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Topological Gravity as Large N Topological Gauge Theory

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Abstract

We consider topological closed string theories on Calabi-Yau manifolds which compute superpotential terms in the corresponding compactified type II effective action. In particular, near certain singularities we compare the partition function of this topological theory (the Kodaira-Spencer theory) to $SU(\infty)$ Chern-Simons theory on the vanishing 3-cycle. We find agreement between these theories, which we check explicitly for the case of shrinking S^3 and Lens spaces, at the perturbative level. Moreover, the gauge theory has non-perturbative contributions which have a natural interpretation in the Type IIB picture. We provide a heuristic explanation for this agreement as well as suggest further equivalences in other topological gravity/gauge systems.

1 Introduction

While it has long been suspected that large N gauge theories might have a string description, it is only recently that physicists have seriously considered the reverse logic of obtaining gravity (or closed string effects) from this limit of gauge theories. That gravity and gauge theory are related in some way is a very old notion. The idea of Kaluza-Klein compactifications, in particular, is a way in which pure gravity leads to gauge theory in lower dimensions. This idea and its extensions have been well studied in the physics literature. The reverse idea of obtaining gravity from gauge dynamics has also been explored, with less clear success. This includes attempts at viewing the graviton as a bound state in a gauge theory (for instance, viewing closed strings as bound states of open strings).

A quite different approach to getting gravity from a gauge theory is to start with a worldvolume theory as some kind of gauge theory (or generalisations) and obtain gravity in the target space as a large N limit [1]. One of the main difficulties in checking these proposals is that, apart from statements protected by supersymmetry, the large N theory is quite difficult to analyze and it has not been possible to verify the validity of these conjectures in full detail.

In this paper we initiate a program of studying topological gravity theories and their connection to large N topological gauge theories in the context of string theory. On the one hand, these provide useful and computable testing grounds for viewing gravity as a large N gauge theory. On the other hand, at least some of the topological gravity theories are related to certain physical amplitudes in compactified string theories. The topological theory computes superpotential terms in the low energy gravitational action of the resulting theory. Thus any proposal for viewing gravity as a large N gauge theory, such as the matrix approach [1], restricted to computation of superpotential terms, will have to descend to a match between a topological gravity theory and a large N topological gauge theory, of the type we are studying in this paper. In this way, we can extract a computable, yet highly non-trivial piece out of string theory and see if it has a large N description in the spirit of the matrix approach.

The organization of this paper is as follows: The topological gravity theories we study arise from topological string theories and are reviewed in Sec.2. In particular, the $N = 2$ topological string computes perturbative contributions to the R^2F^{2g-2} terms of the type IIB effective action compactified on Calabi-Yau threefolds, where R is the curvature and F is the graviphoton field strength (similar statements hold for the $N = 4$ topological strings). It has been known that near the conifold limit of Calabi-Yau 3-folds, these amplitudes are related to the partition function of the $c = 1$ non-critical string at self dual radius. In Sec.3 we will briefly review this and show how this generalises to deformations away from the conifold. These turn out to be related to the amplitudes involving the tachyons and the discrete states of $c = 1$ theory at the self-dual radius. We also consider the case of p times the self dual radius.

Like any string theory the $N = 2$ topological string theory has an intrinsically perturbative definition in terms of a genus expansion. Motivated by certain results in the literature, we propose a large N Chern-Simons theory as a non-perturbative definition of the ($N = 2$) topological string theory (at least near the point where the Calabi-Yau develops a vanishing 3-cycle). In Sec.4 we exactly compute the Chern-Simons partition function on some 3-manifolds which appear as shrinking 3-cycles in Calabi-Yau compactifications. The $N = \infty$ limit can be taken and we find that the resulting free energy reproduces not just the perturbative topological gravity/string theory results but has additional non-perturbative pieces. These are seen to be the effects of pair production of wrapped 3-brane-antibrane states near the conifold. This is something which is not calculable with perturbative string theory and lends support to the claim of the $SU(\infty)$ gauge theory being a complete description of the topological sector of the string theory. We conclude in Sec.5 by explaining some aspects of our results along with a discussion of some of the issues raised in this paper as well as directions for future investigations.

2 Topological String Theories

In this section we will briefly review the two classes of topological string theories. One class corresponds to the underlying 2d conformal field theory having $N = 2$ superconformal symmetry, as, for example, in a sigma model on Calabi-Yau threefolds. The other class corresponds to theories, which before twisting have a small $N = 4$ superconformal symmetry, such as sigma models on $K3$. We shall refer to these two classes as $N = 2$ and $N = 4$ topological strings respectively.

2.1 N=2 Topological Strings

This class of theories was introduced in [2] and further studied in [3–5]. One starts with an $N = 2$ superconformal theory with the two fermionic supercurrents G^+, G^- charged under the $N = 2$ $U(1)$ current. After the "twist", the spin assignments are such that G^+ has dimension 1 and G^- has dimension 2. The structure of the theory after twisting is identical to that of bosonic strings where one thinks of G^+ as the BRST current j_{BRST} and G^- as the anti-ghost field b. Just as is usual in bosonic strings one considers the genus g partition function

$$F_g = \int_{\mathcal{M}_g} < \left|G^- \mu_1 ... G^- \mu_{3g-3}\right|^2 > \tag{2.1}$$

where $\mathcal{M}_g$ denotes the moduli space of genus g surfaces and μ_i are the Beltrami differentials (derivatives of the world sheet metric with respect to moduli). For the partition function to be non-zero without the insertion of any further operators, there should be $3g-3$ units of $U(1)$ charge violation in the measure in order to cancel the $3-3g$ units of charge coming from the G^{-}'s. Since an $N=2$ superconformal theory with central charge $\hat{c}$ has a contribution of $\hat{c}(g-1)$ units of charge at genus g (from the measure, due to twisting) this means that it is for $\hat{c}=3$ that we get the critical case where the free energy is non-zero for all genera. This is the case, in particular, for superconformal theories with Calabi-Yau threefold target spaces. Actually, up to conjugation, there are 2 inequivalent ways to twist an $N=2$ theory, depending on the relative choice of sign for left-moving versus right-moving $U(1)$ currents. Let us refer to these as A versus B topological theories. In the Calabi-Yau realization of these superconformal theories, the partition functions will depend on the moduli of the Calabi-Yau. In particular, the A-twisting gives rise to a partition function which only depends on the Kahler moduli of the Calabi-Yau (and is subject to worldsheet instanton corrections) and arises in questions related to the Coulomb branch of IIA string compactifications. The B-twisting depends only on the complex structure (and is subject to no worldsheet instanton corrections) and similarly appears in questions related to the Coulomb branch of IIB compactifications. Mirror symmetry exchanges the A/B-twisting of a conformal theory corresponding to a manifold and its mirror.

The genus zero partition function of topological theories was studied in detail, beginning with the work [6] (see [7]). The genus one partition function was studied in [4] and was extended to higher genus in [5]. Moreover, it was shown in [5] that the B-twisted topological string gives rise to a theory of topological gravity in 6 dimensions, which was called the *Kodaira-Spencer Theory of Gravity*. It is a theory of variations of the Calabi-Yau metric through variations of the complex structure. The g-th loop vacuum amplitude of the Kodaira-Spencer theory gives rise to the genus g topological partition function F_g.

Let us briefly recall what the Kodaira-Spencer theory is. Let A denote an anti-holomorphic 1-form with values in the holomorphic tangent bundle of the CY 3-fold. In other words, the components of it are denoted by $A_{\bar{i}}^{j}$. It is also convenient to define the $(1,2)$-form A' by

$$A' = A \cdot \Omega \qquad i.e., \qquad A'_{\bar{i}kl} = A_{\bar{i}}^{j}\Omega_{jkl}$$

where Ω denotes the nowhere vanishing holomorphic threeform on the CY 3-fold. Then the Kodaira-Spencer action is defined by

$$S = \int \frac{1}{2}A'\partial^{-1}\overline{\partial}A' + \frac{1}{6}A' \wedge (A \wedge A)' \tag{2.2}$$

where one restricts A to satisfy the gauge condition Then the equation of motion for A gives

$$\overline{\partial} A = [A, A]$$

where the bracket denotes the commutator bracket of A's viewed as vector fields in the CY 3-fold. This equation is the Kodaira-Spencer condition for having a deformation for the $\overline{\partial}$ operator

$$\overline{\partial} \to \overline{\partial} + A\partial$$

satisfying $\overline{\partial}^2 = 0$. This equation and its solution has been extensively studied for Calabi-Yau 3-folds [8] [9].

There is a somewhat heuristic argument [5] which casts the action Eq.(2.2) into a *3-dimensional* Chern-Simons form. Let us Wick rotate our complex coordinates $(z_i, \bar{z}_i)$ on the Calabi-Yau to a (3,3) signature. In other words, if $z_i = u_i + iv_i$, then we take $v_i \to iv_i$ and relabel $z_i \to y_i, \bar{z}_i \to x_i$. The x_i, y_i's are now like coordinates and momenta. (The Kahler form has become a symplectic form.) We can think of the x_i's as parametrising a (3-dimensional) base X and the y_i's the 3-dimensional cotangent fibre Y. The holomorphic 3-form Ω is now the volume form on Y which we can choose to be ϵ_{ijk} in any patch.

Suppressing the indices denoting the dependence on the base X, we can write the gauge condition (adopting three dimensional vector notation on Y) as $\vec{\nabla} \cdot \vec{A}(\vec{y}) = 0$ or locally $\vec{A}(\vec{y}) = \vec{\nabla} \times \vec{C}(\vec{y})$. One can then define the symmetric inner product

$$Tr[A_1 A_2] = \int_Y \vec{A_1}(\vec{y}) \cdot \vec{C_2}(\vec{y}) = \int_Y \vec{A_2}(\vec{y}) \cdot \vec{C_1}(\vec{y}) \qquad (2.3)$$

With this inner product the kinetic term in Eq. (2.2)becomes

$$\int_X Tr[A \wedge d_x A]$$

where A here is a 1-form on X with (implicit) "internal" vector and coordinate (y_i) indices. We can think of these as group indices for the gauge field A on X. $\vec{A}(\vec{y})$ can be thought of as a generator of reparametrisation invariance with the gauge condition restricting it to the group of volume (or Ω) preserving diffeomorphisms of Y.

Using the commutator bracket on Y,

$$[\vec{A_1}, \vec{A_2}] = \vec{\nabla} \times (\vec{A_1} \times \vec{A_2})$$

and the inner product defined above, the cubic term in (2.2) also goes over to

$$\int_X Tr[A \wedge A \wedge A].$$

In other words the Kodaira-Spencer action has taken the form of a Chern-Simons theory with the gauge group of volume preserving diffeomorphisms on Y.

2.2 Open N=2 Topological Strings

One can also consider the open string version of the above, as was done in [3]. In the case of the A twist the boundary condition on the open string is Dirichlet – the endpoints are on a collection of supersymmetric 3-cycles. Moreover it was shown in [3] that if we take the local model of the Calabi-Yau to be T^*M, where M is a real 3-dimensional manifold, and take N copies of M as a supersymmetric cycle in T^*M, then the effective theory describing the target space physics is a $U(N)$ Chern-Simons theory on M where the Chern-Simons coupling k is the string coupling constant g_s. The perturbative expansion of the Chern-Simons effective action has a perturbative open string interpretation. A term in the free energy of the form $\frac{N^h}{k^{2g+h-2}}$ can be associated with a surface with genus g and h boundary components. This follows from the 'tHooft identification of Feynman diagrams in the gauge theory with Riemann surfaces.

The B-version of the twist gives a holomorphic version of Chern-Simons with Neumann boundary conditions with strings propagating on the full Calabi-Yau threefold. This would correspond to a six dimensional gauge theory. The corresponding action is given by [3]:

$$S = \int \Omega \wedge [\frac{1}{2} A\overline{\partial} A + \frac{1}{3} A \wedge A \wedge A] \tag{2.4}$$

where Ω is the holomorphic 3-form of the Calabi-Yau and A is a $U(N)$ holomorphic gauge connection which is an anti-holomorphic 1-form with values in the adjoint of $U(N)$ (the trace over the lie algebra indices is implicit in the above formula).

2.3 What Does the $N = 2$ Topological String Compute?

It is natural to ask what the physical meaning of the topological string amplitudes are, viewed as type II compactifications on the corresponding Calabi-Yau. This has been answered in [5, 10] where it was shown that the amplitude F_g computes those superpotential terms for the $N = 2$ theory on the non-compact R^4 which have only string g loop contributions. In particular if we consider the type IIB string compactified on a Calabi-Yau 3-fold, the B version of the twist computes corrections to superpotential terms involving vector multiplets [5, 10]. More concretely, these are amplitudes involving $(2g - 2)$ graviphotons and 2 gravitons. The A twisted theory gives similar terms for the hypermultiplet [10]. Since the coupling constant of type

II strings is a hypermultiplet (see [11] for a more precise discussion) it follows that the vector multiplet superpotentials do not receive any non-perturbative quantum corrections, whereas the hypermultiplet superpotential terms do.

Let us thus concentrate on vector multiplet superpotentials which are perturbatively exact. The genus g topological partition function computes the correction to the effective action in the four dimensional $N = 2$ theory of the form

$$S = ... + F_g(\{t_i\}) \int R^2 F^{2g-2} \tag{2.5}$$

where F_g is the topological genus g amplitude in Eq.(2.1) which depends on the complex structure moduli $\{t_i\}$ of the Calabi-Yau (the scalars in the vector multiplets), R is the Riemann tensor and F is the $N = 2$ graviphoton field strength (the index contraction as well as the presence of other similar terms is dictated by supersymmetry).

Thus, each genus computation of the Kodaira-Spencer theory corresponds to computing different terms in the effective $N = 2$ field theory. Since in the topological string theory it is natural to sum these up, it is a natural question to ask how this can make sense from the field theory perspective.

From the correction (2.5) it is clear that if we give a constant expectation value to F, i.e. consider constant E and B fields for the graviphoton then the partition function of the topological theory would compute the correction to the R^2 term [12], namely

$$F(\lambda, t_i) \int R^2$$

where

$$F(\lambda, t_i) = \sum_g \lambda^{2g-2} F_g(\{t_i\})$$

with $\lambda^2 \sim \langle F^2 \rangle$. To be a little more precise, the R^2 term we have been considering has two relevant contractions corresponding to the Euler characteristic χ and the signature σ. The full action is

$$S = \ldots + \frac{1}{2}(\chi + \frac{3}{2}\sigma)F(\lambda, t_i) + \frac{1}{2}(\chi - \frac{3}{2}\sigma)\bar{F}(\bar{\lambda}, \bar{t}_i). \tag{2.6}$$

where $\lambda = g_s(E + iB)$.

We should note that it is natural in this context to ask if there are any corrections to the $\int R^2$ term, in the presence of a constant expectation value to F^2, which are not polynomial in F^2. In other words, can we have terms such as $\exp(-\frac{1}{\lambda})$? If so this would be the non-perturbative completion of the Kodaira-Spencer theory. We will later argue that there are indeed such corrections and the topological amplitude should be viewed as a function $F(\lambda)$

whose asymptotic expansion for small λ gives the above genus g expansion. There would be additional $\exp(-\frac{1}{\lambda})$ terms, which are non-perturbative in the string theory, but are nevertheless topological in origin. We will see how the large N Chern-Simons theory will give us this function $F(\lambda)$ at least near certain singularities.

2.4 N=4 Topological Strings

We can generalize topological $N = 2$ strings to topological $N = 4$ strings [13]. If we consider a theory which has $N = 4$ superconformal symmetry, then we can again consider twisting it by choosing an $N = 2$ subalgebra. In this case there is a whole sphere's worth of doing this. Choose one, and consider the four supercurrents $G^{\pm}, \tilde{G}^{\pm}$. The N=4 topological string amplitudes at genus g are defined by

$$F_g = \int_{\mathcal{M}_g} \langle \left| G^-(\mu_1) ... G^-(\mu_{3g-3}) \right|^2 \int [|\tilde{G}^+|^2]^{g-1} \int J_L J_R \rangle \qquad (2.7)$$

Note that the net charge violation is $-(2g-2)$ and so an N=4 theory with $\hat{c} = 2$ will give a critical theory. Examples include superconformal theories with target spaces T^4 or $K3$. This would be an Euclidean example. One could also consider $N = 4$ theories coming from hyperkahler metrics on $T^*\Sigma$ where Σ is a Riemann surface [14]. If $g \leq 1$ this has a Euclidean signature and if $g \geq 1$ it has a $(2,2)$ signature.

It turns out that just as the $N = 2$ topological string was modelled after bosonic string theory ($N = 0$), the $N = 4$ topological string theory is modelled after the $N = 2$ string theory. Note that the critical dimension for the topological $N = 4$ theory is the same as the critical dimension for the $N = 2$ string theory [14]. This is in fact not an accident. As has been shown in [13] the $N = 4$ topological amplitude on a hyperkahler four manifold is identical to that of an $N = 2$ string propagating on the same manifold! This in particular means that the corresponding target space gravity for the $N = 4$ topological string is the target space theory of $N = 2$ strings which is known to be self-dual gravity (the analogue of the A and B twists correspond now to the two distinct ways of writing the self-dual gravity equations– the two "Heavenly equations" of Plebanski).

2.5 Open String Version

One can also consider the open string version of the $N = 4$ topological string, which corresponds to the open $N = 2$ string. Again there are two natural boundary conditions. With Neumann boundary conditions the effective theory lives in four dimensions and is self-dual Yang-Mills theory [15] [16]. With half Dirichlet and half Neumann conditions, one naturally considers supersymmetric two cycles and allows the string endpoints to live on them. For example, consider the local model for the four manifold to be T^*S^2. Consider a collection of N S^2 cycles in this space. In this case one obtains by a simple extension of [16] the target theory living on S^2 and corresponding to the principal chiral model with group $U(N)$ and with action

$$S = \int_{S^2} \mathrm{Tr}(g^{-1}dg)^2$$

This theory has been extensively studied and is known to be an integrable model [17].

It also turns out that, for the $N = 4$ topological string the corresponding gravity theory, i.e. self-dual gravity, can be viewed as the large N limit of the principal chiral model [18]. In particular, self-dual gravity on T^*S^2 can be viewed heuristically as the large N limit of the principal chiral series on S^2. There have also been some checks of this equivalence at the quantum level [19]. This structure parallels that of the $N = 2$ topological string where the role of volume preserving diffeomorphisms is now played by $SU(\infty)$ which is the local group of area preserving diffeomorphisms.

2.6 What Does the $N = 4$ Topological String Compute?

Just as N=2 topological strings compute superpotential amplitudes in type II string theory compactified on Calabi-Yau threefolds, something similar is true for $N = 4$ amplitudes [13]. One can show that they compute terms in the effective action in six dimensions of the form

$$S = F_g \int R^4 F^{4g-4} + \dots$$

where F_g is the genus g $N = 4$ topological partition function in Eq.(2.7). If we consider the four dimensional space to be $T^{4-k} \times R^k$, F_g is related to computations in toroidal compactifications on T^{4-k} down to R^{6+k}. The only case where the F_g's have been computed in detail is for the case where the hyperkahler four manifold is $R^2 \times T^2$ [20]. But, just as in the $N = 2$ case, one can expect to be able to compute the F_g's for $K3$ as well (at least, near an A_n singularity).

3 Non-critical Bosonic Strings and $N = 2$ Topological Strings

As reviewed above, the structure of the $N = 2$ topological strings parallels that of bosonic strings. In fact, that was part of the motivation for studying them, as they provided for certain non-critical bosonic string vacua, a topological string equivalent. As for the critical $N = 2$ topological strings there were a number of hints [5,21–23] that they should get mapped to $c = 1$ non-critical strings. This was in fact established in [24] where it was shown that the $c = 1$ non-critical string corresponding to a conformal field theory on a circle at self-dual radius, which had been specifically studied in [25], is equivalent to the topological $N = 2$ theory at the conifold. This was argued in a number of ways, including the fact that both were determined by the ground ring, which for $c = 1$ at self-dual radius is given by [26]

$$z_1 z_4 - z_2 z_3 = \mu$$

where

$$z_1 = a_L a_R \quad z_4 = b_L b_R \quad z_2 = a_L b_R \quad z_3 = a_R b_L \tag{3.1}$$

and a and b denote the basic positive and negative units of tachyon momenta and in the $c = 1$ terminology μ has the interpretation of the cosmological constant. From the $N = 2$ topological string perspective z_i are the local coordinates describing the Calabi-Yau and μ denotes the complex structure of the local model for the Calabi-Yau. As $\mu \to 0$, the CY develops a singularity and an S^3 shrinks to zero size. This is known as the conifold singularity. As further evidence for this identification it was shown in [24] that the genus 0, 1 and 2 results of the Kodaira-Spencer theory near a conifold, which was studied in [5], gave results in agreement with the genus 0,1 and 2 partition function of the $c = 1$ string at the self-dual radius.

More evidence for this identification was presented in [10]: It was argued in [27] that the 3-branes of type IIB theory wrapped around S^3 would give rise to a massless hypermultiplet in the limit of vanishing S^3 size. As a first check of this, it was argued in [27] that the genus 0 topological string amplitude is consistent with such an interpretation. This check was extended to genus 1 in [28]. More generally it was shown in a beautiful paper [10] that the contribution to $R^2 F^{2g-2}$ term for a single hypermultiplet of mass μ can be computed by a one loop computation (generalizing the Schwinger computation to the $N = 2$ setup) where the hypermultiplet goes around the loop. Given the conjectured identification of the conifold as the locus where we get an extra massless hypermultiplet [27] and the identification of the corresponding topological string with that of the $c = 1$ string at self-dual radius [24], it was checked that the coefficient $F_g(\mu)$ of $R^2 F^{2g-2}$ as a function of g agrees with the genus g partition function of the $c = 1$ string at the self-dual radius. [1] The partition function of the $c = 1$ string at self-dual

[1] An interesting generalisation of this computation to other amplitudes in the heterotic side, which reproduce the $c = 1$ partition function at arbitrary radius, has been made in [29].

radius is given by

$$\mathcal{F}(\mu) = \sum_g \mu^{2-2g} \chi_g \tag{3.2}$$

where χ_g denotes the Euler characteristic of the moduli of genus g Riemann surfaces and is given by

$$\chi_g = \frac{B_g}{2g(2g-2)}$$

where B_g is the g-th Bernoulli number. This perturbative part is given by the large μ expansion of

$$\mathcal{F}(\mu) = \int^{\infty} \frac{ds}{s^3} e^{-is\mu} \left(\frac{s/2}{\sinh \frac{s}{2}} \right)^2 \tag{3.3}$$

This expression has imaginary terms too, like $e^{-2\pi n\mu}$, which correspond to one of many possible non-unitary, non-perturbative completions of the $c = 1$ theory.

3.1 More Detailed Match

There is more to the $c = 1$ non-critical strings than just the partition function. In particular, one can consider the correlation function of physical states. The states are described as follows [26, 30]: Consider the theory at self-dual radius. In this case there is an $SU(2)_L \times SU(2)_R$ current algebra at level one acting on the left/right-movers. The physical states can be labeled by their transformation under this group and it turns out that there is one copy of each state labeled by

$$|j, m, m'\rangle$$

where $j/2$ denotes the spin under both $SU(2)$'s and m, m' are the j_3 quantum numbers, which correspond to left- and right-moving momenta respectively. The main question is, what is the interpretation of such states? If we add the corresponding operators to the action, it will deform the theory and thus our question is equivalent to finding the analog of the (j, m, m') deformations in the type IIB theory near the conifold. This is actually relatively straightforward to study. We started with a theory with a defining equation which can be written as

$$\det M - \mu = 0$$

where

$$M = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \tag{3.4}$$

Note there is an $SL(2,\mathbb{C})_L \times SL(2,\mathbb{C})_R$ action on this manifold given by left/right matrix multiplication on M

$$M \to K_L \cdot M \cdot K_R$$

Let us consider the most general deformation to the manifold. It is convenient to write this deformation as

$$z_1 z_4 - z_2 z_3 - \mu + \sum_j \epsilon_j(z_1, z_2, z_3, z_4) = 0$$

where ϵ_j is a polynomial in z_i consisting of monomials of total degree j. Note that the $SL(2)_L \times SL(2)_R$ action being linear in z_i will act on these monomials. In fact it is easy to show for infinitesimal deformations (and thus using $z_1 z_4 - z_2 z_3 = \mu$) that they form spin $j/2$ representation of either $SL(2)$'s. In particular they are in one to one correspondence with $|j, m, m'\rangle$, where we identify m and m' with the Cartan of the respective $SL(2)$'s:

$$U(1)_L: \qquad (z_1, z_2, z_3, z_4) \to (\alpha z_1, \alpha z_2, \alpha^{-1} z_3, \alpha^{-1} z_4)$$

$$U(1)_R: \qquad (z_1, z_2, z_3, z_4) \to (\beta z_1, \beta^{-1} z_2, \beta z_3, \beta^{-1} z_4)$$

This dictionary allows us to go further in the identification of Kodaira-Spencer amplitudes and the $c = 1$ theory: We can systematically add finite deformations away from the conifold limit to give rise to an actual equation satisfied by a compact Calabi-Yau in a coordinate patch, and this should be equivalent to adding discrete states of the $c = 1$ theory at self-dual radius to the action. Thus, modulo questions of convergence, *computation of* the $c = 1$ *partition function for arbitrary finite deformations by discrete states is equivalent to studying the partition function for* $N = 2$ *topological strings for arbitrary Calabi-Yau 3-folds.* The most general amplitude involving $c = 1$ discrete states at self-dual radius has not been completely solved. The class involving tachyon operators has been studied (see in particular [31] and references therein), and they would correspond to deformations of the defining equations of the form

$$z_1 z_4 - z_2 z_3 + \epsilon(z_1) + \epsilon'(z_4) = \mu$$

where ϵ and ϵ' are arbitrary functions of z_1 and z_4 respectively (we are only writing the deformations to first order here).

3.2 Z_p Orbifolds of $c = 1$

It is natural to ask what would correspond to changing the radius in the $c = 1$ theory, in the geometrical model. There is a natural answer to this, at least when we consider the radius to be $R = pR_0$ where p is an integer and R_0 is the self-dual radius. To see this, note that this can be obtained from the self-dual radius case by modding out by Z_p. The identification of z_i with the ground ring generators (3.1) implies the following action of Z_p on the ground ring, or equivalently on the 3-fold:

$$(z_1, z_2, z_3, z_4) \to (\omega z_1, z_2, z_3, \omega^{-1} z_4)$$

where ω is a p-th root of unity. On the geometrical side, modding out the conifold by this symmetry has been considered (in the same context) in [32]. In particular, it was shown there that it is natural to first rewrite the conifold as

$$z_1 z_4 = \zeta$$

$$\zeta - \mu = z_2 z_3$$

Now modding the first equation by Z_p gives an A_{p-1} type singularity that can be rewritten in terms of the invariant variable by defining $u = z_1^p$ and $v = z_4^p$

$$uv = \zeta^p$$

$$\zeta - \mu = z_2 z_3$$

The first equation which described an A_{p-1} singularity can be deformed to

$$uv = \prod_i (\zeta - \mu_i)$$

$$z_2 z_3 = \zeta - \mu$$

Physically, this translates into p nearly massless hypermultiplets with masses $\hat{\mu}_i = \mu_i - \mu$, corresponding to the p inequivalent S^3 cycles, as was discussed extensively in [32]. To map this to the $c = 1$ theory at p times the self-dual radius, we have to decide what the $\hat{\mu}_i$ are. Since the partition function for this theory as a function of the 'little phase space' has not been worked out in this case, we will first make a prediction based on the equivalent geometrical theory: Since the physics is dominated by the p light charged hypermultiplets, corresponding to the D3 brane wrapped around p inequivalent 3-cycles, the answer will be the same as what we discussed in the context of the Schwinger computation of integrating out p light hypermultiplets. In particular, the free energy of this theory is expected to be

$$F(\{\hat{\mu}_i\}) = \sum_i \mathcal{F}(\hat{\mu}_i)$$

where $\mathcal{F}$ denotes the partition function at self-dual radius given in (3.3). Of course we should rewrite the partition function in terms of the invariant variables, which is simply the p symmetric products involving $\hat{\mu}_i$. Could this simple answer be correct? We will now present evidence in its favor, by showing that for a simple specific choice of $\hat{\mu}_i$ it reproduces the $c=1$ partition function at p times the self-dual radius. The free energy is given by [33]

$$\frac{\partial^2 F}{\partial \mu^2} = \mathrm{Re} \int_0^\infty \frac{ds}{s} e^{-is\mu} (\frac{s/2}{\sinh \frac{s}{2}})(\frac{sp/2}{\sinh \frac{sp}{2}}).$$

or equivalently in an unregularised form [25]

$$\begin{aligned} F &= \sum_{n,m \in \mathbf{Z}_+ + \frac{1}{2}} \log(n + mp + i\mu) \\ &= \sum_{j=1}^{\infty} [\sum_{t=-\frac{p-1}{2}}^{t=\frac{p-1}{2}} j \log(jp + i\mu + t)] \\ &= \sum_{t=-\frac{p-1}{2}}^{t=\frac{p-1}{2}} [\sum_{j=1}^{\infty} j \log(j + \frac{i\mu + t}{p})] \\ &= \sum_{t=-\frac{p-1}{2}}^{t=\frac{p-1}{2}} \mathcal{F}(\hat{\mu}_t), \end{aligned} \tag{3.5}$$

where $\hat{\mu}_k = \frac{\mu + ik}{p}$. Thus the free energy has the expected structure with p equally spaced $\hat{\mu}_k$.

3.3 Other Shrinking 3-Cycles

In the geometrical setup we can also consider other 3-cycles shrinking to zero size. In particular these could be (non-singular) orbifolds of S^3. Examples are given by the Lens spaces $L(p,q)$ and S^3/G where G is an $A-D-E$ discrete subgroup of $SU(2)$. These do not have any natural conformal field theory interpretation analogous to the $c=1$ theory (the modding out suggested by the geometry will not lead to a modular invariant conformal theory), and so do not admit a perturbative description in terms of non-critical bosonic strings. Nevertheless, we find that they make sense as far as the topological string (and IIB compactifications) are concerned and can compute the corresponding amplitudes.

The lens spaces are quotients of S^3,

$$L(p,q): \qquad (z_1, z_2) \sim (\omega z_1, \omega^q z_2), \qquad \omega = e^{\frac{2\pi i}{p}}, \qquad |z_1|^2 + |z_2|^2 = 1. \tag{3.6}$$

Only the $L(p,0)$ are singular. But that is the case corrresponding to the $c=1$ string at p times the self dual radius as we have seen from the action on the ground ring. The spaces $L(p,1)$ are also special. They correspond to simple Z_p orbifolds of S^3 which happens to be the same as orbifolding by the cyclic A_{p-1} discrete subgroup which lies in $SU(2)_L$. One can have D_n, E_n orbifolds of S^3 as well, corresponding to modding out by dihedral and exceptional subgroups of $SU(2)_L$.

String theory in the vicinity of shrinking cycles such as these, exhibits some new features compared to the S^3 case [34]. Geometrically, the important object is the fundamental group of the three cycle. The number of distinct light hypermultiplets is given by the number of distinct irreducible representations of the fundamental group. Moreover, the charge of the hypermultiplet is given by the dimension of the representation.

The Lens spaces $L(p,q)$ have fundamental group Z_p and hence, in string theory there are p light particles all with $U(1)_{RR}$ charge one and mass $\frac{\mu}{p}$. The partition function would be expected to be given by

$$F_{L(p,q)} = p\mathcal{F}(\frac{\mu}{p}) \tag{3.7}$$

In the case of the D_n, E_n orbifolds of S^3, the fundamental groups are the non-abelian dihedral and exceptional groups respectively. This gives rise to a specific prediction for the charges and number of light hypermultiplets. We refer the reader to [34] for details. Accordingly the partition function will reflect this in the various μ_i. In particular let F_G denote the topological partition function for the case corresponding to S^3/G where G corresponds to an $A-D-E$ subgroup of $SU(2)_L$. Then

$$F_G(\mu) = \sum_i \mathcal{F}(\frac{a_i\mu}{d}) \tag{3.8}$$

where the sum i is over the nodes of the corresponding affine Dynkin diagram and a_i are the Dynkin indices associated with the corresponding node, and d is the order of the group, $d=|G|$.

3.4 Schwinger Pair Creation of Branes

The physics of the light, wrapped 3-brane in the IIB theory can actually be described in a much more familiar manner. It was observed in [12] that the perturbative contribution to the R^2F^{2g-2} terms is *exactly* the same as that of a Schwinger type one-loop determinant calculation. This was done by realizing the computation in a heterotic setup and doing a one loop computation using heterotic strings.

To spell matters out : The effective action for the branes in the presence of a constant (self-dual) electromagnetic graviphoton field would vanish because of the $N = 2$ supersymmetry. However, we can consider a non-vanishing effective action with additional R^2 insertions to absorb the fermion zero modes. Thus the analog of Schwinger's computation of the effective action for a constant electromagnetic field in the present case is to study the correction to the R^2 term. To preserve at least half the supersymmetry, the background field needs to be self dual. In Minkowskian space this means

$$\vec{E} = \pm i\vec{B}.$$

As shown first by Schwinger (see for example [35]), one can exactly integrate out the charged field to produce an effective action whose real part is a polynomial in all (even) powers of the field strength. In the case of a boson, the one loop determinant reads as

$$\begin{aligned} F(\vec{E},\vec{B},m) &= \frac{1}{2}\,\mathrm{Tr}\ln\det((i\partial - eA)^2 - m^2) \\ &= \frac{e^2EB}{2\pi^2}\int_\epsilon^\infty \frac{ds}{s^3}e^{-i\frac{sm^2}{2}}\left(\frac{s/2}{\sinh\frac{seE}{2}}\right)\left(\frac{s/2}{\sin\frac{seB}{2}}\right), \end{aligned} \tag{3.9}$$

where $E^2 - B^2 = \vec{E}^2 - \vec{B}^2$ and $EB = \vec{E}\cdot\vec{B}$. In QED, ϵ is an UV cutoff, which will be replaced in string theory by the string scale. Taking a self dual field and redefining $\mu = \frac{m^2}{2eE}$ we have for the free energy

$$F(\mu) = \frac{e^2E^2}{2\pi^2}\int_\epsilon^\infty \frac{ds}{s^3}e^{-is\mu}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 = \frac{e^2E^2}{2\pi^2}\mathcal{F}(\mu). \tag{3.10}$$

This has a perturbative expansion (in inverse powers of μ) which is given in (3.2) and gives the higher polynomial corrections to the Maxwell lagrangian. The first couple of terms in this expansion need an UV cutoff ϵ – they diverge as $\mu^2\ln(\frac{\mu}{\epsilon})$ and $\ln(\frac{\mu}{\epsilon})$ respectively. In string theory this cutoff will be provided by the string scale.

The match of (3.10) with (3.2) is thus the standard Schwinger computation extended to this case. However note that (3.2) is just an asymptotic expansion of $\mathcal{F}(\mu)$ valid for large μ. There is moreover an absorptive part corresponding to pair creation predicted from (3.10). This can be evaluated by extending the (imaginary part of the) integral to the whole real line and closing the contour in the lower half plane to pick up all the non-zero poles of the sinh on the negative imaginary axis. The answer is

$$\mathrm{Im}\,\mathcal{F}(\mu) = ()\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-2\pi n\mu} \tag{3.11}$$

In the type IIB context one should also expect such terms: they are naturally interpreted as the corrections to the R^2 term in the presence of the constant graviphoton field strength from the pair production of light wrapped brane-antibrane states. Note, from Eq. (2.6), that this imaginary part of $\mathcal{F}(\mu)$ is a correction to the signature only. The perturbative (real) part was a correction only to the Euler character. In this case, the parameter $\mu = \frac{m}{2E}$. This follows from the formula ($m = e$) for the relation between graviphoton charge and mass of the wrapped brane, which in turn is proportional to size of the shrinking 3-cycle (and $\frac{1}{g_s}$) by the BPS condition.

The pair production is a process that is not calculable in perturbative string theory but must be computable in a complete description of the theory. We will see below that an $SU(\infty)$ Chern-Simons field theory reproduces both these perturbative and non-perturbative contributions in the case where the shrinking 3-cycles are either S^3 or the Lens spaces. The fact that in the case of S^3 the perturbative part of the Chern-Simons partition function at $N = \infty$ agrees with the effective action near the conifold was already observed in [36] making use of the results of Periwal on the large N limit in Chern-Simons theory [37].

In addition to the formal relation between Kodaira-Spencer theory as volume preserving diffeomorphism Chern-Simons theory, this was part of the motivation for our conjecture. The check we perform for the lens space is new.

4 Chern-Simons Theory and the $N = \infty$ Limit

After briefly summarising the results of [38] we will outline the computation of the partition function on S^3 and S^3/Z_p. Then we will proceed to take the $N = \infty$ limit and interpret the answer.

4.1 Chern-Simons Theory

The Chern-Simons (CS) field theory on an arbitrary 3-dimensional manifold M is defined as

$$Z[M, N, k] = \int [DA] \exp\left[\frac{ik}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\right].$$

Here the N refers to the gauge group which we will take to be $SU(N)$. The theory is topological in that its definition doesn't rely on a background metric. The only other parameter is k – the level which is integer quantised – and plays the role of the (inverse) coupling constant in the theory. To compare the partition function with that of Kodaira-Spencer theory we will actually need to extend this definition beyond integral values of k and this can be done in a canonical way in the cases we are dealing with. In general this is possible for manifolds M for which the Chern-Simons partition function does not involve the "R"-matrix [39]. The manifolds we are dealing with satisfy this property, and thus have an unambiguous analytic continuation valid for all k.

Moreover, the partition function (or with additional Wilson line insertions) is often exactly computable [38]. The central idea is the relation of this three dimensional theory to two dimensional conformal field theories. This essentially follows from the Hamiltonian quantisation of the CS theory on two dimensional "space", a Riemann surface Σ. The resulting Hilbert space is precisely that of the conformal blocks of the corresponding current algebra (at level k) on the Riemann surface. Since the Hamiltonian is identically zero, the partition function on $\Sigma \times S^1$ is, for instance, simply given by the dimension of this vector space. Thus we can immediately read off the answer for T^3 [37]

$$Z(T^3, N, k) = \dim H_{T^2}^{SU(N)} = \frac{(N+k-1)!}{k!(N-1)!}.$$

The process of surgery can be used to relate the partition function of an arbitrary three manifold to that of a simple one like S^3 (possibly with Wilson lines). The process involves removing a solid torus, about a curve C, from a manifold M and gluing it back with a diffeomorphism $U \in SL(2, Z)$ on the boundary T^2 to give a topologically new manifold M'. Then the important result is

$$Z(M') = \sum_j \tilde{U}_0^j Z(M; R_j). \tag{4.1}$$

Here, the R_j refers to the insertion of a Wilson line (in representation R_j) along the curve C. And $\tilde{U}$ is the representation of U on the genus one Hilbert space – the space of characters of the level k $SU(N)$ current algebra. This space is labelled by the roots of a finite number of representations of $SU(N)$ (shifted by the sum of the positive roots $\vec{\rho} = \frac{1}{2}\sum_+ \vec{\alpha_+}$). The trivial representation will be interchangeably labelled as either $\vec{\rho}$ or 0. A further necessary datum is that $Z(S^2 \times S^1; R_j) = \delta_{j0}$.

Now, S^3 in turn can be obtained from gluing two solid tori along their boundary T^2's (with a twist by the $SL(2, Z)$ matrix S which interchanges the two cycles of the T^2). Since gluing two solid tori with no twist gives $S^2 \times S^1$, we have

$$Z(S^3, N, k) = \sum_j \tilde{S}_0^j Z(S^2 \times S^1; R_j) = \tilde{S}_{0,0} \tag{4.2}$$

Thus the main thing we need to know is the action of $SL(2,Z)$ on the space of characters of $SU(N)$ level k current algebras. This has been known for a while (See for e.g. [40]). They have been cast in a particularly useful form in [41,42]. The matrix $\tilde{U}$ acting on the Hilbert space $H_{T^2}^{SU(N)}$ associated with the $SL(2,Z)$ matrix

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{4.3}$$

is given by [41,42]

$$\tilde{U}_{\vec{\alpha},\vec{\beta}} = \frac{[i\,\mathrm{sgn}(b)]^{N(N-1)/2}}{(|b|M)^{\frac{N-1}{2}}} e^{[-\frac{i\pi}{12}(N^2-1)\Phi(U)]} \frac{1}{\sqrt{N}}$$
$$\cdot \sum_{\vec{n}\in\frac{\Lambda^R}{b\Lambda^R}} \sum_{w\in W} (-1)^{|w|} \exp\frac{i\pi}{bM}[a\vec{\alpha}^2 - 2\vec{\alpha}\cdot(M\vec{n}+w(\vec{\beta})) + d(M\vec{n}+w(\vec{\beta}))^2] \tag{4.4}$$

Here $M \equiv N+k$, Λ^R is the root lattice of $SU(N)$ and W the Weyl Group. $\Phi(U)$ is the Rademacher function

$$\Phi(U) = \frac{a+d}{b} - 12\sum_{l=1}^{b-1}((\frac{l}{b}))((\frac{ld}{b}))$$

$$((x)) = 0 \quad x\in\mathbf{Z}; \qquad = x - [x] - \frac{1}{2} \quad otherwise$$

Let us first consider the case of S^3. In this case, the partition function is given by [37]

$$\begin{aligned} Z[S^3,N,k] &= \tilde{S}_{\vec{\rho},\vec{\rho}} \\ &= e^{\frac{i\pi}{8}N(N-1)} \frac{1}{(N+k)^{N/2}} \sqrt{\frac{N+k}{N}} \sum_{w\in W} (-1)^{|w|} \exp[-\frac{2\pi i}{M}\vec{\rho}\cdot w(\vec{\rho})] \\ &= e^{\frac{i\pi}{8}N(N-1)} \frac{1}{M^{N/2}} \sqrt{\frac{N+k}{N}} \prod_{j=1}^{N-1} \{2\sin(\frac{j\pi}{N+k})\}^{N-j} \end{aligned} \tag{4.5}$$

where we have read off from the above expression for the $SL(2,Z)$ matrix S. We have also carried out, in going to the third line, a non-trivial sum over the Weyl group. We note some features of this expression:

$$Z(S^3,N,k=0) = 1$$

$$Z(S^3, N, k) = (\frac{k}{N})^{1/2} Z(S^3, k, N)$$

Also, the structure of Z is very reminiscent of that of a matrix model solved via orthogonal polynomials. These take the general form $(\ ) \prod_{j=1}^{N-1} (R_j)^{N-j}$ which is precisely what we have here with $R_j = 2\sin(\frac{j\pi}{N+k})$. In fact, working backwards, we can write a zero dimensional matrix model which will reproduce this partition function.

If we examine the asymptotic structure of the free energy (large but finite N, k) , we can see that it takes the form

$$F(S^3, N, k) = \sum_{g=0,h=1}^{\infty} \frac{k^h}{N^{2g-2+h}} C_{g,h} + \sum_{g=0}^{\infty} (\frac{1}{N^{2g-2}} + \frac{1}{k^{2g-2}}) \chi_g$$

where we have assumed that $k << N$. For $N << k$, only the first term changes with the roles of N and k interchanged. This reflects the level-rank symmetry of the quantum Chern-Simons theory.

4.2 The Partition Function on S^3/Z_p

We will also be examining the case where the shrinking 3-cycle is S^3/Z_p. (The reader interested purely in the result of the computation may skip to Eq. (4.13).) This is a special subclass of the spaces S^3/G, where G is a discrete subgroup of $SU(2)_L$ – in this case, of the A-series. Moreover it also belongs to a different family of 3-dimensional manifolds, the lens spaces $L(p, q)$. These are also defined in terms of quotients of S^3 as in (3.6). The space we are considering is $L(p, 1)$. It can also be obtained by surgery from two solid tori with a gluing matrix

$$U = ST^pS = \begin{pmatrix} -1 & 0 \\ p & -1 \end{pmatrix}. \tag{4.6}$$

Thus the partition function is given by

$$\begin{aligned} Z(S^3/Z_p, N, k) &= (\tilde{S}\tilde{T}^p\tilde{S})_{\vec{\rho},\vec{\rho}} \\ &= K \sum_{\vec{n} \in \frac{\Lambda^R}{p\Lambda^R}} \sum_{w \in W} (-1)^{|w|} \exp -\frac{i\pi}{pM} [M\vec{n} + w(\vec{\rho}) + \vec{\rho}]^2. \end{aligned} \tag{4.7}$$

where $K = \frac{1}{\sqrt{N}} \frac{e^{\frac{i\pi}{4} N(N-1)}}{(Mp)^{\frac{N-1}{2}}} e^{[-\frac{i\pi}{12}(p-1)(N^2-1)]}$. Consider the case where $M \equiv 0 \pmod p$. Then with

$$\vec{n} = \sum_{i}^{N-1} m_i \vec{\alpha}_i; \qquad m_i \in \{0, 1, \ldots p-1\}$$

we have $\exp -\frac{i\pi M}{p} < \vec{n}, \vec{n} > = 1$ since the $\vec{\alpha}_i$ are the simple roots which obey $< \vec{\alpha}_i, \vec{\alpha}_j >= 2, 1, 0$ depending on whether $i = j$, $i = j \pm 1$ or otherwise. So that (using also $< w(\vec{\rho}), w(\vec{\rho}) >=< \vec{\rho}, \vec{\rho} >$)

$$\begin{aligned} Z(S^3/Z_p, N, k) &= K \sum_{w \in W} (-1)^{|w|} e^{-\frac{2\pi i}{pM} <w(\vec{\rho}), \vec{\rho}>} \\ &\times \sum_{\{m_i\}} \exp -\frac{2\pi i}{p} \sum_i^{N-1} m_i [< \vec{\alpha}_i, w(\vec{\rho}) > + < \vec{\alpha}_i, \vec{\rho} >] \\ &= K \sum_{w \in W} (-1)^{|w|} e^{-\frac{2\pi i}{pM} <w(\vec{\rho}), \vec{\rho}>} \\ &\times \prod_i^{N-1} (\sum_{m_i=0}^{p-1} \exp -\frac{2\pi i}{p} m_i [< \vec{\alpha}_i, w(\vec{\rho}) > + < \vec{\alpha}_i, \vec{\rho} >]). \end{aligned} \tag{4.8}$$

The terms in the product bracket are each of the form $\sum_m e^{-\frac{2\pi i}{p} mn}$ for integer n. This vanishes unless $n \equiv 0 \pmod p$. In other words, the product contributes only if $< w(\vec{\alpha}_i), \vec{\rho} > + < \vec{\alpha}_i, \vec{\rho} > \equiv 0 \pmod p$ for all $i = 1 \ldots N-1$. (Here $w \to w^{-1}$ in the sum over w and we use the property of the inner product to transfer its action onto $\vec{\alpha}_i$.) This puts a condition on the elements $w \in W$ that do contribute. To spell it out we need some properties of the roots and the action of the Weyl Group on them.

The $N-1$ roots lie on a hyperplane in R^N. In terms of an orthonormal basis $\{\vec{e}_i\}$ on R^N, the simple roots are given by

$$\vec{\alpha}_i = \vec{e}_i - \vec{e}_{i+1}$$

and the positive roots by $\mathbf{e}_i - \mathbf{e}_j$ with $i < j$. Therefore

$$\vec{\rho} = \frac{1}{2} \sum_{k=1}^{N} (N - 2k + 1) \vec{e}_k \tag{4.9}$$

The Weyl group acts simply on the the basis $\{\vec{e}_i\}$ as permutations on the indices

$$i \to P_i; \qquad i, P_i \in 1 \ldots N.$$

It is easy then to see that $< \vec{\rho}, \vec{\alpha}_i >= 1$ and $< \vec{\rho}, w(\vec{\alpha}_i) >= P_{i+1} - P_i$. Thus the permutations that have a non-zero contribution are those for which

$$P_{i+1} - P_i \equiv -1 \mod p.$$

What are these permutations like? The results depend sensitively on the value of $k \mod p$. It turns out that in order to reproduce the gravity answer we need to consider the the case when $k \equiv 0 \pmod p$ (the other limits are discussed in the appendix). Since we are considering $N = \infty$ limit at the end, and we have considered the case where $M = N + k = 0 \pmod p$ it implies that we also consider $N = 0 \pmod p$. Divide the integers $1 \ldots N$ into blocks of size p. Consider the permutation on the block $1 \ldots p$

$$\begin{pmatrix} 1 & 2 & \ldots & p \\ p & p-1 & \ldots & 1 \end{pmatrix} \tag{4.10}$$

together with similar permutations on each of the other blocks. This clearly satisfies the above condition. Moreover, starting from this arrangement, we can freely permute the $\frac{N}{p}$ elements, one from each block, which differ from each other by a multiple of p. The resulting permutation also satisfies the condition. Thus the permutation group that survives of the S_N is $(S_{\frac{N}{p}})^p$. But we actually have some more allowed permutations. These are cyclic permutations within each of the blocks starting from the one shown in (4.10), with the restriction that the cyclic permutation C_p must be the same in all the blocks.

For each such allowed permutation the term in brackets in the second line of (4.8) reduces to p and we have

$$Z(S^3/Z_p, N, k) = Kp^{N-1} \sum_{w \in W'} (-1)^{|w|} e^{-\frac{2\pi i}{pM} <w(\vec{\rho}), \vec{\rho}>} \tag{4.11}$$

Where $W' = C_p \otimes (S_{\frac{N}{p}})^p$ is the surviving Weyl group. This sum over W' actually rather neatly factorises. If we express $\vec{\rho}$ in (4.9) in terms of $\vec{\rho}'$'s (where $\vec{\rho}'$ is the sum over the positive roots of $SU(\frac{N}{p})$), the Weyl sum can be shown to reduce to

$$\sum_{c \in C_p} (-1)^{|c|} e^{-\left(\frac{i\pi N}{Mp^2} \sum_{q=0}^{p-1} (p+2q-1)(p+2C(q)-1)\right)} \cdot \left(\sum_{w \in S_{\frac{N}{p}}} (-1)^{|w|} e^{-\frac{2\pi p i}{M} <w(\vec{\rho}'), \vec{\rho}'>} \right)^p . \tag{4.12}$$

Thus we get on comparison with Eq. (4.5), (upto multiplicative factors which are not important for the $N = \infty$ limit)

$$Z(S^3/Z_p, N, k) = (\ ) Z^p(S^3, N/p, k/p). \tag{4.13}$$

This is an extremely simple, closed form expression which will have a natural interpretation when we take the large N limit and compare with the expected result on the Calabi-Yau side.

4.3 The $N = \infty$ Limit

We will be primarily interested in the free energy of the $SU(\infty)$ theory. Let us take this limit for S^3 first. (A similar limit of the exact answer was studied in [37] for the cases of S^3 and T^3.) Its free energy can be written

$$F[S^3, N, k] = \ln Z[S^3, N, k] = \frac{N(N-1)}{2}\ln 2 \\ - \frac{(N-1)}{2}\ln(k+N) - \frac{1}{2}\ln N + \sum_{j=1}^{N-1} j \log[\sin\frac{(k+j)\pi}{N+k}]. \quad (4.14)$$

Let us rather naively take the $N = \infty$ limit dropping all the terms that diverge as positive powers of N (or logarithmically). Then the contribution that survives comes from the last term which then reads as

$$F[S^3, N = \infty, k] = \sum_{j=1}^{\infty} j \log(j+k). \quad (4.15)$$

At first sight, this might seem too naive. We can adopt a more careful procedure, where we take three derivatives with respect to k of Eq.(4.14) and then the limit. This will ensure that we are dealing with well defined convergent series. In fact, even the term $\sum_{j=1}^{\infty} j \log(j+k)$ is divergent as it stands. On doing this it is possible to check that we recover the result in Eq.(4.15) or more precisely, its third derivative

$$\frac{\partial^3 F[S^3, N = \infty, k]}{\partial k^3} = \sum_{j=1}^{\infty} \frac{2j}{(j+k)^3} = \frac{\partial^2}{\partial k^2}(k\psi(k)) \quad (4.16)$$

where $\psi(k) = \frac{d}{dk}\log\Gamma(k)$ with $\Gamma(k)$ being the usual Gamma function. Using the asymptotic Stirling expansion of $\Gamma(k)$, one can also write (after integrating thrice)

$$F[S^3, N = \infty, k] = \frac{1}{2}k^2 \ln k - \frac{B_1}{2}\ln k + \sum_{g=2}^{\infty} \frac{(-1)^{g-1}B_g}{2g(2g-2)}k^{2-2g}. \quad (4.17)$$

This is the same as the genus expansion of the free energy of the $c = 1$ Matrix model at the self dual radius provided we make the analytic continuation $k = i\mu$. As we have seen this is also the correction to the R^2 term in the perturbative effective action (computed by the topological string) near the conifold. This equivalence of the perturbative terms with the S^3 free energy was first observed in [36].

But this was only the large k (μ) asymptotic expansion of $F[S^3, N = \infty, k]$. The full partition function contains non-perturbative information as well. These are terms that go like $e^{-2\pi n\mu}$. To isolate them it helps to write Eq.(4.15)(or the convergent Eq. (4.16)) in an integral representation

$$\frac{\partial^3 F[S^3, N=\infty, \mu]}{\partial \mu^3} = \int_0^\infty ds e^{-is\mu} (\frac{s/2}{\sinh \frac{s}{2}})^2 \tag{4.18}$$

We recognise this as equivalent to Eq. (3.10). In other words the non-perturbative parts are precisely the same as those expected to compute the contribution of brane-antibrane pair production to the R^2 term in the IIB effective action.

We now go onto the case of S^3/Z_p and take the $N = \infty$ limit in (4.13), in a similar manner. Since we had obtained a simple expression in terms of the S^3 partition function, the limit requires no further computation. We see p copies of the S^3 case, but now with mass $\frac{1}{p}$'th what we had before. As discussed earlier, this is precisely what we expect when an S^3/Z_p cycle shrinks in a IIB compactification [34] as given in equation (3.7): there should be p particles becoming light but with charge $\frac{1}{p}$ of the S^3 case. They are distinguished only by their quantum $\mathbf{Z_P}$ charges. As in the S^3 case, the non-perturbative physics of brane-antibrane production is contained in the exponential terms. That things work as expected, gives additional support to the $SU(\infty)$ theory being a full description of the physics described by $N = 2$ topological strings.

5 Heuristic Explanation and Discussion

We have seen that the $N = \infty$ limit of $SU(N)$ Chern-Simons theory on S^3 or lens spaces reproduces those topological terms in the type IIB effective action which are computed by the partition function of the Kodaira-Spencer theory of gravity in 6 dimensional space corresponding to the cotangent space T^*S^3 or the cotangent of lens space. As we discussed in section 2 there is a heuristic explanation [5] for this match if the Chern-Simons gauge group were that of *volume* preserving diffeomorphisms instead of $SU(\infty)$ which corresponds to *area* preserving diffeomorphisms. One explanation for this reduction in the gauge group might be that, since we are considering a shrinking 3-cycle, we are at the fixed point of a rescaling transformation. If we rescaled the metric on the whole manifold by an infinite overall factor it is reasonable to assume that volume preserving diffeomorphisms contract to area preserving diffeomorphisms. This is because we can view R^3 roughly as S^2 times a normal direction, and we can view the normal direction on S^2 as being related to the overall rescaling direction. Thus, in some sense the R^3 shrinks to an S^2 and so volume preserving diffeomorphisms go over to area preserving diffeomorphisms. It would be interesting to make this argument more precise. One more fact supports this explanation of the reduction of group: If one considers T^3, the topological gravity partition function on T^*T^3 is zero. And the $SU(\infty)$ Chern-Simons theory on T^3 though non-trivial, does not admit a closed string genus expansion [37]. This correlates with our explanation that the gauge group of volume preserving diffeomorphisms will not be reducible to $SU(\infty)$ in such cases, since T^3 is not shrinkable to zero size in a Calabi-Yau threefold. Note also, that this suggests that in certain gravity theories we should not expect a naive $SU(\infty)$ gauge theory to give us an equivalent system. In particular, here the relevant group presumably continues to be the group of volume preserving diffeomorphisms. This suggests that, generally, infinite dimensional gauge groups (more exotic than $SU(\infty)$) might be relevant for describing the gravitational equivalents.

It would also be interesting to compute the large N Chern-Simons free energy in the case of S^3/G (G $= D, E$ series of subgroups of $SU(2)$) and compare it with the expected result (3.8). Perhaps there are also other shrinking 3-cycles that could be used as test cases.

We have shown that the genus expansion of Kodaira-Spencer theory of gravity [5] is only an asymptotic expansion valid for small string coupling λ; we saw that there are corrections of the form $\exp[-\frac{1}{\lambda}]$. More generally, we expect there to be a partition function $F(\lambda)$ valid for *all* λ. We saw this to be the case for T^*S^3 and $T^*(S^3/Z_p)$. We believe this is a general property. Namely, for every Calabi-Yau threefold we expect to have a well defined partition function $F(\lambda)$ valid for all λ, whose asymptotic expansion for small λ reproduces the perturbative topological string expansion in terms of Riemann surfaces. It would also be interesting to understand the origin of the non-perturbative corrections in the language of the Kodaira-Spencer theory. Do they correspond to some "topological instantons"?

In this paper we have found a way to relate and compute the partition function of Kodaira-Spencer theory of gravity in terms of the Chern-Simons gauge theory for the special case of a non-compact Calabi-Yau threefold with particular vanishing 3-cycles. In this context, just as we expect the deformations away from the local singularity to be given by amplitudes involving discrete states in the $c = 1$ theory, it will be useful to establish a similar dictionary with observables in the Chern-Simons theory.

It would also be very interesting to extend this to the cases where the Calabi-Yau is compact. Given that in the present case we obtained the gravity partition function by taking a large N Chern-Simons theory on a supersymmetric 3-cycle, the natural guess would be to do a similar thing in the compact case and consider gauge theories corresponding to all possible supersymmetric 3-cycles. The main puzzle about this is that naively this "gauge theory" is sensitive to k and Kahler classes of Calabi-Yau, whereas we are dealing here with the complex structure of the Calabi-Yau, on the gravity side. So, some kind of mirror symmetry may be at work here. Another natural gauge theory to consider is the holomorphic version of Chern-Simons theory in six dimensions [3] whose action is given in (2.4). It may happen that this theory at large N computes the B-model topological gravity partition function on the corresponding Calabi-Yau threefold. However, since this theory is a gauge theory in six dimensionsit is not easy to work with, even though it makes sense since it arises in topological open string theory.

In the circle of ideas relating the $c = 1$ theory, the IIB near the conifold and the Chern-Simons theory, we have a reasonable understanding of the relation between the first two and some understanding of that between the last two. But, it will be very pleasing to have some detailed direct understanding of the connection between the Chern-Simons theory and the $c = 1$ theory. Perhaps, this might be along the lines of Douglas' study [43] of large N Chern-Simons theory in terms of free fermions, a representation familiar in the $c = 1$ context.

In this paper we have mainly concentrated on finding a large N gauge description for the Kodaira-Spencer theory of gravity, which corresponds to $N = 2$ topological strings. As we have briefly indicated, a lot of the structure is parallel to that for $N = 4$ topological strings. In this case a natural theory to consider is the principal chiral model at large N (on S^2) and its relation to self-dual gravity on T^*S^2. Another interesting class to consider is the principal chiral model on T^2. The large N theory should be equivalent to $N = 4$ topological strings on $T^*T^2 = T^2 \times R^2$ (note that for this case we do not need any reduction of the group, so we do not need to have a contractible 2-cycle to apply our considerations). The interesting point is that the partition function of this theory has already been computed for all genera in [20]. Given the integrability of principal chiral models this should lead to an interesting check. One would also like to consider the large N description of self-dual gravity in compact situations such as T^4 or $K3$. In this case one conjecture already exists [20] which states that large N holomorphic Yang-Mills theory in four dimensions may lead to self-dual gravity. Another natural guess would be to consider the large N limit of $U(N)$ self-dual Yang-Mills on T^4 and $K3$ and its relation to self-dual gravity on the corresponding spaces. We also note that the connection between large N QCD_2 [44], (in the topological limit) and $d = 2$ topological gravity coupled to topological sigma models [45] [46] might be viewed as another simple instance of the gravity/gauge theory relations we have been considering. Similar remarks on the connection between large N gauge theories and (quasi) topological string theories have been made by Martinec in the context of N=2 strings [47].

We have explored some relations between large N topological gauge theories and topological gravity/string theory. This is very much in the spirit of the matrix conjecture. In fact, just as the relevant large N gauge theories in the matrix proposal arise from the corresponding open string theory of D-branes, here the Chern-Simons theory (or the principal chiral model) are the large N (dirichlet) open string versions of the corresponding topological closed string theories. Moreover, we have also shown that the computation we are doing has implications for physical compactifications of type IIA and type IIB (i.e. it corresponds to superpotential computations). The question thus arises as to the precise relation of our work to the computation of superpotential terms in a matrix theory approach. The relation to the usual matrix conjecture is not very clear. We have here a bosonic Chern-Simons theory as opposed to a super Yang-Mills. (It was suggested in [48] that Chern-Simons theory being the dirichlet open string version of the A-model, could describe the physics of instantonic 2-branes in the IIA conifold.) But then again, we are only computing the superpotential terms. Thus, perhaps, there is some sense in which the Chern-Simons describes a sector of a full large N theory. It has been noted in the recent paper [49] that the matrix description on Calabi-Yau's might be much simpler than on T^6, in that gravitational degrees of freedom decouple. In particular, one is actually dealing with the theory in the vicinity of the conifold where a natural T-dual description is in terms of 3-branes. [2]

In this context, we must also note that the large N limit that we took is *not* an 'tHooft like limit. In the latter, the ratio $\frac{N}{k}$ would have been held fixed. Nevertheless the limit we took was well-defined – this is similar in spirit to [51] where some non 'tHooft like large N limits in Matrix theory were proposed and seen to give sensible answers. Again, it would be interesting to see if the finite N Chern-Simons gauge theory (which was also computed) has a DLCQ interpretation. In any case, our toy models might shed some light into many of the hard questions that face Matrix theory.

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[2] We also note that an 11-dimensional Chern-Simons like theory has also been proposed as a candidate M-theory [50].

Appendix

In Sec. 4, we evaluated the Lens space partition function for the case where $N, k \equiv 0 \pmod p$. Here we will state a more general result. The case when $N + k \equiv 0 \pmod p$ can be evaluated in quite a similar manner to that in Sec. 4. The sum over the Weyl group again reduces to that for S^3 but with a smaller subgroup. The actual group depends on the congruence properties of N. For $N \equiv q \pmod p$, the final result is upto a multiplicative factor

$$Z\left(S^3/Z_p,\ N \equiv q \pmod p,\ k \equiv -q \pmod p\right) = ()Z^q(S^3, \frac{N-q}{p} + 1, \frac{k+q}{p} - 1)Z^{p-q}(S^3, \frac{N-q}{p}, \frac{k+q}{p}) \quad \text{(A.1)}$$

It is much more difficult to evaluate the partition function in such a simple closed form for general N, k.

It is curious that the large N limit of this expression (which can be easily taken following the discussion in the text), sensitively depends on the congruence properties of k. (From examining the p=2 case in more detail, it seems that the limit is independent, as it should be, of how one takes N to ∞.)

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