

# Comments on String Theory on $AdS_3$

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## Abstract

We study string propagation on  $AdS_3$  times a compact space from an “old fashioned” worldsheet point of view of perturbative string theory. We derive the spacetime CFT and its Virasoro and current algebras, thus establishing the conjectured  $AdS/CFT$  correspondence for this case in the full string theory. Our results have implications for the extreme IR limit of the  $D1 - D5$  system, as well as to 2+1 dimensional BTZ black holes and their Bekenstein-Hawking entropy.

# 1 Introduction

The purpose of this paper is to study string propagation on curved spacetime manifolds that include  $AdS_3$ . We will mostly discuss the Euclidean version also known as  $H_3^+ = SL(2, C)/SU(2)$  (in Appendix A we will comment on the Lorentzian signature version of  $AdS_3$ , which is the  $SL(2, R)$  group manifold). At low energies the theory reduces to  $2 + 1$  dimensional gravity with a negative cosmological constant coupled (in general) to a large collection of matter fields. The low energy action is

$$\mathcal{S} = \frac{1}{16\pi l_p} \int d^3x \sqrt{g} (R + \frac{2}{l^2}) + \dots \quad (1.1)$$

but we will go beyond this low energy approximation. Our analysis has applications to some problems of recent interest:

(a) Brown and Henneaux [1] have shown that any theory of gravity on  $AdS_3$  has a large symmetry group containing two commuting copies of the Virasoro algebra and thus can presumably be thought of as a CFT in space-time. The Virasoro generators correspond to diffeomorphisms which do not vanish sufficiently rapidly at infinity and, therefore, act on the physical Hilbert space. In other words, although three dimensional gravity does not have local degrees of freedom, it has non-trivial “global degrees of freedom.” We will identify them in string theory on  $AdS_3$  as holomorphic (or anti-holomorphic) vertex operators which are integrated over contours on the worldsheet. Similar vertex operators exist in string theory in flat space-time. For example, for any spacetime gauge symmetry there is a worldsheet current  $j$  and  $\oint j(z)$  is a good vertex operator. It measures the total charge (the global part of the gauge symmetry). The novelty here is the large number of such conserved charges, and the fact that, as we will see, they can change the mass of states

(b) There is a well known construction of black hole solutions in  $2 + 1$  dimensional gravity with a negative cosmological constant (1.1), known as the BTZ construction [2]. BTZ black holes can be described as solutions of string theory which are orbifolds of more elementary string solutions [3]. Strominger [4] suggested a unified point of view for all black objects whose near horizon geometry is  $AdS_3$ , including these BTZ black holes and the black strings in six dimensions discussed in [5] and related their Bekenstein-Hawking entropy to the central charge  $c$  of the Virasoro algebra of [1]. The states visible in the low energy three dimensional gravity form a single representation of this Virasoro algebra. Their density of states is controlled by [6, 7]  $c_{\text{eff}} = 1$ , which in general is much smaller than  $c$ . Our analysis shows

that the full density of states of the theory is indeed controlled by  $c$  and originates from stringy degrees of freedom.

(c) Maldacena conjectured [8] (see [9] for related earlier work and [10, 11] for a more precise statement of the the conjecture) that string theory on  $AdS$  times a compact space is dual to a CFT. Furthermore, by studying the geometry of anti-de-Sitter space Witten [11] argued on general grounds that the observables in a quantum theory of gravity on  $AdS$  times a compact space should be interpreted as correlation functions in a local CFT on the boundary. Our work gives an explicit realization of these ideas for the concrete example of strings on  $AdS_3$ . In particular, we construct the coordinates of the spacetime CFT and some of its operators in terms of the worldsheet fields.

(d) For the special case of type IIB string theory on

$$M = AdS_3 \times S^3 \times T^4 \tag{1.2}$$

Maldacena argued that it is equivalent to a certain two-dimensional superconformal field theory (SCFT), corresponding to the IR limit of the dynamics of parallel  $D1$ -branes and  $D5$ -branes (the  $D1/D5$  system). Our discussion proves this correspondence.

(e) In string theory in flat spacetime integrated correlation functions on the worldsheet give S-matrix elements. In anti-de-Sitter spacetime there is no S-matrix. Instead, the interesting objects are correlation functions in the field theory on the boundary [8, 10, 11]. Although the spacetime objects of interest are different in the two cases, we will see that they are computed by following exactly the same worldsheet procedure.

(f) Many questions in black hole physics and the  $AdS/CFT$  correspondence circle around the concept of holography [12]. Our analysis leads to an explicit identification of the boundary coordinates in string theory. We hope that it will lead to a better understanding of holography.

In section 2 we review the geometry of  $AdS_3$  and consider the CFT with this target space (for earlier discussions of this system see [13]- [15] and references therein). We then show how the  $SL(2) \times SL(2)$  current algebra on the string worldsheet induces current algebras and Virasoro algebras in spacetime. This leads to a derivation of the  $AdS/CFT$  correspondence in string theory. In section 3 we extend the analysis to the superstring, and describe the NS and R sectors of the spacetime SCFT. In section 4 we explain the relation between our system and the dynamics of parallel strings and

fivebranes. We discuss both the case of  $NS5$ -branes with fundamental strings and the  $D1/D5$  system. We also relate our system to BTZ black holes. In Appendix A we discuss the geometry of  $AdS_3$  with Lorentzian signature. In Appendix B we discuss string theory on  $\mathcal{M}$  with twisted supersymmetry.

## 2 Bosonic Strings on $AdS_3$

According to Brown and Henneaux [1], any theory of three dimensional gravity with a negative cosmological constant has an infinite symmetry group that includes two commuting Virasoro algebras and thus describes a *two dimensional* conformal field theory in spacetime. In this section we explain this observation in the context of bosonic string theory on

$$AdS_3 \times \mathcal{N} \quad (2.1)$$

where  $\mathcal{N}$  is some manifold (more generally, a target space for a CFT) which together with  $AdS_3$  provides a solution to the equations of motion of string theory.

Of course, such vacua generically have tachyons in the spectrum, but these are irrelevant for many of the issues addressed here (at least up to a certain point) and just as in many other situations in string theory, once the technically simpler bosonic case is understood, it is not difficult to generalize the discussion to the tachyon free supersymmetric case (which we will do in the next section).

We start by reviewing the geometry of  $AdS_3 = H_3^+$ . It can be thought of as the hypersurface

$$-X_{-1}^2 + X_3^2 + X_1^2 + X_2^2 = -l^2 \quad (2.2)$$

embedded in flat  $R^{1,3}$  with coordinates  $(X_{-1}, X_1, X_2, X_3)$ . Equation (2.2) describes a space with constant negative curvature  $-1/l^2$ , and  $SL(2, C) \simeq Spin(1, 3)$  isometry. The space (2.2) can be parametrized by the coordinates

$$\begin{aligned} X_{-1} &= \sqrt{l^2 + r^2} \cosh \tau \\ X_3 &= \sqrt{l^2 + r^2} \sinh \tau \\ X_1 &= r \sin \theta \\ X_2 &= r \cos \theta \end{aligned} \quad (2.3)$$

(where  $\theta \in [0, 2\pi)$  and  $r$  is non-negative) in terms of which the metric takes the form

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + l^2 \left(1 + \frac{r^2}{l^2}\right) d\tau^2 + r^2 d\theta^2 \quad (2.4)$$

Another convenient set of coordinates is

$$\begin{aligned} \phi &= \log(X_{-1} + X_3)/l \\ \gamma &= \frac{X_2 + iX_1}{X_{-1} + X_3} \\ \bar{\gamma} &= \frac{X_2 - iX_1}{X_{-1} + X_3} \end{aligned} \quad (2.5)$$

Note that the complex coordinate  $\bar{\gamma}$  is the complex conjugate of  $\gamma$ . The surface (2.2) has two disconnected components, corresponding to  $X_{-1} > 0$  and  $X_{-1} < 0$ . We will restrict attention to the former, on which  $X_{-1} > |X_3|$ ; therefore, the first line of (2.5) is meaningful. In the coordinates  $(\phi, \gamma, \bar{\gamma})$  the metric is

$$ds^2 = l^2(d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}). \quad (2.6)$$

The metrics (2.4) and (2.6) describe the same space. The change of variables between them is:

$$\begin{aligned} \gamma &= \frac{r}{\sqrt{l^2 + r^2}} e^{-\tau + i\theta} \\ \bar{\gamma} &= \frac{r}{\sqrt{l^2 + r^2}} e^{-\tau - i\theta} \\ \phi &= \tau + \frac{1}{2} \log\left(1 + \frac{r^2}{l^2}\right). \end{aligned} \quad (2.7)$$

The inverse change of variables is:

$$\begin{aligned} r &= l e^{\phi} \sqrt{\gamma \bar{\gamma}} \\ \tau &= \phi - \frac{1}{2} \log(1 + e^{2\phi} \gamma \bar{\gamma}) \\ \theta &= \frac{1}{2i} \log(\gamma/\bar{\gamma}). \end{aligned} \quad (2.8)$$

It is important that both sets of coordinates cover the entire space exactly once – the change of variables between them (2.7) and (2.8) is one to one.

In the coordinates (2.4) the boundary of Euclidean  $AdS_3$  corresponds to  $r \rightarrow \infty$ . It is a cylinder parametrized by  $(\tau, \theta)$ . The change of variables (2.7) becomes for large  $r$ :  $e^\phi \approx re^\tau/l$ ,  $\gamma \approx e^{-\tau+i\theta}$ ,  $\bar{\gamma} \approx e^{-\tau-i\theta}$ . Thus, in the coordinates (2.6) the boundary corresponds to  $\phi \rightarrow \infty$ ; it is a sphere parametrized by  $(\gamma, \bar{\gamma})$ .

## 2.1 Worldsheet Properties of Strings on $AdS_3$

To describe strings propagating on the space (2.2) we need to add a (Neveu-Schwarz)  $B_{\mu\nu}$  field in order to satisfy the equations of motion. From the worldsheet point of view this is necessary for conformal invariance. The necessary  $B$  field is  $B = l^2 e^{2\phi} d\gamma \wedge d\bar{\gamma}$ . Note that it is imaginary. Therefore, the worldsheet theory is not unitary. With a Euclidean worldsheet the contribution of the  $B$  field to the action is real and the theory is not reflection positive. In this respect our system is different from the analytic continuation to flat Euclidean space of strings in flat Minkowski space. The worldsheet Lagrangian with the  $B$  field is

$$\mathcal{L} = \frac{2l^2}{l_s^2} \left( \partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma} \right) \quad (2.9)$$

( $l_s$  is the fundamental string length). Note that with a Euclidean signature worldsheet  $\mathcal{L}$  is real and bounded from below; therefore, the path integral is well defined<sup>1</sup>. Some of the  $SL(2)$  symmetry is manifest in the Lagrangian 2.9; *e.g.* we can shift  $\gamma$  by a holomorphic function. It is convenient to add a one form field  $\beta$  with spin  $(0, 1)$  and its complex conjugate  $\bar{\beta}$  with spin  $(1, 0)$ , and consider the Lagrangian

$$\mathcal{L} = \frac{2l^2}{l_s^2} \left( \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - e^{-2\phi}\beta\bar{\beta} \right). \quad (2.10)$$

Integrating out  $\beta$  and  $\bar{\beta}$  we recover (2.9). As in Liouville theory, at the quantum level the exponent in the last term is renormalized. Similarly, a careful analysis of the measure shows that a dilaton linear in  $\phi$  is generated. Taking these effects into account and rescaling the fields one finds the worldsheet Lagrangian

$$\mathcal{L} = \partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+} \hat{R}^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}\exp\left(-\frac{2}{\alpha_+}\phi\right) \quad (2.11)$$

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<sup>1</sup>This is one of the reasons we limit ourselves to the Euclidean problem of strings on  $H_3^+$ . Had we worked with a Lorentzian signature target space (the  $SL(2, R)$  group manifold), the Euclidean worldsheet action would have had a real part which is not bounded from below and the path integral would have been ill defined.

where  $\alpha_+^2 = 2k - 4$  is related to  $l$ , the radius of curvature of the space (2.2), via:

$$l^2 = l_s^2 k \quad (2.12)$$

The Lagrangian (2.11) leads to the free field representation of  $SL(2)$  current algebra [16] (see also [17, 18]). It uses a free field  $\phi$  and a holomorphic bosonic  $\beta, \gamma$  system [19] (as well as its anti-holomorphic analog  $\bar{\beta}, \bar{\gamma}$ ) with weights  $h(\beta) = 1$ ,  $h(\gamma) = 0$ . The last term in  $\mathcal{L}$  (2.11) can be thought of as a screening charge. Correlation functions in the CFT that are dominated by the region  $\phi \rightarrow \infty$  (such as bulk correlation functions [18] can be studied by perturbing in this term; this leads to a prescription similar to that used in Liouville theory. Generic correlation functions are non-perturbative in the screening charge.

We can repeat a similar analysis in the  $r, \theta, \tau$  variables. After introducing new fields  $\alpha$  and  $\bar{\alpha}$  the Lagrangian for large  $r$  becomes

$$\mathcal{L} = \frac{1}{r^2} \partial r \bar{\partial} r + \alpha \bar{\partial}(\tau - i\theta) + \bar{\alpha} \partial(\tau + i\theta). \quad (2.13)$$

In this limit  $\log r$  is a free field which is a sum of a holomorphic and an anti-holomorphic field. Similarly,  $\tau$  and  $\theta$  are free fields with holomorphic and anti-holomorphic components. However, the equations of motion also guarantee that  $\tau - i\theta$  is holomorphic. This is consistent with the fact that for large  $r$  it is related to the holomorphic field  $\gamma \approx e^{-\tau+i\theta}$  (see (2.7)).

A related description of CFT on Euclidean  $AdS_3$  is obtained by constructing the worldsheet Lagrangian using the  $r, \theta, \tau$  coordinates and performing a T-duality transformation on  $\theta$  [20]. In terms of the dual coordinate  $\tilde{\theta}$  there is no  $B$  field; instead there is a dilaton field which is linear in  $\log r$ . The Lagrangian is

$$\mathcal{L} = \partial \tau \bar{\partial} \tau + \frac{1}{r^2} \partial \tilde{\theta} \bar{\partial} \tilde{\theta} + \frac{1}{r^2 + 1} \partial r \bar{\partial} r - 2i \partial \tilde{\theta} \bar{\partial} \tau. \quad (2.14)$$

Note that it has an imaginary term reflecting the lack of unitarity of the system. In terms of  $\hat{\tau} = \tau - i\tilde{\theta}$  it is

$$\partial \hat{\tau} \bar{\partial} \hat{\tau} + \frac{r^2 + 1}{r^2} \partial \tilde{\theta} \bar{\partial} \tilde{\theta} + \frac{1}{r^2 + 1} \partial r \bar{\partial} r. \quad (2.15)$$

This description of the theory is similar but not identical to that of (2.10), (2.11), (2.13)

For large  $r$  the theory becomes free and the corrections to free field theory can be treated as a screening charge.

The theory has an affine  $SL(2, R) \times SL(2, R)$  Lie algebra symmetry at level  $k$ , generated by worldsheet currents  $J^A(z)$ ,  $\bar{J}^A(\bar{z})$ , which satisfy the OPE:

$$J^A(z)J^B(w) = \frac{k\eta^{AB}/2}{(z-w)^2} + \frac{i\eta_{CD}\epsilon^{ABC}J^D}{z-w} + \dots, \quad A, B, C, D = 1, 2, 3 \quad (2.16)$$

where  $\eta^{AB}$  is the metric on  $SL(2, R)$  (with signature  $(+, +, -)$ ) and  $\epsilon_{ABC}$  are the structure constants of  $SL(2, R)$ . A similar formula describes the operator products of the worldsheet currents with the other chirality,  $\bar{J}^A(\bar{z}) = (J^A(z))^*$ . The level of the affine Lie algebra,  $k$ , is related to the cosmological constant via eq. (2.12). The central charge of this model is:

$$c = \frac{3k}{k-2}. \quad (2.17)$$

It will be useful for our purposes to recall the free field (2.11) realization of  $SL(2, R)$  current algebra ([16]). The worldsheet propagators that follow from (2.11) are:  $\langle \phi(z)\phi(0) \rangle = -\log|z|^2$ ,  $\langle \beta(z)\gamma(0) \rangle = 1/z$ . The current algebra is represented by (normal ordering is implied):

$$\begin{aligned} J^3 &= \beta\gamma + \frac{\alpha_+}{2}\partial\phi \\ J^+ &= \beta\gamma^2 + \alpha_+\gamma\partial\phi + k\partial\gamma \\ J^- &= \beta. \end{aligned} \quad (2.18)$$

Interesting vertex operators are

$$V_{jm\bar{m}} = \gamma^{j+m}\bar{\gamma}^{j+\bar{m}}\exp\left(\frac{2j}{\alpha_+}\phi\right). \quad (2.19)$$

The exponents of  $\gamma$  and  $\bar{\gamma}$  can be both positive and negative. The only constraint that follows from single valuedness on  $AdS_3$  is that  $m - \bar{m}$  must be an integer. Obviously,  $m - \bar{m}$  is the momentum in the  $\theta$  direction. One can check that  $j$ ,  $m$  and  $\bar{m}$  are the values of the  $j$  quantum number of  $SL(2, R)$ , and the  $J^3$  and  $\bar{J}^3$  quantum numbers, respectively<sup>2</sup>. The scaling dimension of  $V_{jm\bar{m}}$  is  $h = -j(j+1)/(k-2)$ .

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<sup>2</sup>Our group is really the infinite multiple cover of  $SL(2, R)$  (see Appendix A) and therefore  $j, m, \bar{m}$  are not restricted to be half integers.



Which  $SL(2, R)$  representations should we consider? The affine  $SL(2, R)$  algebra does not have unitary representations. This should not bother us because, as we said above, our worldsheet theory is not unitary. The problem that we are interested in is string theory and therefore we should use the  $SL(2, R)$  representations which lead to a unitary string spectrum. One way to do this is the following. Consider the affine  $U(1) \subset SL(2, R)$  generated by  $J^3$  (the “timelike direction”) and decompose each  $SL(2, R)$  representation in terms of the coset  $SL(2, R)/U(1)$  and the  $U(1)$  representation. In constructing a string vacuum we need the  $SL(2, R)/U(1)$  coset to be unitary. The conditions for that were analyzed in [21] with the conclusion

$$-1 < j < \frac{k}{2} - 1, \quad 2 < k. \quad (2.20)$$

Imposing the constraint (2.20) in string theory gives rise to a unitary theory (see *e.g.* [15] and references therein).

## 2.2 Spacetime Properties of Strings on $AdS_3$

Due to the presence of the worldsheet affine  $SL(2)$  Lie algebra (2.16) the spacetime theory has three conserved charges

$$\begin{aligned} L_0 &= - \oint dz J^3(z) \\ L_1 &= - \oint dz J^+(z) \\ L_{-1} &= - \oint dz J^-(z) \end{aligned} \quad (2.21)$$

which satisfy the  $SL(2, R)$  algebra  $[L_n, L_m] = (n - m)L_{n+m}$  ( $n, m = 0, \pm 1$ ). The observations of [1] lead one to expect that (2.21) should be extended to an infinite dimensional Virasoro algebra with central charge:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (2.22)$$

Our next task is to derive (2.22) in string theory and compute the central charge  $c$ .

As a warmup exercise, consider the following related problem. Take the worldsheet CFT on the manifold  $\mathcal{N}$  to contain an affine Lie algebra  $\widehat{G}$  for a compact group  $G$ , generated by currents  $K^a$  satisfying the OPE:

$$K^a(z)K^b(w) = \frac{k'\delta^{ab}/2}{(z-w)^2} + \frac{if^{ab}_c K^c}{z-w} + \cdots; \quad a, b, c = 1, \dots, \dim G \quad (2.23)$$

with  $k'$  the level of  $\widehat{G}$ . Normally, this leads to the existence in the space-time theory of  $\dim G$  conserved charges

$$T_0^a = \oint dz K^a(z) \quad (2.24)$$

satisfying the algebra

$$[T_0^a, T_0^b] = if^{ab}_c T_0^c. \quad (2.25)$$

However, in our case the spacetime theory is a two dimensional CFT and we expect the charges  $T_0^a$  to correspond to the zero modes of an infinite symmetry – an affine Lie algebra *in spacetime*, generated by charges  $T_n^a$  satisfying the commutation relations

$$[T_n^a, T_m^b] = if^{ab}_c T_{n+m}^c + \frac{\tilde{k}}{2} n \delta^{ab} \delta_{n+m,0} \quad (2.26)$$

$$[L_m, T_n^a] = -n T_{n+m}^a$$

where  $\tilde{k}$  is the level of affine  $\widehat{G}$  in spacetime. We will next construct the operators  $T_n^a$ , verify the first line of (2.26), and compute  $\tilde{k}$ . Later, when we define the  $\{L_m\}$  we will also verify the second line.

The second line of (2.26) with  $m = 0$  states that the operators  $T_n^a$  carry  $-n$  units of  $L_0$  or  $n$  units of  $J^3$  (2.21). Thus, in order to construct them we need to generalize the definition (2.24) by multiplying the integrand  $K^a(z)$  by a vertex operator that carries  $J^3$  but has worldsheet scaling dimension zero and is holomorphic, so that it can be integrated over  $z$ . There is a unique candidate, the field (2.19)  $V_{j=0,m,\bar{m}=0} = \gamma^m$  with integer  $m$ . Thus, we define

$$T_n^a = \oint dz K^a(z) \gamma^n(z) \quad (2.27)$$

and compute the commutator using standard techniques:

$$[T_n^a, T_m^b] = \oint dw \oint dz K^a(z) \gamma^n(z) K^b(w) \gamma^m(w) \quad (2.28)$$

where the integral over  $z$  is taken as usual along a small contour around  $w$ , and the integral over  $w$  is taken around some origin 0. The only source of singularities in the contour integral of  $z$  around  $w$  comes from the OPE

of currents (2.23) (the OPE of  $\gamma$ 's is regular). The second term in the OPE (2.23) gives a first order pole that is easily integrated to give:

$$if_{abc} \oint dw \oint dz K^c(w) \gamma^{n+m}(w) \frac{1}{z-w} = if_{abc} \oint dw K^c \gamma^{n+m} = if_{abc} T_{n+m}^c \quad (2.29)$$

The first term in (2.23) gives a second order pole and needs to be dealt with separately:

$$\begin{aligned} \frac{k' \delta^{ab}}{2} \oint dw \oint dz \frac{\gamma^n(z) \gamma^m(w)}{(z-w)^2} &= \frac{k' \delta^{ab}}{2} \oint dw \partial_w (\gamma^n) \gamma^m \\ &= \frac{nk' \delta^{ab}}{2} \oint dw \gamma^{n+m-1} \partial_w \gamma \end{aligned} \quad (2.30)$$

The r.h.s. of (2.30) is central – it commutes with the generators  $T_n^a$ , (2.27), and more generally with all physical vertex operators in the theory. Therefore, this charge is not carried by the excitations of the string but only by the vacuum. The charge is non-vanishing only for  $n+m=0$  because otherwise  $\oint dw \gamma^{m+n-1} \partial_w \gamma = \frac{1}{m+n} \oint dw \partial_w \gamma^{m+n} = 0$ . For  $n+m=0$  the integral

$$p \equiv \oint dz \frac{\partial_z \gamma}{\gamma} \quad (2.31)$$

can be nonzero. It counts the number of times  $\gamma$  winds around the origin when  $z$  winds once around  $z=0$ . Since  $\gamma$  is a single valued function of  $z$ ,  $p$  must be an integer<sup>3</sup>.

To understand the meaning of the integer  $p$ , recall the spacetime interpretation of the free field  $\gamma$ , (2.7). In the spacetime CFT at  $r \rightarrow \infty$  (2.7),  $\gamma = e^{-\tau+i\theta}$  is a coordinate on the sphere with two punctures (corresponding to  $\tau = \pm\infty$ ). Therefore,  $p$  measures the number of times the string worldsheet wraps around  $\theta$ . We interpret string theory on  $AdS_3$  as having  $p$  stretched fundamental strings at  $r \rightarrow \infty$ . The excitations of the vacuum described by vertex operators correspond to small fluctuations of these infinitely stretched strings and this is the reason they do not carry the charge (2.31). String vacua with different values of  $p$  correspond to different sectors of the theory.

It is important that our target space is simply connected. Therefore, there cannot be any winding perturbative string states and hence  $p$  commutes with all vertex operators describing perturbative states.

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<sup>3</sup>For Lorentzian signature  $\gamma$  is real (see Appendix A), and there is no natural definition of winding. This is another reason for studying the Euclidean version of the theory.

Collecting all the terms (2.29), (2.30) we find that the  $T_n^a$  satisfy the algebra (2.26), with

$$k_{\text{spacetime}} \equiv \tilde{k} = pk' \quad (2.32)$$

Thus, the affine Lie algebra structure is lifted from the worldsheet to spacetime and the level of the affine Lie algebra in spacetime is equal to  $p$  times that on the worldsheet.

A few comments are in order here:

(a) The fact that  $p$  is a positive integer is important to get a unitary realization of  $\hat{G}$  in spacetime.

(b) We see that in string theory on  $AdS_3$  there is a close correspondence between worldsheet and spacetime properties. A left-moving affine Lie algebra on the worldsheet gives rise to a left-moving affine Lie algebra in spacetime, *etc.* This correspondence, seen here and in many other aspects of our analysis below, is reminiscent of analogous phenomena in theories of worldsheets for worldsheets and related ideas [22]-[24].

(c) The derivation and, in particular, the treatment of the integral in (2.30) makes it clear that one should think of  $\gamma$  as a holomorphic coordinate in spacetime, in agreement with the geometric analysis of eqs. (2.6 – 2.8). Note that  $\gamma$  depends holomorphically on the worldsheet coordinates,  $\bar{\partial}\gamma = 0$ . This is another example of the worldsheet – spacetime connection mentioned in item (b). (d) The discussion above is very reminiscent of the construction of DDF states in string theory (see [25] for details). One can think of the operators  $T_n^a$  (2.27) as a spectrum generating algebra.

We are now ready to turn to the original problem of finding the spacetime Virasoro algebra. We proceed in complete analogy with (2.27) – (2.30) but there are a few new elements. On general grounds we expect the Virasoro generators to be given by

$$L_n = \oint dz \left( a_3 J^3 \gamma^n + a_- J^- \gamma^{n+1} + a_+ J^+ \gamma^{n-1} \right) \quad (2.33)$$

The operators (2.33) are very similar to photon vertex operators in a three dimensional curved space. As usual, only one of the three polarizations in (2.33) is physical. First, we have to impose BRST invariance, *i.e.* require the operator in brackets to be primary under the worldsheet Virasoro algebra. This gives rise to the constraint

$$na_3 + (n+1)a_- + (n-1)a_+ = 0 \quad (2.34)$$

Furthermore, the fact that “longitudinal photons” are BRST exact and decouple leads to the identification

$$(a_3, a_-, a_+) \simeq (a_3, a_-, a_+) + \alpha(1, -\frac{1}{2}, -\frac{1}{2}) \quad (2.35)$$

for all  $\alpha$ , corresponding to gauge invariance in spacetime. A natural solution to the above constraints which reduces to (2.21 for  $n = 0, \pm 1$  is:

$$-L_n = \oint dz \left[ (1 - n^2) J^3 \gamma^n + \frac{n(n-1)}{2} J^- \gamma^{n+1} + \frac{n(n+1)}{2} J^+ \gamma^{n-1} \right] \quad (2.36)$$

To see that the operators  $L_n$  satisfy the Virasoro algebra (2.22) as well as (2.26) it is convenient to use the gauge invariance (2.35) to transform (2.36) to the equivalent form:

$$-L_n = \oint dz \left[ (n+1) J^3 \gamma^n - n J^- \gamma^{n+1} \right] \quad (2.37)$$

and compute:

$$\begin{aligned} [L_n, L_m] &= \oint dw \oint dz \left[ (n+1) J^3 \gamma^n - n J^- \gamma^{n+1} \right] (z) \\ &\quad \times \left[ (m+1) J^3 \gamma^m - m J^- \gamma^{m+1} \right] (w) \end{aligned} \quad (2.38)$$

There are four terms to evaluate; the residues of single poles in the OPE are of three different kinds:  $J^3 \gamma^{n+m}$ ,  $J^- \gamma^{n+m+1}$  and  $\gamma^{n+m-1} \partial \gamma$ . The numerical factors conspire so that the algebra closes. Using the OPE's

$$\begin{aligned} J^3(z) \gamma^n(w) &= \frac{n \gamma^n(w)}{z-w} + \dots \\ J^-(z) \gamma^n(w) &= \frac{n \gamma^{n-1}(w)}{z-w} + \dots \\ J^3(z) J^-(w) &= -\frac{J^-(w)}{z-w} + \dots \\ J^3(z) J^3(w) &= -\frac{k/2}{(z-w)^2} + \dots \end{aligned} \quad (2.39)$$

one finds that (2.38) leads to the algebra (2.22) with the central charge in spacetime given in terms of the level of  $SL(2, R)$ ,  $k$ , (2.16), and the charge  $p$ , (2.31):

$$c_{\text{spacetime}} = 6kp \quad (2.40)$$

Thus, for fixed  $SL(2, R)$  level  $k$ , as  $p$  increases the spacetime central charge  $c_{\text{spacetime}} \rightarrow \infty$ , which is the semiclassical limit in the spacetime CFT. We will see later that the string coupling is proportional to  $1/\sqrt{p}$ ; thus the theory indeed becomes more and more weakly coupled as  $p \rightarrow \infty$ . Similarly, as  $k \rightarrow \infty$  for fixed  $p$ , the curvature of  $AdS_3$  goes to zero and the gravity approximation to (aspects of) the full string theory becomes better and better.

Note that the Virasoro algebra acts as holomorphic reparametrization symmetry on  $\gamma$ . Indeed, one can verify using (2.37), (2.39) that:

$$[L_n, \gamma(z)] = -\gamma^{n+1}(z) \quad (2.41)$$

which implies that one can think of  $L_n$  as

$$L_n = -\gamma^{n+1} \frac{\partial}{\partial \gamma}. \quad (2.42)$$

The second line of (2.26) is also a straightforward consequence of (2.27), (2.37), (2.39). Note that our derivation of the Virasoro and affine Lie algebras was performed in the free field limit of (2.11), in which one can ignore the screening charge  $\beta\bar{\beta}\exp(-2\phi/\alpha_+)$ . This is accurate at the boundary of  $AdS_3$ ,  $\phi \rightarrow \infty$ . One can check that the affine Lie and Virasoro generators (2.27), (2.37) do not commute with the screening charge. This means that there are corrections to these generators which form a power series in  $\exp(-2\phi/\alpha_+)$ .

In the presence of both  $SL(2, R)$  (2.16) and  $G$  (2.23) affine Lie algebras on the worldsheet one can define a second Virasoro algebra in spacetime – the Sugawara stress tensor of the  $\hat{G}$  generated by  $T_n^a$  (2.27). This second Virasoro algebra should be thought of as a part of the total Virasoro algebra (2.36). In the three dimensional string theory the reason for this is that all degrees of freedom must couple to three dimensional gravity. In particular, if the spacetime theory is unitary, the central charges must satisfy the inequality

$$c_{\text{spacetime}} = 6kp \geq \frac{k' p \dim G}{k' p + Q} \quad (2.43)$$

where the right hand side is the Sugawara central charge for  $\hat{G}$ , and  $Q$  is the quadratic Casimir of  $G$  in the adjoint representation. The inequality (2.43) becomes trivial in the weak coupling limit  $p \rightarrow \infty$ , but for large string coupling  $p \simeq 1$  it provides a constraint on the parameters of the theory. Of course, the whole discussion of unitarity in the bosonic string is quite

confusing because of the instability which is signaled by the tachyon. Below we will apply a similar discussion to stable vacua of string theory.

The form of the spacetime central charge (2.40) is interesting. In the original work of Brown and Henneaux [1] this central charge was computed using low energy gravity and was found to be

$$c_{\text{spacetime}} = \frac{3l}{2l_p} \quad (2.44)$$

where  $l$  is the radius of curvature of  $AdS_3$  (see (2.4)) and  $l_p$  is the three dimensional Planck length ( $l_p \equiv G_3$ , the three dimensional Newton constant). The calculation of [1] is expected to be reliable in the semiclassical regime  $l \gg l_p$ ; (2.44) should be thought of as the leading term in an expansion in  $l_p/l$ . In our case  $l$  is related to the level of the  $SL(2, R)$  affine algebra,  $k$  (2.12), while  $l_p$  is given in terms of the fundamental string coupling  $g$  and the volume of the compactification manifold  $\mathcal{N}$  (2.1),  $V_{\mathcal{N}}$  (measured in string units), by:

$$\frac{1}{l_p} = \frac{V_{\mathcal{N}}}{g^2 l_s} \quad (2.45)$$

Thus, the formula (2.40) for the central charge implies in this case that the string coupling is quantized. More precisely, the three dimensional Planck scale satisfies:

$$l_s/l_p = 4p\sqrt{k} \quad (2.46)$$

The three dimensional string coupling  $g_3^2 \sim l_p/l_s \sim 1/(p\sqrt{k})$  is small if either  $k$  or  $p$  are large. As we will see later, the higher dimensional string coupling is typically *large*<sup>4</sup> for small  $p$ , and it decreases as  $p \rightarrow \infty$ , where perturbative string theory is reliable. The fact that the string coupling is fixed in string theory on  $AdS_3$  in a given sector of the theory (*i.e.* for given  $k, p$ ) implies that the dilaton is massive and its potential has a unique minimum.

The string coupling behaves like  $g \sim 1/\sqrt{p}$ , which is reminiscent of the coupling between mesons in large  $N$  gauge theory (where  $g \sim 1/\sqrt{N}$ ). Perhaps one can think of the closed strings on  $AdS_3$  as “mesons” constructed out of “quarks.”

Another (related) useful analogy is WZW models. The WZW Lagrangian for a compact group  $G$  at level  $P$  is proportional to  $P$  (at the fixed point where the infinite conformal and affine Lie symmetries appear). The interactions between physical states are of order  $1/\sqrt{P}$ ;  $P$  is quantized due

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<sup>4</sup>Essentially because the volume of  $\mathcal{N}$  (2.1) typically grows as  $V_{\mathcal{N}} \sim k^a$ , with  $a > 1/2$ .

to non-perturbative effects (it must be a positive integer). Similarly, in the string theory described above the spacetime action is proportional to  $p$ , while interactions are proportional to  $1/\sqrt{p}$ . The fact that  $p$  is quantized is non-perturbative in the string loop expansion<sup>5</sup>, and presumably related to the appearance of the infinite symmetry (2.22), (2.26) in spacetime.

Physical states in the theory fall into representations of the Virasoro algebra (2.36). A large class of such states is obtained by taking a primary of the worldsheet conformal algebra on  $\mathcal{N}$  (2.1), and dressing it with a conformal primary from the  $AdS_3$  sector. In what follows we will describe this dressing first for the case of vanishing worldsheet spin and then for non-zero spin.

Let  $W_N$  be a spinless worldsheet operator in the CFT on  $\mathcal{N}$ , with scaling dimension  $\Delta_L = \Delta_R = N$  (which of course need not be integer). We can form a physical vertex operator by “dressing”  $W_N$  by an  $AdS_3$  vertex operator  $V_{jm\bar{m}}$  (2.19). The physical vertex operator

$$V_{\text{phys}}(j, m, \bar{m}) = W_N V_{jm\bar{m}} \quad (2.47)$$

must have worldsheet dimension one:

$$N - \frac{j(j+1)}{k-2} = 1 \quad (2.48)$$

Stability of the vacuum requires the solutions of (2.48) to have real  $j$  (see below). Furthermore, the unitarity condition (2.20) shows that only operators with  $N < (k/4) + 1$  can be dressed using (2.47).

To determine the transformation properties of the spacetime field corresponding to  $V_{\text{phys}}$  under the spacetime conformal symmetry, we need to compute the commutator  $[L_n, V_{\text{phys}}]$ . A straightforward calculation using the form<sup>6</sup> (2.37) of  $L_n$  and the free field realization of  $SL(2, R)$ , (2.11), leads to:

$$[L_n, V_{\text{phys}}(j, m, \bar{m})] = (nj - m)V_{\text{phys}}(j, m + n, \bar{m}) \quad (2.49)$$

To understand the meaning of eq. (2.49), recall the following result from CFT. Given an operator  $A^{(h\bar{h})}(\xi, \bar{\xi})$  with scaling dimensions  $(h, \bar{h})$ , we can expand it in modes:

$$A^{(h\bar{h})}(\xi, \bar{\xi}) = \sum_{m\bar{m}} A_{m\bar{m}}^{(h\bar{h})} \xi^{-m-h} \bar{\xi}^{-\bar{m}-\bar{h}}. \quad (2.50)$$

<sup>5</sup>However, the discussion after eq. (2.31) makes it clear that if  $p$  is non-integer the theory is non-perturbatively inconsistent.

<sup>6</sup>One can also derive this relation by using the representation (2.36) and the operator product  $J^\pm(z)V_{j,m,\bar{m}}(w,\bar{w}) = (m \mp j)V_{j,m\pm 1,\bar{m}}(w,\bar{w})/(z-w) + \dots$



The precise values of  $m$  and  $\bar{m}$  depend, as usual, on the sector – the operator insertion at  $\xi = 0$ . In the identity sector  $m + h, \bar{m} + \bar{h} \in Z$ . The mode operators  $A_{m\bar{m}}^{(h\bar{h})}$  satisfy the following commutation relations with the Virasoro generators:

$$[L_n, A_{m\bar{m}}^{(h\bar{h})}] = [n(h-1) - m] A_{n+m, \bar{m}}^{(h\bar{h})}. \quad (2.51)$$

Comparing (2.49) with (2.51) we see that we should identify the physical vertex operators  $V_{\text{phys}}(j, m, \bar{m})$  with modes of primary operators in the spacetime CFT,  $A_{m\bar{m}}^{(h\bar{h})}$ . The scaling dimension in spacetime of the operator  $V_{\text{phys}}(j, m, \bar{m})$  is  $h = \bar{h} = j + 1$ :

$$V_{\text{phys}}(j, m, \bar{m}) \leftrightarrow A_{m\bar{m}}^{(h\bar{h})}; \quad h = \bar{h} = j + 1 \quad (2.52)$$

Note that due to (2.20) there are bounds on the scaling dimensions arising from single particle states:  $0 < h < k/2$ . Equation (2.48) furthermore relates the spectrum of scaling dimensions to the structure of the compact CFT on  $\mathcal{N}$ . Tachyons correspond to solutions of (2.48) with complex  $j = -\frac{1}{2} + i\lambda$ , and we see (2.52) that they give rise to complex scaling dimensions in the spacetime CFT. According to (2.48) worldsheet operators  $W_N$  with  $N < 1 - \frac{1}{4(k-2)}$  in the CFT on  $\mathcal{N}$  correspond to tachyons. Since the identity operator is such an operator that always exists, bosonic string theory in the background (2.1) is always unstable, just like in flat space.

The operators (2.47) give rise to spacetime primaries with spin zero, *i.e.*  $h = \bar{h}$ , (2.52). One expects in general to find many primaries with non-zero spin (in spacetime). These are obtained by coupling worldsheet operators with non-zero *worldsheet* spin to the  $AdS_3$  sector. Consider, for example, a worldsheet primary  $Z_{N, \bar{N}}$  in the CFT on  $\mathcal{N}$ , with scaling dimensions  $\Delta_L = N$ ,  $\Delta_R = \bar{N}$ . We will assume, without loss of generality, that  $N < \bar{N}$ . We cannot couple  $Z_{N, \bar{N}}$  directly to  $V_{j\bar{m}\bar{m}}$  as in (2.47), since this would violate worldsheet level matching. In order to consistently couple  $Z$  to  $AdS_3$ , we need a primary of the worldsheet conformal algebra on  $AdS_3$  that has spin  $n = \bar{N} - N \in Z$ , and is thus a descendant of  $V_{j\bar{m}\bar{m}}$  (2.19) (under the  $SL(2, R)$  current algebra) at level  $n$ .

There are many such descendants; to illustrate the sort of spacetime states they give rise to, we will study a particular example, the operator  $(\partial_z \gamma)^n V_{j\bar{m}\bar{m}}$ . It is not difficult to check that this operator is a conformal primary with  $\Delta_R = -j(j+1)/(k-2)$  and  $\Delta_L = \Delta_R + n$ . Therefore, as in (2.47), we can form the physical operator

$$V_{\text{phys}}^{(n)}(j, m, \bar{m}) = Z_{N, \bar{N}} (\partial_z \gamma)^n V_{j\bar{m}\bar{m}} \quad (2.53)$$

with

$$\bar{N} - \frac{j(j+1)}{k-2} = 1 \quad (2.54)$$

Under the right moving spacetime Virasoro algebra  $\bar{L}_n$ , the operator (2.53) transforms, as before (2.47), as a primary with dimension  $\bar{h} = j + 1$ . The addition of  $(\partial\gamma)^n$  does change the transformation of (2.53) under the left moving Virasoro algebra. Using eq. (2.41) we have:  $[L_s, (\partial_z\gamma)^n] = -n(s+1)\gamma^s(\partial_z\gamma)^n$  and, therefore,

$$[L_s, V_{\text{phys}}^{(n)}(j, m, \bar{m})] = [s(j-n) - (m+n)] V_{\text{phys}}^{(n)}(j, m+s, \bar{m}) \quad (2.55)$$

Comparing to (2.51) we see that the left moving spacetime dimension of our operator is  $h = j + 1 - n$  and the modes  $m$  are shifted by  $n$  units. This adds another entry to our spacetime – worldsheet correspondence: operators with spin  $n$  on the worldsheet give rise to operators with spin  $n$  in the spacetime CFT.

One might wonder what happens for  $j + 1 - n < 0$  when the left-moving scaling dimension might become negative. The answer is that the unitarity constraint (2.20) does not allow this to happen. Indeed,  $j < (k-2)/2$  implies using (2.54) that  $n \leq \bar{N} < (j+3)/2$ . Therefore  $j + 1 - n > (j-1)/2$ ; it can become negative only for  $j < 1$ . Furthermore, since  $N \geq 0$  and  $\bar{N} \geq N + 1 \geq 1$ , (2.54) implies that  $j \geq 0$ . For  $j < 1$  we have  $n < 2$ , which leaves only the case  $n = 1$ ; therefore,  $j + 1 - n = j \geq 0$ .

Note that it is not surprising that we had to use the constraint (2.20) to prove that the scaling dimensions in spacetime cannot become negative, since both have to do with the unitarity of our string theory in spacetime.

One can also study the transformation properties of physical states under the spacetime affine Lie algebra  $\hat{G}$ , (2.27). For example, if the operator  $W_N$  in (2.47) transforms under the worldsheet affine Lie algebra (2.23) in a representation  $R$ :

$$K^a(z)W_N(w, \bar{w}) = \frac{t^a(R)}{z-w}W_N(w, \bar{w}) + \dots \quad (2.56)$$

where  $t^a(R)$  are generators of  $G$  in the representation  $R$ , then the physical vertex operator (2.47) satisfies the commutation relations:

$$[T_n^a, V_{\text{phys}}(j, m, \bar{m})] = t^a(R)V_{\text{phys}}(j, m+n, \bar{m}) \quad (2.57)$$

*i.e.* it is in the representation  $R$  of the spacetime affine Lie algebra.

Correlation functions of physical operators  $V_{\text{phys}}$  satisfy in this case the Ward identities of two dimensional CFT. To prove this one uses (2.49) and the fact that  $L_n|0\rangle = 0$  for  $n \geq -1$ . This last condition can be understood by thinking of the Virasoro generators (2.33) as creating physical states from the vacuum. Since energies of states are positive definite, we can think of  $L_n$  with  $n < -1$  as creation operators, and of  $L_{n>1}$  as annihilation operators. The latter must therefore annihilate the vacuum. Note that the identification of observables in three dimensional string theory with two dimensional CFT correlators found here provides a proof (for the case of  $AdS_3$ ) of the map between string theory in anti-de-Sitter background and boundary CFT proposed in [10, 11].

We see that string theory on  $AdS_3$  has many states which are obtained by applying the holomorphic vertex operators  $\oint dz V_{\text{phys}}^{(L)}(j = 0, m, \bar{m} = 0)$  and their anti-holomorphic analogs to the vacuum. Examples include the generators of the spacetime affine Lie (2.27) and Virasoro (2.36) algebras. More generally, since the worldsheet and spacetime chiralities of operators are tied in this background, the chiral algebra of the spacetime CFT is described by states of this form. As we stressed above, these states are *not* standard closed string states. This new sector in the Hilbert space must be kept even at large  $k, p$ , where the theory becomes semiclassical and the weakly coupled string description is good. Clearly, these chiral states should also appear in the discussion of [10, 11].

### 3 Superstrings on $AdS_3$

Bosonic string theory on  $AdS_3$  contains tachyons, which as we saw means that some of the operators in the spacetime CFT have complex scaling dimensions, and thus the theory is ultimately inconsistent. In this section we generalize the discussion to the spacetime supersymmetric case, which as we will see gives rise to consistent, unitary superconformal field theories in spacetime. We will work in the Neveu-Schwarz-Ramond formalism [19].

There are two steps in generalizing the discussion of section 2 to the supersymmetric case. The first is introducing worldsheet fermions and enlarging the worldsheet gauge principle from  $N = 0$  to  $N = 1$  supergravity. This is usually straightforward, but it does not solve the tachyon problem. The second step involves introducing spacetime fermions by performing a chiral GSO projection. This leads to spacetime supersymmetry and a host of new issues, some of which will be explored below in the context of superstring theory on the manifold  $\mathcal{M}$  (1.2).

### 3.1 Fermionic Strings on $AdS_3$

Following the logic of section 2, we are interested in fermionic string propagation in a spacetime of the form (2.1). The  $\sigma$  model on  $AdS_3 \times \mathcal{N}$  has  $N = 1$  superconformal symmetry with  $c = 15$  which we gauge to construct the string vacuum. The worldsheet SCFT with target space  $AdS_3$  will be taken as before to have an affine  $SL(2, R)$  Lie algebra symmetry at level  $k$ , generated by worldsheet currents  $J^A$  satisfying the OPE (2.16). We will also assume that the SCFT on  $\mathcal{N}$  has in addition an affine Lie algebra symmetry  $\widehat{G}$  corresponding to some compact Lie group  $G$ , at level  $k'$ , with currents  $K^a$  and OPE's (2.23). Under the  $N = 1$  superconformal algebra on the worldsheet, the currents  $J^A, K^a$  are upper components of superfields, whose lower components are free fermions  $\psi^A$  ( $A = 1, 2, 3$ ) and  $\chi^a$  ( $a = 1, \dots, \dim G$ ), respectively (see *e.g.* [26] for a detailed discussion of superstring propagation on group manifolds). The currents  $J^A, K^a$  can be written as sums of “bosonic” currents  $j^A, k^a$  whose levels are  $k + 2$ , and  $k' - Q$  (recall that  $Q$  is the quadratic Casimir of  $G$  in the adjoint representation, as in (2.43)), which commute with the free fermions, and contributions from the free fermions which complete the levels to  $k$  and  $k'$ :

$$\begin{aligned} J^A &= j^A - \frac{i}{k} \epsilon^A_{BC} \psi^B \psi^C \\ K^a &= k^a - \frac{i}{k'} f^a_{bc} \chi^b \chi^c \end{aligned} \quad (3.1)$$

We are using the convenient but unconventional normalization of the free fermions,

$$\begin{aligned} \langle \psi^A(z) \psi^B(w) \rangle &= \frac{k \eta^{AB} / 2}{z - w}, \quad A, B = 1, 2, 3 \\ \langle \chi^a(z) \chi^b(w) \rangle &= \frac{k' \delta^{ab} / 2}{z - w}, \quad a, b = 1, \dots, \dim G \end{aligned} \quad (3.2)$$

As in the bosonic case, the worldsheet currents (3.1) lead to spacetime symmetries. To construct the corresponding charges, recall that in fermionic string theory, physical states are obtained from superconformal primaries with dimension  $h = 1/2$  by taking their upper component (by applying the  $N = 1$  supercharge  $G_{-1/2} = \oint dw G(w)$ ), and integrating the resulting dimension one operator. In the present case, the gauged worldsheet supercurrent  $G$  is given by

$$G(z) = \frac{2}{k} (\eta_{AB} \psi^A j^B - \frac{i}{3k} \epsilon_{ABC} \psi^A \psi^B \psi^C) + \frac{2}{k'} (\chi^a k_a - \frac{i}{3k'} f_{abc} \chi^a \chi^b \chi^c) + G_{\text{rest}} \quad (3.3)$$

where  $G_{\text{rest}}$  is the contribution to  $G$  of the degrees of freedom that complete (3.1), (3.2) to a critical string theory. The relevant dimension  $\frac{1}{2}$  superconformal primaries are  $\psi^A$  and  $\chi^a$ . The corresponding upper components are  $J^A$ ,  $K^a$  (3.1), in terms of which the charges have the same form as in the bosonic case (2.21), (2.24), respectively. In particular, they satisfy the  $SL(2, R) \times G$  algebra.

The global symmetry algebra can again be extended to a semi-direct product of Virasoro and affine  $G$  (2.22), (2.26). The  $\widehat{G}$  affine Lie algebra generators are:

$$T_n^a = \oint dz \{G_{-1/2}, \chi^a \gamma^n(z)\} \quad (3.4)$$

The Virasoro generators  $L_n$  are:

$$\begin{aligned} -L_n &= \oint dz G_{-1/2}, (1 - n^2) \psi^3 \gamma^n + \frac{n(n-1)}{2} \psi^- \gamma^{n+1} \\ &\quad + \frac{n(n+1)}{2 \psi^+ \gamma^{n-1}(z)} \\ &= \oint dz [(1 - n^2) J^3 \gamma^n + \frac{n(n-1)}{2} J^- \gamma^{n+1} + \frac{n(n+1)}{2} J^+ \gamma^{n-1}] \end{aligned} \quad (3.5)$$

In the second line of (3.5) we used the fact that all the terms in which  $G_{-1/2}$  acts on  $\gamma$  cancel. Note that this way of writing  $L_n$  is the same as (2.36), which can also be simplified as (2.37). In fact, this result should have been anticipated because (2.36) only uses the presence of  $SL(2)$ . We emphasize that in (3.4), (3.5)  $G_{-1/2}$  is a *worldsheet* supercharge, while  $T_n^a$ ,  $L_n$  are *spacetime* symmetry generators. It is not difficult to verify by direct calculation that the generators (3.4), (3.5) satisfy the algebra (2.22), (2.26), with the central charges (2.32), (2.40).

Just as in the bosonic case, one can construct physical states which are primaries of the conformal algebra (3.5). For simplicity, we describe the construction for spinless operators (both on the worldsheet and in space-time). These are obtained by taking a primary of the  $N = 1$  worldsheet superconformal algebra on  $\mathcal{N}$ ,  $W_N$ , with scaling dimension  $\Delta_L = \Delta_R = N$ , and dressing it with a superconformal primary on  $AdS_3$ . The corresponding vertex operator in the  $-1$  picture is:

$$V_{\text{phys}}(j, m, \bar{m}) = e^{-\phi - \bar{\phi}} W_N V_{j m \bar{m}} \quad (3.6)$$

where  $\phi$  and  $\bar{\phi}$  are the bosonized super-reparametrization ghosts<sup>7</sup> [19]. The commutation relations of the operators (3.6) with the Virasoro generators

<sup>7</sup>In section 2 we denoted by  $\phi$  the radial direction in  $AdS_3$  (2.5). It should be clear from the context which  $\phi$  we mean everywhere below.

$[L_n, V_{\text{phys}}]$  are similar to the bosonic case (2.49). The resulting scaling dimensions are:

$$h = \bar{h} = j + 1; \quad N - \frac{j(j+1)}{k} = \frac{1}{2} \quad (3.7)$$

In particular, states with  $N < \frac{1}{2} - \frac{1}{4k}$  (tachyons) give rise to complex scaling dimensions. The lowest such state is the identity  $W_N = 1$ . Its presence in the spectrum implies that the fermionic string on  $AdS_3 \times \mathcal{N}$  is an unstable vacuum of string theory, just like the bosonic theory of section 2. However, in this case there is a well known solution to the problem. One can eliminate the tachyons from the spectrum by performing a chiral GSO projection. We will next describe this projection for the particular case of  $AdS_3$ . Rather than being very general, we will do that in the context of an example: superstring theory on  $AdS_3 \times S^3 \times T^4$  (1.2).

### 3.2 Superstrings on $\mathcal{M} = AdS_3 \times S^3 \times T^4$

In addition to  $AdS_3$ , the manifold  $\mathcal{M}$  includes now a three-sphere, or equivalently the  $SU(2)$  group manifold. The worldsheet theory is the  $N = 1$  superconformal WZW model on  $S^3$ . We use the notation of (3.1), (3.2). The  $AdS_3$  fermions and currents are denoted by  $(\psi^A, J^A)$ , while those corresponding to  $SU(2)$  are  $(\chi^a, K^a)$ . The levels of  $SL(2)$  and  $SU(2)$  current algebras are  $k$  and  $k'$ , respectively.

The total central charge of the  $AdS_3 \times S^3$  part of the worldsheet SCFT is:

$$c = \frac{3(k+2)}{k} + \frac{3}{2} + \frac{3(k'-2)}{k'} + \frac{3}{2} \quad (3.8)$$

The first and third terms on the r.h.s. of (3.8) are the contributions of the bosonic  $\sigma$  models on  $AdS_3$  and  $S^3$ ; the second and fourth are due to worldsheet fermions. Criticality of the fermionic string in the background (1.2) implies that  $c = 9$ , which leads to a relation between the levels of the current algebras:

$$k' = k \quad (3.9)$$

The  $T^4$  in  $\mathcal{M}$  corresponds to an  $N = 1$ ,  $U(1)^4$  SCFT; four canonically normalized (compact) free scalar fields  $Y^i$  and fermions  $\lambda^i$ ,  $i = 1, 2, 3, 4$ . The energy-momentum tensor  $T(z)$  and supercurrent  $G(z)$  of this system are:

$$T(z) = \frac{1}{k}(j^A j_A - \psi^A \partial \psi_A) + \frac{1}{k}(k^a k_a - \chi^a \partial \chi_a) + \frac{1}{2}(\partial Y^i \partial Y_i - \lambda^i \partial \lambda_i)$$

$$\begin{aligned}
G(z) = & \frac{2}{k} \left( \psi^A j_A - \frac{i}{3k} \epsilon_{ABC} \psi^A \psi^B \psi^C \right) \\
& + \frac{2}{k} \left( \chi^a k_a - \frac{i}{3k} \epsilon_{abc} \chi^a \chi^b \chi^c \right) + \lambda^i \partial Y_i
\end{aligned} \tag{3.10}$$

So far our treatment of string theory in the background (1.2) was a special case of the discussion of the previous subsection and, in particular, the resulting spacetime theory is tachyonic. We would like next to perform a chiral GSO projection and remove the tachyons, in the process making the vacuum supersymmetric. We expect to be able to perform different projections, corresponding to different boundary conditions for the spacetime supercharges. We start with the construction of the vacuum corresponding to the NS sector of the spacetime SCFT, and then turn to the Ramond vacuum.

### 1) The Neveu-Schwarz Sector of the Spacetime Theory

A well known sufficient condition for spacetime supersymmetry is the enhancement of the  $N = 1$  superconformal symmetry of the worldsheet theory to  $N = 2$  superconformal. This requires in particular the existence of a conserved  $U(1)_R$  current in the worldsheet theory, under which the supercurrent  $G$  splits into two parts,  $G = G^+ + G^-$  with charges  $\pm 1$ . It will turn out that this standard route is *not* the way to proceed here<sup>8</sup>. We will next construct the spacetime supercharges directly, and study the resulting superalgebra.

It is convenient to start by pairing the ten free worldsheet fermions (*i.e.* choosing a complex structure) and bosonizing them into five canonically normalized scalar fields,  $H_I$ ,  $I = 1, \dots, 5$ , which satisfy  $\langle H_I(z) H_J(w) \rangle = -\delta_{IJ} \log(z - w)$ :

$$\partial H_1 = \frac{2}{k} \psi^1 \psi^2, \quad \partial H_2 = \frac{2}{k} \chi^1 \chi^2, \quad i\partial H_3 = \frac{2}{k} \psi^3 \chi^3, \tag{3.11}$$

$$\partial H_4 = \lambda^1 \lambda^2, \quad \partial H_5 = \lambda^3 \lambda^4$$

All the fields except for  $H_3$  satisfy the standard reality condition  $H_I^\dagger = H_I$ . Because of the negative norm of  $\psi^3$ ,  $H_3$  instead satisfies  $H_3^\dagger = -H_3$ . Note that the standard “fermion number” current  $J = i \sum_I \partial H_I$  is not suitable for extending the  $N = 1$  superconformal algebra (3.10) to  $N = 2$ , since the OPE of  $J(z)$  with  $G(w)$  contains a double pole from the second and fourth terms in  $G$  (see Appendix B for a discussion of the  $N = 2$  superconformal structure on the worldsheet).

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<sup>8</sup>In Appendix B we describe some features of the theory obtained by utilizing an  $N = 2$  superconformal symmetry on the worldsheet, and its relation to the spacetime SCFT studied in this section.

Ignoring this complication and proceeding, following the most naive version of [19], we attempt to construct supercharges of the form

$$Q = \oint dz e^{-\frac{\phi}{2}} S(z); \quad S(z) = e^{\frac{i}{2} \sum_I \epsilon_I H_I} \quad (3.12)$$

where  $\epsilon_I = \pm 1$ . In flat space all 32 supercharges (3.12) are BRST invariant and due to the requirement of mutual locality between different supercharges, which is necessary to have a sensible worldsheet theory, one keeps only the sixteen supercharges that satisfy

$$\prod_{I=1}^5 \epsilon_I = 1 \quad (3.13)$$

Since we did not use an  $N = 2$  superconformal algebra on the worldsheet to construct the supercharges, in our case BRST invariance of (3.12) is not guaranteed. Indeed, due to the second and fourth terms in  $G$  (3.10) one finds that only the supercharges that satisfy in addition to (3.13),

$$\prod_{I=1}^3 \epsilon_I = 1 \quad (3.14)$$

are physical. Thus, this system has eight spacetime supercharges (from each worldsheet chirality). The supercharges that survive (3.13), (3.14) can be labeled by three signs, say  $\epsilon_1, \epsilon_2$  and  $\epsilon_4 (= \epsilon_5)$ . The meaning of these signs is revealed by looking at the transformation of the supercharges under the bosonic symmetries of the vacuum,  $SL(2, R) \times SU(2) \times SO(4)$ , with the last factor rotating the four fermions  $\lambda^i$ . The supercharges are in the  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mathbf{0})$  of this symmetry. Thinking about the  $SL(2, R)$  charges as the global part of a spacetime Virasoro algebra, we see that we have four pairs of supercharges  $Q_{\pm \frac{1}{2}}^{\epsilon_2, \epsilon_4}$  in the  $(\frac{1}{2}, \frac{1}{2}, \mathbf{0})$  of  $SU(2) \times SO(4)$ .

Using the technology of [19] one finds that the anticommutators of  $Q_r^{\epsilon_2, \epsilon_4}$  ( $r = \pm 1/2, \epsilon_2, \epsilon_4 = \pm 1$ ) close on the  $SL(2, R) \times SU(2)$  charges (2.21), (2.24). The superalgebra formed by these objects is the global  $N = 4$  superconformal algebra in the NS sector:

$$\begin{aligned} \{Q_r^i, \bar{Q}_s^j\} &= 2\delta^{ij} L_{r+s} - 2(r-s) \sigma_{ij}^a T_{r+s}^a \\ \{Q_r^i, Q_s^j\} &= 0 = \{\bar{Q}_r^i, \bar{Q}_s^j\}, \quad i, j = 1, 2, \quad r, s = \pm 1/2 \\ [T_0^a, Q_r^i] &= -\frac{1}{2} \sigma_{ij}^a Q_r^j, \quad [T_0^a, \bar{Q}_r^i] = \frac{1}{2} \sigma_{ij}^{a*} \bar{Q}_r^j \end{aligned}$$



$$[L_n, Q_r^i] = (\frac{n}{2} - r)Q_{n+r}^i, \quad [L_n, \bar{Q}_r^i] = (\frac{n}{2} - r)\bar{Q}_{n+r}^i, \quad n = 0, \pm 1 \quad (3.15)$$

where we have formed out of our supercharges (3.12) two  $SU(2)$  doublets (for given  $SL(2, R)$  weight  $r$ ).  $Q_r^i$  in (3.15) corresponds in the language of (3.12) to  $\epsilon_1 = 2r$ ,  $\{\epsilon_2 = \pm 1\} \leftrightarrow \{i = 1, 2\}$ ,  $\epsilon_4 = 1$ , and  $\bar{Q}_r^i$  is the same but with  $\epsilon_4 = -1$ ;  $\sigma^a$  are Pauli matrices. The commutation relations of the supercharges with  $L_n$ ,  $n = 0, \pm 1$ , and  $T_0^a$ ,  $a = 1, 2, 3$ , are determined by recalling that the supercharges have scaling dimension  $h = 3/2$  in space-time and (2.51), and that they transform in the **2** of the spacetime  $SU(2)$  symmetry and (2.57).

As we saw in the previous subsection, the bosonic part of the superalgebra (3.15), namely  $SL(2) \times SU(2)$ , is extended to a semi-direct product of a  $c = 6kp$  Virasoro algebra and an affine  $SU(2)$  at level  $kp$ . This clearly means that the  $N = 4$  supercharges  $Q_r$  which we have constructed only for  $r = \pm 1/2$ , actually exist with an arbitrary  $r \in Z + 1/2$ . One way of finding them is to act with  $L_n, T_n^a$  (3.4), (3.5) on  $Q_{\pm 1/2}$ . The resulting structure is the full  $N = 4$  superconformal algebra in spacetime.

Note also that, as in the bosonic case, there is a correlation between chirality on the worldsheet and in spacetime. The spacetime dynamics is that of a  $(4, 4)$  superconformal field theory, with the right moving chiral algebra in spacetime arising from the right movers on the worldsheet via formulae like (3.4), (3.5), (3.12), and similarly for the left movers.

There are now two kinds of physical states. Bosons are described by vertex operators of the form (3.6). The fermion vertices are straightforward generalizations of those described in [19]. They are related to the bosons by supersymmetry (3.12) – (3.15). Additional bosonic states appear from the worldsheet RR sector.

We will only comment briefly on the spectrum of physical states, leaving a more detailed analysis to future work (see also section 4.3). Consider the vertex operators (3.6). The string theory has eight towers of oscillator states coming from all three sectors in (1.2): the  $AdS_3$ ,  $SU(2)$  and  $T^4$  parts of the worldsheet SCFT. Roughly speaking, four of the eight towers can be thought of as describing descendants of the  $N = 4$  superconformal algebra in spacetime (3.15). The four remaining towers correspond to descendants of the  $U(1)^4$  affine Lie algebra generated by the operators

$$\alpha_m^i = \oint dze^{-\phi} \lambda^i \gamma^m \quad (3.16)$$

which satisfy the commutation relations (2.26)

$$[\alpha_n^i, \alpha_m^j] = pn\delta^{ij}\delta_{n+m,0} \quad (3.17)$$

Examples of low lying physical states are the scalar fields  $B^{\bar{i}\bar{i}}$  obtained by Kaluza-Klein reduction of the metric and  $B$  field from ten down to six dimensions on  $T^4$ . The  $l = 2j$  partial wave of  $B^{\bar{i}\bar{i}}$  on the sphere transforms in the  $(\mathbf{j}, \mathbf{j})$  representation of the  $SU(2)_L \times SU(2)_R$  isometry of  $S^3$  and is described by the vertex operator

$$B^{\bar{i}\bar{i}}(j; m, m', \bar{m}, \bar{m}') = e^{-\phi - \bar{\phi}} \lambda^i \bar{\lambda}^{\bar{i}} V_{jm\bar{m}} V'_{j'm'\bar{m}'} \quad (3.18)$$

where  $V'_{j'm'\bar{m}'}$  is the vertex operator of the  $SU(2)$  WZW model with isospin  $j'$ ,  $j'_3 = m'$ ,  $j'_3 = \bar{m}'$ , and we have set  $j = j'$  in (3.18) to satisfy (3.7). The scaling dimension of the operator  $B^{\bar{i}\bar{i}}(j)$  is (3.7),  $h = \bar{h} = 1 + j = 1 + l/2$ .  $j$  takes the values<sup>9</sup>  $j = l/2$ ,  $l = 0, 1, 2, \dots, k-2$ . Applying the superconformal and  $U(1)^4$  generators to (3.18) generates the spectrum of the theory.

The states obtained by acting on the massless vertex operators (3.18) with the spectrum generating operators (3.16) can be alternatively described by replacing  $\lambda^i$  in (3.18) by excited state vertex operators in the (4, 4) supersymmetric worldsheet theory on  $T^4$ . This confirms that there are four towers of such (single particle) states.

Spacetime fermion vertices are obtained as usual [19] by acting on the boson vertex operators, *e.g.* (3.18), with the spacetime supercharges (3.12) – (3.14). This gives rise to vertex operators in the  $-3/2$  picture, which can be brought to the standard  $-1/2$  picture by applying the picture changing operator of [19].

Some of the resulting spacetime fermions correspond to chiral operators in the spacetime SCFT in the sense that their  $SU(2)$  quantum number coincides with their dimension. For example, the superpartners of the partial waves of the six dimensional massless scalar  $B^{\bar{i}\bar{i}}$  (3.18) are complex fermions  $F^{a\bar{i}}(j + \frac{1}{2}; r, r', \bar{m}, \bar{m}')$  carrying a spinor index  $a$  under  $SO(4)$  and spin  $j + \frac{1}{2}$  under  $SU(2)$ .  $r$  and  $r'$  are the eigenvalues of  $-L_0$  and  $T_0^3$ . Since  $B^{\bar{i}\bar{i}}(j) \sim Q_{-1/2} F^{a\bar{i}}(j + \frac{1}{2})$ , the scaling dimension of the fermions  $F$  in the spacetime SCFT is  $h_F = j + \frac{1}{2}$ . Thus, these operators have the property that their scaling dimension and  $SU(2)$  spin coincide – they are chiral in spacetime. This gives rise to  $k-1$  (complex) chiral operators with  $SU(2)$  spins  $j = n/2$ ,  $n = 1, 2, \dots, k-1$ . Of course, we can also apply the supercharges with the opposite spacetime chirality and construct bosons  $B^{a\bar{a}}$

<sup>9</sup>Note that this is consistent with (2.20) since we have to replace  $k \rightarrow k+2$  there to account for the difference between the full and bosonic  $SL(2)$  level.

which are chiral under both the left moving and the right moving spacetime superconformal algebras.

From the general representation theory of  $N = 4$  SCFT, in a unitary theory with  $c = 6kp$  we expect to find chiral operators in small representations with  $SU(2)$  spins  $j \leq kp/2$ . The states with  $k/2 \leq j \leq kp/2$  correspond in string theory to multiparticle states. For finite  $p$  the spectrum of multiparticle chiral states is truncated at  $j = kp/2$ . This is reminiscent of a similar phenomenon in WZW theory. Classically, the WZW Lagrangian describes an infinite number of primaries of  $\hat{G}$ , while quantum effects restrict the possible representations, *e.g.* in the case of  $G = SU(2)$  to  $j \leq k/2$ . Like here, the simplest way to see this restriction is to impose unitarity of the quantum theory. The truncation of the spectrum of multi-particle chiral operators in string theory on  $\mathcal{M}$  has been recently discussed in [27].

## 2) The Ramond Sector of the Spacetime Theory

Having understood the string vacuum corresponding to the NS sector of the spacetime SCFT we next turn to the Ramond vacuum. In this vacuum we expect to find integer modded spacetime supercharges  $Q_n^i, \bar{Q}_m^j$  with  $i, j = 1, 2$  and  $n, m \in \mathbb{Z}$ , satisfying the algebra (3.15) (and a similar structure from the other spacetime chirality).

Since the Euclidean  $AdS_3$  space (2.2) is simply connected, it has only one spin structure. Spinors do not change sign when transported around any point, say  $\gamma = 0$ , with any  $\phi$  (in the notation (2.5) – (2.8)). The change of variables (2.7) leads to a change of sign when spinors are transported around  $\gamma = 0$ , *i.e.* under  $\theta \rightarrow \theta + 2\pi$ . In terms of a SCFT on the boundary, string theory on  $AdS_3$  thus corresponds to the NS sector. If we want to describe the R sector we need to introduce two spin fields in the boundary field theory. We can put them at  $\gamma = 0$  and  $\gamma = \infty$ , *i.e.* at  $\tau = \pm\infty$ . These two points on the boundary can be connected by a line through the bulk. The line going through the bulk is the (analytic continuation to Euclidean space of the) worldline of an  $M = J = 0$  black hole. With this line omitted from the space, the latter is no longer simply connected and there can be a non-trivial spin structure. In the R sector, fermions are antiperiodic under  $\gamma \rightarrow e^{2\pi i}\gamma$ .

To construct the Ramond sector of the spacetime SCFT using worldsheet methods, we use the isomorphism (a.k.a. spectral flow) [28] of the NS and R  $N = 4$  superalgebras (3.15). Given generators that satisfy the NS algebra (3.15), we can define a one parameter set of algebras labeled by a variable  $\eta$ , which for  $\eta = 1$  (say) goes over to the NS algebra, and for  $\eta = 0$  to the

Ramond one. Some of the generators in (3.15) have  $\eta$  dependent modes. If we denote the generators of the algebras by  $L_n(\eta)$ , *etc.*, the generators of the algebra at  $\eta$  are given in terms of the  $\eta = 0$  (Ramond) generators by [28]:

$$\begin{aligned} T_n^3(\eta) &= T_n^3(0) - \frac{\eta kp}{2} \delta_{n,0}; & T_{n\pm\eta}^\pm(\eta) &= T_n^\pm(0) \\ Q_{n+\frac{\eta}{2}}^1(\eta) &= Q_n^1(0); & Q_{n-\frac{\eta}{2}}^2(\eta) &= Q_n^2(0) \\ L_n(\eta) &= L_n(0) - \eta T_n^3(0) + \eta^2 \frac{kp}{4} \delta_{n,0} \end{aligned} \quad (3.19)$$

One can verify that the generators (3.19) indeed satisfy the  $N = 4$  superconformal algebra (3.15).

Therefore, the operators that we have constructed before, (3.4), (3.5), (3.12) – (3.14), that were interpreted as generating an NS  $N = 4$  superalgebra (3.15), can be reinterpreted as generators of the Ramond superalgebra using the dictionary (3.19) with  $\eta = 1$ . Denoting the charges which generate the Ramond superalgebra by  $\tilde{L}_n$ ,  $\tilde{T}_n^a$ ,  $\tilde{Q}_n$ , *etc.*, we have for example:

$$\begin{aligned} \tilde{T}_n^3 &= T_n^3 + \frac{kp}{2} \delta_{n,0} \\ \tilde{T}_n^+ &= T_{n+1}^+; & \tilde{T}_n^- &= T_{n-1}^- \\ \tilde{L}_n &= L_n + T_n^3 + \frac{kp}{4} \delta_{n,0} \end{aligned} \quad (3.20)$$

The shifts in  $T_0^3$  and  $L_0$  (first and third lines of (3.20)) are due to the fact that the Hilbert space also transforms non-trivially under spectral flow. For example, the NS vacuum, which is annihilated by  $L_0$ ,  $T_0^3$ , is mapped to a Ramond ground state with  $\tilde{L}_0 = kp/4 (= c/24)$  and the largest possible  $SU(2)$  charge,  $\tilde{T}_0^3 = kp/2$ .

The excitations of the Ramond ground states are given, as before, by vertex operators such as (3.18). Using the redefinition of the Virasoro generators (3.20), as well as the commutation relations (2.49), (2.57), one finds that physical operators such as  $B^{i\bar{i}}(j; m, m', \bar{m}, \bar{m}')$  (3.18) satisfy the commutation relations

$$[\tilde{L}_n, B^{i\bar{i}}(j; m, m', \bar{m}, \bar{m}')] = (nj - (m - m')) B^{i\bar{i}}(j; n + m, m', \bar{m}, \bar{m}') \quad (3.21)$$

Comparing to (2.51) we see that the fields still have the same scaling dimensions  $h = \bar{h} = j + 1$ , but their modes are shifted by  $m'$  (the  $SU(2)$  charge  $T_0^3$ ). The chiral operators  $B^{a\bar{a}}(j + \frac{1}{2}; r, r', \bar{r}, \bar{r}')$  with  $j \in Z + \frac{1}{2}$  acquire zero modes, corresponding to  $r = r'$  (and thus  $\tilde{L}_0 - kp/4 = 0$ ) and/or  $\bar{r} = \bar{r}'$ . Therefore, the Ramond vacuum is highly degenerate.

## 4 Applications of String Theory on $\mathcal{M}$

In this section we explain the relation of string theory on  $AdS_3$  discussed here to other problems of recent interest.

### 4.1 Relation to the Theory of the NS Fivebrane

Callan, Harvey and Strominger (CHS) [29] found the classical supergravity fields around  $k$   $NS5$ -branes. One can wrap the fivebranes on a four-torus of arbitrary volume<sup>10</sup>  $vl_s^4$ , parametrized by the coordinates  $x^i$ ,  $i = 1, 2, 3, 4$ ; the fivebranes then become  $k$  strings whose worldsheet is in the  $(\gamma, \bar{\gamma})$  plane. One can extend the CHS solution by adding  $p$  fundamental strings (which are “smeared over the four-torus”) parallel to the fivebranes. The supergravity solution corresponding to  $p$  fundamental strings (and  $k = 0$ ) was found in [30]; the solution with general  $p$  and  $k$  was found in [31]. The dilaton, NS  $B_{\mu\nu}$  field and metric corresponding to this collection of fivebranes and strings are:

$$\begin{aligned} \frac{1}{g_{eff}^2(r)} &= e^{-2\Phi} = \frac{1}{g^2} f_5^{-1} f_1 \\ H &= 2ik\epsilon_3 + \frac{2ig^2p}{v} f_5 f_1^{-1} *_6 \epsilon_3 \\ ds^2 &= f_1^{-1} d\gamma d\bar{\gamma} + dx_i dx^i + f_5 (dr^2 + r^2 d\Omega_3^2) \end{aligned} \quad (4.1)$$

The two contributions to the Kalb-Ramond field in the second line of (4.1) correspond to the  $k$  fivebranes and  $p$  strings, respectively.  $g$  is the (arbitrary) string coupling at infinity,  $*_6$  is the Hodge dual in the six dimensions  $\gamma, \bar{\gamma}, r, \Omega_3$ , and

$$\begin{aligned} f_1 &= 1 + \frac{g^2 l_s^2 p}{vr^2} \\ f_5 &= 1 + \frac{l_s^2 k}{r^2}. \end{aligned} \quad (4.2)$$

The first line of eq. (4.2) takes into account the smearing of the fundamental string charge over the four-torus. It is valid (for a torus which is roughly square) for  $r \gg v^{\frac{1}{4}} l_s$ .

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<sup>10</sup>We will mostly ignore other moduli of the torus, and possible background RR fields. The full moduli space will be discussed in section 4.3.

Note that in the classical limit,  $g \rightarrow 0$ , the solution goes over to that of CHS [29]. In this limit the  $k$   $NS5$ -branes affect the background fields because they are heavy (their tension scales like  $\frac{1}{g^2}$ ), while the effect of the fundamental strings (whose tension is of order one) goes to zero. If we want to retain the effect of the fundamental strings in the classical limit, we have to take  $p \rightarrow \infty$  with  $g^2 p$  fixed. Intuitively, the mass of the  $p$  strings then scales like  $\frac{1}{g^2}$  and, therefore, they can affect the background.

We can now study the near horizon geometry of (4.1), which corresponds to distances  $r$  which satisfy:

$$\frac{g^2 l_s^2 p}{v r^2} \gg 1; \quad \frac{l_s^2 k}{r^2} \gg 1 \quad (4.3)$$

Since the validity of (4.2) requires  $r \gg v^{\frac{1}{4}} l_s$ , we conclude that to study the near horizon region (4.3) in a weakly coupled theory we must have  $p \gg v^{\frac{3}{2}}, k \gg v^{\frac{1}{2}}$ . In the limit (4.3) the configuration (4.1) turns to:

$$\begin{aligned} \frac{1}{g_0^2(r)} &= e^{-2\Phi_0} = \frac{p}{vk} \\ H_0 &= 2ik(\epsilon_3 + * \epsilon_3) \\ ds_0^2 &= k \frac{r^2}{l_s^2} d\gamma d\bar{\gamma} + k l_s^2 \left( \frac{1}{r^2} dr^2 + d\Omega_3^2 \right) + dx_i dx^i \end{aligned} \quad (4.4)$$

where we have rescaled  $\gamma$  and  $\bar{\gamma}$ . We would like to make a few comments about this solution:

(a) Unlike the solution (4.1), here the string coupling is a constant independent of  $r$ . Its value is independent of the coupling at infinity,  $g$ . Thus the dilaton is a fixed scalar.

(b) The number of strings  $p$  enters only in the string coupling constant. Furthermore, the string coupling depends on  $p$  in exactly the way that was needed above in order for the fundamental strings to affect the background, *i.e.*  $g^2 \sim \frac{1}{p}$ . The six dimensional string coupling

$$\frac{1}{g_6^2} = \frac{v}{g_0^2} = \frac{p}{k} \quad (4.5)$$

is independent of  $v$ .

(c) The moduli space of classical solutions such as (4.1) is subject to some stringy identifications. For example, the action of T-duality on the

four-torus includes the transformation  $v \rightarrow 1/v$ . Therefore, we can limit ourselves to  $v \geq 1$ . (d) The configuration (4.4) is precisely the one we studied<sup>11</sup> in section 3. Here we see how it is embedded in a CHS-like solution (4.1) which is asymptotically flat. This provides further evidence for the interpretation of  $p$  defined in (2.31) as the number of fundamental strings in the background. (e) It is important to identify the range of validity of the analysis in section 3. The worldsheet theory is weakly coupled for  $k \gg 1$ . However, most of our analysis in section 3 treats the CFT exactly and, therefore, does not depend on this condition. For the strings to be weakly coupled we need  $g_0^2(r) = \frac{vk}{p} \ll 1$ .

## 4.2 Relation to the D1/D5 System

The field configuration corresponding to  $k$   $NS5$ -branes and  $p$  fundamental strings (4.1) is mapped under S-duality into that describing  $k$   $D5$ -branes and  $p$   $D$ -strings ([33]):

$$\begin{aligned} \frac{1}{\widehat{g}_{eff}^2(r)} &= g_{eff}^2(r) = e^{-2\widehat{\Phi}} = g^2 f_1^{-1} f_5 \\ \widehat{H} &= H = 2ik\epsilon_3 + \frac{2ipg^2}{v} f_5 f_1^{-1} *_6 \epsilon_3 \\ \widehat{ds}^2 &= e^{-\Phi} ds^2 = \frac{1}{g} f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} d\gamma d\bar{\gamma} + \frac{1}{g} f_1^{\frac{1}{2}} f_5^{-\frac{1}{2}} dx_i dx^i \\ &\quad + \frac{1}{g} f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dr^2 + r^2 d\Omega_3^2). \end{aligned} \tag{4.6}$$

Its near horizon limit is

$$\begin{aligned} \frac{1}{\widehat{g}_0^2(r)} &= e^{-2\widehat{\Phi}_0} = \frac{kv}{p} \\ \widehat{H}_0 &= 2ik(\epsilon_3 + *_6 \epsilon_3) \\ \widehat{ds}_0^2 &= \frac{r^2}{l_s^2} \sqrt{\frac{v}{kp}} d\gamma d\bar{\gamma} + \sqrt{\frac{p}{kv}} dx_i dx^i + \sqrt{\frac{kp}{v}} l_s^2 \left( \frac{1}{r^2} dr^2 + d\Omega_3^2 \right). \end{aligned} \tag{4.7}$$

A few comments are in order:

(a) After rescaling all the coordinates in (4.6) by  $g^{\frac{1}{2}}$  we find the D1/D5 solution of [33, 34] with the identifications  $Q_5 = k$  and  $Q_1 = p$ .

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<sup>11</sup>The role of  $SL(2) \times SU(2)$  in describing the near-horizon geometry of (4.1) was pointed out in [31, 32].

(b) The direct relation between (4.4) and (4.7) was obtained in [27]. Our minor addition to their calculation is the observation that S-duality commutes with taking the near horizon limit of (4.1).

(c) In the  $D$ -brane picture the volume of the four-torus and the six and ten dimensional string couplings are

$$\begin{aligned}\hat{v} &= \frac{p}{k} \\ \frac{1}{\hat{g}_0^2} &= \frac{kv}{p} \\ \frac{1}{\hat{g}_6^2} &= \frac{\hat{v}}{\hat{g}_0^2} = v.\end{aligned}\tag{4.8}$$

The free continuous parameter of the  $NS$  problem,  $v$ , is now interpreted as the six dimensional coupling constant, while the volume of the four torus  $\hat{v}$  is fixed in terms of  $p$  and  $k$ .

(d) The parameter space of the problem is subject to discrete identifications. For example, T-duality includes the transformation  $p \leftrightarrow k$ . Therefore, we can limit ourselves to  $p \geq k$ .

(e) String loop corrections are small in the  $D$ -brane picture when  $\hat{g}_0^2 = \frac{p}{kv} \ll 1$ . The worldsheet theory is weakly coupled (the low energy supergravity is a good approximation) when also  $\frac{pk}{v} \gg 1$ . Clearly, there is no situation where both the  $NS$  and  $D$  descriptions are simultaneously weakly coupled.

Maldacena [8] proposed that string theory in the near-horizon background (4.6) describes in spacetime the CFT obtained by studying the extreme IR dynamics of  $p$   $D$ -strings and  $k$   $D5$ -branes (see also [27],[35]-[40] for more recent work on this correspondence). This system corresponds to a  $(4,4)$  SCFT with central charge<sup>12</sup>  $c = 6kp$ . It has two decoupled sectors which are sometimes referred to as the CFT's of the Coulomb and Higgs branches. The former is obtained by quantizing the motion of the  $D$ -strings away from the  $D5$ -branes in the directions transverse to both; the latter corresponds to the  $D$ -strings being absorbed by the  $D5$ -branes, becoming  $p$  small instantons in  $U(k)$  and growing to finite size instantons.

The theory of the Higgs branch (which we will refer to as the  $D1/D5$  SCFT) is of interest for applications, such as the calculation of the Bekenstein-Hawking entropy of five dimensional black holes [5], as well as three dimen-

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<sup>12</sup>Actually,  $c = 6(kp + 1)$ , but a  $c = 6$  part of the theory is free and decoupled; it plays no role in the subsequent discussion and thus will be ignored.



sional ones [4], the matrix description of certain non-critical string theories and the  $(2,0)$  theory [41] *etc.*, and is the one that, according to [8], is described by string theory on (4.6).

As we saw here, there are actually two weakly coupled descriptions of the  $D1/D5$  SCFT, each useful in a different region in parameter space. The theory depends on the discrete parameters  $p, k$ , and on continuous moduli like  $v$ . This parameter space is subject to discrete identifications such as  $v \rightarrow \frac{1}{v}$ ,  $p \leftrightarrow k$ , *etc.* These identifications can be used to restrict to  $p \geq k$ ,  $v \geq 1$ . For some range of parameters ( $p \gg vk$ ) the  $NS$  description is weakly coupled and useful. For  $p \ll vk$  the  $D$  description is weakly coupled but it requires an understanding of string theory in RR background fields. If also  $pk \gg v$ , one can use supergravity to understand many aspects of the physics. Most of the existing work on this system is in this regime. For generic  $p, k, v$  the theory is strongly coupled in all of the above descriptions.

### 4.3 Comparison of String Theory on $\mathcal{M}$ and the $\sigma$ Model on $T^{4kp}/S_{kp}$

In the previous section we described some of the features of the spacetime SCFT corresponding to fundamental string theory on  $\mathcal{M}$  (1.2); as explained above, it is the same SCFT as the  $D1/D5$  system. This SCFT is expected to be equivalent to a  $(4,4)$   $\sigma$  model on the target space

$$\mathcal{P} = (\tilde{T}^4)^{kp}/S_{kp} \quad (4.9)$$

at some value of the moduli which resolve the orbifold singularity [42, 5, 43] (see also [44]).  $\tilde{T}^4$  must be distinguished from the four-torus  $T^4$  on which the fundamental strings propagate. In this subsection we will comment on the relation between the spacetime SCFT corresponding to string theory on  $\mathcal{M}$  and the  $\sigma$  model on  $\mathcal{P}$  (4.9); a precise study of the relation is left for future work. Since the NS and R sectors of any  $(4,4)$  SCFT are equivalent by spectral flow (3.19), we will restrict our comments to the NS sector of the spacetime theory.

First, note that the two theories have the same chiral algebra. In addition to the  $(4,4)$  superconformal algebra they also share a  $U(1)^4$  affine Lie superalgebra (actually two copies of  $U(1)^4$  from the two chiralities). In string theory on  $\mathcal{M}$  this symmetry is generated by the vertex operators  $\oint e^{-\phi} \lambda^i \gamma^m$  (3.16). In the  $\sigma$  model on  $\mathcal{P}$  there is a “diagonal”  $\tilde{T}^4$  which is invariant under the orbifold action, and is insensitive to the blow up deformations; the symmetry comes from SCFT on that  $\tilde{T}^4$ .

Furthermore, like the  $\sigma$  model on  $\mathcal{P}$ , the spacetime SCFT obtained in string theory on  $\mathcal{M}$  appears to be unitary (see [15] and references therein). In our description the no ghost theorem follows from the explicit construction of the Hilbert space of the theory. To complete the proof of unitarity one needs to show that the set of states satisfying the bound (2.20) is closed under OPE.

The moduli spaces of the two CFT's agree as well. The moduli space of  $(4, 4)$  superconformal  $\sigma$  models on  $\mathcal{P}$  is twenty dimensional. Sixteen of the moduli correspond to the metric  $G_{ij}$  and antisymmetric tensor field  $B_{ij}$  on  $\tilde{T}^4$ . The remaining four moduli are certain blowing up modes of  $\mathcal{P}$ . This space has singular subspaces fixed under various elements of  $S_{kp}$ . A standard CFT analysis shows that the only element of  $S_{kp}$  whose fixed point set can be blown up by a marginal operator is the  $Z_2$  that exchanges two  $\tilde{T}^4$ 's. All other blowing up operators are irrelevant.

The fixed point set of the  $Z_2 \in S_{kp}$  is a connected  $4(kp - 1)$  dimensional manifold. The marginal operators that blow up the  $Z_2$  singularity have vanishing momentum along this set (higher momentum leads to higher scaling dimension). They are isomorphic to the blowing up modes of a *single*  $Z_2$  singularity in CFT on  $T^4/Z_2$ . Therefore these blowing up modes extend the Narain moduli space  $SO(4, 4)/SO(4) \times SO(4)$ .  $(4, 4)$  supersymmetry guarantees [45, 46] that the space is locally  $SO(5, 4)/SO(5) \times SO(4)$ . The full moduli space is:

$$\mathcal{H} \backslash SO(5, 4)/SO(5) \times SO(4) \quad (4.10)$$

$\mathcal{H}$  is a discrete duality group that determines the global structure of the moduli space. It contains the T-duality group  $SO(4, 4; Z)$ ; a natural guess is  $\mathcal{H} = SO(5, 4; Z)$ .

In string theory on  $\mathcal{M}$  one finds the same twenty moduli; the sixteen moduli  $G_{ij}$ ,  $B_{ij}$  correspond to the operators (3.18) with  $j = m = \bar{m} = 0$ . Changing these spacetime moduli corresponds to adding to the worldsheet Lagrangian the term  $(G_{ij} + B_{ij}) \int d^2z \partial Y^i \bar{\partial} Y^j$ . Thus, the size and shape of  $\tilde{T}^4$  is directly related to that of  $T^4$ .

The four remaining moduli are related to the chiral fields described by the vertex operators:

$$e^{-\phi - \bar{\phi}} (\psi^3 - \frac{1}{2} \gamma \psi^- - \frac{1}{2} \gamma^{-1} \psi^+) (\bar{\psi}^3 - \frac{1}{2} \bar{\gamma} \bar{\psi}^- - \frac{1}{2} \bar{\gamma}^{-1} \bar{\psi}^+) V_{jm\bar{m}} V'_{j\bar{m}'\bar{m}'} \quad (4.11)$$

One can show that (4.11) corresponds to a chiral primary of the  $(4, 4)$  superconformal symmetry (3.15), with  $h = \bar{h} = j = \bar{j}$ , for all  $0 < j \in Z/2$ .

The highest components of the field (4.11) with  $j = 1/2$  are four singlets under  $SU(2)_R \times SU(2)_L$  which are truly marginal in spacetime. They are described on the worldsheet by RR vertex operators.

The fact that the moduli space of string theory on  $\mathcal{M}$  is given by (4.10) can be understood by noting that type II string theory on  $T^4$  (or M-theory on  $T^5$ ) has the moduli space of vacua

$$SO(5, 5; Z) \backslash SO(5, 5) / SO(5) \times SO(5) \quad (4.12)$$

Compactifying the remaining six dimensions on  $AdS_3 \times S^3$  gives a mass to five of the twenty five scalars parametrizing (4.12) and restricts the moduli space to (4.10). The discrete duality group  $SO(5, 5; Z)$  is reduced as well. Thinking of  $\mathcal{M}$  as the near horizon geometry of a system of  $NS5$ -branes and fundamental strings (as in section 4.1), the U-duality group is the subgroup of  $SO(5, 5; Z)$  that leaves the  $k$  fivebranes and  $p$  strings invariant. This clearly includes the T-duality group  $SO(4, 4; Z)$ . It is possible that the discrete symmetry of the near horizon theory is larger, as mentioned above.

The connection between the  $T^4$  and  $\tilde{T}^4$  parameters can be made more precise as follows. As an example, take  $T^4$  to be a square torus with sides  $R$  and volume  $v = R^4$ , with  $1 \ll v \ll p/k$  such that the  $T^4$  is large but the description of section 3 is still weakly coupled. To calculate the volume of  $\tilde{T}^4$  we would like to consider string states (3.6) which carry momentum along the  $T^4$ . These are states of the general form  $\exp(-\phi - \bar{\phi}) \exp(i\vec{p} \cdot \vec{Y}) W_N V_{jm\bar{m}}$ , with  $W_N$  an operator constructed out of the non-zero modes of the worldsheet fields as in (3.6). The components of  $\vec{p}$  are quantized in integer multiples of  $1/R$ . We would like to compute the spacetime scaling dimensions of the corresponding operators and, in particular, the spacing between subsequent momentum modes.

Substituting into (3.7) we have:

$$N - \frac{j(j+1)}{k} + \frac{1}{2} |\vec{p}|^2 = \frac{1}{2} \quad (4.13)$$

Solving for  $j$  we find:

$$j = \frac{1}{2} \left( -1 + \sqrt{1 + 4k(N - \frac{1}{2}) + 2k|\vec{p}|^2} \right) \quad (4.14)$$

We are interested in the dependence of the spectrum on  $\vec{p}$  for small  $|\vec{p}|$ . To leading order we have

$$j = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4k(N - \frac{1}{2})} + \frac{1}{2} \frac{k|\vec{p}|^2}{\sqrt{1 + 4k(N - \frac{1}{2})}} + \dots \quad (4.15)$$

Thus the spacetime scaling dimension is proportional to  $|\vec{p}|^2$ , and therefore the volume of  $\tilde{T}^4$  is proportional to  $v$ . The precise coefficient of proportionality depends on  $N$ . This is probably due to the presence of the blowing up modes and requires more study<sup>13</sup>.

We see that the volume  $v$  discussed in section 4.1 is indeed the modulus controlling the volume of the torus  $\tilde{T}^4$  (4.9). This identification was made in the region of validity of the description of section 3,  $v \ll p/k$ . As discussed in the previous subsections, when  $v$  grows and eventually becomes much larger than  $p/k$ , the description of the system given in section 3 becomes strongly coupled and we have to pass to the  $D$  picture of subsection 4.2. There, the volume of  $T^4$ ,  $\hat{v}$ , is fixed at  $p/k$  and the parameter  $v$  corresponds to another modulus of the SCFT on  $\mathcal{P}$ , perhaps one of the four blowing up modes.

Next we turn to the chiral fields of both theories. Consider first the  $\sigma$  model on  $\mathcal{P}$ . Denote the  $\sigma$  model fields by  $Z_A^i$ , with  $i = 1, \dots, 4$  a vector index in the  $\tilde{T}^4$ , and  $A = 1, \dots, kp$ . The (left and right moving) fermion superpartners of  $Z_A^i$  will be denoted by  $\Psi_A^{a\alpha}$ ,  $\bar{\Psi}_A^{b\beta}$ , respectively, where  $a, b$  are spinor indices of the  $SO(4)$  acting on the  $\tilde{T}^4$  and  $\alpha, \beta$  are spinor indices in  $SU(2)$ . The orbifold in (4.9) acts on the index  $A$ ; the SCFT has an untwisted sector and various twisted sectors. We will focus on the untwisted sector; the twisted sectors can be discussed analogously.

The basic chiral operators of dimension  $(h, \bar{h}) = (1/2, 0)$  and  $(0, 1/2)$  in the untwisted sector are  $\sum_A \Psi_A^{a\alpha}$ ,  $\sum_A \bar{\Psi}_A^{b\beta}$ . The upper components of these operators are  $\sum_A \partial Z_A^i$  and  $\sum_A \bar{\partial} Z_A^i$ , respectively. These chiral superfields live in the decoupled  $\tilde{T}^4$  sector and generate the  $U(1)^4$  affine Lie superalgebra which we have identified in the string context before.

The first non-trivial chiral operators have dimension  $(1/2, 1/2)$ , and are given by<sup>14</sup>:

$$\Psi_A^{a\alpha} \bar{\Psi}_A^{b\beta} \quad (4.16)$$

The operators (4.16) have spin  $(1/2, 1/2)$  under  $SU(2)_L \times SU(2)_R$  and transform as  $(\frac{1}{2}, \frac{1}{2})$  under  $SO(4)$ . More generally, we can define chiral operators with  $(h, \bar{h}) = (l/2, l/2)$ :

$$\Psi_{A_1}^{a\alpha_1} \Psi_{A_2}^{a_2\alpha_2} \Psi_{A_3}^{a_3\alpha_3} \dots \Psi_{A_l}^{a_l\alpha_l} \bar{\Psi}_{A_1}^{b\beta_1} \bar{\Psi}_{A_2}^{b_2\beta_2} \bar{\Psi}_{A_3}^{b_3\beta_3} \dots \bar{\Psi}_{A_l}^{b_l\beta_l} \quad (4.17)$$

<sup>13</sup>It is interesting to note that if we take  $N$  to have the largest value compatible with the unitarity bound (2.20),  $j(\vec{p} = 0) = (k-1)/2$ , we find from (4.15) that  $j(\vec{p}) = j(0) + |\vec{p}|^2/2 + \dots$  which seems to imply that  $\tilde{T}^4$  has the same volume as  $T^4$ .

<sup>14</sup>Here and below a sum over repeated indices is implied.

symmetrized over  $(\alpha_1, \dots, \alpha_l)$  and  $(\beta_1, \dots, \beta_l)$ . The operators (4.17) have  $SU(2)_L \times SU(2)_R$  spin  $(l/2, l/2)$ . We have summed over  $a_2, \dots, a_l$  in (4.17) since it is sufficient to identify operators corresponding to low representations of  $SO(4)$  in the SCFT on  $\mathcal{P}$  and string theory on  $\mathcal{M}$ . Higher representations of  $SO(4)$  can then be obtained by acting with the affine  $U(1)^4$  Lie superalgebra, which has been identified in both theories. Note also that, due to the Fermi statistics of the  $\Psi$ 's, one must have  $l \leq kp$ .

The corresponding chiral operators in string theory on  $\mathcal{M}$  are the lowest components of superfields whose highest components are the scalars with  $(-1, -1)$  picture vertex operators (see (3.18))

$$e^{-\phi-\bar{\phi}} \lambda^i \bar{\lambda}^{\bar{i}} V_{jm\bar{m}} V'_{j'm'\bar{m}'} \quad (4.18)$$

These fields have  $SU(2)_R \times SU(2)_L$  spin  $(\tilde{j}, \tilde{j})$  and spacetime scaling dimensions  $(h, \bar{h}) = (\tilde{j}, \tilde{j})$ , with  $\tilde{j} \equiv j + \frac{1}{2}$ . These states are in one to one correspondence with the chiral primaries (4.17), with  $j = l/2$ . Note that while  $l$  in (4.17) is bounded by  $kp$ , the string construction only gives rise to operators with  $l \leq k - 2$ , since unitarity of the worldsheet  $SU(2)$  affine Lie algebra requires  $j \leq (k - 2)/2$ . The remaining states are supposed to arise from multiparticle states or bound states at threshold; it would be nice to understand precisely how that happens.

In the orbifold limit the  $\sigma$  model on the space  $\mathcal{P}$  has a large chiral algebra of operators with  $L_0 = m \in Z/2$ ,  $\bar{L}_0 = 0$ . For example, one can consider products of the  $N = 4$  superconformal generators of each  $\tilde{T}^4$  in (4.9), symmetrized to impose the permutation symmetry [38]. One can ask what happens to these states with scaling dimension  $(h, \bar{h}) = (m, 0)$  when one turns on the blowing up moduli  $\alpha$ . Typically, one expects the dimensions  $h$  and  $\bar{h}$  to shift, while preserving the spin  $h - \bar{h}$ . For small  $\alpha$ , the resulting  $\bar{h}(\alpha)$  will be small and, similarly,  $h(\alpha)$  will be approximately equal to its orbifold limit value.

We have seen that operators with spin larger than two correspond in string theory on  $\mathcal{M}$  to massive string modes, such as (2.53). If we take  $k$  to be large to make supergravity reliable, operators with non-zero spin (on the worldsheet and in spacetime) have  $\bar{N} \geq 1$  and therefore due to (3.7) large  $j$  and scaling dimension in spacetime,  $h, \bar{h} \sim \sqrt{k\bar{N}}$ , (2.54). Therefore, the spacetime SCFT obtained from string theory on  $\mathcal{M}$  has the peculiar property that for large  $k$  and  $p$  states<sup>15</sup> with  $2 < |L_0 - \bar{L}_0| \ll \sqrt{k}$  must have  $L_0, \bar{L}_0 \geq \sqrt{k}$ . In particular, in the language of SCFT on  $\mathcal{P}$ , operators

<sup>15</sup>Of course, here we are referring to single particle states. Multi-particle states with spin higher than two but scaling dimensions much smaller than  $\sqrt{k}$  can and do exist.

which in the orbifold limit have scaling dimensions  $(h, \bar{h}) = (m, 0)$  with  $1 \ll m \ll \sqrt{k}$  have after the blow up  $h, \bar{h} \gg m$ , but  $h - \bar{h} = m$ . This means that the deformation from the orbifold limit is *large*. This is consistent with expectations based on six dimensional supergravity in which all the light states have small spin [38]. The remaining states have large dimension  $\geq \sqrt{k}$  and hence are stringy in nature.

#### 4.4 BTZ Black Holes and Fundamental String States

Since string theory on  $\mathcal{M}$  reduces to Einstein gravity in the low energy limit, we know that it should contain BTZ black holes [2]

which are parametrized by their mass  $M$  and angular momentum  $J$ . The Lorentzian signature black hole metric is given by:

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \\ N^2 &= (r/l)^2 - 8l_p M + (4l_p J/r)^2 \\ N^\phi &= -4l_p J/r^2 \end{aligned} \tag{4.19}$$

One can think of the black holes (4.19) as excitations of the vacuum described by the  $M = J = 0$  black hole, *i.e.* the Ramond vacuum of the spacetime SCFT [2, 4]. The mass and spin of the black holes are given in terms of the Virasoro generators in the Ramond sector  $L_0, \bar{L}_0$  by:

$$Ml = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0 \tag{4.20}$$

where we have defined  $L_0$  such that it vanishes on the Ramond vacuum (by subtracting  $c/24 = kp/4$  from  $\tilde{L}_0$  (3.20)). Some of the solutions (4.19), namely those with  $J = \pm Ml$ , preserve half of the supersymmetries of the  $M = 0$  vacuum [47]. In the spacetime SCFT these are states with either  $L_0 = 0$  or  $\bar{L}_0 = 0$ .

Note that the correspondence (4.20) means that from the spacetime gravity point of view the lowest lying BTZ black holes are very light. Since the low lying states in the spacetime CFT have  $L_0 \sim 1$ , the lowest mass BTZ black holes have  $M \sim 1/l$  and are much lighter than the natural mass scale of the theory  $1/l_p$ . In fact, comparing (2.40) and (2.44) we see that  $M \sim 1/(l_p kp)$ . Note also that the scales  $l$  and  $l_p$  of our system depend differently on the parameters  $k, p, v$  of the model in the two different regions corresponding to weak  $NS$  and  $D$  coupling discussed in the previous

subsections:

$$\begin{aligned}
 D: \quad l &= 2\pi l_s \left( \frac{kp}{v} \right)^{\frac{1}{4}}, \quad l_p = \frac{\pi}{2} l_s (kp)^{-\frac{3}{4}} v^{-\frac{1}{4}} \\
 NS: \quad l &= l_s \sqrt{k}, \quad l_p = \frac{l_s}{4p\sqrt{k}}
 \end{aligned}
 \tag{4.21}$$

The Bekenstein-Hawking entropy of BTZ black holes with mass  $M$  and angular momentum  $J$  has the usual form in terms of the area  $A$  of the event horizon:

$$\begin{aligned}
 S = \frac{A}{4l_p} &= \pi \sqrt{\frac{l(lM + J)}{2G_3}} + \pi \sqrt{\frac{l(lM - J)}{2G_3}} \\
 &= 2\pi \sqrt{kpL_0} + 2\pi \sqrt{kp\bar{L}_0}
 \end{aligned}
 \tag{4.22}$$

How can one describe BTZ black holes in the framework of our previous analysis? The states we are looking for should have finite masses (4.20) in the weak coupling limit  $p \gg 1$ ,  $M \sim L_0/l_s\sqrt{k}$ . Thus, we would like to identify them with fundamental string states. By the analysis of the previous sections, we can associate to every perturbative string state a value of  $L_0$  and  $\bar{L}_0$  and, therefore, (4.20) a mass and spin.

Of course, string states should only be thought of as black holes if their horizon area is larger than their size, which is of order  $l_s$ . In fact, while one can construct large black holes with  $A \gg l_s, l_p$  from multi-particle perturbative string states at weak string coupling, the perturbative description is *not* valid for such black holes. Indeed, substituting (2.46) in (4.22) one finds that  $A \sim l_s \sqrt{L_0/p}$ ; thus large black holes necessarily have  $L_0 \gg p$ . The corresponding energies are of order  $1/g_6^2$  (with the six dimensional string coupling  $g_6^2 \propto k/p$ ), and the perturbative string picture is not expected to be reliable at such high energies. It is nevertheless possible that one can use the large symmetry of this system to obtain useful information about the physics of large black holes.

BTZ black holes with  $Ml = \pm J$  (*i.e.* vanishing  $L_0$  or  $\bar{L}_0$  (4.20)) correspond in string theory to multi-particle states constructed out of the chiral algebra modes  $L_{-n}$ ,  $T_{-n}^a$ ,  $\alpha_{-n}^i$ , *etc.* Most of the black holes correspond to massive string states, have non-zero  $L_0, \bar{L}_0$  and break the supersymmetry completely.

The above construction of BTZ black holes in string theory on  $\mathcal{M}$  allows one to compute the Bekenstein-Hawking entropy of these objects. Since

BTZ black holes are described in our framework as multiparticle states in the spacetime SCFT, we can apply the standard result from CFT [50], to compute the entropy  $S$ :

$$S = 2\pi\sqrt{\frac{cL_0}{6}} + 2\pi\sqrt{\frac{c\bar{L}_0}{6}} \quad (4.23)$$

The central charge of the spacetime theory is  $c = 6kp$  (2.40); thus (4.23) agrees with the form (4.22) we found by using the area formula before.

This argument is due to Strominger [4] (see also [53]). Our analysis supplements that of [4] in two respects:

(a) The formula (4.23) only applies to CFT's for which the lowest dimension operator has  $h = \bar{h} = 0$ . In the context of gravity on  $AdS_3$  it was applied to the CFT living on the boundary of  $AdS_3$ , but it was not clear whether this condition applies (see [35, 51] for recent discussions). In fact it has been argued that the boundary (S)CFT is a (super-)Liouville theory [52], for which  $c$  should be replaced [6, 7] by  $c_{\text{eff}} = 1$  in (4.23). Our string theory on  $\mathcal{M}$  is unitary and its lowest dimension operator has  $h = 0$  (the identity operator). Therefore, the conditions for applying (4.23) are satisfied here, at least for weak coupling (*i.e.* for large enough  $p$ ).

(b) We showed that the states contributing to the density of states (4.23) are fundamental string states, most of which are furthermore massive; therefore, one cannot reduce to supergravity without losing the microscopic interpretation of (4.22).

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## Appendix A. The Geometry of Lorentzian $AdS_3$

The Lorentzian signature version of  $AdS_3$  is obtained by analytically continuing the Euclidean version described by eq. (2.2). The inequivalent continuations correspond to replacing  $X_3 = iX_0$  and  $X_1 = iX_0$  in (2.2). Clearly



there should not be any difference between them. The first corresponds to setting  $\tau = it$  in (2.3) and (2.4). It leads to<sup>16</sup>

$$ds^2 = \frac{1}{1+r^2} dr^2 - (1+r^2) dt^2 + r^2 d\theta^2. \quad (\text{A.1})$$

The second corresponds to treating  $\gamma$  and  $\bar{\gamma}$  as two independent real coordinates and letting  $u = e^{-\phi}$  in (2.5) be both positive and negative (now  $u = 0$  is not a boundary of the space). The metric is

$$ds^2 = \frac{1}{u^2} du^2 + u^2 d\gamma d\bar{\gamma}. \quad (\text{A.2})$$

Because of the reality properties of  $\gamma$  and  $\bar{\gamma}$ , we can no longer use (2.7) to relate them. Instead, these two coordinate systems are related by the transformations

$$\begin{aligned} u &= \sqrt{1+r^2} \cos t + r \cos \theta \\ \gamma &= \frac{\sqrt{1+r^2} \sin t + r \sin \theta}{\sqrt{1+r^2} \cos t + r \cos \theta} \\ \bar{\gamma} &= \frac{\sqrt{1+r^2} \sin t + r \sin \theta}{\sqrt{1+r^2} \cos t + r \cos \theta} \end{aligned} \quad (\text{A.3})$$

and the inverse map

$$\begin{aligned} r^2 &= \frac{[u^2(\gamma\bar{\gamma} - 1) + 1]^2}{4u^2} + \frac{(\gamma + \bar{\gamma})^2 u^2}{4} = \frac{[u^2(\gamma\bar{\gamma} + 1) + 1]^2}{4u^2} \\ &\quad + \frac{(\gamma - \bar{\gamma})^2 u^2}{4} - 1 \\ \sin t &= \frac{u(\gamma - \bar{\gamma})}{2\sqrt{1+r^2}} \\ \sin \theta &= \frac{u(\gamma + \bar{\gamma})}{2r} \end{aligned} \quad (\text{A.4})$$

where in the last two expressions we use  $r$  from the first. Note that the expression for  $r^2$  is always non-negative and, therefore, the square root can be taken. Similarly, the two ways of writing  $r^2$  guarantee that  $\sin t$  and  $\sin \theta$  are in  $[-1, 1]$ . This shows that for  $t \in [0, 2\pi)$  and  $u \in (-\infty, \infty)$  the change of variables (A.3), (A.4) is one to one.

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<sup>16</sup>We set  $l = 1$  in this appendix.

The relation to the  $SL(2, R)$  group manifold is obtained by parametrizing it by the Gauss decomposition

$$g = \begin{pmatrix} 1 & \bar{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} \gamma\bar{\gamma}u + \frac{1}{u} & \bar{\gamma}u \\ \gamma u & u \end{pmatrix} \quad (\text{A.5})$$

where  $\gamma$  and  $\bar{\gamma}$  are two independent real numbers. The metric on the group is

$$ds^2 = \frac{1}{2} \text{Tr}(g^{-1}dg)^2 = \frac{1}{u^2}(du)^2 + u^2 d\gamma d\bar{\gamma} \quad (\text{A.6})$$

which is readily identified with (A.2). Here it is clear that the  $SL(2, R)$  group manifold is described by  $u \in (-\infty, \infty)$ .

For physics we would like the time  $t$  to be non-compact. Therefore, although in the original space there is a closed time-like curve corresponding to  $t \in [0, 2\pi)$ , we must consider the infinite cover of this space with  $t \in (-\infty, \infty)$ . Now we see that the transformations (A.3), (A.4) are no longer one to one. Points which differ by  $t \rightarrow t + 2\pi$  are mapped to the same  $u, \gamma, \bar{\gamma}$ .

## Appendix B. Twisted Strings on $\mathcal{M}$

In section 3 we discussed superstring theory on the manifold  $\mathcal{M}$  (1.2). Our construction of spacetime supercharges did not follow the usual route of identifying a global  $N = 2$  superconformal symmetry on the worldsheet and using the  $U(1)_R$  current inside that  $N = 2$  algebra to construct the spacetime supercharges in the standard way. In this appendix we will describe the string vacuum that is obtained by following the usual path. This vacuum seems to be related to the Ramond vacuum of the spacetime SCFT discussed above by twisting the spacetime supersymmetry and treating it as a BRST charge. This interpretation of the construction below is conjectural and requires a much better understanding.

To enhance the  $N = 1$  superconformal algebra (3.10) to  $N = 2$  we must find a  $U(1)_R$  current,  $J_{N=2}$ , which is part of the  $N = 2$  algebra. In our case this  $U(1)_R$  current can be chosen to be<sup>17</sup> [54]:

$$J_{N=2}(z) = -\frac{2(k+2)}{k^2}\psi^+\psi^- + \frac{2(k-2)}{k^2}\chi^+\chi^- + \frac{2}{k}j^3 - \frac{2}{k}k^3 + \frac{2}{k}\psi^3\chi^3 + i\lambda^1\lambda^2 + i\lambda^3\lambda^4 \quad (\text{B.1})$$

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<sup>17</sup>In this appendix we normalize  $\psi^+\psi^- = i\psi^1\psi^2$ , etc. This differs by a factor of two from the normalization used in the text.

(B.1) is unique up to global symmetries<sup>18</sup>. It can be obtained by decomposing

$$M \simeq \frac{SL(2, R)}{U(1)} \times \frac{SU(2)}{U(1)} \times U(1)^2 \times U(1)^4 \quad (\text{B.2})$$

Each of the factors in (B.2) has a natural complex structure; (B.1) (as well as the other  $N = 1$  superconformal generators (3.10)) can be written as a sum of the corresponding currents:

$$\begin{aligned} J_{N=2} &= J_{N=2}^{(1)} + J_{N=2}^{(2)} + J_{N=2}^{(3)} + J_{N=2}^{(4)} \\ J_{N=2}^{(1)} &= -\frac{2}{k} (\psi^+ \psi^- - J^3) \\ J_{N=2}^{(2)} &= \frac{2}{k} (\chi^+ \chi^- - K^3) \\ J_{N=2}^{(3)} &= \frac{2}{k} \psi^3 \chi^3 \\ J_{N=2}^{(4)} &= i\lambda^1 \lambda^2 + i\lambda^3 \lambda^4 \end{aligned} \quad (\text{B.3})$$

where  $J^3$  and  $K^3$  are the total  $SL(2, R)$  and  $SU(2)$  currents defined in (3.1). Note that the choice of the complex structure (B.1), (B.3) breaks  $SL(2, R) \times SU(2) \rightarrow U(1) \times U(1)$ . It is also useful for future purposes to note that the currents  $J^3$  and  $K^3$  have non-singular OPE's with the  $U(1)_R$  currents  $J_{N=2}^{(i)}$  (B.3).

$T$ ,  $G$ ,  $J_{N=2}$ , (3.10), (B.1) generate together an  $N = 2$  superconformal algebra. The supercurrent  $G$  (3.10) splits into two parts,  $G = G^+ + G^-$  with charges  $\pm 1$  under  $J$ . To construct the spacetime supercharges [19], we bosonize the  $U(1)_R$  current (B.1):

$$J_{N=2} = i\partial(H + H_1 + H_2 + H_3) \quad (\text{B.4})$$

where  $H, H_{1,2,3}$  are chiral scalar fields obeying

$$\begin{aligned} i\partial H &= \frac{2}{k} (-\psi^+ \psi^- + \chi^+ \chi^- + J^3 - K^3) \\ i\partial H_1 &= \frac{2}{k} \psi^3 \chi^3, \quad \partial H_2 = \lambda^1 \lambda^2, \quad \partial H_3 = \lambda^3 \lambda^4 \end{aligned} \quad (\text{B.5})$$

The fields  $H_I$ ,  $I = 1, 2, 3$  are normalized canonically,  $\langle H_I(z) H_J(w) \rangle = -\delta_{IJ} \log(z - w)$ , while  $\langle H(z) H(w) \rangle = -2 \log(z - w)$ . The spacetime supercharges are the zero-modes  $Q_\alpha^\pm$ ,  $\bar{Q}_{\bar{\alpha}}^\pm$  of the eight mutually local BRST

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<sup>18</sup> Assuming that we do not want to mix the  $SL(2, R) \times SU(2)$  and  $T^4$  parts of the theory.

invariant spin-fields:

$$\begin{aligned} S_\alpha^\pm &= e^{-\frac{\phi}{2} \pm \frac{i}{2} H + \frac{i}{2} H_1} S_\alpha, & Q_\alpha^\pm &= \oint dz S_\alpha^\pm \\ \bar{S}_{\bar{\alpha}}^\pm &= e^{-\frac{\phi}{2} \pm \frac{i}{2} H - \frac{i}{2} H_1} \bar{S}_{\bar{\alpha}}, & \bar{Q}_{\bar{\alpha}}^\pm &= \oint dz \bar{S}_{\bar{\alpha}}^\pm \end{aligned} \quad (B.6)$$

$S_\alpha, \bar{S}_{\bar{\alpha}}$  are spinors in the  $\mathbf{2}, \bar{\mathbf{2}}$  of the  $SO(4)$  symmetry acting on the fermions  $\lambda^i$ :

$$S_\alpha = e^{\pm \frac{i}{2}(H_2 + H_3)}, \quad \bar{S}_{\bar{\alpha}} = e^{\pm \frac{i}{2}(H_2 - H_3)} \quad (B.7)$$

The spacetime superalgebra is

$$\begin{aligned} \{Q_\alpha^+, Q_\beta^-\} &= \delta_{\alpha\beta}(J^3 - K^3) \\ \{\bar{Q}_{\bar{\alpha}}^+, \bar{Q}_{\bar{\beta}}^-\} &= \delta_{\bar{\alpha}\bar{\beta}}(J^3 + K^3) \\ \{Q_\alpha^-, \bar{Q}_{\bar{\beta}}^+\} &= \{Q_\alpha^+, \bar{Q}_{\bar{\beta}}^-\} = \gamma_{\alpha\bar{\beta}}^i P_i \end{aligned} \quad (B.8)$$

All other (anti-) commutators vanish.  $P_i$  is the four-vector of momenta along the  $T^4$ , and  $J^3, K^3$  are the zero-modes of the total CSA currents (3.1).

The first line of the superalgebra (B.8) looks similar to the Ramond sector superalgebra (3.15) in the zero mode sector. We have four supercharges  $Q_\alpha^\pm$  which one can attempt to identify<sup>19</sup> with  $Q_0^i, \bar{Q}_0^j$  in (3.15).  $J^3 - K^3$  on the r.h.s. of the first line of (B.8) is then interpreted as  $L_0$ , which is just the way the Ramond sector  $L_0$  is expected to be related to the NS generators  $L_0 = -J^3$  and  $T_0^3 = K^3$ , as in the last line of eq. (3.20). It is important for the above interpretation of  $Q_\alpha^\pm$  and  $J^3 - K^3$  that the supercharges commute with the bosonic generators  $J^3, K^3$ .

The second line of (B.8) appears to describe another copy of the same structure as the first line, with four more supercharges,  $\bar{Q}_{\bar{\alpha}}^\pm$  which square to the bosonic generator  $J^3 + K^3$ . At first sight it looks like we should interpret this as the Ramond superalgebra corresponding to the other chirality in spacetime, but this appears to be inconsistent for the following reasons:

(a) The third line of (B.8) would then say that left and right moving supercharges in spacetime have non-zero anticommutators. Such terms indeed arise in two dimensional (4,4) supersymmetric theories, *e.g.* when the latter are obtained by compactifying  $N = 1$  supersymmetric six dimensional theories; the charges  $P_i$  in (B.8) correspond to central charges in the superalgebra. However, such central charges are inconsistent with the *conformal* symmetry that string theory on  $AdS_3$  is supposed to possess.

<sup>19</sup>The symmetry structure implies that the index  $\pm$  on  $Q_\alpha^\pm$  corresponds to the index  $i$  on  $Q, \bar{Q}$  in (3.15), while  $\alpha = +$  and  $\alpha = -$  correspond to  $Q$  and  $\bar{Q}$  in (3.15).

(b) We saw in the text that in string theory on  $AdS_3$  the worldsheet and spacetime chiralities are related. It would be strange to get both left and right movers in the spacetime SCFT from the same chirality on the worldsheet. A related problem is that if this had happened, we would have had trouble interpreting the eight additional supercharges and translation generators  $\bar{J}^3, \bar{K}^3$  arising from the other worldsheet chirality.

A clue towards the correct interpretation of the vacuum in question comes from noticing that while the superalgebra (B.8) looks symmetric under interchange of  $J^3 - K^3$  and  $J^3 + K^3$ , this is misleading. The choice of the complex structure (B.1), which contains the term (B.3)  $2(J^3 - K^3)/k$ , picks one over the other. This leads to an asymmetry of the spectrum of excitations, which must satisfy the GSO projection (*i.e.* have integer  $U(1)$  charges under (B.1). There is another vacuum in which we flip the relative sign between  $J^3$  and  $K^3$  in (B.1) and in which the roles of  $J^3 \pm K^3$  are reversed.

Therefore, we would like to propose that what we are actually describing here is the two possible Ramond vacua corresponding to different twists (3.19) of the NS vacuum of section 3. Of course, we cannot be describing both vacua of the theory at the same time. Thus, to make sense of the theory we are instructed to do the following. If the  $U(1)_R$  current  $J_{N=2}$  contains the combination  $J^3 - K^3$  as in (B.1), we define the theory by restricting physical states to the cohomology of the operators  $Q_\alpha^\pm$ :

$$Q_\alpha^\pm |\text{phys}\rangle = 0, \quad |\text{phys}\rangle \sim |\text{phys}\rangle + Q |\text{anything}\rangle \quad (\text{B.9})$$

Because of (B.8) all such states have  $J^3 - K^3 = 0$ . We then interpret the four remaining supercharges  $\bar{Q}_\alpha^\pm$  together with  $J^3 + K^3$  as forming the zero mode sector of the left moving  $N = 4$  superconformal algebra (3.15). Obviously, if  $J_{N=2}$  contains the combination  $J^3 + K^3$ , we reverse the roles of  $Q$  and  $\bar{Q}$  and of  $J^3 \pm K^3$ .

Note that this procedure resolves the difficulties mentioned above. The non-zero anticommutators in the third line of (B.8) are no longer relevant since all the states are killed by  $Q$ . In fact, (B.8) implies that physical states have  $P^i = 0$ . The reason is that we expect the spacetime theory to be unitary, and positivity of the norm of physical states implies that they satisfy

$$(J^3)^2 \geq (K^3)^2 + |\vec{P}|^2 \quad (\text{B.10})$$

Since  $J^3 = K^3$ , one must have  $P^i = 0$ . The doubling of the superalgebra compared to what one expects in the Ramond sector is avoided by imposing the condition (B.9) on physical states. And, the correlation between the

worldsheet and spacetime chirality is restored: left moving symmetries on the worldsheet give left moving symmetries in spacetime, and vice-versa.

To summarize, string theory on  $\mathcal{M}$  with the GSO projection related to the  $U(1)_R$  current (B.1) is non-unitary (see below). One can restrict to a unitary sub-sector by restricting to the cohomology of  $Q_\alpha^\pm$  (B.9). This unitary sub-sector describes in spacetime the *chiral ring* or zero mode sector of the  $D1/D5$  SCFT, obtained by restricting to states satisfying  $L_0 = T_0^3$  (and similarly  $\bar{L}_0 = \bar{T}_0^3$ ). The projection (B.9) is an analog of the twist one does in  $N = 2$  SUSY theories in two dimensions, which leads to a topological  $N = 2$  theory whose physical states are in one to one correspondence with the chiral ring of the original theory.

To verify this interpretation we next turn to the spectrum of excitations of the theory. Due to the spacetime supersymmetry we can restrict our attention to the Neveu-Schwarz sector of the worldsheet theory (*i.e.* concentrate on spacetime bosons); we will study the spectrum in the  $-1$  picture of [19] and continue to work chirally. As discussed above, we will also set the momenta along the  $T^4$ ,  $P_i$ , to zero.

Before the projection (B.9) the system appears to contain tachyons, analogs of those that exist in fermionic string theory in flat space. They are described by the vertex operators (we consider only the holomorphic part of the vertex operators):

$$T_{jmj'm'} = e^{-\phi} V_{jm} V'_{j'm'} \quad (\text{B.11})$$

where, as in the text,  $V_{jm}$  is a primary operator of  $SL(2)$  affine Lie algebra with quadratic Casimir  $-j(j+1)$  and  $J^3 = m$ ;  $V'_{j'm'}$  is an  $SU(2)$  primary with similar notation. Unitarity restricts the allowed values of  $j', m'$ ,

$$j' \leq \frac{k}{2} - 1, \quad |m'| \leq j' \quad (\text{B.12})$$

The mass shell condition on  $T$  is in this case

$$\frac{j'(j'+1)}{k} - \frac{j(j+1)}{k} = \frac{1}{2} \quad (\text{B.13})$$

The GSO projection provides a constraint on  $m, m'$ :

$$\frac{m' - m}{k} \in \mathbb{Z} + \frac{1}{2} \quad (\text{B.14})$$

Thus, these states disappear from the spectrum in the flat space limit  $k \rightarrow \infty$ . Many of the solutions of (B.11) – (B.14) have complex  $j = -(1/2) + i\lambda$

and, therefore, correspond to tachyons; an example is states with  $j' = m' = 0$ ,  $m = (2n+1)k/2$  for integer  $n$ . Therefore, before the projection (B.9), the string vacuum in question is unstable and the spacetime dynamics in it is not unitary. The projection (B.9) eliminates the tachyons (B.11) since due to (B.14) none of these modes satisfy  $J^3 = K^3$ .

The low lying states of the theory are “transverse photons.” As usual there are eight physical polarizations, four along the  $T^4$  and four living in  $SL(2, R) \times SU(2)$ . The photons polarized along the  $T^4$  are described by the vertex operators

$$W_{jm}^i = e^{-\phi} \lambda^i V_{jm} V'_{jm} \quad (\text{B.15})$$

where we set  $j = j'$  so that the total scaling dimension of (B.15) is one, and  $m = m'$  to enforce (B.9),  $J^3 = K^3$ . The four remaining light transverse photons are described by vertex operators of the form

$$\begin{aligned} W = e^{-\phi} [ & a_+ \psi^+ V_{jm-1} V'_{jm} + a_- \psi^- V_{jm+1} V'_{jm} + \\ & b_+ \chi^+ V_{jm} V'_{jm-1} + b_- \chi^- V_{jm} V'_{jm+1} + (c_3 \psi^3 + d_3 \chi^3) V_{jm} V'_{jm} ] \end{aligned} \quad (\text{B.16})$$

BRST invariance and the freedom to add BRST commutators to  $W$  imply in the usual way that four of the six independent polarizations in (B.16) are physical.

In addition to the photons (B.15), (B.16), the spectrum also includes eight towers of oscillator states obtained *e.g.* by replacing  $\lambda^i$  in eq. (B.15) by an  $N = 1$  superconformal primary with scaling dimension  $h = N + 1/2$ . This leads to the standard exponential density of states at level  $N$ . The  $SL(2, R)$  and  $SU(2)$  Casimirs obey in this case the relation:

$$\frac{j(j+1)}{k} = \frac{j'(j'+1)}{k} + N \quad (\text{B.17})$$

Of course, one still has to impose the projection  $J^3 = K^3$  and the bound (2.20). All operators (B.15) – (B.17) satisfy  $L_0 = T_0^3$  and, therefore, are chiral. Their form and degeneracies are in agreement with the discussion of chiral operators in section 3.

Note that unlike the discussion in the text (sections 2, 3), here we *cannot* find an infinite super-Virasoro algebra in spacetime. The reason is that, as discussed above, this vacuum describes the topological dynamics of the chiral ring, on which the infinite dimensional algebra does not act.

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