

AdS/CFT dualities involving large 2d $N=4$ superconformal symmetry

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Abstract

We study the duality between string theory on $AdS_3 \times S^3 \times S^3$ and two-dimensional conformal theories with large $N = 4$ superconformal algebra \mathcal{A}_γ . We discuss configurations of intersecting

branes which give rise to such near-horizon geometries. We compute the Kaluza-Klein spectrum and propose that the boundary superconformal theory can be described by a sigma model on a suitable symmetric product space with a particular choice of anti-symmetric two-form.

1 Introduction

Among the AdS/CFT dualities [29, 22, 49] the case of AdS_3/CFT_2 is distinguished because generically one has better control of both sides of the correspondence. On the one side, one deals with $2d$ superconformal field theories (SCFT) which have been quite extensively studied. On the other side, one has a three-dimensional AdS gravity which is quite simple by itself. In addition, the near-horizon limit generically involves (after suitable dualities) exact WZW models for the various parts that make up the near-horizon configuration, so one can go beyond the supergravity approximation by considering string theory on AdS_3 [20, 5, 27].

It is thus quite natural to try to fully explore these cases and hopefully learn some lessons for the higher dimensional cases as well. To that end one can try to formulate the correspondence in a setting such that there is as much control over the theory as possible. This can happen either if we are dealing with a theory with very little structure, like a free fermion, or with a theory with a large symmetry group. In this paper we will follow the second route. We will study the correspondence in the case the boundary superconformal theory has the maximal possible linearly realized symmetry, i.e. the symmetry algebra is the large (or double) $N = 4$ superconformal algebra \mathcal{A}_γ [42, 39]. This algebra contains two commuting affine $\widehat{SU(2)}$ Lie algebras (in contrast, the small $N = 4$ algebra contains only one affine $\widehat{SU(2)}$ Lie algebra). It also contains the finite dimensional superalgebra $D^1(2, 1, \gamma/(1 - \gamma))$. Conformal models with this symmetry algebra are characterized by two integers, the two levels k^+ and k^- of the two affine $\widehat{SU(2)}$ algebras ($\gamma = k^-/(k^+ + k^-)$).

One might expect that these models would be easy to analyze because of the large amount of symmetry. However, this turns out not

to be the case as the structure of the superconformal algebra is quite non-trivial. In supersymmetric theories the analysis of BPS sector of the theory is usually tractable. For instance, in theories with the $N = 2$ or the small $N = 4$ symmetry algebra there is a linear relation between the conformal dimension and charge of a BPS state. This implies that the BPS states form a ring. In the case of \mathcal{A}_γ there is still a Bogomolnyi bound. The relation, however, between the conformal weight and the charges is non-linear. This complicates the analysis. In particular, for the brane realization that we will study the non-linear piece is sub-leading in $1/N$ ¹ and therefore corresponds to string loop corrections. Thus, we find the novel situation that the mass formula for BPS states receives quantum corrections. This leads to the possibility that states that satisfy a bound in supergravity cease to do so in string theory. In addition, BPS states do not form a ring (but they may form a module over a ring).

AdS/CFT dual pairs can be obtained by considering a configuration of branes and taking a limit in which a decoupled worldvolume theory is obtained. This limit is at the same time a near-horizon limit for the corresponding supergravity configuration. The near-horizon isometry superalgebra becomes the worldvolume superconformal algebra. A brane configuration that leads to a dual superconformal theory with symmetry algebra \mathcal{A}_γ was identified in [6]. It consists of a “non-standard” intersection of two M5 branes with an M2 brane. In the near-horizon limit the geometry contains $AdS_3 \times S^3 \times S^3$. The isometry groups of the two spheres become the R-symmetry $SU(2)$ s, and the radii of the spheres are related to the levels of the $\widehat{SU(2)}$ s. Brane configurations with the same near-horizon limit exist in all string theories. In particular, in IIB string theory a configuration that leads to this geometry consists of an overlap of two D1-D5 systems (similarly the M-theory configuration may be thought of as an overlap of two M2-M5 systems). The near-horizon limit in the case of M-branes is the standard low-energy limit. In the case of IIB branes, however, one needs to consider an ultra-low energy limit.

Another motivation for studying this system is that it can be viewed as the master configuration from which one can reach other systems in-

¹ N can stand for $k \equiv k^+ + k^-$, k^+/k^- or k^-/k^+ . In the large N limit the number of branes of certain types always becomes large. When we talk about $1/N$ corrections in later sections it will be clear from the context what is meant by N .

volving in their near-horizon limit a factor of AdS_2 or AdS_3 and one or two factors of S^2 and/or S^3 by either adding branes and/or taking appropriate limits [6]. Therefore, a complete understanding of the $AdS_3 \times S^3 \times S^3$ configuration may lead to a unified picture of all the other cases as well.

In this paper we begin a detailed analysis of the $AdS_3 \times S^3 \times S^3/\mathcal{A}_\gamma$ duality. Previous work can be found in [14]. One of our aims is to identify the dual superconformal theory. We will propose, generalizing the proposal of [14, 19], that the dual SCFT is a sigma model with target space the symmetric product $\text{Sym}^{k^-}(U(2))$, with k/k^- units of $H = dB$ flux and certain discrete “gauge fields” associated to the permutation group S_{k^-} turned on. The latter are needed to make the conformal field theory well-defined. An alternative description is obtained by exchanging the roles of k^+ and k^- .

We start our analysis by studying the supergravity solutions. The first problem one encounters is that the near-horizon geometry contains an extra non-compact isometry. In order to be able to reduce the theory to three dimensions we need to compactify this direction. The corresponding identifications induce a novel UV/IR relation between the worldvolume theories of the two overlapping brane configurations. Furthermore, the requirement that the Brown-Henneaux central charge [8] matches the central charge of the \mathcal{A}_γ theory fixes the product of the radius of the extra circle and the string coupling constant. The ratio of the two is a modulus of the solution. For fixed string coupling constant, the radius can be made very small at large N .

To obtain some information about the spectrum of the conformal field theory we compute the Kaluza-Klein (KK) spectrum of supergravity on $adS^3 \times S^3 \times S^3 \times S^1$. Since the radius of S^1 can be made very small in the large N limit we start from $9d$ supergravity. The KK spectrum is obtained by using the group theory method developed in [28, 4]. The rather tedious computation yields a quite simple spectrum. The KK states fit in short multiplets of $D^1(2, 1, \alpha)$, but many short multiplets can be combined into long multiplets and we get only limited information about the set of BPS states at a generic point in the moduli space.

Having obtained the single particle KK spectrum the next step is to obtain the chiral spectrum of the dual conformal field theory. The main

question is which of the single and multiparticle states correspond to chiral states of the boundary SCFT. Because of the non-linearity of the BPS bound this is a difficult question, especially since the non-linearity is invisible in supergravity. We will consider various possible answers to this question and discuss which ones are consistent with the above mentioned proposal for a candidate boundary SCFT. Notice that to verify directly which of the single and multiparticle states are actually massive we would need to calculate $1/N$ corrections as the BPS bound contains terms subleading in $1/N$.

Another way to obtain information about the boundary superconformal theory is to use D-brane perturbation theory. The boundary theory is then the infrared limit of the worldvolume gauge theory. This is expected to be (a deformation of) a sigma model with target space the moduli space of vacua of the worldvolume gauge theory. One may study the D-brane system by introducing a probe brane. A new feature in our case is that the theory on the probe contains new couplings due to loops of open strings that cannot be ignored even at low-energies since they involve massless particles. These couplings, however, are subleading in $1/N$. Thus, only in the large N limit one expects that the \mathcal{A}_γ theory can be described by a perturbative sigma model. Indeed, we find that $1/N$ is what controls the loop expansion of the sigma models with \mathcal{A}_γ symmetry.

The paper is organized as follows. In section 2 we present the relevant supergravity solutions, both in M-theory and in type IIB. The main properties of the large $N = 4$ superconformal algebra \mathcal{A}_γ as well as the closely related non-linear algebra $\tilde{\mathcal{A}}_\gamma$ are recalled in section 3. Section 4 contains the computation of the Kaluza-Klein spectrum. In section 5 we discuss the multiparticle spectrum and we present our proposal for the boundary SCFT. Section 6 contains some remarks on the the D-brane analysis of the system, and section 7 a discussion of σ -models with \mathcal{A}_γ symmetry. Open problems are discussed in section 8. In the appendix we present our conventions.

2 Solitonic description

In this section we review supergravity solutions which in the “near-horizon” limit yield a solution of the form $adS_3 \times S^3 \times S^3 \times S^1$. Supergravities in 10 dimensions (i.e. type IIA, type IIB and type I) as well as 11d supergravity have solutions describing intersecting branes that in an appropriate limit approach a geometry that contains $adS_3 \times S^3 \times S^3$.² The type IIA solution can be obtained from the M-theory solution by reduction. The type I solution can be obtained from either the IIA or the IIB solution. The M-theory and IIB solution seem not to be related by dualities (at least in an obvious manner) so we will discuss them separately.

Before presenting the solutions we briefly discuss their connection to solutions involving a factor of adS_2 and/or S^2 's in the near-horizon limit. There is a simple way, the wave/monopole rule[6], to generate solutions that contain a factor of adS_2 and/or S^2 in their near-horizon limit starting from a solution that contains adS_3 and/or S^3 . To get a factor of adS_2 one has to add a wave to the solution, after which one has to T-dualize it (or reduce it if we start from eleven dimensions) in the direction of the wave. To get a solution that involves an S^2 one has to add a KK monopole, after which one has to T-dualize (or reduce) in the nut-direction.³ In addition, the wave/monopole rule allows one to determine the isometry superalgebra of the new solution (since one can obtain both the killing spinors and the bosonic isometries starting from the killing spinors and the bosonic isometries of the solution that involves adS_3 and/or S^3), and therefore the symmetry algebra of the dual superconformal theory. All solutions found in this way involve exact CFT's in their near-horizon limit. (Actually one can trace the origin of the wave/monopole rule to the relation between the exact CFT's associated to adS_3 , adS_2 and S^3 , S^2). Details can be found in [6]. The solutions can be further generalized to include rotation [18]. This does not change the near-horizon configuration as the effect of the rotation can be removed by a coordinate transformation. Further

²A solution of this form appeared first in the context of the heterotic string theory in [1], and as a near-horizon limit of a configuration of intersecting branes in type I supergravity in [9].

³Notice that the “near-horizon” limit of the final configuration is a “very-near-horizon” limit[45] of the original one. This is so because the T-duality that connects the two configuration involves explicit factors of α' .

solutions with the same near-horizon geometry can be found in [34].

2.1 M-theory case

Consider the configuration

$$\begin{array}{cccccccccc} M5_1 & 1 & & 3 & 4 & 5 & 6 & & & \\ M5_2 & 1 & & & & & & 7 & 8 & 9 & 10 \\ M2 & 1 & 2 & & & & & & & & \end{array}$$

The explicit solution belonging to this configuration is given by [47, 16]

$$\begin{aligned} ds^2 = & (H_T)^{\frac{1}{3}} (H_F^{(1)} H_F^{(2)})^{\frac{2}{3}} \{ (H_T H_F^{(1)} H_F^{(2)})^{-1} (-dt^2 + dx_1^2) \\ & + (H_T)^{-1} dx_2^2 + (H_F^{(1)})^{-1} (dx_3^2 + \cdots + dx_6^2) \\ & + (H_F^{(2)})^{-1} (dx_7^2 + \cdots + dx_{10}^2) \}, \end{aligned} \quad (2.1)$$

$$F_{012I} = -\partial_I (H_T)^{-1}, \quad F_{2m'n'p'} = \epsilon_{m'n'p'q'} \partial_{q'} H_F^{(1)}, \quad F_{2mnp} = \epsilon_{mnpq} \partial_q H_F^{(2)},$$

where I runs over all $m \in \{3, 4, 5, 6\}$ and $m' \in \{7, 8, 9, 10\}$. $H_F^{(1)}(x')$ and $H_F^{(2)}(x)$ are harmonic functions in the relative transverse directions,

$$H_F^{(1)} = 1 + \frac{Q_F^{(1)}}{r'^2}, \quad H_F^{(2)} = 1 + \frac{Q_F^{(2)}}{r^2}, \quad (2.2)$$

where $r^2 = x_3^2 + \cdots + x_6^2$, $r'^2 = x_7^2 + \cdots + x_{10}^2$ and $Q_F^{(i)} = N_F^{(i)} l_p^2$, $i = 1, 2$. $N_F^{(i)}$ is equal (up to a numerical constant) to the number of coincident fivebranes. $H_T(x, x')$ satisfies [47, 17]

$$\left(H_F^{(1)}(x') \partial_x^2 + H_F^{(2)}(x) \partial_{x'}^2 \right) H_T(x, x') = 0. \quad (2.3)$$

This equation can be solved by

$$H_T = \left(1 + \frac{Q_T^{(1)}}{r'^2} \right) \left(1 + \frac{Q_T^{(2)}}{r^2} \right) \quad (2.4)$$

The charges $Q_T^{(i)}$ are equal to $N_T^{(i)} l_p^2$, where the quantities $N_T^{(1)}$ and $N_T^{(2)}$ are membrane densities in (x^3, x^4, x^5, x^6) and (x^7, x^8, x^9, x^{10}) , respectively. Since there are two harmonic functions associated with the membrane one may interpret the solution as an overlap of two M2-M5

systems. In the near horizon limit the solution will only depend on the product $N_T^{(1)} N_T^{(2)}$ which we will denote by N_T .

We now consider the low energy limit, in which we keep the masses of stretched membranes and the lengths in the x_2 direction fixed in Planck units (this means that we keep fixed the string coupling constant in the corresponding type IIA configuration)

$$l_p \rightarrow 0, \quad U = \frac{r^2}{l_p^3} = \text{fixed}, \quad U' = \frac{r'^2}{l_p^3} = \text{fixed}. \quad (2.5)$$

The geometry becomes

$$\begin{aligned} \frac{ds^2}{l_p^2} = & (Q_3)^{-1} U U' (-dt^2 + dx_1^2) + Q_4 dx_2^2 \\ & + \frac{Q_1}{4} \frac{dU^2}{U^2} + \frac{Q_2}{4} \frac{dU'^2}{U'^2} + Q_1 d\Omega_{(1)}^2 + Q_2 d\Omega_{(2)}^2 \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} Q_1 &= \left(\frac{N_T}{N_F^{(1)}} \right)^{1/3} (N_F^{(2)})^{2/3}, & Q_2 &= \left(\frac{N_T}{N_F^{(2)}} \right)^{1/3} (N_F^{(1)})^{2/3}, \\ Q_3 &= (N_T)^{2/3} \left(N_F^{(1)} N_F^{(2)} \right)^{1/3}, & Q_4 &= \left(\frac{N_F^{(1)} N_F^{(2)}}{N_T} \right)^{2/3} \end{aligned} \quad (2.7)$$

We introduce new variables

$$u^2 = l^2 \frac{U U'}{Q_3}, \quad \lambda = \frac{l}{2} \left(\sqrt{\frac{Q_1}{Q_2}} \log U - \sqrt{\frac{Q_2}{Q_1}} \log U' \right), \quad l = \sqrt{\frac{Q_1 Q_2}{Q_1 + Q_2}}. \quad (2.8)$$

The metric becomes

$$\begin{aligned} \frac{ds^2}{l_p^2} = & \left[\left(\frac{u}{l} \right)^2 (-dt^2 + dx_1^2) + l^2 \frac{du^2}{u^2} \right] \\ & + d\lambda^2 + Q_1 d\Omega_{(1)}^2 + Q_2 d\Omega_{(2)}^2 + Q_4 dx_2^2 \end{aligned} \quad (2.9)$$

This is a metric for $adS_3 \times S^3 \times S^3 \times E^2$. The field strengths are equal to

$$F_{\kappa\mu\nu 2} = 2l^{-1} Q_4^{1/2} l_p^3 \epsilon_{\kappa\mu\nu}, \quad F_{\alpha\beta\gamma 2} = 2N_F^{(2)} l_p^3 \epsilon_{\alpha\beta\gamma}, \quad F_{\alpha'\beta'\gamma' 2} = 2N_F^{(1)} l_p^3 \epsilon_{\alpha'\beta'\gamma'}, \quad (2.10)$$

where $\kappa, \mu, \nu \in \{0, 1, u\}$ are adS_3 indices, $\epsilon_{\kappa\mu\nu}$ is the volume form of the adS_3 , α and α' are indices for the two S^3 factors, respectively, and $\epsilon_{\alpha\beta\gamma}$ and $\epsilon_{\alpha'\beta'\gamma'}$ are volume forms for the corresponding unit spheres. The field strengths are covariantly constant. One can check that the explicit factors of l_p in the solution cancel against the factors of l_p in Newton's constant, so we set $l_p = 1$ from now on.

Let us briefly recall the analysis of supersymmetry from [6]. One can easily check that the solution (2.1) preserves 1/4 of the supersymmetries. In the near-horizon limit this is enhanced by a factor of 2. The Killing spinors are products of the geometric Killing spinors on AdS_3 and the three-spheres as we now discuss. To analyze the supersymmetry it is most convenient to choose a basis for the Dirac matrices that is adapted to the geometry of the space. Such basis is given in (18) of [6]. The 11d spinors ϵ are also decomposed correspondingly: $\epsilon = \eta \otimes \rho \otimes \rho' \otimes \xi \otimes \chi$, where η is a spinor on AdS_3 , ρ and ρ' are spinors on the two S^3 's, ξ is a spinor on the two-dimensional space spanned by x_2 and λ , and χ is an extra two-component spinor. All spinors are two-component ones, so we get that ϵ has 32 components as it should. We refer to [6] for details, here we only give the final solution. The killing spinor equations are satisfied if η , ρ , ρ' are geometric killing spinors on AdS_3 and the two S^3 's:

$$\begin{aligned} D_\mu \eta \pm \frac{1}{2} \frac{Q_4^{1/2}}{l} \gamma_\mu \eta &= 0, \\ D_\alpha \rho \pm \frac{i}{2} \frac{N_F^{(2)}}{Q_1^{3/2}} \gamma_\alpha \rho &= 0, \\ D_{\alpha'} \rho' \pm \frac{i}{2} \frac{N_F^{(1)}}{Q_2^{3/2}} \gamma_{\alpha'} \rho' &= 0. \end{aligned} \quad (2.11)$$

In addition, there is one projection on the $\xi \otimes \chi$ spinors, $\mathcal{P}(\xi \otimes \chi) = \frac{1}{2}(1 + \Gamma)(\xi \otimes \chi) = 0$, where

$$\Gamma = \frac{l}{Q_4^{1/2}} i \gamma^2 \gamma^\lambda \otimes \left(\frac{N_F^{(2)}}{Q_1^{3/2}} \sigma_3 - \frac{N_F^{(1)}}{Q_2^{3/2}} \sigma_1 \right) \quad (2.12)$$

where γ^2, γ^λ are gamma matrices in the Euclidean x_2, λ space, and σ_1, σ_3 are Pauli matrices. One can check that Γ is traceless and $\Gamma^2 = 1$, so the projection breaks 1/2 of the supersymmetry. In (2.11) the signs are correlated, namely the 11d Killing spinor is either a product of

Killing spinors that solve the equations (2.11) with the plus sign or the ones that solve the equations with the minus sign. Equations (2.11) have maximal number of solutions. Therefore, the near-horizon solution preserves 16 supercharges. The form of the killing spinors plus the bosonic symmetries already imply that the isometry superalgebra is $D^1(2, 1, \alpha)$. This has been explicitly verified in [18] by constructing the isometry superalgebra from the killing spinors.

From the M-theory solution (2.1) we can obtain a solution of IIA supergravity describing an intersection of two solitonic fivebranes over a fundamental string upon reduction over x_2 . Its “near-horizon” limit can be obtained from (2.9):

$$\begin{aligned} ds^2 &= \left[\left(\frac{u}{l} \right)^2 (-dt^2 + dx_1^2) + l^2 \frac{du^2}{u^2} \right] \\ &\quad + Q_4^{1/2} d\lambda^2 + N_F^{(2)} d\Omega_{(1)}^2 + N_F^{(1)} d\Omega_{(2)}^2 \\ H_{\kappa\mu\nu} &= 2l^{-1} \epsilon_{\kappa\mu\nu}, \quad H_{\alpha\beta\gamma} = 2N_F^{(2)} \epsilon_{\alpha\beta\gamma}, \quad H_{\alpha'\beta'\gamma'} = 2N_F^{(1)} \epsilon_{\alpha'\beta'\gamma'}, \\ e^{-2\phi} &= \frac{N_T}{N_F^{(1)} N_F^{(2)}} \end{aligned} \quad (2.13)$$

where

$$\frac{1}{l^2} = \frac{1}{N_F^{(1)}} + \frac{1}{N_F^{(2)}} \quad (2.14)$$

and⁴ $g_s = (R_2)^{3/2}$, where R_2 is the radius of x_2 .⁵

Notice that the values of the fields in (2.13) are the canonical ones such that there are exact CFT's associated with each factor, namely an $SL(2, R)$ WZW model for the adS_3 part and two $SU(2)$ WZW models at level $N_F^{(1)}$ and $N_F^{(2)}$, respectively, for the two S^3 's. This implies a quantization condition for $N_F^{(i)}$, $i = 1, 2$. In the original solution this quantization was due to a quantization of the magnetic fluxes over the two S^3 's.

⁴We use the convention to leave a factor of g_s^2 in Newton's constant, so in the full solution the dilaton field vanishes asymptotically. S-duality acts as $g_s \rightarrow 1/g_s$, $\alpha' \rightarrow \alpha' g_s$, and T-duality as $R \rightarrow \alpha'/R$, $g_s \rightarrow g_s \sqrt{\alpha'}/R$, see [44] for details.

⁵Notice that $N_T^{(i)}$, $i = 1, 2$ are the membrane densities in eleven dimensional Planck units. Since the 11d Planck scale differs from the string scale by factors of g_s , the IIA solution that describes an intersection of a number of solitonic fivebranes with a density (in 10d units) of fundamental strings will involve extra factors of g_s in the dilaton field.

2.2 IIB case

The type IIB configuration that has a “near-horizon” limit of the form $adS_3 \times S^3 \times S^3$ can be thought of as an overlap of two D1-D5 systems,

$$\begin{array}{ccccccccc} D5_1 & 1 & 2 & 3 & 4 & 5 & & & & \\ D1_1 & 1 & & & & & & & & \\ D5_2 & 1 & & & & & 6 & 7 & 8 & 9 \\ D1_2 & 1 & & & & & & & & \end{array}$$

The harmonic functions of the first D1-D5 system depend on the relative transverse directions of the second D1-D5 system and vice versa. The explicit form of the solution is

$$\begin{aligned} ds^2 &= (H_1^{(1)} H_5^{(1)})^{-1/2} (H_1^{(2)} H_5^{(2)})^{-1/2} (-dt^2 + dx_1^2) \\ &+ \left(\frac{H_1^{(1)}}{H_5^{(1)}} \right)^{1/2} (H_1^{(2)} H_5^{(2)})^{1/2} (dr^2 + r^2 d\Omega_{(1)}^2) \\ &+ \left(\frac{H_1^{(2)}}{H_5^{(2)}} \right)^{1/2} (H_1^{(1)} H_5^{(1)})^{1/2} (dr'^2 + r'^2 d\Omega_{(2)}^2) \\ H_{01I} &= -\partial_I (H_1^{(1)} H_1^{(2)})^{-1}, \quad H_{m'n'p'} = \epsilon_{m'n'p'q'} \partial_{q'} H_5^{(1)} \quad H_{mnp} = \epsilon_{mnpq} \partial_q H_5^{(2)} \\ e^{-2\phi} &= \left(\frac{H_5^{(1)}}{H_1^{(1)}} \right) \left(\frac{H_5^{(2)}}{H_1^{(2)}} \right), \quad r^2 = x_2^2 + \dots + x_5^2, \quad r'^2 = x_6^2 + \dots + x_9^2 \end{aligned} \quad (2.15)$$

where I runs over all $m \in \{2, 3, 4, 5\}$ and $m' \in \{6, 7, 8, 9\}$. The harmonic functions are equal to

$$\begin{aligned} H_1^{(1)} &= 1 + \frac{Q_1^{(1)} \alpha'}{r'^2}, & H_5^{(1)} &= 1 + \frac{Q_5^{(1)} \alpha'}{r'^2} \\ H_1^{(2)} &= 1 + \frac{Q_1^{(2)} \alpha'}{r^2}, & H_5^{(2)} &= 1 + \frac{Q_5^{(2)} \alpha'}{r^2} \end{aligned} \quad (2.16)$$

where $Q_k^{(i)} = g_s N_k^{(i)}$, $k = 1, 5$, $i = 1, 2$. $N_5^{(i)}$, $i = 1, 2$ are (up to a numerical constant) the number of D5 branes. $N_1^{(i)}$, $i = 1, 2$, are D1 brane densities. As in the M-theory solution, in the near horizon limit the solution only depends on the product $N_1^{(1)} N_1^{(2)}$ which we will denote by N_1 .

Let us discuss the low-energy limit. We consider the limit

$$\alpha' \rightarrow 0, \quad U = \frac{r}{\alpha'} = \text{fixed}, \quad U' = \frac{r'}{\alpha'} = \text{fixed},$$

$$\tilde{t} = \frac{t}{\sqrt{\alpha'}} = \text{fixed}, \quad \tilde{x}_1 = \frac{x_1}{\sqrt{\alpha'}} = \text{fixed}. \quad (2.17)$$

The scaling of x_1, t is necessary so that all terms in the metric get an overall factor of α' . This limit can be interpreted as an ultra-low energy limit.

The metric becomes

$$\frac{ds^2}{\alpha'} = Q_3^{-1} U^2 U'^2 (-d\tilde{t}^2 + d\tilde{x}_1^2)$$

$$+ Q_1 \frac{dU^2}{U^2} + Q_2 \frac{dU'^2}{U'^2} + Q_1 d\Omega_{(1)}^2 + Q_2 d\Omega_{(2)}^2 \quad (2.18)$$

where

$$Q_1 = g_s \sqrt{\frac{N_1 N_5^{(2)}}{N_5^{(1)}}}, \quad Q_2 = g_s \sqrt{\frac{N_1 N_5^{(1)}}{N_5^{(2)}}}, \quad Q_3 = g_s^2 \sqrt{N_1 N_5^{(1)} N_5^{(2)}}. \quad (2.19)$$

This metric is of the form $adS_3 \times S^3 \times S^3 \times R$. To see this we further change variables

$$u = \frac{l}{\sqrt{Q_3}} U U', \quad \lambda = l \left(\sqrt{\frac{Q_1}{Q_2}} \log U - \sqrt{\frac{Q_2}{Q_1}} \log U' \right), \quad l = \sqrt{\frac{Q_1 Q_2}{Q_1 + Q_2}}. \quad (2.20)$$

The solution takes the form (we drop the tilde from t, x_1)

$$\frac{ds^2}{\alpha'} = \left[\left(\frac{u}{l} \right)^2 (-dt^2 + dx_1^2) + l^2 \frac{du^2}{u^2} \right] + d\lambda^2 + Q_1 d\Omega_{(1)}^2 + Q_2 d\Omega_{(2)}^2 \quad (2.21)$$

$$H_{\kappa\mu\nu} = 2l^{-1} \frac{Q_3}{Q_1 Q_2} \alpha' \epsilon_{\kappa\mu\nu}, \quad H_{\alpha\beta\gamma} = 2g_s N_5^{(2)} \alpha' \epsilon_{\alpha\beta\gamma},$$

$$H_{\alpha'\beta'\gamma'} = 2g_s N_5^{(1)} \alpha' \epsilon_{\alpha'\beta'\gamma'}$$

$$e^\phi = \frac{Q_1 Q_2}{Q_3}$$

where $\kappa, \mu, \nu = 0, 1, u$, are adS_3 indices, $\epsilon_{\kappa\mu\nu}$ is the volume form of adS_3 , α and α' are indices on the two spheres, and $\epsilon_{\alpha\beta\gamma}, \epsilon_{\alpha'\beta'\gamma'}$ are the corresponding unit volume forms. All factors of α' cancel at the end, so we set $\alpha' = 1$ from now on.

The metric in (2.21) is of the form $adS_3 \times S^3 \times S^3 \times R$. The adS_3 radius is equal to l and the radii of the two spheres are equal to $Q_1^{1/2}$ and $Q_2^{1/2}$, respectively. We would like to compactify the λ coordinate, but leave the rest of the configuration intact. A similar discussion applies to the M-theory configuration in section 2.1. This can be achieved by the following identification

$$U \sim Ue^L, \quad U' \sim U'e^{-L} \quad (2.22)$$

This identification leaves invariant the $adS_3 \times S^3 \times S^3$ part of the metric and implies

$$\lambda \sim \lambda + L(Q_1 + Q_2)^{1/2} \quad (2.23)$$

Since U and U' can be thought of as a cut-off energy of the two D1-D5 systems, the identification (2.22) imposes some kind of UV-IR identification between the two D1-D5 systems. At this stage the parameter L seems a free parameter. However, its value is fixed to a specific value in order to get the correct central charge for the boundary conformal field theory.

The value of the central charge of the boundary conformal field theory is given by [8, 25]

$$c = \frac{3l}{2G_N^{(3)}} \quad (2.24)$$

In our case we get (ignoring all numerical factors)

$$c \sim g_s^{-2} Q_3^2 L = g_s^2 N_1 N_5^{(1)} N_5^{(2)} L. \quad (2.25)$$

The level k of a current algebra originating from the isometries of the internal space is proportional to the square of the radius of that space; this can be seen either via KK reduction or directly in string theory. In particular, the level k of the $SU(2)$ current algebra associated to a three-sphere of radius R is given by

$$k = \frac{R^2}{4lG_N^{(3)}}. \quad (2.26)$$

Therefore, k^\pm are given by

$$k^+ = \frac{c}{6} \left(1 + \frac{N_5^{(2)}}{N_5^{(1)}}\right), \quad k^- = \frac{c}{6} \left(1 + \frac{N_5^{(1)}}{N_5^{(2)}}\right). \quad (2.27)$$

From this we deduce that $k^+/k^- = N_5^{(2)}/N_5^{(1)}$. This is consistent with the values $k^+ = N_1 N_5^{(2)}$ and $k^- = N_1 N_5^{(1)}$ that one obtains by requiring that k^+ and k^- are the levels of the affine $\widehat{SU(2)}$ Lie algebra of the dual conformal field theory for each separate D1-D5 system. The central charge of the \mathcal{A}_γ algebra is equal to $c = 6k^+k^-/(k^+ + k^-)$. Therefore,

$$L \sim \frac{1}{g_s^2(k^+ + k^-)} \frac{k^+k^-}{N_1 N_5^{(1)} N_5^{(2)}} \quad (2.28)$$

In the large k^+ (or k^-) limit one of the D5 systems decouples. The central charge goes over to $c = 6k^-$, and $g_s^2 L \rightarrow 0$ so there are no identifications, as it should.

The S-dual configuration consists of an intersection of two IIB NS5-F1 systems. The metric, antisymmetric tensor and dilaton are exactly as in the IIA configuration described in the previous section.

3 The \mathcal{A}_γ and $\tilde{\mathcal{A}}_\gamma$ SCFT

In this section we review the $N = 4$ double SCFTs and discuss some of their properties. The $N = 4$ double SCFTs are based on the one-parameter family of $N = 4$ super conformal algebras (SCA) \mathcal{A}_γ . These algebras contain, among other sub-structures, two commuting $\widehat{SU(2)}$ affine Lie algebras and a finite sub-superalgebra $D(2, 1, \gamma/(1 - \gamma))$, where $\gamma \in \mathbb{R}$. They are parametrized by the central charge of the Virasoro algebra c and the parameter γ or equivalently by the levels k^+ and k^- of $\widehat{SU(2)}^+$ and $\widehat{SU(2)}^-$. The parameters are related by

$$c = \frac{6k^+k^-}{k} \quad \text{with} \quad k \equiv k^+ + k^- = \frac{c}{6\gamma(1 - \gamma)} \quad (3.1)$$

Besides the Virasoro generators, the algebra is generated by the two sets of $\widehat{SU(2)}^\pm$ generators $A^{\pm i}$, a $\widehat{U(1)}$ generator U , all with dimensions

1; four supersymmetry generators G^a with dimensions $3/2$ and four fermionic generators Q^a with dimensions $1/2$. The operator product expansions of these fields are given in Appendix A.

We are interested in the unitary representations of this algebra [23, 35, 36], where k^+ and k^- are positive integers, and in particular in the primary states in the Neveu-Schwarz (NS) sector. We thus consider the unitary highest weight states $|hws\rangle$ defined by ⁶

$$\begin{aligned} (L)_n|hws\rangle &= (A^{\pm i})_n|hws\rangle = (U)_n|hws\rangle \\ &= (A^{\pm +})_0|hws\rangle = 0 \quad \text{for } n = 1, 2, \dots, \quad i = 1, 2, 3 \\ (G^a)_r|hws\rangle &= (Q^a)_r|hws\rangle = 0 \quad \text{for } r = \frac{1}{2}, \frac{3}{2}, \dots, \quad a = 1, 2, 3, 4 \end{aligned} \quad (3.2)$$

Thus each hws is characterized by its conformal dimension h , $U(1)$ charge u , and the spins l^\pm under the two $SU(2)$ s. The unitarity of the representation implies that

$$\begin{aligned} l^\pm &= 0, \frac{1}{2}, 1, \dots, \frac{1}{2}(k^\pm - 1) \\ kh &\geq (l^+ - l^-)^2 + k^-l^+ + k^+l^- + u^2 \end{aligned} \quad (3.3)$$

In supersymmetric conformal field theories based on the $N = 2$ or small $N = 4$ algebra, the states which saturate some bound for the conformal dimension, are in short multiplets and form a ring, called “chiral ring”. Indeed also for the double $N = 4$ algebras the hws for which $kh = (l^+ - l^-)^2 + k^-l^+ + k^+l^- + u^2$ satisfy the chirality (or massless) condition

$$(\tilde{G}^+)^{-1/2}|hws\rangle = 0 \quad (3.4)$$

where \tilde{G} is defined in eq. (3.5). It is important to notice that in the case of the \mathcal{A}_γ algebra, the “chiral” states do not form any obvious ring, due to the quadratic dependence on the $U(1)$ charge and the $SU(2)$ spins. We will come back to this issue after having introduced the $\tilde{\mathcal{A}}_\gamma$ algebra.

The \mathcal{A}_γ algebra is related [21] to another SCA called $\tilde{\mathcal{A}}_\gamma$, or non-linear \mathcal{A}_γ , with no dimension $1/2$ generators, but composite operators

⁶By convention, the $SU(2)$ indices i, j run over $1, 2, 3$ or $+, -, 3$, and the indices a, b over $1, 2, 3, 4$ or $+, -, +K, -K$; see ref. [23] for more details.

in the operator product expansion (OPE). We will denote all operators belonging to this algebra by a tilde. Indeed, starting from the \mathcal{A}_γ algebra, if one introduces the operators ⁷

$$\begin{aligned}\tilde{L} &= L + \frac{1}{k}(UU + \partial Q^a Q_a) \\ \tilde{G}_a &= G_a + \frac{2}{k}UQ_a - \frac{2}{3k^2}\epsilon_{abcd}Q^b Q^c Q^d + \frac{4}{k}Q^b(\alpha_{ba}^{+i}\tilde{A}_i^+ - \alpha_{ba}^{-i}\tilde{A}_i^-) \\ \tilde{A}^{\pm i} &= A^{\pm i} - \frac{1}{k}\alpha_{ab}^{\pm i}Q^a Q^b \\ \tilde{Q} &= Q \qquad \qquad \qquad \tilde{U} = U\end{aligned}\tag{3.5}$$

one can show that the \tilde{U} and \tilde{Q} operators decouple completely from the others and form a free algebra. The other operators form a SCA with central charge $\tilde{c} = c - 3$ with respect to which \tilde{G} and \tilde{A} are primary operators with dimension $3/2$ and 1 respectively. The \tilde{A}^\pm are $\widehat{SU(2)}^\pm$ affine Lie algebra generators with levels

$$\tilde{k}^\pm = k^\pm - 1\tag{3.6}$$

The unitary *hws* representations of the two algebras are in “one-to-one” correspondence. Indeed to each representation of \mathcal{A}_γ with h, l^\pm, u there corresponds a representation of $\tilde{\mathcal{A}}_\gamma$ with

$$\tilde{h} = h - \frac{u^2}{k}, \qquad \tilde{l}^\pm = l^\pm\tag{3.7}$$

Inversely, to each representation of $\tilde{\mathcal{A}}_\gamma$ there corresponds an infinite set of representations of \mathcal{A}_γ built by adding one free boson and four free fermions and choosing some u compatible with the radius of the $\widehat{U(1)}$ generator U .

To compare with the spectrum that we obtain from the supergravity analysis, we are interested in the explicit form of the “chiral” multiplets in the $\tilde{\mathcal{A}}_\gamma$ algebra. The *hws* states are now defined by

$$\begin{aligned}(\tilde{L})_n|hws\rangle &= (\tilde{A}^{\pm i})_n|hws\rangle \\ &= (\tilde{A}^{\pm +})_0|hws\rangle = 0 \text{ for } n = 1, 2, \dots, i = 1, 2, 3 \\ (\tilde{G}^a)_r|hws\rangle &= 0 \quad \text{for } r = \frac{1}{2}, \frac{3}{2}, \dots, a = 1, 2, 3, 4\end{aligned}\tag{3.8}$$

⁷See Appendix A for the definition of the α and ϵ symbols.

The “chiral” condition is again given by eq. (3.4). The conformal dimension of a “chiral” state is now

$$k\tilde{h}_c = (l^+ - l^-)^2 + k^-l^+ + k^+l^- \quad (3.9)$$

The explicit form of the “chiral” multiplet is the same as the one of the $D^1(2, 1, \alpha)$ given in (4.1)-(4.3). As in the case of the linear algebra, these states do not form a ring. However, since

$$\begin{aligned} k\tilde{h}_c(l^+, l^-) + k\tilde{h}_c(m^+, m^-) \\ = k\tilde{h}_c(l^+ + m^+, l^- + m^-) - 2(l^+ - l^-)(m^+ - m^-) \end{aligned} \quad (3.10)$$

if either $l^+ = l^-$ or $m^+ = m^-$, the OPE of the two “chiral” operators corresponding to these states contains no singular terms and the products of these two operators can give rise to another “chiral” operator. This shows that in contrast to the case of the usual $N = 2$ or $N = 4$ algebra, the chiral operators do not form a ring but they could form a module over a ring. The latter ring is generated by all “chiral” fields of the form $l^+ = l^-$.

We conclude this section by discussing some general constraints on the spectrum of chiral operators in any $\tilde{\mathcal{A}}_\gamma$ theory that were derived in [32]. Let N_{l^\pm, \bar{l}^\pm} be the number of chiral operators with left- and right-moving $SU(2)$ quantum numbers $(l^+, l^-; \bar{l}^+, \bar{l}^-)$. Consider the generating function

$$f(p, q, r, s) = \sum_{l^\pm, \bar{l}^\pm} N_{l^\pm, \bar{l}^\pm} p^{2l^+} q^{2l^-} r^{2\bar{l}^+} s^{2\bar{l}^-} \quad (3.11)$$

According to [32] certain linear combinations of the multiplicities N_{l^\pm, \bar{l}^\pm} should give rise to $SU(2)$ modular invariants. These linear combinations are contained in another generating function defined by

$$\begin{aligned} g(x, y) \\ = \sum_{l, \bar{l}=0}^{(\tilde{k}^+ + \tilde{k}^-)/2} x^{2l} y^{2\bar{l}} \sum_{l^\pm, \bar{l}^\pm=0}^{\tilde{k}^\pm/2} (-1)^{2l^- + 2\bar{l}^-} \delta_{\tilde{k}^- + 2l^+ - 2l^-, 2l} \delta_{\tilde{k}^- + 2\bar{l}^+ - 2\bar{l}^-, 2\bar{l}} N_{l^\pm, \bar{l}^\pm} \end{aligned} \quad (3.12)$$

It is easy to see that

$$g(x, y) \equiv \sum_{l, \bar{l}=0}^{(\tilde{k}^+ + \tilde{k}^-)/2} x^{2l} y^{2\bar{l}} n_{l, \bar{l}} = (xy)^{\tilde{k}^-} f(x, -x^{-1}, y, -y^{-1}). \quad (3.13)$$

The numbers $n_{l,\bar{l}}$ should correspond to a modular invariant of $SU(2)$. For example, the diagonal modular invariant appears when the function f satisfies

$$\begin{aligned} f(x, -x^{-1}, y, -y^{-1}) \\ = A((xy)^{-\tilde{k}^-} + (xy)^{1-\tilde{k}^-} + \cdots + (xy)^{\tilde{k}^+-1} + (xy)^{\tilde{k}^+}) \end{aligned} \quad (3.14)$$

for some integer A .

4 Kaluza-Klein Spectrum

To compute the Kaluza-Klein Spectrum of $10d$ supergravity on $adS_3 \times S^3 \times S^3 \times S^1$ we use representation theory and follow the method explained in [4]. In other words, we will only compute the quantum numbers of the KK modes under the $SO(4) \times SO(4)$ isometry group of $S^3 \times S^3$, and deduce the conformal weights of the KK states from that. In the large k limit (and with g_s fixed) the radius of S^1 is small so we start from nine dimensional supergravity.

The first step in this procedure is to determine the relevant AdS supergroup, which is a symmetry of nine-dimensional supergravity on $AdS_3 \times S^3 \times S^3$. KK modes have to fall into multiplets of this AdS supergroup. As we discussed in section 2, in the case at hand the relevant supergroup is $D^1(2, 1, \alpha) \times D^1(2, 1, \alpha)^8$ [6]. Indeed, in general the chiral algebra of the boundary CFT is the Hamiltonian reduction of the affine Lie superalgebra made from the AdS supergroup, and the $\tilde{\mathcal{A}}_\gamma$ SCA can be obtained via Hamiltonian reduction from $\widehat{D^1(2, 1, \alpha)}$ [7]. Also notice that the bosonic subalgebra of $D^1(2, 1, \alpha) \times D^1(2, 1, \alpha)$ is $SU(2)^4 \times SL(2, R)^2$, which is the isometry group of $AdS_3 \times S^3 \times S^3$.

The next step is to study the representations of $D^1(2, 1, \alpha)$. We can think of $D^1(2, 1, \alpha)$ as being generated by the generators $L_{\pm 1}, L_0, G_{\pm 1/2}^a, A_0^{\pm i}$ of the \mathcal{A}_γ algebra. The representation theory of $D^1(2, 1, \alpha)$ mirrors that of \mathcal{A}_γ . There are long and short representations. Representations are labeled by the $SU(2)$ quantum numbers l^\pm and the conformal weight h , and will be denoted by (l^+, l^-, h) . Unitarity implies the bound

⁸Recall that $\alpha = \gamma/(1 - \gamma) = k^+/k^-$

$h \geq \gamma l^- + (1 - \gamma)l^+$. When this bound is saturated the corresponding representation is a short representation, which will be denoted by $(l^+, l^-)_S$. Each $D^1(2, 1, \alpha)$ representation can be decomposed in terms of representations of the $SU(2) \times SU(2)$ subgroup whose representations will be labeled by (j, j') . Generic short representations contain 8 $SU(2) \times SU(2)$ representations, namely

$$\begin{aligned}
 (l^+, l^-)_S \rightarrow & 2(l^+, l^-) + 2\left(l^+ - \frac{1}{2}, l^- - \frac{1}{2}\right) + \left(l^+ - \frac{1}{2}, l^- + \frac{1}{2}\right) \\
 & + \left(l^+ + \frac{1}{2}, l^- - \frac{1}{2}\right) + (l^+ - 1, l^-) + (l^+, l^- - 1). \quad (4.1)
 \end{aligned}$$

These 8 $SU(2) \times SU(2)$ representations can be organized according to the action of $G_{-1/2}^a$ in the following way

$$\begin{array}{ccccc}
 & & (l^+, l^-) & & \\
 (l^+ - \frac{1}{2}, l^- - \frac{1}{2}) & & (l^+ + \frac{1}{2}, l^- - \frac{1}{2}) & & (l^+ - \frac{1}{2}, l^- + \frac{1}{2}) \\
 & (l^+, l^- - 1) & (l^+ - 1, l^-) & & (l^+, l^-) \\
 & & (l^+ - \frac{1}{2}, l^- - \frac{1}{2}). & &
 \end{array} \quad (4.2)$$

The states on the first line have conformal weight $h = \gamma l^- + (1 - \gamma)l^+$, and the conformal weight increases by $\frac{1}{2}$ as we move down one line. This result (4.1) is only valid if $l^\pm \geq 1$. For other values of l^\pm the decomposition reads

$$\begin{aligned}
 \left(\frac{1}{2}, l^-\right)_S & \equiv 2\left(\frac{1}{2}, l^-\right) + 2\left(0, l^- - \frac{1}{2}\right) + \left(0, l^- + \frac{1}{2}\right) \\
 & + \left(1, l^- - \frac{1}{2}\right) + \left(\frac{1}{2}, l^- - 1\right) \\
 (0, l^-)_S & \equiv (0, l^-) + \left(\frac{1}{2}, l^- - \frac{1}{2}\right) + (0, l^- - 1) \\
 \left(\frac{1}{2}, \frac{1}{2}\right)_S & \equiv 2\left(\frac{1}{2}, \frac{1}{2}\right) + 2(0, 0) + (0, 1) + (1, 0) \\
 \left(0, \frac{1}{2}\right)_S & \equiv \left(0, \frac{1}{2}\right) + \left(\frac{1}{2}, 0\right) \\
 (0, 0)_S & \equiv (0, 0) \quad (4.3)
 \end{aligned}$$

and similarly for $l^+ \leftrightarrow l^-$.

The KK spectrum can now be determined as in [4]. We start with nine-dimensional supergravity, determine using representation theory

the $SU(2)^4$ quantum numbers of all KK states, and organize those quantum numbers in terms of short representations of $D^1(2, 1, \alpha) \times D^1(2, 1, \alpha)$. Short representations of $D^1(2, 1, \alpha) \times D^1(2, 1, \alpha)$ are simply the tensor product of two short representations $(l^+, l^-)_S$ and $(\bar{l}^+, \bar{l}^-)_S$, and will be denoted by $(l^+, l^-; \bar{l}^+, \bar{l}^-)_S$. Each of these contains 64 $SU(2)^4$ representations, and it is a non-trivial check on the correctness of the set of $SU(2)^4$ quantum numbers of the KK states to see if they organize in appropriate groups of 64. Once we organize the KK states in short representations we also know their conformal weights. The reason that we expect only short representations to appear in the KK spectrum is that all KK fields originated from massless fields in nine-dimensions. Thus, they saturate the inequality $m^2 \geq 0$ and it is natural to identify this with the bound on the conformal weight of a $D^1(2, 1, \alpha)$ representation.

We omit the details of the calculation, but it turns out that the KK spectrum can indeed be organized in terms of short representations and the final result for the KK spectrum reads

$$\begin{aligned} & \oplus_{l^+ \geq 0, l^- \geq 1/2} (l^+, l^-; l^+, l^-)_S + \oplus_{l^+ \geq 1/2, l^- \geq 0} (l^+, l^-; l^+, l^-)_S \quad (4.4) \\ & + \oplus_{l^+, l^- \geq 0} \left(\left(l^+, l^-; l^+ + \frac{1}{2}, l^- + \frac{1}{2} \right)_S + \left(l^+ + \frac{1}{2}, l^- + \frac{1}{2}; l^+, l^- \right)_S \right) \end{aligned}$$

An important remark is that the highest weight states with quantum numbers $(l^+, l^-; l^+, l^-)_S$ are bosonic, whereas $(l^+, l^-; l^+ + \frac{1}{2}, l^- + \frac{1}{2})_S$ and $(l^+ + \frac{1}{2}, l^- + \frac{1}{2}; l^+, l^-)_S$ are fermionic.

We included in (4.4) two short representations that do not correspond to propagating degrees of freedom in the bulk but to nontrivial fields on the boundary. These are the short multiplets $(0, 0; \frac{1}{2}, \frac{1}{2})_S$ and $(\frac{1}{2}, \frac{1}{2}; 0, 0)_S$, which contain all higher modes of the \mathcal{A}_γ algebra, in particular the stress-energy tensor of the boundary theory. They arise via a suitable supersymmetric generalization of [8].

5 Multiparticle spectrum and boundary SCFT

In the previous section we determined the KK spectrum of single particle states in supergravity, see (4.4). We organized the spectrum in terms

of short representations of $D^1(2, 1, \alpha) \times D^1(2, 1, \alpha)$. The first issue we want to address in this section is which of the single and multiparticle states correspond to chiral operators of the boundary SCFT. Because of the nonlinearity of the bound (3.9) this is a rather difficult question, especially since the nonlinear part of (3.9) is invisible in supergravity where k^\pm are very large. A priori there are at least four options:

- (i) all multiparticle states correspond to chiral operators of the boundary SCFT.
- (ii) only products of one single particle state with arbitrary many states of the form $(l, l; \bar{l}, \bar{l})_S$ are massless; this is inspired by equation (3.10) and the discussion below it, in which we argued the most natural structure on the space of chiral operators is that of a module over a ring.
- (iii) Only the single particle states correspond to chiral operators of the boundary SCFT.
- (iv) Except for powers of the representations $(0, 0; \frac{1}{2}, \frac{1}{2})_S$, $(\frac{1}{2}, \frac{1}{2}; 0, 0)_S$ and $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_S$, there are no multiparticle states corresponding to chiral operators. The motivation for this is that long representations of $D^1(2, 1, \alpha)$, as far as their $SU(2) \times SU(2)$ quantum numbers go, contain two short $D^1(2, 1, \alpha)$ representations

$$(l^+, l^-)_{\text{long}} = (l^+, l^-)_S + \left(l^+ + \frac{1}{2}, l^- + \frac{1}{2} \right)_S. \quad (5.1)$$

From this we see that the entire KK spectrum (4.4) can be written as the sum of $(0, 0; \frac{1}{2}, \frac{1}{2})_S$, $(\frac{1}{2}, \frac{1}{2}; 0, 0)_S$ and $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_S$, and $(\text{long}) \otimes (\text{long})$ representations. The conformal weight of these long representations is not protected by anything and they can therefore correspond to nonchiral operators. In particular the spectrum of chiral operators can jump as we move around in the moduli space.

Unfortunately, there only a few things we can do to decide whether one of the options (i)-(iv) gives the right spectrum of chiral operators. One possibility is to examine whether the resulting spectrum is in agreement with the modular invariance constraints discussed at the end of section 3. Another possibility is to compute the spectrum of chiral operators of known \mathcal{A}_γ theories and to match those to the KK spectrum in the hope of identifying the precise boundary theory. The first possibility is somewhat problematic to apply. First of all, we do not know where to truncate the spectrum, and furthermore, there is no obvious map from a unitary \mathcal{A}_γ theory to a unitary $\tilde{\mathcal{A}}_\gamma$ with finite multiplicities. Still, if we assume that the spectrum of chiral primaries

of an \mathcal{A}_γ theory should satisfy the same constraints as the spectrum of an $\tilde{\mathcal{A}}_\gamma$ and if we truncate the spectrum to keep only states with $l_{\text{total}}^\pm < k^\pm/2$ and $\bar{l}_{\text{total}}^\pm < k^\pm/2$ we see that option (i) is not compatible with modular invariance, but, quite surprisingly, options (ii), (iii) and (iv) are compatible with modular invariance. Nevertheless, we should probably not attach too much value to these observations.

What about the spectrum of known \mathcal{A}_γ theories? Almost all known \mathcal{A}_γ and $\tilde{\mathcal{A}}_\gamma$ theories are associated to certain so-called Wolf spaces [42, 43, 37, 41]. A preliminary investigation shows that the spectra of chiral operators of such Wolf space theories are quite distinct from the Kaluza-Klein spectrum (4.4). Another natural series of \mathcal{A}_γ theories to consider is that of orbifolds of known \mathcal{A}_γ theories. In the case of AdS/CFT dualities involving the small $N = 4$ algebra, the conformal field theory is a sigma model on a symmetric product $\text{Sym}^k(M_4)$ [48, 46, 31, 4, 11, 3, 30, 40]. Symmetric products appear naturally as the configuration space of unordered branes, so here we are led to consider S_N orbifolds of \mathcal{A}_γ theories. In fact, the KK spectrum in (4.4) is highly reminiscent of that of an orbifold theory. However, the \mathcal{A}_γ theories depend on two integers k^\pm , and the central charge in general is fractional, so it is not a priori clear what type of orbifold to write down for general k^\pm . The KK spectrum looks like that of a sigma model on a space of the form $\text{Sym}^{k^+}(\text{Sym}^{k^-}(M))$, but clearly this cannot be true. To get a clue one can first consider various limiting cases. In the limit $k^- \gg k^+$, one of the two D1-D5 (or M2-M5 in the M-theory case) systems approximately decouples and one expects the considerations of the standard D1-D5 system to apply. In our case, the D5 brane is wrapped on $S^1 \times S^3$. This suggests that the theory is related to a sigma model on $\text{Sym}^{k^-}(U(2))$, where we used the fact that $U(2)$ is topologically $S^1 \times S^3$. As we will see in section 7, classical sigma models with \mathcal{A}_γ symmetry have as target space a hyperkähler manifold with torsion and compatible $U(2)$ action. Clearly, $\text{Sym}^{k^-}(U(2))$ is hyperkähler, and has a natural $U(2)$ action. The fact that we need torsion implies that we should consider a $U(2)$ WZW model. This leads to the proposal:

For general k^+, k^- the SCFT is a supersymmetric sigma model with target space $\text{Sym}^{k^-}(U(2))$, with k/k^- units of $H = dB$ flux and certain discrete “gauge fields” associated to the permutation group S_{k^-} turned on.

Since the original brane configuration is invariant under exchange of k^+ and k^- , this proposal requires the equivalence of the respective sigma models on $\text{Sym}^{k^-}(U(2))$ and $\text{Sym}^{k^+}(U(2))$. As explained in section 7, only one of the two sigma models can be weakly coupled for given k^+, k^- , and in this sense this duality is reminiscent of level-rank duality. It would be interesting to find a direct proof of this duality.

It may seem puzzling that we allow a fractional flux of $H = dB$ in the conformal field theory. A tentative description of this theory is as follows. In order to define a sigma model with target space M and $H \neq 0$, we include a term

$$S_{wz} = 2\pi i \int_{X, \partial X = \Sigma} H \quad (5.2)$$

in the action. Here, X is a three-manifold in target space whose boundary is the world-sheet Σ . This term should be independent of the choice of X and this requires that $H \in H^3(M, \mathbf{Z})$. It is well-known that this statement is modified if there are also chiral world-sheet fermions coupled to background gauge fields A in the action. In this case it is $H' = dB - t\text{CS}(A)$ rather than H which should be in $H^3(M, \mathbf{Z})$, where $\text{CS}(A)$ refers to the Chern-Simons three-form, and t is proportional to the number of chiral fermions. In our case there are no continuous gauge fields A , but there are discrete S_{k^-} gauge fields. By this we mean that every world-sheet embedded in the non-singular part of $\text{Sym}^{k^-}(U(2))$ naturally carries an S_{k^-} bundle. Given a three-manifold X with boundary Σ and some S_{k^-} bundle, a topological action can be defined as in [13]. The different topological actions are classified by elements α of $H^3(BS_{k^-}, U(1))$, where BS_{k^-} is the classifying space of the finite group S_{k^-} . We choose an element α such that the action for S^3/\mathbf{Z}_{k^-} (with the obvious \mathbf{Z}_{k^-} bundle which is also an S_{k^-} bundle) is equal to $\exp(-2\pi i/k^-)$. (We have not shown that this is possible for general k^- , but for small k^- we verified that such an element exists). We now include in the path integral the topological action associated to $k\alpha$. This depends only on $k \bmod k^-$. In particular, if k is a multiple of k^- , this topological action is trivial. The topological action together with the standard sigma model action with the Wess-Zumino term (5.2) with fractional dB flux yields a sigma model that does not depend on the choice of the three-manifold X , and is well-defined for all values of k^+, k^- .

The level of the $SU(2)$ current algebra associated to the diagonal

$SU(2)$ action on $\text{Sym}^{k^-}(U(2))$ is given by the integral of H over the orbit of $SU(2) = S^3$ in the symmetric product. This S^3 is in homology equal to k^- times a 3-cycle of the form⁹ S^3/\mathbf{Z}_{k^-} . It is with respect to this latter cycle that units of flux are defined, so the level of $SU(2)$ is $k^-(k/k^-) = k$.

Conformal invariance of the sigma model will impose constraints on the metric on $\text{Sym}^{k^-}(U(2))$. For special values of k there exists an exact SCFT description of the sigma model, and our proposal coincides with the conjecture in [14, 19]. Namely, if k is a multiple of k^- , $k = qk^-$, the topological action is trivial and the sigma model can be described by the orbifold conformal field theory $(U(2)_q)^{k^-}/S_{k^-}$, where $U(2)_q$ is the level q , $U(2)$, $N = 1$ super WZW model. This is indeed a sigma model with target space $\text{Sym}^{k^-}(U(2))$ and q units of 3-form flux. We expect the sigma model for arbitrary rational q to be closely related to an analytic continuation of the exact SCFT's that exist for integer q . Indeed, if we analytically continue the central charge of $(U(2)_q)^{k^-}/S_{k^-}$ to rational q , we find the right value

$$c = k^-(3/2 + 3(q-2)/q + 3/2) = 6k^-(q-1)/q = 6k^+k^-/k \quad (5.3)$$

where the first factor of $3/2$ comes from the supersymmetric $U(1)$ factor, the second factor from the bosonic part of the $SU(2)$ WZW theory and the last one from the corresponding fermions. The truncation of spins can also be analytically continued to arbitrary $q = k/k^-$. In the $SU(2)_{q-2}$ theory, the maximal spin of a primary field is $(q-2)/2$. Thus in the orbifold the maximal spin is $k^-(q-2)/2$. This is half-integer precisely when q is a multiple of $1/k^-$.

In the remaining of this section we analyze the spectrum of chiral primaries for some special cases with $k = qk^-$ and compare the result to the KK spectrum obtained from supergravity. The conformal weights of nonchiral operators can change as we vary the moduli, only the conformal weights of chiral operators are protected, and the chiral operators with a small conformal weight should be present in the KK spectrum in order for the duality to be valid.

First we consider some general aspects of S_N orbifolds of \mathcal{A}_γ theo-

⁹To see this one can e.g. look at the orbit of $SU(2)$ through $(1, g_0, \dots, g_0^{k^- - 1})$ where $g_0 = \text{diag}(e^{2\pi i/k^-}, e^{-2\pi i/k^-}) \in U(2)$. After modding out by S_{k^-} this orbit is a k^- -fold cover of $SU(2)/\mathbf{Z}_{k^-}$, where \mathbf{Z}_{k^-} is generated by g_0 .

ries. The RR sector of an S_N orbifold can be decomposed in various subsectors as in [12]. Consider a chiral RR state in the Z_{p_1} twisted sector with quantum numbers l_1^+, l_1^- , and another one in the Z_{p_2} twisted sector with quantum numbers l_2^+, l_2^- . We suppress the dependence on $U(1)$ momenta in this discussion. The conformal weight of chiral operators in the R sector is given by

$$h_R(l^+, l^-, k^+, k^-) = \frac{1}{k} \left((l^+ + l^-)^2 + \frac{1}{4} k^+ k^- \right). \quad (5.4)$$

Thus the two states mentioned above have conformal weight $h_R(l_1^+, l_1^-, p_1 k^+, p_1 k^-)$ and $h_R(l_2^+, l_2^-, p_2 k^+, p_2 k^-)$, if k^+ and k^- are the levels of the original theory that we are orbifolding. If we combine the two states to make one in the $p_1 + p_2$ twisted sector, we do not always get a new chiral operator. We do have the following inequality

$$\sum_{i=1}^2 h_R(l_i^+, l_i^-, p_i k^+, p_i k^-) \geq h_R(l_1^+ + l_2^+, l_1^- + l_2^-, k^+(p_1 + p_2), k^-(p_1 + p_2)) \quad (5.5)$$

but equality only holds if

$$\frac{l_1^+ + l_1^-}{p_1} = \frac{l_2^+ + l_2^-}{p_2}. \quad (5.6)$$

Thus only specific states in the twisted sectors can combine to give chiral primaries in the full theory. A second subtlety is that states that obey the equality (5.4) are not necessarily primaries of the original theory. Descendants that satisfy (5.4) will give rise to chiral primaries in Z_p twisted sectors for sufficiently large p .

Let us now apply this to the S_{k-} orbifold of the theory with $k^+ = q - 1$, $k^- = 1$. The latter theory has only massless representations and the quantum numbers of the chiral primary operators in the RR sector (labeled by $(l^+, l^-; \bar{l}^+, \bar{l}^-)$) are [23]

$$\left(\frac{j+1}{2}, 0; \frac{j+1}{2}, 0 \right), \quad \left(\frac{j+1}{2}, 0; \frac{j}{2}, \frac{1}{2} \right), \quad \left(\frac{j}{2}, \frac{1}{2}; \frac{j+1}{2}, 0 \right), \\ \left(\frac{j}{2}, \frac{1}{2}; \frac{j}{2}, \frac{1}{2} \right), \quad 0 \leq j \leq q-2. \quad (5.7)$$

The first and fourth state are bosonic, the second and third state are fermionic. They satisfy $l^+ + l^- = \bar{l}^+ + \bar{l}^- = \frac{j+1}{2}$. In case k^- is prime, the only way to combine these chiral primary operators in order to get a chiral primary operator in the orbifold theory is to take states with $l^+ + l^- = \bar{l}^+ + \bar{l}^- = \frac{j+1}{2}$ in the Z_{j+1} twisted sector. This follows from condition (5.6). By means of spectral flow [10] we find a set of NS states generated by

$$\begin{aligned} & \left(\frac{j}{2}, \frac{j}{2}, \frac{j}{2}, \frac{j}{2} \right)_s, \quad \left(\frac{j}{2}, \frac{j}{2}, \frac{j+1}{2}, \frac{j+1}{2} \right)_s, \quad \left(\frac{j+1}{2}, \frac{j+1}{2}, \frac{j}{2}, \frac{j}{2} \right)_s, \\ & \left(\frac{j+1}{2}, \frac{j+1}{2}, \frac{j+1}{2}, \frac{j+1}{2} \right)_s, \quad 0 \leq j \leq q-2. \end{aligned} \quad (5.8)$$

All these states are present in the KK spectrum (4.4), and form a natural ring because they satisfy $l^+ = l^-, \bar{l}^+ = \bar{l}^-$, see section 3. Besides the states generated by (5.8), there will in general be additional chiral primaries as explained above. However, the spins of most of these states become very large as $k^\pm \rightarrow \infty$, and we do not expect to see any sign of them in the supergravity spectrum. This can be seen as follows. If we combine states from twisted sectors they should satisfy $\sum p_i = k^-$ and $(l_i^+ + l_i^-)/p_i = r$ where r is some fixed number independent of i . The spin of the combined state satisfies $l^+ + l^- = rk^-$. After spectral flow the spins in the NS sector obey $l^+ - l^- = (r - 1/2)k^-$. If we send k^- to infinity, we need $r = 1/2$ to keep l^\pm finite. Most of the additional states have $r \neq 1/2$ and are invisible in supergravity. States with $r = 1/2$ can be used to build arbitrary multiparticle states in the RR sector. The same is true in the NS sector because $r = 1/2$ implies $l^+ = l^-$. All this supports the picture that the chiral primaries carry the structure of a module over a ring, the ring being generated by the chiral primaries with $l^+ = l^-$.

Finally, we illustrate the appearance of additional chiral primaries due to the existence of descendants that satisfy (5.4). Consider the case with $q = 2$. The $N = 1$ $U(2)_2$ WZW theory is a theory consisting of one free boson and four free fermions. In the RR sector of this theory there are several descendants that satisfy (5.4). Their left moving part is of the form

$$t_{a_n a_{n-1} \dots a_1} Q_{-n}^{a_n} Q_{-(n-1)}^{a_{n-1}} \dots Q_{-1}^{a_1} |\Omega\rangle \quad (5.9)$$

where $|\Omega\rangle$ is a ground state in the Ramond sector, and the tensor t is chosen so the $SU(2)$ spins of the state (5.9) are maximal. These

descendants give rise to the RR chiral primaries

$$\begin{aligned} & \left(\frac{n+1}{2}, \frac{n}{2}; \frac{n+1}{2}, \frac{n}{2} \right), \left(\frac{n+1}{2}, \frac{n}{2}; \frac{n}{2}, \frac{n+1}{2} \right), \\ & \left(\frac{n}{2}, \frac{n+1}{2}; \frac{n+1}{2}, \frac{n}{2} \right), \left(\frac{n}{2}, \frac{n+1}{2}; \frac{n}{2}, \frac{n+1}{2} \right), \end{aligned} \quad (5.10)$$

in the Z_p twisted sector when $p > n$.

6 D-brane analysis

In the case of the IIB configuration, one can study the boundary SCFT using D-brane perturbation theory. Indeed, this is how the boundary SCFT was identified in the case of the D1-D5 system. We consider the dynamics in the substringy regime, so the relevant degrees of freedom are the ones that come from open strings stretching between the various branes. By considering strings with boundary conditions dictated by the D-brane configuration one can easily obtain the massless degrees of freedom. 11 strings yield a $2d$ vector multiplet and $5_1 5_1$ and $5_2 5_2$ yield $6d$ vector multiplets. 15_1 and 15_2 strings yield $6d$ hypermultiplets. Finally there are $5_1 5_2$ strings that yield a single complex fermionic field, localized in the intersection of the 5_1 – 5_2 system. One could study the system from the point of view of either brane by introducing a corresponding probe brane and studying its worldvolume theory. Consider for instance the case of a probe D1 brane positioned in (x_0, x_1) . The 15_1 and 15_2 strings imply that the worldvolume theory contains two hypermultiplets. These hypermultiplets interact in a non-local manner through the $5_1 5_2$ strings. To lowest order one has the interaction coming from the one-loop diagram $15_1 - 5_1 5_2 - 5_2 1$. One could obtain these interactions by integrating out the $5_1 5_2$ strings. These interactions cannot be ignored even at low-energies as $5_1 5_2$ strings contain a massless degree of freedom. For the same reason, integrating out these strings introduces a singularity in the worldvolume theory. This complicates the analysis of the vacuum structure of the theory. We will not attempt such an analysis here. We only note that we also encountered some, perhaps related, non-locality in the supergravity description of the system. There we saw that in order to compactify the λ coordinate we had to identify large scales in the one D1-D5 system with small scales

of the other D1-D5 system (see (2.22)). In both cases, in the large k^+ limit keeping k^- fixed (or vice versa) the non-locality goes away. In the same limit there is a weakly coupled sigma model realization of \mathcal{A}_γ as we discuss in the next section.

7 σ Model

In this section we describe a natural class of σ -models with \mathcal{A}_γ symmetry and some of their properties. This class includes the ones that we proposed as duals of the string theory on $adS^3 \times S^3 \times S^3 \times S^1$. Previous work in this direction includes the realizations of \mathcal{A}_γ theories via Wolf spaces [42, 43, 37, 41], and on more general sigma models in [15].

It will be easiest to work in $N = 1$ superspace. The \mathcal{A}_γ algebra can be written in $N = 1$ superspace [41] because it is a linear algebra, and contains then the super stress-energy tensor of weight $3/2$, three spin-one supercurrents and four superfields of spin $1/2$. To realize this algebra we consider a generic $N = (1, 1)$ sigma model

$$S = \int d^2z d^2\theta (g_{\mu\nu} + b_{\mu\nu}) D_+ X^\mu D_- X^\nu \quad (7.1)$$

with

$$D_+ = \partial_\theta + \theta \partial, \quad D_- = \partial_{\bar{\theta}} + \bar{\theta} \bar{\partial}. \quad (7.2)$$

Suppose the action is invariant under a symmetry $\delta_\epsilon X^\mu$, with ϵ satisfying $D_- \epsilon = 0$. Then by varying the action with an arbitrary unconstrained parameter the variation will be of the form

$$\delta S = \int d^2z d^2\theta (D_- \epsilon)(2J) \quad (7.3)$$

where J is the Noether current of the symmetry. On-shell it satisfies $D_- J = 0$. Since we want to realize three spin one and four spin one-half symmetries in the sigma model we write as ansatz for the corresponding symmetries

$$\delta X^\mu = \epsilon J_{(a)\nu}^\mu D_+ X^\nu, \quad a = 1, 2, 3 \quad (7.4)$$

$$\delta X^\mu = \epsilon U_{(a)}^\mu, \quad a = 0, 1, 2, 3 \quad (7.5)$$

where we have three antisymmetric tensors $J_{(a)}$ and four vector fields $U_{(a)}$. As is well known, these transformation are symmetries of the action if $J_{(a)}$ and $U_{(a)}$ are covariantly constant with respect to the covariant derivative with torsion,

$$\nabla_\rho^+ J_{(a)\nu}^\mu = \nabla_\rho^+ U_{(a)}^\mu = 0. \quad (7.6)$$

In that case we find corresponding Noether currents

$$\Sigma_a = -U_{(a)\mu} D_+ X^\mu \quad (7.7)$$

$$S_a = -\frac{1}{2} J_{(a)\mu\nu} D_+ X^\mu D_+ X^\nu \quad (7.8)$$

Furthermore, the stress energy tensor is

$$T = -\frac{1}{2} G_{\mu\nu} D_+ X^\mu \partial X^\nu - \frac{1}{12} H_{\rho\mu\nu} D_+ X^\rho D_+ X^\mu D_+ X^\nu \quad (7.9)$$

with $H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}$; it generates the coordinate transformations

$$\delta X^\mu = \epsilon \partial X^\mu + \frac{1}{2} D_+ \epsilon D_+ X^\mu. \quad (7.10)$$

Next, we consider the Poisson brackets of these Noether currents. Ideally, these should give us the OPE's of \mathcal{A}_γ . However, some terms in the OPE's can come from higher order contractions between fields in the Noether currents and can therefore not be seen in the Poisson brackets. The parts of the OPE's that come from tree level contractions should be correctly reproduced by the Poisson brackets, and this implies several geometric constraints on the sigma model. We will omit the details of this analysis. One of the interesting equations one encounters is that the antisymmetric tensors $J_{(a)}$ have to satisfy, in the sense of matrix multiplication,

$$J_{(a)} \cdot J_{(b)} = -\delta_{ab} + \frac{k^- - k^+}{k} \epsilon_{abc} J_{(c)}. \quad (7.11)$$

However, the algebra of J 's is not associative unless

$$\frac{(k^- - k^+)^2}{k^2} = 1 \quad (7.12)$$

which means $k^+ = 0$ or $k^- = 0$. This is clearly not what we want. The point is that (7.12) only needs to be valid up to higher order

corrections. We can therefore allow the situation where $k^+ \gg k^-$ and k^-/k^+ is what controls the quantum corrections (or $k^- \gg k^+$ and k^+/k^- controls the quantum corrections), because then (7.12) is valid up to quantum corrections. This is also consistent with the fact that for a sigma model interpretation the central charge should be $3/2$ times the dimension of the target space up to quantum corrections. Indeed, for $k^+ \gg k^-$, the central charge is $c = 6k^-(1 + \mathcal{O}(k^-/k^+))$, which shows that the target space should have dimension $4k^-$, and similarly with the roles of k^+ and k^- reversed. An analogous situation would appear if we would consider a WZW model for a group of dimension $4k^-$ at level k^+ . This is all consistent with the realizations related to Wolf spaces, where one of the levels is related to the the dimension of the Wolf space, and the other to the level of the underlying WZW model.

Altogether this suggests that for $k^+ \gg k^-$, \mathcal{A}_γ theories can potentially be described by weakly coupled sigma models on spaces of dimension $4k^-$. As we decrease k^+ , the quantum corrections become stronger and stronger, until they are so large that the target space is no longer visible. As we then continue parameters to $k^+ \ll k^-$, another weakly coupled sigma model description appears, namely one where the target space has dimension $4k^+$.

Thus, a necessary condition for the sigma model under consideration to have a \mathcal{A}_γ symmetry is that the corresponding classical sigma model should have an \mathcal{A}_γ symmetry with $k^+ = 0$ or $k^- = 0$. The geometric conditions for this are that the target space should be a hyperkähler manifold with torsion with compatible $U(2)$ action (as defined in [33]). This should be true for both choices of torsion in the covariant derivatives $\nabla^\pm \sim \partial + \Gamma \pm H$. It was shown in [33] that if M is hyperkähler with torsion and compatible $U(2)$ action, $M/U(2)$ is quaternionic kähler with torsion and all quaternionic kähler manifolds can be obtained this way. We therefore see a close relation between quaternionic manifolds and the \mathcal{A}_γ algebra, but to actually construct a sigma model we should not use the quaternionic space as target space but rather the hyperkähler space it descended from.

A space which satisfies the above requirements is the sigma model with target space $(U(2))^{k^-}/S_{k^-}$. This has an obvious (diagonal) action of $U(2)$ from the left and right; once we choose a nonzero 3-form flux together with suitable discrete gauge fields and appropriate metric for

this diagonal $U(2)$, the $U(2)$ action is compatible with the hyperkähler structure with torsion. Thus our candidate SCFT has the right properties to correspond to an exact conformal invariant \mathcal{A}_γ theory.

It would be interesting to understand the sigma models with \mathcal{A}_γ symmetry in some more detail, and in particular to find the geometric interpretation of the notion of chiral primaries and of the ring structure of chiral primaries with $l^+ = l^-$. Quaternionic manifolds have various interesting cohomologies [38, 26], whose relation to the chiral operators remains to be explored.

8 Discussion and Open Problems

In this paper we studied the AdS/CFT duality in the case the boundary SCFT has the large $N = 4$ superconformal symmetry. We have presented supergravity solutions that in their near-horizon limit contain $AdS_3 \times S^3 \times S^3$, and computed the corresponding KK spectrum. We proposed that the boundary SCFT is (possibly a deformation of) a sigma model with target space $\text{Sym}^{k^-}(U(2))$, k/k^- units of 3-form flux, and suitable discrete gauge fields, and found that this is consistent with the KK spectrum and has the right qualitative properties.

There are many issues that deserve further study. It will be interesting to analyze in more detail the IIB configuration using D-brane perturbation theory. In particular, to consider a probe brane and to obtain all couplings to the first non-trivial order. As noted in section 6, certain couplings are expected to first appear at one-loop level. Since there are massless states running in these loops, one cannot ignore these couplings even in low energies. These couplings, however, are of order $1/N$. Knowing in detail the worldvolume theory will presumably also help us understand the meaning of the identification (2.22) needed in supergravity in order to compactify one of the “near-horizon” coordinates. Furthermore, the moduli space of the gauge theory should be related to $\text{Sym}^{k^-}(U(2))$.

Another issue is to provide a more precise formulation of the boundary SCFT, and more stringent tests of the conjectured duality. In particular, the precise meaning of the discrete fluxes and the correct

treatment of the singularities (for a recent discussion see [40]) deserves further clarification. A more detailed comparison of the spectrum of the boundary SCFT with that of supergravity is also desirable. This requires a more detailed understanding of the spectrum of the boundary SCFT. A useful tool may be the index recently proposed in [30] which plays the role of the elliptic genus for \mathcal{A}_γ theories. Once the spectra are better understood one could go on and compare correlation functions in the two descriptions.

In summary, it appears that for generic k^+, k^- there is no useful semi-classical description. The \mathcal{A}_γ σ -models are only weakly coupled for $k^+ \gg k^-$ (or vice versa), the worldvolume gauge theory contains non-local interactions that are only suppressed at large N , and the supergravity solution implies a $3d$ solution only if there are identifications of order $1/N$. Nevertheless, the large amount of symmetry and exact knowledge of the theory at $k = qk^-$ may be sufficient to provide a non-perturbative solution.

Independently of the considerations regarding the AdS/CFT duality, it is an interesting question to investigate which theories flow in the infrared to a fixed point with \mathcal{A}_γ symmetry. It is also interesting to further study the σ -models with \mathcal{A}_γ symmetry, and in particular to understand the geometric notion of chiral primaries.

There is an interesting connection between our considerations and the issue of creation of branes when branes cross each other. Consider the case of two D5 branes intersecting over a string, say in 12345 and 16789, respectively, and with an electric field along the string. After T-duality along x_1 one gets two $D4$ branes with a relative velocity in the circle direction. As shown in [2], an open string is created when the two $D4$ branes cross each other. This phenomenon is U-dual to the creation of a single D3-brane when a D5-brane crosses an NS5-brane[24]. Lifting the IIA configuration to M-theory we get that an M2-brane is created when two M5 branes cross each other. In particular, the branes are positioned as in the M-theory configuration we described in section 2. Thus, the fivebranes interact through the creation (or annihilation) of M2 branes, and the boundary SCFT should capture some of the dynamics of this phenomenon.

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A \mathcal{A}_γ and $\tilde{\mathcal{A}}_\gamma$ conventions

The OPE of the \mathcal{A}_γ algebra generators are ¹⁰

$$\begin{aligned}
 G_a(z)G_b(w) &= \frac{2c/3\delta_{ab}}{(z-w)^3} + \frac{2M_{ab}(w)}{(z-w)^2} + \frac{2L(w)\delta_{ab} + \partial M_{ab}(w)}{z-w} + \dots \\
 M_{ab} &\equiv -\frac{4}{k} [k^- \alpha_{ab}^{+i} A_i^+ + k^+ \alpha_{ab}^{-i} A_i^-] \\
 A^{\pm i}(z)G_a(w) &= \alpha^{\pm i}{}_a{}^b \left(\frac{G_b(w)}{z-w} \mp \frac{2k^\pm Q_b(w)}{k(z-w)^2} \right) + \dots \\
 A^{\pm i}(z)A^{\pm j}(w) &= \frac{\epsilon^{ijk} A_k^\pm(w)}{z-w} - \frac{k^\pm \delta^{ij}}{2(z-w)^2} + \dots \\
 Q_a(z)G_b(w) &= \frac{2(\alpha_{ab}^{+i} A_i^+(w) - \alpha_{ab}^{-i} A_i^-(w)) + \delta_{ab} U(w)}{(z-w)} + \dots \\
 A^{\pm i}(z)Q_a(w) &= \frac{\alpha^{\pm i}{}_a{}^b Q_b(w)}{z-w} + \dots \\
 U(z)G_a(w) &= \frac{Q_a(w)}{(z-w)^2} + \dots \\
 Q_a(z)Q_b(w) &= -\frac{k\delta_{ab}}{2(z-w)} + \dots \\
 U(z)U(w) &= -\frac{k}{2(z-w)^2} + \dots
 \end{aligned} \tag{A.1}$$

¹⁰Those with the Virasoro algebra generators are as usual.

where in complex notations $i = \{+, -, 3\}$, $a = \{+, -, +K, -K\}$ and the non-vanishing values (up to symmetry) of the various symbols are

$$\begin{aligned}
 \delta_{+-} &= \delta_{+K-K} = \frac{1}{2} & \epsilon^{+-}_3 &= -2i & \epsilon^{3\pm}_{\pm} &= \mp i \\
 \alpha^{\pm 3}_{+-} &= -\frac{i}{4} & \alpha^{\pm 3}_{+K-K} &= \mp \frac{i}{4} & \alpha^{+-}_{++K} &= \frac{i}{2} \\
 \alpha^{++}_{--K} &= -\frac{i}{2} & \alpha^{-+}_{-+K} &= -\frac{i}{2} & \alpha^{--}_{+-K} &= \frac{i}{2}
 \end{aligned} \tag{A.2}$$

ϵ^{abcd} can be defined by

$$\alpha^{\pm iab} \alpha^{\pm}_{icd} = \frac{1}{4} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \pm \epsilon^{abcd}) \tag{A.3}$$

For the $\tilde{\mathcal{A}}_\gamma$ algebra, the non trivial OPEs are

$$\begin{aligned}
 \tilde{A}^{\pm i}(z) \tilde{G}_a(w) &= \frac{\alpha^{\pm i}_a{}^b \tilde{G}_b(w)}{z-w} + \dots \\
 \tilde{A}^{\pm i}(z) \tilde{A}^{\pm j}(w) &= \frac{\epsilon^{ijk} \tilde{A}^{\pm}_k(w)}{z-w} - \frac{\tilde{k}^{\pm} \delta^{ij}}{2(z-w)^2} + \dots
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 &\tilde{G}_a(z) \tilde{G}_b(w) \\
 &= \frac{4\tilde{k}^+ \tilde{k}^- \delta_{ab}}{k(z-w)^3} + \frac{2\tilde{L}(w) \delta_{ab}}{(z-w)} - \frac{8(\tilde{k}^- \alpha_{ab}^{+i} \tilde{A}_i^+(w) + \tilde{k}^+ \alpha_{ab}^{-i} \tilde{A}_i^-(w))}{k(z-w)^2} \\
 &\quad - \frac{4\partial(\tilde{k}^- \alpha_{ab}^{+i} \tilde{A}_i^+(w) + \tilde{k}^+ \alpha_{ab}^{-i} \tilde{A}_i^-(w))}{k(z-w)} \\
 &\quad - \frac{8(\alpha^{+i} \tilde{A}_i^+(w) - \alpha^{-i} \tilde{A}_i^-(w))_{c(a} (\alpha^{+j} \tilde{A}_j^+(w) - \alpha^{-j} \tilde{A}_j^-(w))_{b)}^c}{k(z-w)} + \dots
 \end{aligned}$$

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