alpha_1+\alpha_2}(q,t) = c^{\omega_2}_{\alpha_2}(q,t) = c^{\omega_2}_{-\alpha_2}(q,t) \\ &= c^{\omega_2}_{-3\alpha_1-\alpha_2}(q,t) = c^{\omega_2}_{-3\alpha_1-2\alpha_2}(q,t) = 1\,, \\ c^{\omega_2}_{2\alpha_1+\alpha_2}(q,t) &= c^{\omega_2}_{\alpha_1+\alpha_2}(q,t) = c^{\omega_2}_{\alpha_1}(q,t) = c^{\omega_2}_{-\alpha_1}(q,t) \\ &= c^{\omega_2}_{-\alpha_1-\alpha_2}(q,t) = c^{\omega_2}_{-2\alpha_1-\alpha_2}(q,t) = \frac{\langle q, q^3t^{-1}\rangle}{\langle q, q^3t^{-1}\rangle}\,, \\ c^{\omega_2}_0(q,t) &= \frac{\langle q^3, qt^{-1}, q^5t^{-1}, q^4t^{-2}\rangle}{\langle q, q^3t^{-1}, q^4t^{-1}, q^5t^{-2}\rangle}\,, \\ c^{\omega_2\prime}_0(q,t) &= \frac{\langle q^4, qt^{-1}\rangle}{\langle q, q^4t^{-1}\rangle}\,, \\ c^{\omega_2\prime\prime}_0(q,t) &= -\,\frac{\langle q^2, qt\rangle}{\langle q, q^2t^{-1}\rangle}\,. \end{aligned} \tag{A.33}$$

Note that again we find perfect agreement with both Theorems 2.5 and 2.6. The construction of the generator $T_2(z)$ illustrates two important features. First, the $\mathfrak{sl}_2$ string built on, e.g., $Y^{\omega_2}_{\alpha_1}(z)$, requires 4 terms as compared to the 3-dimensional $U_q(\mathfrak{sl}_2)$ representation which occurs at this point in the **14** of $U_q(G_2)$. This illustrates the necessity for extending the **14** by a **1**. Secondly, the $\mathfrak{sl}_2$ strings built on $Y^{\omega_2}_{\alpha_1}(z)$ and $Y^{\omega_2}_{\alpha_2}(z)$ intersect at the point $Y^{\omega_2\prime}_0(z)$. For consistency of the $\mathfrak{sl}_2$ pasting procedure we therefore need to find the same coefficient $c^{\omega_2\prime}_0(q,t)$ regardless of which $\mathfrak{sl}_2$ path we choose to reach $Y^{\omega_2\prime}_0(z)$. This can indeed be verified.

Note that

$$
\begin{aligned}
Y^{\omega_2}_{3\alpha_1+2\alpha_2}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_2(zqt^{-1}):\,, \\
Y^{\omega_2}_{3\alpha_1+\alpha_2}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_3(zqt^{-1}):\,, \\
Y^{\omega_2}_{2\alpha_1+\alpha_2}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_4(zqt^{-1}):\,, \\
Y^{\omega_2}_{\alpha_1+\alpha_2}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_5(zqt^{-1}):\,, \\
Y^{\omega_2}_{\alpha_1}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_6(zqt^{-1}):\,, \\
Y^{\omega_2}_{\alpha_2}(z) &= \, :\Lambda_2(zq^{-1}t)\Lambda_5(zqt^{-1}):\,, \\
Y^{\omega_2}_{0}(z) &= \, :\Lambda_1(zq^{-1}t)\Lambda_7(zqt^{-1}):\,, \\
Y^{\omega_2\prime}_{0}(z) &= \, :\Lambda_2(zq^{-1}t)\Lambda_6(zqt^{-1}):\,, \\
Y^{\omega_2\prime\prime}_{0}(z) &= \, :\Lambda_3(zq^{-1}t)\Lambda_5(zqt^{-1}):\,, \\
Y^{\omega_2}_{-\alpha_2}(z) &= \, :\Lambda_3(zq^{-1}t)\Lambda_6(zqt^{-1}):\,, \\
Y^{\omega_2}_{-\alpha_1}(z) &= \, :\Lambda_2(zq^{-1}t)\Lambda_7(zqt^{-1}):\,, \\
Y^{\omega_2}_{-\alpha_1-\alpha_2}(z) &= \, :\Lambda_3(zq^{-1}t)\Lambda_7(zqt^{-1}):\,, \\
Y^{\omega_2}_{-2\alpha_1-\alpha_2}(z) &= \, :\Lambda_4(zq^{-1}t)\Lambda_7(zqt^{-1}):\,, \\
Y^{\omega_2}_{-3\alpha_1-\alpha_2}(z) &= \, :\Lambda_5(zq^{-1}t)\Lambda_7(zqt^{-1}):\,, \\
Y^{\omega_2}_{-3\alpha_1-2\alpha_2}(z) &= \, :\Lambda_6(zq^{-1}t)\Lambda_7(zqt^{-1}):\,,
\end{aligned}
\tag{A.34}
$$

so that we can also write

$$
T_2(z) \;=\; \sum_{i<j} c_{ij}(q,t) \, :\Lambda_i(zq^{-1}t)\Lambda_j(zqt^{-1}):\,, \tag{A.35}
$$

with appropriately chosen coefficients $c_{ij}(q,t)$ (some of which are vanishing). In fact, an explicit examination of all the contractions shows

$$
T_2(zqt^{-1}) \;=\; \lim_{w\to zq^2t^{-2}} f^{11}_{2\alpha_1+\alpha_2,\alpha_1+\alpha_2}\Big(\frac{w}{z}\Big)\, T_1(z)T_1(w)\,, \tag{A.36}
$$

which again confirms Conjecture 3.3.

The commutation relations are given by

$$
\begin{aligned}
& f_{11}(\frac{w}{z})T_1(z)T_1(w) - f_{11}(\frac{z}{w})T_1(w)T_1(z) \;= \\
& \quad \frac{\langle q, t^{-1}\rangle}{\langle qt^{-1}\rangle}\Big(\delta(q^2t^{-2}\frac{w}{z})T_2(wqt^{-1}) - \delta(q^{-2}t^2\frac{w}{z})T_2(zqt^{-1})\Big) \\
& \quad + \frac{\langle q^2, t^{-1}, q^4t^{-1}, q^3t^{-2}\rangle}{\langle q^2t^{-1}, q^3t^{-1}, q^4t^{-2}\rangle} \\
& \quad \cdot \Big(\delta(q^8t^{-4}\frac{w}{z})T_1(wq^4t^{-2}) - \delta(q^{-8}t^4\frac{w}{z})T_1(zq^4t^{-2})\Big) \\
& \quad + \frac{\langle q, t^{-1}, q^4t^{-1}, q^2t^{-2}, q^6t^{-2}, q^5t^{-3}\rangle}{\langle qt^{-1}, q^2t^{-1}, q^4t^{-2}, q^5t^{-2}, q^6t^{-3}\rangle}\Big(\delta(q^{12}t^{-6}\frac{w}{z}) - \delta(q^{-12}t^6\frac{w}{z})\Big),
\end{aligned}
$$

$$
\begin{aligned}
& f_{12}(\frac{w}{z})T_1(z)T_2(w) - f_{12}(\frac{z}{w})T_2(w)T_1(z) \;= \\
& \quad \frac{\langle q^3, t^{-1}\rangle}{\langle q^3t^{-1}\rangle}\Big(\delta(q^7t^{-3}\frac{w}{z})T_{V(2\omega_1)}(wq^2t^{-1}) - \delta(q^{-7}t^3\frac{w}{z})T_{V(2\omega_1)}(wq^{-2}t)\Big) \\
& \quad + \frac{\langle q^3, t^{-1}, q^5t^{-1}, q^2t^{-2}\rangle}{\langle q^2t^{-1}, q^3t^{-1}, q^5t^{-2}\rangle} \\
& \quad \cdot \Big(\delta(q^{11}t^{-5}\frac{w}{z})T_1(wqt^{-1}) - \delta(q^{-11}t^5\frac{w}{z})T_1(wq^{-1}t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
& f_{22}(\frac{w}{z})T_2(z)T_2(w) - f_{22}(\frac{z}{w})T_2(w)T_2(z) \;= \\
& \quad \frac{\langle q^3, t^{-1}\rangle}{\langle q^3t^{-1}\rangle}\Big(\delta(q^6t^{-2}\frac{w}{z})T_{V(3\omega_1)}(wq^3t^{-1}) - \delta(q^{-6}t^2\frac{w}{z})T_{V(3\omega_1)}(zq^3t^{-1})\Big) \\
& \quad + \frac{\langle q^3, q^3, t^{-1}, qt^{-1}, q^5t^{-1}, q^2t^{-2}\rangle}{\langle q, q^2t^{-1}, q^3t^{-1}, q^3t^{-1}, q^5t^{-2}\rangle} \\
& \quad \cdot \Big(\delta(q^{10}t^{-4}\frac{w}{z})T_{V'(2\omega_1)}(wq^5t^{-2}) - \delta(q^{-10}t^4\frac{w}{z})T_{V'(2\omega_1)}(zq^5t^{-2})\Big) \\
& \quad + \frac{\langle qt, q^2, q^3, t^{-1}, q^4t^{-1}, qt^{-2}\rangle}{\langle q, qt^{-1}, q^2t^{-1}, q^3t^{-1}, q^4t^{-2}\rangle} \\
& \quad \cdot \Big(\delta(q^8t^{-4}\frac{w}{z})T_2(wq^4t^{-2}) - \delta(q^{-8}t^4\frac{w}{z})T_2(zq^4t^{-2})\Big) \\
& \quad + \frac{\langle q^3, t^{-1}, q^4t^{-1}, q^5t^{-1}, qt^{-2}, q^2t^{-2}, q^6t^{-2}, q^3t^{-3}\rangle}{\langle qt^{-1}, q^2t^{-1}, q^3t^{-1}, q^3t^{-2}, q^4t^{-2}, q^5t^{-2}, q^6t^{-3}\rangle} \\
& \quad \cdot \Big(\delta(q^{12}t^{-6}\frac{w}{z}) - \delta(q^{-12}t^6\frac{w}{z})\Big),
\end{aligned}
\tag{A.37}
$$

where

$$\begin{aligned} T_{V(3\omega_1)}(z) &= :Y_1(zq^{-2})Y_1(z)Y_1(zq^2): + \ldots\,, \\ T_{V(2\omega_1)}(z) &= :Y_1(zq^{-1})Y_1(zq): + \ldots\,, \\ T_{V'(2\omega_1)}(z) &= :Y_1(zq^{-4}t)Y_1(zq^4t^{-1}): + \ldots\,, \end{aligned} \tag{A.38}$$

are $\mathcal{W}_{q,t}[G_2]$ generators corresponding to irreducible $U_q(\widehat{G_2})$ representations that decompose under $U_q(G_2)$ as $\mathbf{77}' \oplus \mathbf{27} \oplus 2(\mathbf{14})$, $\mathbf{27} \oplus \mathbf{7}$ and $\mathbf{27} \oplus \mathbf{14} \oplus \mathbf{1}$, respectively.

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