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Supersymmetric Fivebrane Solitons

Jerome P. Gauntlett[a], Neil D. Lambert[b] and Peter C. West[b]

[a]Department of Physics
Queen Mary and Westfield College
University of London
Mile End Road
London E1 4NS, UK

[b] Department of Mathematics
King's College
University of London
The Strand
London WC2R 2LS, UK

Abstract

We study the conditions for the fivebrane worldvolume theory in D=11 to admit supersymmetric solitons with non-vanishing self-dual three-form. We construct some new soliton solutions consisting of "superpositions" of calibrated surfaces, self-dual strings and instantons.

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1 Introduction

An interesting feature of soliton solutions of brane worldvolume theories is that they contain their own spacetime interpretation [1, 2, 3]. A simple example is the self-dual string soliton of [3] which has the spacetime interpretation of a membrane ending on a fivebrane. Since the corresponding supergravity solutions are typically not fully localised (see [4] for a review) the worldvolume theory provides a sensitive tool to study the properties of brane intersections. In particular they have a number of applications within string theory and also Yang-Mills theory.

In [5] a study was initiated of interpreting general static spacetime configurations of intersecting branes (*e.g.*, supergravity solutions) in terms of solitons on the worldvolume. In that paper we only considered configurations of intersecting fivebranes which correspond to solitons on the worldvolume with vanishing self-dual three-form. These configurations can be interpreted as a single static fivebrane with a non-planar worldvolume. Furthermore one finds that the supersymmetric worldvolume solitons correspond to fivebranes whose spatial worldvolumes form calibrated surfaces [6, 7, 5][1]. In particular we showed that the differential equations for calibrated surfaces derived in [13] are equivalent to the preservation of some of the worldvolume supersymmetries.

In this paper we continue the study of supersymmetric fivebrane solitons by considering configurations with non-vanishing three-form. From a mathematical point of view this is a natural generalisation of calibrations. From a physical point of view, one expects to find supersymmetric solitons that include configurations corresponding to intersecting fivebranes and membranes and M-waves in spacetime. Here we shall explore several interesting cases, all in D=11 Minkowski space, leaving further analysis and applications to future work.

The solitons constructed here may be thought of as arising from three "building blocks". The first of these has the spacetime interpretation as a membrane intersecting a fivebrane which we may denote as

$$\begin{array}{lcccccc} M5: & 1 & 2 & 3 & 4 & 5 & \\ M2: & & & & & 5 & 6 \end{array} \qquad (1)$$

On the worldvolume of the fivebrane this solution has a single scalar X^6 active, depending on the four coordinates $\sigma^m = \{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$. It also

[1] For another connection between calibrations and intersecting branes, see [8, 9, 10, 11, 12].

has non-vanishing H_{05m} and, as a consequence of the self-duality constraint imposed on H, H_{mnp}. The resulting configuration on the worldvolume is a self-dual string soliton parallel to the σ^5 direction [3].

A second building block solution which we will use can be thought of as an M-wave intersecting a fivebrane

$$\begin{array}{llllll} M5: & 1 & 2 & 3 & 4 & 5 \\ MW: & & & & & 5 \end{array} \tag{2}$$

This solution has no scalars active. We will see below that the three-form for this configuration has non-vanishing H_{0mn} and H_{5mn}, where $m = 1, 2, 3, 4$. In fact H_{5mn} is self-dual as a two-form and gives rise to a string-like soliton in the σ^5 direction. By dimensional reduction we simply obtain an instanton in the worldvolume Dirac-Born-Infeld (DBI) theory of a D4-brane. This corresponds to a D0-brane in a D4-brane [14] which is the configuration that is obtained by reducing (2) along the 5 direction. In contrast to the self-dual string above, these strings do not carry any charge with respect to the field H and we therefore also refer to these instanton solutions as neutral strings.

The final basic building block solutions which we will use are the calibrated surfaces corresponding to intersecting fivebranes. The simplest example is provided by two intersecting fivebranes

$$\begin{array}{llllllll} M5: & 1 & 2 & 3 & 4 & 5 & & \\ M5: & & & 3 & 4 & 5 & 7 & 8 \end{array} \tag{3}$$

The corresponding worldvolume solution has two scalars X^7, X^8 active that depend on the two worldvolume coordinates σ^1, σ^2. The general solution can be viewed as a single fivebrane wrapped around a calibrated surface. In this case the relevant calibration is Kähler so that the embedding $X^7(\sigma^1, \sigma^2)$, $X^8(\sigma^1, \sigma^2)$ defines a Riemann surface. However our analysis will include considerably more complicated intersections such as

$$\begin{array}{lcccccccccc}
M5: & 1 & 2 & 3 & 4 & 5 & & & & & \\
M5: & & & 3 & 4 & 5 & & 7 & 8 & & \\
M5: & & 2 & & 4 & 5 & & 7 & & 9 & \\
M5: & 1 & 2 & & & 5 & & 7 & 8 & & \\
M5: & 1 & & 3 & & 5 & & 7 & & 9 & \\
M5: & & 2 & 3 & & 5 & & & 8 & 9 & \\
M5: & 1 & & & 4 & 5 & & & 8 & 9 & \\
M5: & & & & & 5 & & 7 & 8 & 9 & 10 \\
M5: & & 2 & 3 & & 5 & & 7 & & & 10 \\
M5: & & & 3 & 4 & 5 & & & & 9 & 10 \\
M5: & & 2 & & 4 & 5 & & & 8 & & 10 \\
M5: & 1 & & 3 & & 5 & & & 8 & & 10 \\
M5: & 1 & & & 4 & 5 & & 7 & & & 10 \\
M5: & 1 & 2 & & & 5 & & & & 9 & 10
\end{array} \qquad (4)$$

The corresponding worldvolume solitons have four scalars active depending on four worldvolume coordinates and it can be viewed as a single fivebrane wrapped on a Cayley four-fold of $\mathbf{R}^8$ [5].

In [5] we used orthogonal configurations of fivebranes to motivate the search for supersymmetric solitons. This point of view suggests which scalar fields should be active and which projections to impose on the spinor parameters. The logic in this paper is similar. In particular we note that the orthogonal configurations above can be combined (usually breaking more supersymmetry) and this suggests that the corresponding worldvolume solitons can similarly be "superposed". Adding a membrane in the 5,6 directions to (3) and (4) and other configurations in [5] suggests that we can add a self-dual string to the corresponding calibrated surface. Similarly adding a wave in the 5 direction suggests that we can add instantons to the calibrated surfaces. We shall see that this is indeed the case and that moreover it is possible to combine all three.

With this in mind here we will only consider calibrated surfaces M of dimension n with $n \leq 4$. *i.e.*, the fivebrane worldvolume has the form $\mathbf{R}^{6-n} \times M$. These have the feature that there is at least one common flat direction that is an isometry (*e.g.*, 5 in (4)) and one overall transverse direction (*e.g.*, 6 in (4)) (of the cases considered in [5] only the five-dimensional special Lagrangian manifolds are excluded).

To help illustrate this point let us list the spacetime configurations for the simplest example of a calibration in this class (3). For this case the corresponding soliton solutions that we discuss can be pictured as

$$\begin{array}{lcccccccc} M5: & 1 & 2 & 3 & 4 & 5 & & & \\ M5: & & & 3 & 4 & 5 & & 7 & 8 \\ M2: & & & & & 5 & 6 & & \end{array} \tag{5}$$

where we have added a membrane,

$$\begin{array}{lcccccccc} M5: & 1 & 2 & 3 & 4 & 5 & & & \\ M5: & & & 3 & 4 & 5 & & 7 & 8 \\ MW: & & & & & 5 & & & \end{array} \tag{6}$$

where we have added an M-wave, and finally

$$\begin{array}{lcccccccc} M5: & 1 & 2 & 3 & 4 & 5 & & & \\ M5: & & & 3 & 4 & 5 & & 7 & 8 \\ M2: & & & & & 5 & 6 & & \\ MW: & & & & & 5 & & & \end{array} \tag{7}$$

where both have been added.

Another type of solution which we will consider, consists of adding membranes intersecting at a point to a fivebrane

$$\begin{array}{lccccccc} M5: & 1 & 2 & 3 & 4 & 5 & & \\ M2: & & & & & 5 & 6 & \\ M2: & & & & 4 & & & 7 \end{array} \tag{8}$$

which corresponds to two intersecting self-dual strings.

The analysis in this paper is divided into two parts. The first part obtains static soliton solutions using the Hamiltonian formalism. The general procedure for finding supersymmetric states in this formalism is described in the next section. In section 2.1 we go on to consider adding self-dual strings to calibrated surfaces. In section 2.2 we discuss instanton configurations and discuss their interpretation as neutral strings. Next in section 2.3 we consider instantons on calibrated surfaces. In section 2.4 we then consider superpositions of neutral strings and self-dual strings on a calibrated surface. In section 2.5 we will describe the case of two orthogonally intersecting self-dual strings (8), although in this case we will not include the dependence of the fields on the 4,5 directions in (8).

The second part of the paper focuses on obtaining soliton solutions using the manifestly covariant formalism. The general supersymmetry conditions for this formalism are derived in section 3. In section 3.1 we consider time-dependent supersymmetric states corresponding to travelling waves along a fivebrane wrapped on a calibrated surface. These states simply correspond to a ripple in the "shape" of the calibrated surface, travelling along a flat direction *i.e.,* the string intersection such as 5 in (4). In section 3.2 we repeat the construction of instantons on a calibrated surface to obtain neutral strings, including a generalisation to non-static instantons. Section 3.3 considers self-dual and neutral strings together on a flat fivebrane. In section 3.4 we reconsider two intersecting self-dual strings, this time including the dependence on the 4,5 transverse directions in (8). Lastly we conclude in section 4 with some comments.

2 Supersymmetry in the Hamiltonian Formalism

Let us begin by describing the Lagrangian formulation of the fivebrane in D=11 Minkowski space [15]. The bosonic variables are scalars X^μ, $\mu = 0,1,2,...,10$, a closed three-form H_{ijk}, $i = 0,1,2,...,5$, and an auxiliary scalar field a. The role of a is to impose a non-trivial self-duality constraint on H. The supersymmetry and κ-symmetry transformations of the fermions are given by

$$\delta\theta = \frac{1}{2}(1+\Gamma)\kappa + \epsilon \tag{9}$$

where ϵ is a constant 32 component D=11 spinor. The matrix Γ is given by

$$\begin{aligned}\Gamma = \frac{1}{\sqrt{-\det(g+\tilde{H})}}[\quad & \frac{1}{5!}\frac{1}{(\partial a)^2}(\partial_i a\Gamma^i)\Gamma_{i_1\ldots i_5}\epsilon^{i_1\ldots i_5 j}\partial_j a + \partial_i a\Gamma^i\Gamma_j t^j \\ - \quad & \frac{\sqrt{-g}}{2(-(\partial a)^2)^{\frac{1}{2}}}\partial_i a\Gamma^i\Gamma_{jk}\tilde{H}^{jk}] ,\end{aligned} \tag{10}$$

where

$$\begin{aligned} g_{ij} &= \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} , \\ \tilde{H}^{ij} &= \frac{1}{(-(\partial a)^2)^{\frac{1}{2}}}(*H)^{ijk}\partial_k a , \\ t^i &= \frac{1}{8(\partial a)^2}\varepsilon^{ij_1j_2k_1k_2l}\tilde{H}_{j_1j_2}\tilde{H}_{k_1k_2}\partial_l a , \\ \Gamma_i &= \partial_i X^\mu \gamma_\mu . \end{aligned} \tag{11}$$

Here g_{ij} is the pull back of the eleven-dimensional flat metric, $*H$ is the Hodge dual of H with respect to g and γ^μ are flat D=11 gamma-matrices. Note that $\Gamma^2 = 1$ and hence $\frac{1}{2}(1 \pm \Gamma)$ are projection operators. For a bosonic configuration the corresponding variation of the bosonic fields X^μ and H

automatically vanish. Thus bosonic configurations will preserve some supersymmetry if and only if (9) vanishes. This is equivalent to the condition[2]

$$(1-\Gamma)\epsilon = 0 \ . \tag{12}$$

For most of this paper we will study static configurations for which the Hamiltonian formalism is most convenient. We will work in static gauge, $X^i = \sigma^i$ and in addition choose the gauge $a = t$. In this case $t^0 = \tilde{H}^{0a} = 0$, where $a = 1, \ldots, 5$ is a spatial index. Furthermore, one can check that $\Gamma^\dagger = \Gamma$. Using this we have

$$\begin{aligned} ||(1-\Gamma)\epsilon|| = \epsilon^\dagger(1-\Gamma^\dagger)(1-\Gamma)\epsilon &\geq 0 \\ \Longleftrightarrow \epsilon^\dagger(1-\Gamma)\epsilon &\geq 0 \quad . \end{aligned} \tag{13}$$

Choosing a spinor satisfying $\epsilon^\dagger\epsilon = 1$ we thus deduce the Bogomol'nyi bound on any static configuration

$$1 \geq \epsilon^\dagger \Gamma \epsilon \ , \tag{14}$$

with equality for supersymmetric configurations. The bound can be rewritten in the form

$$\sqrt{\det(g+\tilde{H})} \geq \epsilon^\dagger \Gamma^0 \left[\Gamma_a t^a - \frac{\sqrt{g}}{2} \Gamma_{ab} \tilde{H}^{ab} + \frac{1}{5!} \Gamma_{a_1 \ldots a_5} \epsilon^{a_1 \ldots a_5} \right] \epsilon \ , \tag{15}$$

where

$$\begin{aligned} \tilde{H}^{ab} &= \frac{1}{3!}\frac{1}{\sqrt{g}} \varepsilon^{abc_1c_2c_3} H_{c_1c_2c_3} \ , \\ t_f &= \frac{1}{4!} \varepsilon^{abcde} H_{abc} H_{def} \ , \end{aligned} \tag{16}$$

and we use the convention that $\varepsilon^{12345} = 1$ (and hence $\varepsilon_{12345} = \det g$).

One expects that this condition should provide a bound on the energy. The energy functional of static configurations in static gauge are given by [17, 18]

$$\mathcal{E}^2 = \det(g_{ab} + \tilde{H}_{ab}) + t_a t_b m^{ab} \ , \tag{17}$$

where

$$m^{ab} = g^{aa'} g^{bb'} [\partial_{a'} \mathbf{X} \cdot \partial_{b'} \mathbf{X} + (\partial_{a'} \mathbf{X} \cdot \partial_c \mathbf{X}) \delta^{cd} (\partial_{b'} \mathbf{X} \cdot \partial_d \mathbf{X})] \ . \tag{18}$$

Noting that $m^{ab} = \partial_a \mathbf{X} \cdot \partial_c \mathbf{X} g^{cb}$ we see that m^{ab} is a positive definite matrix and hence we deduce that the energy for all static configurations of the

[2]This form for the preservation of supersymmetry of was first discussed in [6] and was subsequently considered in [7, 16, 5, 8].

fivebrane is also bounded by the right hand side of (15). For configurations where $t_a t_b m^{ab} = 0$, the condition for the bound on the energy being saturated is the same as the condition for preservation of supersymmetry (*i.e.*, saturation of (15)). In the rest of the paper we will mainly consider this case. However, since the preservation of some supersymmetry generally implies that the energy is minimised it should be possible to derive a better bound on the energy such that when it is saturated it is equivalent to preservation of supersymmetry even when $t_a t_b m^{ab} \neq 0$, but we shall not pursue this here.

2.1 Self-Dual Strings on Calibrated Surfaces

Let us now begin our construction of supersymmetric solutions by superimposing a self-dual string on a calibrated surface. First consider a configuration of fivebranes only with at least one overall string intersection, *e.g.*, (3) or (4). On the first fivebrane worldvolume these correspond to configurations with the scalars $X^I, I = 7, \ldots, 10$ being non-trivial functions of the world-volume coordinates $\sigma^\alpha, \alpha = 1, \ldots, 4$. It was shown in [5] that configurations which preserve supersymmetry correspond to calibrated fivebrane world-volumes, the calibration depending on the particular case being considered. Note that if all of the scalars are excited then the calibrated surface M is four dimensional and the spatial part of the fivebrane world-volume takes the form $\mathbb{R} \times M$. In general if not all of the four scalars are excited then this becomes $\mathbb{R}^{5-n} \times M$ where the calibrated surface M is now n dimensional with $n = 2, 3, 4$.

Since we can add a membrane to these fivebrane intersections while preserving supersymmetry, we expect to find supersymmetric self dual strings superposed on the corresponding calibrated fivebrane worldvolume. A concrete example is given by adding a membrane to (3) as pictured in (5). The two fivebranes correspond to a spatial worldvolume of the fivebrane given by $\mathbb{R}^3 \times \Sigma$ where Σ is a two dimensional Riemann surface lying in the σ^3, σ^4 directions. We then expect to be able to add a self-dual string along the σ^5 direction.

Let us now consider the conditions required for the self-dual string to preserve supersymmetry and solve the equations of motion. We first write the scalar coordinate not appearing in the calibrated surface as

$$X^6 \equiv X \ , \qquad (19)$$

and we will demand $\partial_{0,5} X = \partial_{0,5} X^I = 0$. The induced spatial worldvolume metric can be written as

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \partial_\alpha X \partial_\beta X \ , \quad g_{\alpha 5} = 0 \ , \quad g_{55} = 1 \ , \tag{20}$$

where

$$\bar{g}_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha X^I \partial_\beta X^J \delta_{IJ} \ . \tag{21}$$

Our ansatz for a supersymmetric solution will include taking taking $\bar{g}$ to be the induced metric on $R^{4-n} \times M$ where M is an n dimensional calibrated surface. As discussed in detail in [5] this by itself leads to a supersymmetric solution where both the number of supersymmetries preserved and the type of calibration are determined by certain projections on the supersymmetry parameters. These projections are the same as those for the corresponding spacetime configurations of orthogonally intersecting fivebranes. Here we will need that the condition for supersymmetry to be preserved is given by

$$\gamma^0 \bar{\Gamma}_{1234} \gamma_5 \epsilon = \sqrt{\bar{g}} \epsilon \ , \tag{22}$$

where $\bar{\Gamma}_\alpha = \gamma_\alpha + \partial_\alpha X^I \gamma_I$. By multiplying both sides by $\epsilon^\dagger$ and imposing the relevant projection operators we obtain the calibration Ω via

$$\Omega_{\alpha_1\alpha_2\alpha_3\alpha_4} = \epsilon^\dagger \gamma^0 \bar{\Gamma}_{\alpha_1\alpha_2\alpha_3\alpha_4} \gamma_5 \epsilon \ . \tag{23}$$

The ansatz for including a membrane is obtained by "superposing" this with the ansatz for the self dual string. Specifically, we take

$$\tilde{H}^{\alpha 5} = \pm \frac{\sqrt{\bar{g}}}{\sqrt{g}} \bar{g}^{\alpha\beta} \partial_\beta X \ , \tag{24}$$

or equivalently

$$H = \pm \bar{*} dX \ , \tag{25}$$

where $\bar{*}$ is the Hodge dual with respect to the metric $\bar{g}$. Since H is closed, X must be a harmonic function[3] on the geometry specified by $\bar{g}$. Note that in general this is not quite the same as saying that it is harmonic on the calibrated surface M. Recall that if not all of the scalars X^I are excited then the geometry determined by $\bar{g}$ is actually $\mathbb{R}^{4-n} \times M$.

To further illustrate this point consider the configuration (5), *i.e.*, n=2. If we let $\hat{g}$ specify the metric of the calibrated surface, in this case the Riemann surface Σ, and if we let $i = 1, 2$ and $a = 3, 4$ (for here only), then X satisfies

$$\partial_a \partial_a X + \frac{1}{\sqrt{\hat{g}}} \partial_i (\sqrt{\hat{g}} \hat{g}^{ij} \partial_j) X = 0 \ . \tag{26}$$

[3] We note that related solutions were argued to solve the DBI equations of motion in [8].

If $\partial_a X = 0$ then this is equivalent to X being harmonic on the calibrated surface $\hat{g}$. Solutions are provided by the real parts of holomorphic functions, but it should be noted that the string is then delocalised in the 3, 4 directions. A localised string is obtained by solving the more general equation (26).

To analyse the conditions for preserved supersymmetry we first observe that $t^a = 0$ and hence from (17) that $\mathcal{E} = \sqrt{\det(g_{ab} + \tilde{H}_{ab})}$. Noting that

$$\det g = \det \bar{g}(1 + (\overline{\partial X})^2) \ , \tag{27}$$

where we have introduced the notation $(\overline{\partial X})^2 = \bar{g}^{\alpha\beta}\partial_\alpha X \partial_\beta X$, we can use the the expansion

$$\det(g_{ab} + \tilde{H}_{ab}) = \det g(1 + \frac{1}{2}\tilde{H}^2) + t^2 \tag{28}$$

to show that $\mathcal{E}^2$ is a perfect square with

$$\mathcal{E} = \sqrt{\det(g_{ab} + \tilde{H}_{ab})} = \sqrt{\bar{g}}(1 + (\overline{\partial X})^2) \ . \tag{29}$$

This should be compared to the right-hand side of (15). Using the fact that $\Gamma_a = \bar{\Gamma}_a + \partial_a X \gamma_6$, we have

$$\Gamma_{12345} = \bar{\Gamma}_{1234}\gamma_5 + \partial_\alpha X \bar{g}^{\alpha\beta}\bar{\Gamma}_\beta \bar{\Gamma}_{1234}\gamma_{56} \ , \tag{30}$$

and

$$-\frac{\sqrt{g}}{2}\Gamma^{ab}\tilde{H}_{ab} = \mp\sqrt{\bar{g}}\bar{g}^{\alpha\beta}\partial_\beta X \left(\partial_\alpha X \gamma_6\gamma_5 + \bar{\Gamma}_\alpha \gamma_5\right) \ . \tag{31}$$

If in addition to demanding (22) we also insist that

$$\gamma^{056}\epsilon = \pm\epsilon \ , \tag{32}$$

which is the supersymmetry condition for a self-dual string without the calibrated surface, then the right side of the supersymmetry condition precisely becomes (29). Thus the ansatz preserves supersymmetry. To determine the amount of supersymmetry one needs to consider the specific calibration. In general one expects that the addition of the membrane should break a further half of the supersymmetry. However, in some cases the projection is a consequence of the projections imposed by the surface being calibrated. From a spacetime point of view, this corresponds to configurations of fivebranes where the membrane can be "added for free" (see section 2 of [5] for examples). To fully specify the solution, one needs to find harmonic functions on calibrated surfaces (or solve the analogue of (26).

2.2 Instantons and Neutral Strings

It is well known that a D0-brane bound to a D4-brane manifests itself as an instanton in the D4-brane world volume theory [14]. For a single D4-brane it was shown in [18] that Abelian instantons saturate a Bogomol'nyi bound in the full DBI theory. It was further argued in [18] that the bound should also hold in the non-Abelian theory, corresponding to a collection of coincident D4-branes, and this was confirmed in [19].

As a spacetime configuration a D0-brane intersecting a D4-brane uplifts to eleven dimensions as a gravitational pp-wave or M-wave intersecting a fivebrane according to the pattern (2). We thus expect this configuration to be realised on the fivebrane worldvolume theory as an "instanton string". We shall confirm this and then show that it can be superposed with calibrated surfaces and self-dual strings in the following subsections.

To construct the instanton string, we set all of the scalar fields to zero, and hence the induced metric is flat, $g_{ab} = \delta_{ab}$. The ansatz for the H-field is taken to be

$$H_{5\alpha\beta} = F_{\alpha\beta} \ , \tag{33}$$

where $F = dA$ with F an (anti-) self-dual field strength. We then have

$$t_\alpha = 0 \ , \qquad t_5 = \pm\frac{1}{4}F^2 \ , \tag{34}$$

and hence $t_a t_b m^{ab} = 0$. As in [18] $\det(\delta + F)$ is a perfect square and the energy is given by

$$\mathcal{E} = \sqrt{\det(g + \tilde{H})} = 1 + \frac{1}{4}F^2 \ . \tag{35}$$

For this configuration to preserve supersymmetry, this should be equal to the right hand side of (15). Substituting the ansatz, the latter is given by

$$\epsilon^\dagger\gamma^0[\gamma_5 t_5 + \gamma_{12345} \mp \frac{1}{2}\gamma^{\alpha\beta}F_{\alpha\beta}]\epsilon \ . \tag{36}$$

If we impose the projectors

$$\gamma^{05}\epsilon = \pm\epsilon \ , \qquad \gamma^{012345}\epsilon = \epsilon \ , \tag{37}$$

we see that the last term in (36) vanishes and the Bogomol'nyi bound is saturated. The configuration breaks one half of the world-volume supersymmetry corresponding to one quarter of the spacetime supersymmetry.

Thus we find this configuration corresponds to (anti-) self-dual $U(1)$ gauge fields on the space transverse to the string. Explicit solutions can be constructed using a complex structure J on R^4 via

$$A_\alpha = J_\alpha{}^\beta \partial_\beta \phi \tag{38}$$

where ϕ is a harmonic function on R^4 [8]. If the Kähler-form is anti-self dual the field strength is self dual and vice-versa. Note that, unlike the D4-brane, there is no known non-Abelian extension of the classical fivebrane theory and thus the classical BPS solutions are restricted to Abelian instantons.

It is worth emphasising that these instanton strings differ from the self-dual strings of [3] in that they have vanishing H-charge

$$Q = \int_{S^3} H = 0 \ , \tag{39}$$

where S^3 is the sphere at transverse spatial infinity. Thus we will also refer to these instanton strings as "neutral strings"[4].

Note that the instanton can have an arbitrary dependence on σ^5, while still maintaining $dH = 0$. In other words the σ^5 direction specifies a one-dimensional path in the moduli space of instantons. For example in the solutions (38), we can take

$$\phi = \frac{f(\sigma^5)}{|\sigma^\alpha - h^\alpha(\sigma^5)|^2} \tag{40}$$

where $|\sigma|^2 = \sum_{\alpha=1}^4 \sigma^\alpha \sigma^\alpha$. In other words the instanton is free to change its location, h, and also its "amplitude", f, along the length of the string.

2.3 Instantons on Calibrated Surfaces

We now discuss an interesting generalisation by considering instantons on calibrated surfaces. As in the last section we let $X^I(\sigma^\alpha)$, $I = 7,8,9,10$, $\alpha = 1,2,3,4$ specify a calibrated surface with induced metric $\bar{g}$. The spatial worldvolume metric therefore takes the form

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} \ , \quad g_{55} = 1 \ , \quad g_{5\alpha} = 0 \ . \tag{41}$$

For a neutral string along the 5 direction, the only non-zero component of H_{abc} is as in (33)

$$H_{5\alpha\beta} = F_{\alpha\beta} \ , \tag{42}$$

[4]Neutral strings were discussed in the context of the linearised approximation to the fivebrane theory compactified on a torus in [20].

and we take F_{ab} to be (anti-) self-dual on the calibrated surface specified by $\bar{g}$

$$\frac{1}{2\sqrt{\bar{g}}}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} = \pm F_{\alpha\beta} \ , \tag{43}$$

This leads to

$$\tilde{H}_{\alpha\beta} = \pm F_{\alpha\beta} \ , \quad t_5 = \pm\frac{1}{4}\sqrt{\bar{g}}F^2, \quad t_\alpha = 0 \ . \tag{44}$$

Since $t_a t_b m^{ab} = 0$ and using (28) we have

$$\mathcal{E} = \sqrt{\det(g+\tilde{H})} = \sqrt{\bar{g}}(1+\frac{1}{4}F^2) \ . \tag{45}$$

To see that the solution preserves supersymmetry we want to compare this to the right hand side of (15). We will impose the projections for both the calibration and the instanton:

$$\gamma^0\bar{\Gamma}_{1234}\gamma_5\epsilon = \sqrt{\bar{g}}\epsilon \ , \quad \gamma^{05}\epsilon = \pm\epsilon \ . \tag{46}$$

In [5] examples of calibrations were given for which (46) followed from the intersecting fivebranes alone, *i.e.*, one could add an M-wave 'for free'. However one could always consider imposing (46) as an addition constraint breaking another half of the supersymmetry. A little algebra now shows that these conditions imply

$$\frac{1}{2}\sqrt{\bar{g}}\epsilon_{\alpha\beta\gamma\delta}\bar{\Gamma}^{\gamma\delta}\epsilon = \mp\bar{\Gamma}_{\alpha\beta}\epsilon \ . \tag{47}$$

Substituting the ansatz into the right-hand side of (15) we get

$$\epsilon^\dagger\Big[\pm t_5 + \frac{1}{2}\sqrt{\bar{g}}\gamma_0\bar{\Gamma}^{\alpha\beta}\tilde{H}_{\alpha\beta} + \sqrt{\bar{g}}\Big]\epsilon \ . \tag{48}$$

Upon imposing the projectors on ϵ we note that the second term vanishes by the self-duality of F and we obtain $\pm t_5 + \sqrt{\bar{g}}$ which is the same as (45).

The number of smooth instanton strings is thus given by the number of (anti-) self-dual two forms on the calibrated submanifold. An interesting case is for a Cayley 4-fold M. It was shown in [13] that the Kähler two-forms ω_e associated to each of the complex structures of R^8, J_e, $e \in S^6$, are anti-self dual when restricted to M.

By dimensional reduction, the construction we have presented implies that instantons on D4-brane worldvolumes wrapped around calibrated surfaces are also supersymmetric. It seems likely that this will remain true for the $U(N)$ non-Abelian extension of the DBI theory corresponding to N coincident D4-branes. It was shown in [13] that any Cayley four-fold naturally admits anti-self dual $SU(2)$ Yang-Mills fields. Such instantons correspond to D0-branes on two superposed D4-branes wrapped around the Cayley four-fold. It would be interesting to investigate this in more detail.

2.4 Self-Dual Strings and Neutral Strings on Calibrated Surfaces

We now generalise the above cases by considering the addition of both self-dual and neutral strings (instantons) along the direction σ^5, to a calibrated surface with metric $\bar{g}_{ab}$ in the $(\sigma^1, \sigma^2, \sigma^3, \sigma^4, X^7, X^8, X^9, X^{10})$ plane. We will see that generically $m^{ab}t_a t_b \neq 0$ and the resulting energy expression exhibits new features.

The ansatz consists of superposing a neutral string with the self-dual string on a calibrated surfaces considered in section three. Thus the metric is given by

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \partial_\alpha X \partial_\beta X \ , \quad g_{\alpha 5} = 0 \ , \quad g_{55} = 1 \ , \tag{49}$$

and the three-form is specified by

$$\begin{aligned} \tilde{H}^{\alpha\beta} &= \kappa \frac{\sqrt{\bar{g}}}{\sqrt{g}} \bar{F}^{\alpha\beta} \ . \\ \tilde{H}^{\alpha 5} &= \lambda \frac{\sqrt{\bar{g}}}{\sqrt{g}} \bar{g}^{\alpha\beta} \partial_\beta X \ , \end{aligned} \tag{50}$$

where κ, λ are signs and again F is (anti-) self-dual in the $\bar{g}_{\alpha\beta}$ metric:

$$\frac{1}{2}\frac{1}{\sqrt{\bar{g}}}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \kappa \bar{F}^{\alpha\beta} \ , \tag{51}$$

and the bar on F denotes that we have raised the indices using the metric $\bar{g}$ (note that we again have (42)). This leads to the following expressions for t_a

$$t_5 = \frac{\kappa}{4}\sqrt{\bar{g}}\bar{F}^2 \ , \quad t_\alpha = \kappa\lambda\sqrt{\bar{g}}\bar{g}^{\beta\gamma}\partial_\beta X F_{\alpha\gamma} \ . \tag{52}$$

and the bar on F^2 indicates we have contracted the indices here with $\bar{g}$ rather than g. Using (28) and (27) and the lemma

$$F_{\alpha\beta}\bar{g}^{\beta\gamma}F_{\gamma\delta} = -\frac{1}{4}\bar{g}_{\alpha\delta}\bar{F}^2 \tag{53}$$

we obtain

$$\sqrt{\det(g + \tilde{H})} = \sqrt{\bar{g}}(1 + (\overline{\partial X})^2 + \frac{1}{4}(\bar{F}^2)) \tag{54}$$

To check the preservation of supersymmetry we will impose the projections

$$\gamma^0 \bar{\Gamma}_{1234}\gamma_5\epsilon = \sqrt{\bar{g}}\epsilon \ , \quad \gamma^{05}\epsilon = \kappa\epsilon \ , \quad \gamma^{056}\epsilon = \lambda\epsilon \ , \tag{55}$$

As before the Γ-matrices are given by $\Gamma_a = \bar{\Gamma}_a + \partial_a X \gamma_6$, where $\bar{\Gamma}_a$ are the Γ-matrices for the calibrated surface. Using $\bar{\Gamma}_{\alpha\beta}\tilde{H}^{\alpha\beta}\epsilon = 0$ and the facts $\partial_a X t^a = 0$, $t^\alpha = \lambda\sqrt{\bar{g}}\partial_\beta X \tilde{H}^{\alpha\beta}$, it is straightforward to show that the right hand side of the supersymmetry condition (15) is the same as (54).

For the generic case the energy is no longer given by $\sqrt{\det(g+\tilde{H})}$ since $t_a t_b m^{ab} \neq 0$. Instead we find

$$\begin{aligned}\mathcal{E}^2 &= \det(g+\tilde{H}) + t_a t_b m^{ab} \\ &= \det\bar{g}[1 + (\overline{\partial X})^2 + \frac{1}{4}(\overline{F}^2)]^2 + \det\bar{g}(\delta^{\alpha\beta} - \bar{g}^{\alpha\beta})\partial_\gamma X \partial_\delta X \bar{F}_\alpha{}^\gamma \bar{F}_\beta{}^\delta \ .\end{aligned} \tag{56}$$

Thus the energy contains a non-linear term arising from the interaction of the self-dual string, neutral string and calibration. In the solutions constructed before we saw that the strings behaved very much like additional fields living on the calibrated surface. In those cases the energy was precisely what one would expect for a scalar field or Maxwell field living on the calibrated submanifold. However here we see that the energy of the combined self-dual string, neutral string and calibration configuration is greater than simply the sum of the two string energies on a background calibrated surface.

2.5 Two Intersecting Self-Dual Strings

Consider two membranes orthogonally intersecting a fivebrane according to the pattern (8). We expect that this should manifest itself as two intersecting self-dual strings in the fivebrane worldvolume theory. Here we will construct such supersymmetric solutions. It should be pointed out in advance that the two self-dual strings will be delocalised in the directions tangent to the other string. *i.e.*, in the 4, 5 directions on the fivebrane. In section 3.4 we will consider the most general solution using the covariant formalism.

In these solutions two scalars in the fivebrane worldvolume theory are active, X^6, X^7, which we will denote by X and Y respectively. These scalars are functions of the worldvolume coordinates σ^α, $\alpha = 1, 2, 3$. We first observe that

$$\det g = 1 + (\partial X)^2 + (\partial Y)^2 + (\partial X)^2(\partial Y)^2 - (\partial X \cdot \partial Y)^2 \ , \tag{57}$$

where *e.g.*, $(\partial X)^2 = \partial X \cdot \partial X = \partial_\alpha X \partial_\alpha X$. The ansatz for the three-form is given by

$$\tilde{H}^{\alpha 5} = \frac{\lambda}{\sqrt{g}}\partial_\alpha X, \qquad \tilde{H}^{\alpha 4} = \frac{\kappa}{\sqrt{g}}\partial_\alpha Y \ . \tag{58}$$

where κ, λ are again signs. Closure of the three-form implies that X and Y are harmonic on $\mathbb{R}^3$:

$$\partial^2 X = \partial^2 Y = 0 \ . \tag{59}$$

This ansatz implies that the only non-zero component of t^a is given by

$$t^\alpha = (\partial X \times \partial Y)^\alpha \ , \tag{60}$$

where we are using the usual vector cross product $(\partial X \times \partial Y)^\alpha = \epsilon_{\alpha\beta\gamma}\partial_\beta X \partial_\gamma Y$. This implies that $t_a t_b m^{ab}$ vanishes and from (18),(28) we find that

$$\mathcal{E} = \sqrt{\det(g+\tilde{H})} = \left[1 + (\partial X)^2 + (\partial Y)^2\right] \ . \tag{61}$$

The condition for preserved supersymmetry (15) now reads

$$\sqrt{\det(g+\tilde{H})} = \epsilon^\dagger \Gamma^0 \left[(\partial X \times \partial Y)^\alpha \Gamma_\alpha + \partial_\alpha X \Gamma_{\alpha 5} - \partial_\alpha Y \Gamma_{\alpha 4} + \Gamma_{12345}\right]\epsilon \ . \tag{62}$$

The right hand side can be recast in the form

$$\begin{aligned}\epsilon^\dagger\gamma^0[(\partial X \times \partial Y)\gamma_i(1+\gamma_{4567}) + \partial_i X \gamma_i(\gamma_{123456} - \lambda\gamma_5) + \partial_i Y \gamma_i(\gamma_{123457} - \kappa\gamma_4) \\ +\gamma_{12345} - \lambda(\partial X)^2\gamma_{65} - \kappa(\partial Y)^2\gamma_{74} - (\partial X \cdot \partial Y)(\lambda\gamma_{75} + \kappa\gamma_{64})]\epsilon \ .\end{aligned} \tag{63}$$

If we now impose the supersymmetry projections

$$\gamma^{056}\epsilon = \lambda\epsilon \ , \qquad \gamma^{047}\epsilon = \kappa\epsilon \ , \qquad \gamma^{012345}\epsilon = \epsilon \ , \tag{64}$$

then we find the first three terms in (63) vanish and that the remaining terms combine precisely to give $\sqrt{\det(g+\tilde{H})}$. Thus the configuration preserves 1/4 of the worldvolume supersymmetry which is what one expects for spacetime configurations that preserves 1/8 of the supersymmetry. Again we note that the projections are exactly the same as in the spacetime configurations.

It is very likely that the solution can be generalised to include more membranes. For example, if we added a membrane in the 3,8 directions and demanded that the active scalars are only dependent on the σ^1, σ^2 directions, we expect the solution to be determined by three harmonic functions. Note that in this case, such functions have logarithmic divergences.

3 Supersymmetry in the Covariant Formalism

Let us now turn our attention to obtaining supersymmetric solitons using the covariant formalism [21]. We will first present some details of the general formalism, analogous to those in section two, before using it to discuss some of the previous solutions, including some generalisations to include time-dependence.

It is convenient to use a different choice of notation. We now let $a, b, c, .. = 0, 1, 2, 3, 4, 5$ refer to tangent indices and $m, n, p, ... = 0, 1, 2, 3, 4, 5$ be world indices of the fivebrane worldvolume. In this formalism one first introduces a self-dual three-form h_{abc} and then constructs from it another three-form

$$H_{abc} = (m^{-1})_a{}^d h_{dbc} \ , \tag{65}$$

where

$$m_a{}^b = \delta_a{}^b - 2k_a{}^b \ , \quad k_a{}^b = h_{acd}h^{bcd} \ . \tag{66}$$

Note that, as a consequence of self-duality, $h_{eab}h^{ecd} = \delta_{[a}{}^{[c}k_{b]}{}^{d]}$. From this one can see that H is indeed totally anti-symmetric; $H_{abc} = H_{[abc]}$. The importance of the three-form H_{abc} is that its equation of motion simply states that H is closed. It is useful to note that, because of the self-duality of h, one has the formula [21]

$$(m^{-1})_a{}^b = Q^{-1}(\delta_a{}^b + 2k_a{}^b) \ , \tag{67}$$

where $Q = 1 - \frac{2}{3}k_a{}^b k_b{}^a$. Because the self-dual three-form h is not closed it is not a convenient field for discussing the physics. Instead one is primarily interested in the closed, but not self-dual, three-form H. However H does split up into its self-dual and anti-self-dual parts as [21]

$$H^+_{abc} = Q^{-1}h_{abc} \ , \quad H^-_{abc} = 2Q^{-1}k_a{}^d h_{bcd} \ . \tag{68}$$

Thus to obtain a formula for h in terms of H we need only evaluate the function Q in terms of H. To this end we write $H = H^+ + H^-$ and note that since $H^{+2} = H^{-2} = 0$

$$\begin{aligned} H^2 &= 2H^+_{abc}H^{-\ abc} \\ &= 4Q^{-2}h_{abc}k^a{}_d h^{dbc} \\ &= 4Q^{-2}k_a{}^d k^a{}_d \\ &= 6Q^{-2}(1-Q) \ . \end{aligned} \tag{69}$$

The unique non-singular solution to this quadratic equation is

$$Q = -\frac{3}{H^2}\left[1 - \sqrt{1 + \frac{2}{3}H^2}\right] \ . \tag{70}$$

Note that if H is self-dual then $Q = 1$. Thus whenever we see the three-form h we may replace it by the identity

$$h_{abc} = -\frac{3}{2H^2}\left[1 - \sqrt{1 + \frac{2}{3}H^2}\right](H_{abc} + \frac{1}{3!}\epsilon_{abcdef}H^{def}) \ . \tag{71}$$

where $\epsilon^{012345} = 1$. The supersymmetry projector in this formalism has been described in [21] and takes the form

$$\begin{aligned} \Gamma &= -\frac{1}{6!}\frac{1}{\sqrt{-g}}\epsilon^{m_1\dots m_6}\Gamma_{m_1\dots m_6} + \frac{1}{3}h^{m_1m_2m_3}\Gamma_{m_1m_2m_3} \\ &\equiv \Gamma_{(0)} + \Gamma_{(h)} \ , \end{aligned} \tag{72}$$

where g is the determinant of the induced metric on the fivebrane and, as before, $\Gamma_m = \partial_m X^\mu \Gamma^{\underline{a}}$. Here $\Gamma^{\underline{a}}$, $\underline{a} = 0, 1, 2, ..., 10$, are flat eleven-dimensional Γ-matrices. Clearly when the three-form is zero and we consider only static solutions, Γ is the same for both formalisms. One can show the following properties of Γ

$$\Gamma^2 = 1 \ , \quad \Gamma^\dagger_{(0)} = \Gamma_{(0)} \ . \tag{73}$$

However $\Gamma_{(h)}$, and hence Γ also, is not Hermitian. As before we may derive a bound from the inequality[5]

$$\begin{aligned} ||\epsilon(1-\Gamma)||^2 &= \epsilon(1-\Gamma)(1-\Gamma^\dagger)\epsilon^\dagger \geq 0 \\ \Longleftrightarrow \epsilon(1+\frac{1}{2}\Gamma_{(h)}\Gamma^\dagger_{(h)})\epsilon^\dagger &\geq \epsilon(\Gamma_{(0)} + \Gamma_{(h)} + \Gamma^\dagger_{(h)})\epsilon^\dagger \ . \end{aligned} \tag{74}$$

Unfortunately, although the left hand side is manifestly positive definite, it is not clear what its interpretation is. Physically one expects that $1+\frac{1}{2}\Gamma_{(h)}\Gamma^\dagger_{(h)}$ is related to the energy but we cannot check this as the Hamiltonian has not yet been constructed in the covariant formalism variables. In any case supersymmetric configurations satisfy $\epsilon(1-\Gamma) = 0$ and so saturate this bound.

To use this formalism to construct supersymmetric solutions we will gauge fix the fivebrane. The procedure for how to do this was discussed in detail in [5] for the purely scalar case. Now we need to repeat this analysis including the field h_{abc}. Firstly we gauge fix the fivebrane by choosing its worldvolume coordinates to be equal to the spacetime coordinates $\sigma^0 = X^0, ..., \sigma^5 = X^5$. This leaves the remaining five spacetime coordinates as scalar zero modes on the fivebrane worldvolume $X^{a'}$, $a' = 6, 7, 8, 9, 10$. The induced metric g_{mn} on the fivebrane then takes the form

$$\begin{aligned} g_{mn} &= \eta_{mn} + \partial_m X^{a'}\partial_n X^{a'} \\ &= \eta_{ab} e_m^{\ a} e_n^{\ b} \ . \end{aligned} \tag{75}$$

Next we need to consider the spinors. The thirty-two component eleven-dimensional spinor indices $\underline{\alpha}$ naturally split up into two sixteen component indices $\alpha = 1, 2, 3, ..., 16$ and $\alpha' = 1, 2, 3, ..., 16$. The M-fivebrane preserves

[5] Note that the spinor notation is that of [5] and differs from section 2.

half of the thirty-two spacetime supersymmetries, which leaves sixteen supersymmetries ϵ^α in the worldvolume theory. The sixteen spinor modes $\Theta^{\alpha'}$ become fermionic Goldstone fields in the worldvolume theory. Furthermore these indices can be reduced into $Spin(1,5)$ and $Sp(4) \cong SO(5)$ indices which we denote as (α, i) with $\alpha, i = 1,2,3,4$. In particular superscript indices decompose as $\alpha \to {}^{\alpha i}$ and $\alpha' \to {}_\alpha^{\ i}$ and subscript indices decompose as $\alpha \to {}_{\alpha i}$ and $\alpha' \to {}^\alpha_{\ i}$. For example the worldvolume supersymmetries ϵ^α and fermions $\Theta^{\alpha'}$ are written as $\epsilon^{\alpha i}$ and $\Theta_\alpha^{\ i}$ respectively. One can always tell in which sense we mean a particular index, depending on whether or not there are i, j indices present. For the sake of clarity we will try to use as few spinor indices as possible without being ambiguous. For a more complete discussion of the spinors we refer the reader to [5].

Finally we must split up the eleven-dimensional Γ-matrices into a six-dimensional form. For these we take

$$(\Gamma^{a'})_{\underline{\alpha}}^{\ \underline{\beta}} = (\gamma^{a'})_i^{\ j}\begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & -\delta^\alpha_\beta \end{pmatrix}\ , \quad (\Gamma^a)_{\underline{\alpha}}^{\ \underline{\beta}} = \delta_i^j \begin{pmatrix} 0 & (\gamma^a)_{\alpha\beta} \\ (\tilde{\gamma}^a)^{\alpha\beta} & 0 \end{pmatrix}\ . \tag{76}$$

Here $\gamma^{a'}$ are a set of five-dimensional Euclidean γ-matrices. The $\tilde{\gamma}$-matrices are simply related to the γ-matrices by $\tilde{\gamma}^a = \gamma^a$ for $a \neq 0$ and $\tilde{\gamma}^0 = -\gamma^0$. The matrices γ^a must be chosen so that the Γ^a satisfy a six-dimensional Clifford algebra. A convenient choice of representation [5] consists of taking γ^a to be five-dimensional γ-matrices for $a \neq 0$ and $\gamma^0 = 1$. Note also that since they act on different indices, $(\gamma^{a'})_i^{\ j}$ and $(\gamma^a)_{\alpha\beta}$ commute with each other.

It was shown in [5] that to preserve supersymmetry we need only look for zero modes of

$$\hat{\delta}\Theta^{\gamma'} = -\frac{1}{2}\epsilon^\alpha \Gamma_\alpha^{\ \gamma'}\ , \tag{77}$$

i.e., we need to consider the off diagonal components of Γ. After some algebra one then arrives at the following expression

$$\begin{aligned}
\hat{\delta}\Theta_\beta^{\ j} = \quad & \frac{1}{2}\epsilon^{\alpha i}\Big\{\det(e^{-1})\partial_m X^{c'}(\gamma^m)_{\alpha\beta}(\gamma_{c'})_i^{\ j} \\
& -\frac{1}{3!}\det(e^{-1})\partial_{m_1}X^{c'_1}\partial_{m_2}X^{c'_2}\partial_{m_3}X^{c'_3}(\gamma^{m_1m_2m_3})_{\alpha\beta}(\gamma_{c'_1c'_2c'_3})_i^{\ j} \\
& +\frac{1}{5!}\det(e^{-1})\partial_{m_1}X^{c'_1}\ldots\partial_{m_5}X^{c'_5}(\gamma^{m_1\ldots m_5})_{\alpha\beta}(\gamma_{c'_1\ldots c'_5})_i^{\ j} \\
& -h^{m_1m_2m_3}\partial_{m_2}X^{c'_2}\partial_{m_3}X^{c'_3}(\gamma_{m_1})_{\alpha\beta}(\gamma_{c'_2c'_3})_i^{\ j} \\
& -\frac{1}{3}h^{m_1m_2m_3}(\gamma_{m_1m_2m_3})_{\alpha\beta}\delta_i^{\ j}\Big\}\ ,
\end{aligned} \tag{78}$$

where γ-matrices always appear in tangent frame (*i.e.*, $\gamma_m = \delta_m^a\gamma_a$) and we use the definition

$$\gamma^{a_1 a_2 a_3 \dots a_{2n}} = \gamma^{[a_1}\tilde{\gamma}^{a_2}\gamma^{a_3}\dots\gamma^{a_{2n}]} \ . \tag{79}$$

Clearly this agrees with the expression given in [5] when $h = 0$. It is also easy to see, using the self-duality of h, that this expression agrees with the one given in [3] when only one scalar is active.

3.1 Travelling-waves on Calibrated Surfaces

Before reconsidering some of the solutions discussed in section 2, we shall first discuss supersymmetric travelling wave solutions. In the simplest setting of a flat fivebrane we expect either purely left-moving or right-moving transverse oscillations moving at the speed of light to preserve supersymmetry. This is simply the fivebrane analogue of the string travelling waves discussed in detail in [22]. In fact we will be more general and show that such travelling waves exist on calibrated fivebrane worldvolumes. Specifically, the waves propagate along a flat direction σ^5 of the fivebrane and are simply fluctuations in the "shape" of the calibrated surface.

We again suppose that we have a calibrated surface in the $\sigma^1, \dots, \sigma^4, X^7, \dots, X^{10}$ plane with no dependence on the coordinate σ^5 and some spinor zero modes ϵ of (78). Let us suppose that we can introduce the projector

$$\epsilon\gamma^0\gamma^5 = \pm\epsilon \ , \tag{80}$$

and still preserve some supersymmetry. Some examples of intersecting fivebrane configurations for which this projector automatically follows from the calibration are given in [5]. To describe these waves it is helpful introduce the light-cone coordinates

$$u = \frac{1}{\sqrt{2}}(\sigma^5 \mp \sigma^0) \ , \quad v = \frac{1}{\sqrt{2}}(\sigma^5 \pm \sigma^0) \ , \tag{81}$$

and from now on we assume that $a, b, c, \dots$ and $m, n, p, \dots$ take the values $1, 2, 3, 4$ only. The flat metric then has the following non-zero components

$$\eta_{uv} = 1, \quad \eta_{mn} = \delta_{mn} \ . \tag{82}$$

In these coordinates the projector is just $\epsilon\gamma^u = 0$.

Next we turn on a dependence on the coordinate u only. To be more specific we now allow for all of the scalars, including the scalar X^6, to be functions of u but not v, *i.e.*, $\partial_v X^{a'} = 0$. The supersymmetry condition (78) is now, recalling that our spinors ϵ are zero modes when $\partial_u X^{a'} = 0$,

$$\begin{aligned}\hat{\delta}\Theta = \quad & \frac{1}{2}\epsilon\Big\{\det(e^{-1})\partial_u X^{a'}\gamma^u\gamma_{a'} \\ & -\frac{1}{3!}\det(e^{-1})\partial_u X^{c_1'}\partial_{m_2}X^{c_2'}\partial_{m_3}X^{c_3'}\gamma^{um_2m_3}\gamma_{c_1'c_2'c_3'} \\ & +\frac{1}{5!}\det(e^{-1})\partial_u X^{c_1'}\partial_{m_2}X^{c_2'}\ldots\partial_{m_5}X^{c_5'}\gamma^{um_2\ldots m_5}\gamma_{c_1'\ldots c_5'}\Big\} \quad (83)\end{aligned}$$

It is not hard to see that, on account of (76) and (79), $\gamma^{ua_1a_2} = \gamma^u\gamma^{a_1a_2}$ and $\gamma^{ua_1a_2a_3a_4} = \gamma^u\gamma^{a_1a_2a_3a_4}$. Thus the projector $\epsilon\gamma^u = 0$ clearly implies that $\hat{\delta}\Theta = 0$ and supersymmetry is again preserved for any dependence on u.

Lastly one can check that the equation of motion, *i.e.*, the Laplacian with respect to the induced metric, continues to vanish. In addition the six-volume of the fivebrane is unaffected

$$\det g = -\det\bar{g}\ , \quad (84)$$

where $\bar{g}_{mn}$ is the metric of the calibrated surface with $\partial_u X^{a'} = 0$. Thus these configurations are area minimising in the sense that their six-volume is constant and is the same as the static calibrated surface, although the spatial part of the volume form is not constant.

3.2 Neutral Strings on Calibrated Surfaces

Let us now see how one can describe neutral strings on a calibrated surface of section 2 in the covariant formalism. In the u, v coordinates defined above h takes on the form

$$h_{uva} = V_a\ , \quad h_{uab} = F_{ab}\ , \quad h_{vab} = G_{ab}\ . \quad (85)$$

Self-duality then implies that $h_{abc} = \mp\epsilon_{abcd}V^d$ and also that $F_{ab} = \pm\frac{1}{2}\epsilon_{abcd}F^{cd}$ and $G_{ab} = \mp\frac{1}{2}\epsilon_{abcd}G^{cd}$ and again $a, b, c... = 1, 2, 3, 4$. The matrix m takes the form

$$m = \begin{pmatrix} 1+4V^2 & -2F^2 & 8V_cF^{bc} \\ -2G^2 & 1+4V^2 & -8V_cG^{bc} \\ -8V^cG_{ac} & 8V^cF_{ac} & (1-4V^2)\delta_a^{\ b} + 8V_aV^b \end{pmatrix} . \quad (86)$$

To describe neutral strings we will set $V_a = 0$. If we assume that ϵ is a preserved supersymmetry for a calibrated surface then from (78) we now have

$$\begin{aligned}\hat{\delta}\Theta \quad = \quad & -\frac{1}{2}\epsilon\Big[F^{mn}\gamma_{vmn} + G^{mn}\gamma_{umn} + F^{mn}\gamma_v\partial_m X^I\partial_n X^J\gamma_{IJ} \\ & +G^{mn}\gamma_u\partial_m X^I\partial_n X^J\gamma_{IJ}\Big]\ . \quad (87)\end{aligned}$$

Again we suppose that we may consider the additional projector $\epsilon\gamma^u = 0$ without breaking all the supersymmetries. Just as with (83) the contribution of the F_{mn} terms vanishes automatically. However, since $\epsilon\gamma_u \neq 0$, we see that we must set $G^{mn} = 0$ to preserve supersymmetry.

Finally we need to find the form H. In this case using (65) one simply finds that the only non-zero component is

$$H_{umn} = F_{mn} \ . \tag{88}$$

Clearly the closure of H asserts that F is an arbitrary function of u and satisfies the standard Bianchi identity. Note that in the Hamiltonian formalism F was an arbitrary function of σ^5. From the covariant picture, we see that this can be achieved by taking a time slice of a non-static configuration. The most general configuration corresponds to an abelian instanton in the transverse space which can change its "moduli" along the length of the string as in equations (38) and (40), with σ^5 replaced by u. Thus the left- and right-moving supersymmetric modes of the string live in the moduli space of abelian instantons. This resonates with the description of the non-critical six-dimensional string as a sigma model on the moduli space of non-abelian instantons [23].

3.3 Self-Dual Strings and Neutral Strings

It is insightful to also consider the case of self-dual and neutral strings in the covariant formalism. (We shall not consider the most general case of adding a calibrated surface here). In the u, v coordinates defined above h takes on the form (85) and we now consider $V_a \neq 0$. As above we must set $G_{ab} = 0$ in order to preserve any supersymmetry. However the appearance of the self-dual string requires that one of the fivebrane scalars are active, say $X = X^6$, and we assume that $\partial_{u,v}X = 0$. The condition for supersymmetry to be preserved by this configuration is then

$$0 = \epsilon\Big[\frac{1}{2}\det(e^{-1})\gamma^m\partial_m X\gamma_6 - \gamma^{uv}(V^m\gamma_m + \det(e^{-1})V_m\gamma^m) - \frac{1}{2}\gamma^u\gamma^{mn}F_{mn}\Big] \ . \tag{89}$$

Just as was the case for a single self-dual string [3], we restrict to spinors which satisfy

$$\epsilon\gamma^{uv}\gamma_6 = \epsilon \ , \tag{90}$$

and set

$$V_a = \frac{1}{2}\frac{1}{1+\det(e)}\delta_a{}^n\partial_n X \ . \tag{91}$$

To include the neutral strings we further impose the supersymmetry projector $\epsilon\gamma^u = 0$.

Our next step is to calculate the three-form H_{nmp} and demand that it is closed. In the tangent frame we find

$$\begin{aligned} H_{uva} &= (1+4V^2)^{-1}V_a \ , \\ H_{abc} &= \mp(1-4V^2)^{-1}\epsilon_{abcd}V^d \ , \\ H_{uab} &= (1-4V^2)^{-1}F_{ab} + 8(1-16V^4)^{-1}(V_aV^cF_{bc} - V_bV^cF_{ac}) \ , \\ H_{vab} &= 0 \ . \end{aligned} \tag{92}$$

However things simplify considerably when we construct H in the world frame and substitute (91). We find

$$\begin{aligned} H_{uvm} &= \frac{1}{4}\partial_m X \ , \\ H_{mnp} &= \mp\frac{1}{4}\epsilon_{mnpq}\delta^{qr}\partial_r X \ , \\ H_{umn} &= K_{mn} \ , \\ H_{vmn} &= 0 \ , \end{aligned} \tag{93}$$

where $K_{mn} = (1-4V^2)\delta^a_m\delta^b_n F_{ab}$. Thus the closure of H leads to the simple equations

$$\delta^{mn}\partial_m\partial_n X = 0 \ , \quad \partial_v K_{mn} = 0 \ , \quad \partial_{[m}K_{np]} = 0 \ . \tag{94}$$

Thus X is harmonic on the flat transverse space and K_{mn} can be interpreted as a (anti-) self-dual field strength of a (possibly u-dependent) vector potential.

3.4 Two Intersecting Self-Dual Strings

As a final solution in the covariant formalism we now reconsider two interacting self-dual strings. In particular we will consider the dependence of each string on the relative transverse coordinates. The dynamics of this solution and its relation to the Seiberg-Witten effective action is discussed in [24].

The configuration that we are interested in may be written as

$$\begin{array}{lccccccc} M5: & 1 & 2 & 3 & 4 & 5 & & \\ M2: & & & & 4 & & 6 & \\ M2: & & & & & 5 & & 7 \\ M5: & 1 & 2 & 3 & & & 6 & 7 \end{array} \tag{95}$$

for which (8) is a special case (after relabelling) with only one fivebrane present. The supersymmetry projectors for the two membranes are

$$\epsilon\gamma^{05}\gamma_7 = \eta\epsilon \ , \quad \epsilon\gamma^{04}\gamma_6 = -\eta\epsilon \ , \tag{96}$$

where $\eta = \pm 1$. Note that as a result of these two projectors we may add another fivebrane in "for free"

$$\epsilon\gamma^{45}\gamma_{67} = -\epsilon \ , \tag{97}$$

which we have already included in (95). Thus we expect to obtain two self-dual strings (or a single non-trivial string) on a Riemann surface corresponding to (3).

Rather than use the light cone coordinates of the previous sections it is helpful instead to introduce complex notation

$$z = \sigma^4 + i\sigma^5 \ , \quad s = X^6 + iX^7 \ , \tag{98}$$

with the derivatives $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$. The projectors can then simply be written as

$$\epsilon\gamma_{0z} = \eta\epsilon\gamma_{\bar{s}} \ , \quad \epsilon\gamma^{z}\gamma_s = 0 \ . \tag{99}$$

In total this configuration preserves one quarter of the fivebrane's world-volume supersymmetry. In this section the indices $a, b, c, ... = 0, 1, 2, 3$ are in tangent frame, the indices $\mu, ... = 0, 1, 2, 3$ are in world frame and we also take $i, j, k, ... = 1, 2, 3$ in the world frame. For simplicity we will only consider the solution to second order in the spatial derivatives ∂_i. This will simplify our calculations, however it is reasonable to hope that the end result is valid to all orders.

Our next step is to decompose h into a four-dimensional vector V_a and anti-symmetric tensor $\mathcal{F}_{ab}$ as follows (all indices are in the tangent frame)

$$h_{abz} = \mathcal{F}_{ab} \ , \quad h_{ab\bar{z}} = \bar{\mathcal{F}}_{ab} \ , \quad h_{az\bar{z}} = iV_a \ . \tag{100}$$

Self-duality implies that $h_{abc} = 2\epsilon_{abcd}V^d$ and $\mathcal{F}_{ab} = \frac{i}{2}\epsilon_{abcd}\mathcal{F}^{cd}$. For the convenience of the reader we list the components of the vielbein $e_m^{\ a}$ for the geometry resulting from s

$$\begin{aligned} e_\mu^{\ a} &= \delta_\mu^{\ a} - \frac{1}{2}\left(\frac{1}{\det e}\right)^2 (\bar{\partial}s\partial s\partial_\mu\bar{s}\partial^a\bar{s} + \partial\bar{s}\bar{\partial}\bar{s}\partial_\mu s\partial^a s) \ , \\ &\quad + \frac{1}{4}\left(\frac{1 + |\partial s|^2 + |\bar{\partial}s|^2}{(\det e)^2}\right)(\partial_\mu s\partial^a\bar{s} + \partial_\mu\bar{s}\partial^a s) \ , \\ e_\mu^{\ z} &= \frac{(X^2 - |\partial s|^2)\bar{\partial}s\partial_\mu\bar{s} + (X^2 - |\bar{\partial}s|^2)\bar{\partial}\bar{s}\partial_\mu s}{X\det e} \ , \end{aligned}$$

$$
\begin{aligned}
e_\mu{}^{\bar z} &= \frac{(X^2-|\partial s|^2)\partial\bar s\partial_\mu s+(X^2-|\bar\partial s|^2)\partial s\partial_\mu\bar s}{X\det e}\ ,\\
e_z{}^{\bar z} &= \frac{\partial s\partial\bar s}{X}\ ,\qquad e_{\bar z}{}^{z}=\frac{\bar\partial\bar s\bar\partial s}{X}\ ,\\
e_z{}^{z} &= e_{\bar z}{}^{\bar z}=X\ ,
\end{aligned}
\tag{101}
$$

where

$$
\begin{aligned}
X^2 &= \frac{1}{2}\left[(1+|\partial s|^2+|\bar\partial s|^2)+\det e\right]\ ,\\
\det e &= \sqrt{(1+|\partial s|^2+|\bar\partial s|^2)^2-4|\partial s|^2|\bar\partial s|^2}\ .
\end{aligned}
\tag{102}
$$

Note that $\det e$ denotes the part of the determinant of the vielbein that is independent of spatial derivatives. Next we note that the projectors (99) imply that there are four independent terms appearing in supersymmetry condition $\hat\delta\Theta=0$ proportional to

$$
\epsilon\gamma_{0iz}\ ,\quad \epsilon\gamma_{0z\bar z}\ ,\quad \epsilon\gamma_{iz\bar z}\ ,\quad \epsilon\gamma_0\ ,
\tag{103}
$$

and their complex conjugates. Thus we may obtain the Bogomol'nyi equations by setting the corresponding coefficients to zero. Using the decomposition (100) this yields

$$
\begin{aligned}
\mathcal{F}_{0i} &= \frac{1}{8}\eta\left(\frac{1+|\partial s|^2-|\bar\partial s|^2}{X^2-|\bar\partial s|^2}\right)\left(\frac{X^2\partial_i s+\partial\bar s\partial s\partial_i\bar s}{X\det e}\right)\ ,\\
V_0 &= +\frac{i}{16}\eta\left(\frac{1+|\partial s|^2-|\bar\partial s|^2}{(X^2-|\bar\partial s|^2)^2}\right)\left[(1+|\partial s|^2+|\bar\partial s|^2)\frac{\bar\partial s\partial_i s\partial^i\bar s}{(\det e)^2}\right.\\
&\qquad\left.+|\bar\partial s|^2\frac{(\partial s\partial_i\bar s\partial^i\bar s-\bar\partial\bar s\partial_i\partial^i s)}{(\det e)^2}\right]+\frac{i}{4}\eta\frac{\bar\partial s}{X^2-|\bar\partial s|^2},\\
V_i &= \frac{1}{16}\eta\bar\partial s\left(\frac{1+|\partial s|^2-|\bar\partial s|^2}{(X^2-|\bar\partial s|^2)^2}\right)\frac{\epsilon_{ijk}\partial^j s\partial^k\bar s}{\det e}\ ,\\
\bar\partial s &= -\partial\bar s\ ,
\end{aligned}
\tag{104}
$$

respectively. Here all indices are raised and lowered with the flat metric.

The next step is to calculate the physical three-form H defined in (65) which most naturally appears in the equations of motion. To help the reader we give here the matrix $m^{-1}=Q^{-1}(1+2k)$

$$
m^{-1}=Q^{-1}\begin{pmatrix}\delta_\mu{}^\nu+2k_\mu{}^\nu & 32i\bar\kappa v_0\bar{\mathcal{F}}_\mu{}^0 & -32i\kappa v_0\mathcal{F}_\mu{}^0\\ -16i\kappa v_0\mathcal{F}^{\nu0} & 1-16v_0^2 & 4\kappa^2\mathcal{F}^2\\ 16i\bar\kappa v_0\bar{\mathcal{F}}^{\nu0} & 4\bar\kappa^2\bar{\mathcal{F}}^2 & 1-16v_0^2\end{pmatrix}\ ,
\tag{105}
$$

where

$$\begin{aligned} k_\mu{}^\nu &= 8v_0^2\delta_\mu{}^\nu + 16v_\mu v^\nu + 4|\kappa|^2\mathcal{F}_{\mu\lambda}\bar{\mathcal{F}}^{\nu\lambda} + 4|\kappa|^2\bar{\mathcal{F}}_{\mu\lambda}\mathcal{F}^{\nu\lambda} , \\ Q &= 1 - 256v_0^2(v_0^2 - 2|\kappa|^2\mathcal{F}_{0i}\bar{\mathcal{F}}^{0i}) . \end{aligned} \tag{106}$$

Despite the complicated form of these Bogomol'nyi equations one finds after a lengthy calculation that the three-form H takes on a relatively simple form. In particular, in the world frame

$$\begin{aligned} H_{iz\bar{z}} &= 0 , \\ H_{0ij} &= 0 , \\ H_{0iz} &= \frac{1}{8}\eta\partial_i s , \qquad H_{0i\bar{z}} = \frac{1}{8}\eta\partial_i\bar{s} , \\ H_{ijz} &= \frac{i}{8}\eta\epsilon_{ijk}\partial^k s , \qquad H_{ij\bar{z}} = -\frac{i}{8}\eta\epsilon_{ijk}\partial^k\bar{s} , \\ H_{0z\bar{z}} &= -\frac{1}{4}\eta\bar{\partial}s , \\ H_{ijk} &= -\frac{i}{8}\eta\epsilon_{ijk}\left(\frac{4\bar{\partial}s + 2\bar{\partial}s\partial_i s\partial^i\bar{s} + \partial s\partial_i\bar{s}\partial^i\bar{s} - \bar{\partial}\bar{s}\partial_i s\partial^i s}{1 + |\partial s|^2 - |\bar{\partial}s|^2}\right) . \end{aligned} \tag{107}$$

Note that s is not holomorphic but instead satisfies $\bar{\partial}s = -\partial\bar{s}$. Indeed one sees that the complete dependence of fields on the relative transverse coordinates of the two self-dual strings is given by the non-holomorphicity of s. The equation of motion for s can then be found by demanding that $dH = 0$. If we set $\partial s = \bar{\partial}s = 0$ we then arrive at the solution in section five, with the equation of motion $\partial^i\partial_i s = 0$ resulting from $\partial_{[i}H_{jkz]} = 0$.

4 Conclusion

In this paper we have examined the conditions for the preservation of the non-linear supersymmetry of the fivebrane for any bosonic configuration in both the Hamiltonian and covariant formalisms. Furthermore we formulated the conditions and field equations for several supersymmetric solitons with a non-zero three-form field. In particular we found that self-dual and neutral strings correspond to harmonic functions and instantons on calibrated surfaces, respectively. To produce a specific solution one typically needs to solve the field equations, *i.e.,* construct harmonic functions or instantons on calibrated surfaces, and it would be interesting to consider some specific cases in more detail. All of the cases we have considered (perhaps setting $\partial_5 = 0$)

may be dimensionally reduced along σ^5 to obtain solutions of the D4-brane worldvolume theory. We expect that they have a natural generalisation to the non-Abelian DBI theory and this would be worth checking.

We have by no means produced an exhaustive list of solutions however it is hoped that the reader will have gained some insight into the general form of the solutions by means of these examples. In addition we hope that the reader will have gained some further understanding of the complicated non-linear theory on the fivebrane worldvolume in both the formalisms discussed.

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