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Chen's Iterated Integral represents the Operator Product Expansion

D. Kreimer

Dept. of Physics, Mainz Univ.
D-55099 Mainz, Germany

dirk.kreimer@uni-mainz.de

Abstract

The recently discovered formalism underlying renormalization theory, the Hopf algebra of rooted trees, allows to generalize Chen's lemma. In its generalized form it describes the change of a scale in Green functions, and hence relates to the operator product expansion. Hand in hand with this generalization goes the generalization of the ordinary factorial $n!$ to the tree factorial $t^!$. Various identities on tree-factorials are derived which clarify the relation between Connes-Moscovici weights and Quantum Field Theory.

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1 Introduction

In this paper we want to explore a close resemblance between a mathematical structure, iterated integrals [1], and a structure from physics, renormalization. Renormalization theory has been recently identified to be rooted in a Hopf algebra structure, which encapsulates its combinatorial properties [2]. Further, renormalization establishes a calculus which generalizes the algebraic approach to the diffeomorphism group, featured by Connes and Moscovici [3, 4]. Structures due to this generalization are relevant for the practitioner of perturbative Quantum Field Theory (pQFT) [5].

Here, we will study all of these aspects in some detail, featuring in particular the role of renormalization schemes, the renormalization group and the operator product expansion (OPE).

To proceed, we push forward two parallel developments. One is the use of toy models at a more and more sophisticated level, which provide clarifying examples to exhibit underlying ideas on which the reader can check formal developments, which will establish a set-up which allows to recover the standard notions established by physicistes, the before mentioned existence of renormalization schemes, renormalization groups and operator product expansions.

In this manner, we will show that the Hopf algebra of rooted trees not only describes the combinatorics of renormalization, but also analytical structure: the behaviour under variations of scales.

Throughout this paper we assume that the results and notions of [2, 3, 5, 6] are familiar. We will nevertheless summarize some basic notions and conventions.

1.1 Notation

The Hopf algebra of rooted trees (which are possibly decorated multiplicative generators) is denoted by $\mathcal{H}$, with coassociative coproduct Δ, antipode S and counit $\bar{e}$. The multiplication in the algebra is denoted by m. It is commutative, and hence $S^2 = 1$. The unit of the algebra is denoted by e. The counit is denoted by $\bar{e}$, with $\bar{e}(e) = 1$ and 0 otherwise. The underlying number-field is assumed to be $\mathbf{Q}$.

The fertility of a vertex of a rooted tree is the number of outgoing edges. The root is always drawn as the uppermost vertex, and all edges are oriented away from the root.

Rooted trees t are graded by their number of vertices $\#(t)$. For a product of rooted trees $\prod_i t_i$ we define $\#(\prod_i t_i) = \sum_i \#(t_i)$. Obviously, $\#(e) = 0$.

We abbreviate the coproduct using Sweedler's notation: $\Delta(t) = \sum t_{(1)} \otimes t_{(2)} \; \forall t \in \mathcal{H}$.

1.2 Summary of sections

This paper is organized as follows. In section two we introduce the iterated integral. In particular, we focus on iterated integrals which have a divergence at the upper boundary, which is a choice motivated by the renormalization problem in QFT, formulated in momentum space. We show how we get well-defined iterated integrals as renormalized Green functions, and show that a variation of scales amounts to an application of Chen's Lemma.

In section three we focus on the multiplicativity of renormalization. Crucial is the formulation of multiplicativity constraints, which are sufficient to derive the multiplicativity of counterterms. We present a boundary operator d_R for any renormalization scheme R and show that all renormalization schemes can be treated on the same footing on the expense of introducing tree-indexed parameters.

Section four applies those results to a restricted class of Feynman diagrams, those which represent trees with the same decoration at each vertex. Such classes were considered already in [5]. Here, we use them to exemplify the results of section three.

Section five proves some identities for rooted trees which were conjectured in [5] and which are useful in understanding the relation to noncommutative geometry.

Section six gives the principal reason why operator product expansions are related to Chen's Lemma.

Conclusions finish the paper. It is the main objective of this paper to introduce and exhibit some essential properties of the Hopf algebra approach underlying renormalization, using iterated integrals as a convenient toy all along on which the reader can test the relevant notions. The translation to proper Green functions is a notational exercise which can be conveniently spelled out whenever needed, as, for example, in [8].

While this paper introduces essential conceptual properties, details will be presented in future work.

2 Iterated Integrals and Renormalization

The crucial feature of renormalization is the fact that it is governed by its underlying Hopf algebra structure of rooted trees. Bare Green functions as they appear in a perturbative approach to QFT based on polynomial interactions provide a representation of this Hopf algebra. Apart from a systematic access to the renormalization problem of such QFTs the Hopf algebra also allows to study other representations and hence to define models for the renormalization problem which deliver handy tools to study more advanced topics, for example the change of scales and renormalization schemes. In this section, we will largely consider iterated integrals for that purpose.

2.1 The iterated integral

We will start our considerations by reminding ourselves of some basic properties of iterated integrals [1, 7]. We specialize to the case of a single function $f(x)$ with associated one-form $f(x)dx$ on the real line.

Then, iterated integrals built with the help of f are parametrized by an integer n, and two real numbers a, T say. They are defined by

$$F^{[0]}_{a,T} = 1, \forall a, T \in \mathbf{R}, \tag{1}$$

$$F^{[n]}_{a,T} = \int_a^T f(x) F^{[n-1]}_{a,x} dx, \forall n > 0. \tag{2}$$

Hence we can write them as an integral over the simplex

$$F^{[n]}_{a,T} = \int_{a\leq x_1<\ldots<x_n\leq T} f(x_1)\ldots f(x_n)dx_1\ldots dx_n. \quad (3)$$

We can easily generalize this to the case of different functions f_i, and, defining a string of integers $I = (i_1, \ldots, i_n)$, we can define

$$F^{I}_{a,T} = \int_{a\leq x_1<\ldots<x_{n-1}\leq x_n} f_{i_1}(x_1)\ldots f_{i_n}(x_n)dx_1\ldots dx_n, \quad (4)$$

where i_k are integers, taken from some index set $\mathcal{I}$, labelling the available forms $f_{i_k}(x_k)dx_k$.

It is a well-known fact [7] that such integrals fulfil a convolution

$$F^{I}_{a,T} = F^{I}_{a,s} + F^{I}_{s,T} + \sum_{I=(I'I'')} F^{I'}_{a,s}F^{I''}_{s,T}, \quad (5)$$

where the sum is over the $n-1$ partitions of the string I into two non-empty substrings I', I''. We shall dubb (5) Chen's Lemma, following [7].

2.2 Renormalization of iterated integrals

Let us motivate our interest in iterated integrals and Chen's Lemma. Consider again the trivial case of only one f and let us assume it behaves for large $x \gg b$ as $f(x) \equiv f(\epsilon; x) \sim x^{-1-\epsilon}$, for $0 < \epsilon \ll 1$. Let us then define

$$G^{[t_2]}_{b,\infty} = \left[\int_b^\infty \left(\int_x^\infty f(y)dy\right) f(x)dx\right]. \quad (6)$$

Then, in the limit $\epsilon \to 0$, this expression is ill-defined. It has the structure of a nested y-integration, furnishing a subdivergence (in ()-brackets) in the jargon of renormalization theory, which is nested inside the final x-integration, which diverges as well and thus provides an overall divergence (in []-brackets).

To such a combination of ill-defined integrations we associate a rooted tree $t = t_2(f, f)$, as in Fig.(1), following the guidance of [2, 3].

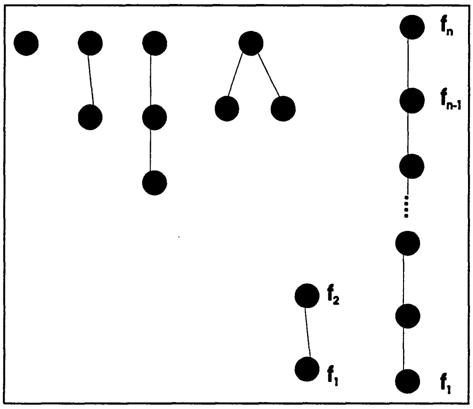

Figure 1: Rooted trees describe nested integrations. We give the trees $t_1, t_2, t_{3_1}, t_{3_2}$ from left to right and a decorated tree without sidebranchings, $B_+^n(e)$, with decorations f_n to f_1. Also, we explicitly give the tree $t_2(f_1, f_2)$ for the case $n = 2$.

Generalizing to arbitrary n, we define, $\forall b \in \mathbf{R}_+$, functions

$$G_{b,\infty}^{[e]} = 1, \quad (7)$$

$$G_{b,\infty}^{[t_n]} = -\left[\int_b^\infty \left(G_{x,\infty}^{[t_{n-1}]}\right) f(x)dx\right], \ \forall n \geq 1. \quad (8)$$

To them, we assign the rooted tree $t_n := B_+^n(e)$ of n vertices without side-branching, as in Fig.(1), and understand that the empty tree, the unit e of the Hopf algebra $\mathcal{H}$, is associated to $G_{b,\infty}^{[e]} = 1$. As a decorated rooted tree t_n carries the same decoration f at each vertex.

It is straightforward to see that the trees t_n form a closed sub-Hopf algebra $\mathcal{H}_{Chen}$ of $\mathcal{H}$, which is a Hopf algebra based on rooted trees without sidebranchings:

$$\Delta[t_n] = t_n \otimes e + e \otimes t_n + \sum_{i=1}^{n-1} t_i \otimes t_{n-i} \quad (9)$$

$$S[t_n] = -t_n - \sum_{i=1}^{n-1} S[t_i]t_{n-i}. \quad (10)$$

This remains true for decorated rooted trees and in the same spirit, we can assign a decorated rooted tree t_I to any function

$$G_{b,\infty}^{[t_I]} = -\int_b^\infty G_{x,\infty}^{[t_I']} f_{i_n}(x)dx, \quad (11)$$

where t'_I is the rooted tree $B_-(t_I)$, providing an index string which has the n-th entry i_n of I deleted. The Hopf algebra of decorated rooted trees without sidebranchings is still denoted by $\mathcal{H}_{Chen}$.

Note that the functions $G^{[t_I]}_{b,\infty}$ can be regarded as iterated integrals in their own right:

$$G^{[t_I]}_{b,\infty} = \lim_{\lambda\to\infty} F^I_{\lambda,b}. \tag{12}$$

All the functions f_i, $i \in \mathbf{N}$, are assumed to behave as $\lim_{x\to\infty} f_i(x) = c_i x^{j_i-\epsilon}$, for some constant c_i and some integer j_i with $j_i \geq -1$. Hence, in the limit $\epsilon \to 0$, these iterated integrals are ill-defined, due to a divergence at the upper boundary. We will have to renormalize them. The Hopf algebra will allow us to find a way to make sense out of the expressions $G^{[t_I]}_{b,\infty}$ at $\epsilon = 0$. We will discuss further aspects of the behaviour at infinity in some detail later. All renormalization properties discussed below extend in an obvious way to the renormalization at endpoints different from infinity, if it so happens that the functions $f_i(x)$ have singularities at such endpoints. The renormalization procedure is a very natural operation, as we will see, and one can define applications largely extending the task of eliminating UV divergences. A recent review on its relations to many branches of science can be found in [9]. An obvious application is to the configuration space of n distinct points, which can be tested out by differential forms which diverge at (sub-)diagonals. Such an application will be described in [11].

We can multiply the functions and add the functions $G^{[t_I]}_{b,\infty}$ freely. Hence, if $\phi : \mathbf{R}_+ \times \mathcal{H}_{Chen} \to V$ is the map which assigns to any decorated rooted tree $t_I \in \mathcal{H}_{Chen}$ and positive real number b the function $G^{[t_I]}_{b,\infty}$, then this gives us a representation, parametrized by b,

$$\phi(b; t_I t_J) = \phi(b; t_I)\phi(b; t_J). \tag{13}$$

We also set $\phi(b; 0) = 0$ and $\phi(b; e) = 1$, $\forall b$, in accordance with (1). For a chosen b, we further write $\phi_b : \mathcal{H}_{Chen} \to V$, $t \to \phi_b(t) := \phi(b; t)$. The target space V can be considered as the ring $\mathbf{R}[\epsilon^{-1}, [\epsilon]]$ of Laurent series with poles of finite order. The parameter b is from now on denoted as the scale of the representation. The functions $\phi(b, t) = G^{[t]}_{b,\infty}$ can be considered as role models for bare Green functions. They depend on an external scale b. We will utilize this dependence to define the renormalization procedure.

Let R_a be the map which sends $\phi_b \to \phi_a$. We thus evaluate at a different scale. Essentialy, we claim, it is this change of external scale(s) which allows us to renormalize in a non-trivial manner. To the bare Green function $\phi_b : \mathcal{H}_{Chen} \to V$, which defines a representation of $\mathcal{H}_{Chen}$, we associate another function $S_{R_a}(\phi_b) : \mathcal{H}_{Chen} \to V$ by

$$S_{R_a}(\phi_b)(t) := -R_a[\phi_b(t) + m[(S_{R_a} \otimes id)(\phi_b \otimes \phi_b)P_2\Delta(t)]], \tag{14}$$

defined for any monomial t of decorated rooted trees, $t \neq e$. For $t = e$ we set $S_{R_a}(\phi_b)(e) = 1$. In the above, P_2 denotes the projector $(id - e\bar{e}) \otimes (id - e\bar{e})$ which annihilates any appearance of the unit e.

The resulting map $S_{R_a}(\phi_b)$ is independent of b by the definition of R_a which eliminates any dependence on the scale b. Two examples, $t = t_1(f_i)$ and $t = t_2(f_i, f_j)$ might be useful:

$$S_{R_a}(\phi_b)(t_1(f_i)) = -\phi_a(t_1(f_i)) = \int_a^\infty f_i(x)dx, \tag{15}$$

$$\begin{aligned} S_{R_a}(\phi_b)(t_2(f_i, f_j)) &= -\phi_a(t_2(f_i, f_j)) + \phi_a(t_1(f_i))\phi_a(t_1(f_j)) \\ &= -\int_a^\infty \int_x^\infty f_i(y)dyf_j(x)dx \\ &\quad + \int_a^\infty f_i(x)dx \int_a^\infty f_j(y)dy \\ &= \int_a^\infty \int_a^x f_i(y)dyf_j(x)dx. \end{aligned} \tag{16}$$

Now, we define a function $\Gamma : \mathbf{R}_+ \times \mathbf{R}_+ \times \mathcal{H}_{Chen} \to V$ by

$$\Gamma_{a,b}(t) = \sum S_{R_a}(\phi_b)(t_{(1)})\ \phi_b(t_{(2)}) \tag{17}$$

$$= m[(S_{R_a} \otimes id)(\phi_b \otimes \phi_b)\Delta(t)] \tag{18}$$

$$= m[(\phi_{R_a} \otimes \phi_b)(S \otimes id)\Delta(t)], \tag{19}$$

where we define $\phi_{R_a} : \mathcal{H}_{Chen} \to V$ by $\phi_{R_a} = S_{R_a}(\phi_b \circ S)$. Note that this renormalized function has naturally the structure of a ratio, comparing to scalar functions of rooted trees with the help of the antipode. This has far reaching consequences [13, 9].

ϕ_{R_a} is still independent of b by the definition of R_a. Note further that $\Gamma_{a,b}(t)$ exists in the limit $\epsilon \to 0$ when we integrate to infinity and can be regarded as the renormalized iterated integral associated to the bare iterated integral $G^t_{b,\infty}$. The equality between (18) and (19) follows because of $S_{R_a} \circ \phi_b = S_{R_a} \circ \phi_b \circ S^2 = \phi_{R_a} \circ S$,using the definition of ϕ_{R_a} and $S^2 = id$.

Proposition 1 $\Gamma_{a,b}(t) = F^I_{a,b}$, *where t is the decorated rooted tree with n vertices corresponding to the string I.*

Straightforward (for example, use (5) and that $G^t_{b,\infty}$ is itself an iterated integral from b to ∞).

Example.

$$\Gamma_{a,b}(t(f_i)) = \left[-\int_b^\infty + \int_a^\infty\right] f_i(x)dx = \int_a^b f_i(x)dx, \tag{20}$$

$$\begin{aligned}\Gamma_{a,b}(t(f_i, f_j)) = & \left[\int_b^\infty \int_x^\infty - \int_b^\infty \int_a^\infty \right. \\ & \left. - \int_a^\infty \int_x^\infty + \int_a^\infty \int_a^\infty \right] f_i(y)dy f_j(x)dx \\ = & \int_a^b \int_a^x f_i(y)dy f_j(x)dx. \end{aligned} \tag{21}$$

We now read this as an instructive example for renormalization. The role of a bare Green function, demanding renormalization, is played by $G^t_{b,\infty} \equiv \phi_b(t)$. It provides $n-1$ subdivergences, as all integrations diverge at the upper boundary.

Then, $S_{R_a}[\phi(t_I)]$ delivers a counterterm such that $\Gamma_{a,b}(t_I)$ is a quantity which is renormalized: it contains only well-defined integrations and the limit $\epsilon \to 0$ can be taken at the level of integrands.

Note that if the bare Green function would be independent of the external scale furnished by the parameter b, then $\phi_a(t) = \phi_b(t)$ and as a consequence, $\Gamma_{a,b}(t) \equiv 0$.[1]

This can be utilized to show that $\Gamma_{a,b}(t)$ is determined by the coefficient of logarithmic divergence at infinity. It is thus natural to look for a representation of $\mathcal{H}$ in terms of residues in the sense of [12, 9].

[1] This makes dimensional regularization a succesful regularization scheme in practice: it annihilates scale independent terms from the beginning, and is hence extremely economic.

For the moment, we note that the presence of a second scale a is unavoidable if we want to go from bare functions to renormalized ones. $\Gamma_{a,b}(t)$ is essentially the ratio of two representations, one parametrized by b, the other by a. We call a the renormalization point. For all possible values of the external scale b there is one place, the diagonal $b = a$, at which $\Gamma_{b,b}(t) = 0$. Further,

$$R_a(\Gamma_{a,b}(t)) = R_a[(S_{R_a} \star id)\phi_b](t) = \bar{e}(t), \tag{22}$$

showing that S_{R_a} is the inverse of the identity in the range V_{R_a} of R_a in V. This inverse is taken with respect to the induced convolution in $\mathcal{H}^\star_{Chen} \otimes V$

$$[\psi \star \phi](t) = \sum \psi(t_{(1)})\phi(t_{(2)}), \tag{23}$$

valid for all maps $\psi, \phi : \mathcal{H}_{Chen} \to V$.

Let us summarize: We start with a bunch of ill-defined integrations, labelled by a decorated rooted tree t from which we can determine the bare integral demanding renormalization.

We then construct the analytic expressions determined by the counterterm map $S_{R_a} : \mathcal{H}^\star_{Chen} \otimes V \to \mathcal{H}^\star_{Chen} \otimes V$. This gives rise to a renormalized iterated integral $\Gamma_{a,b}(t)$ which only involves well-defined integrations. It assigns a well-defined analytic expression to any decorated rooted tree. This expression necessary vanishes along the diagonal $a = b$. In contrast to this, ϕ_b associated the ill-defined bare integral $\phi_b(t)$ to any rooted tree t. This transition from $\phi_b(t)$ to $\Gamma_{a,b}(t)$ is what renormalization typically achieves.

A final remark in this section concerns the solution to the Knizhnik-Zamolodchikhov (K-Z) equation in two variables, based on forms $\frac{dz}{z}, \frac{dz}{1-z}$, say. See [7] for a review.

Consider the K-Z equation

$$\frac{dF}{dz} = \left(\frac{a}{z} - \frac{b}{1-z}\right) F. \tag{24}$$

Here, a and b are two *noncommuting* variables which actually provide a free Lie algebra on two elements. Arbitrary words out of the two-letter alphabet $\{a, b\}$ are considered and no relation amongst such words exist. Let W be the set of all words.

The length of such a word w is $l(w)$ and the i'th letter of w is $w(i)$. Let us consider the following expression

$$G(u,v) = \sum_{w \in W} \int_{\Delta(u,v)} \prod_{i:w(i)=a} \frac{dz_i}{z_i} \prod_{i:w(i)=b} \frac{dz_i}{(1-z_i)} \tag{25}$$

$$\Delta(u,v) = u > z_{l(w)} > \ldots > z_1 > v. \tag{26}$$

$G(u,v)$ is known to be a solution to the K-Z equation in the interval $]0,1[$.

$G(u,v)$ contains multiple zeta values [17] (MZV's) for $(u,v) = (1,0)$, whenever the limits $u \to 1$, $v \to 0$ are defined. But whenever a word starts with b or ends with a these limits do not exist.

Hence this solution is a series $\sum_w wF^w_{v,u}$ over all words built out of two noncommuting variables a, b multiplying iterated integrals $F^w_{v,u}$ in the interval $]0,1[$ which possibly diverge at both endpoints of the interval $]0,1[$. Let W' be the words which neither start with b nor end with a.

Renormalization can be applied to $G(u,v)$ such that it extends to either boundary, and applying it successively using the renormalization schemes R_0 (which removes all words ending with a) and R_1 (which removes all words beginning with b) at the lower and upper boundaries leaves us with the K-Z associator

$$\phi_{KZ} = \sum_{w \in W'} \int_{\Delta(u,v)} \prod_{i:w(i)=a} \frac{dz_i}{z_i} \prod_{i:w(i)=b} \frac{dz_i}{(1-z_i)} \tag{27}$$

as the renormalized 'Green function'. An iterated integrals accompanying the word $b^{m_1}a^{n_1}b^{m_2}\ldots a^{n_{k-1}}b^{m_k}a^{n_k}$ must be regarded as a representative of the rooted tree $B_+^{m_1}(e)$ for the renormalization at the upper boundary 1 (where b is the variable assigned to $dz/(1-z)$). For the renormalization at the lower boundary it represents $B_+^{n_k}(e)$ (where a is the variable assigned to dz/z). Then, the renormalized iterated integral assigned to $G(u,v)$ extends to $[0,1]$, where it is the above K-Z associator.

2.3 Change of Scales

It is most interesting to consider the behaviour if we change the renormalization point $a \to a'$, which will lead us to the group law underlying the evolution of functions of rooted trees quite generally, which is the group law of the Butcher group (comp. [13, 9] and references there). One gets

$$\Gamma_{a,b}(t_I) = \sum_{I=(I',I'')} \Gamma_{a,a'}(t_{I'})\Gamma_{a',b}(t_{I''}). \tag{28}$$

Proof: this is just (5) for iterated integrals [7]. Nevertheless, let us derive it from the Hopf algebra structure of $\mathcal{H}$. At this stage we should actually use $\mathcal{H}_{Chen}$. But as we will see that nothing in the following derivation depends on the peculiarities of this sub-Hopf algebra of $\mathcal{H}$, we directly use the latter instead. Hence define the following operator

$$U : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to V \tag{29}$$

$$U = m[m \otimes m](\phi_{R_a} \otimes \phi_{a'} \otimes \phi_{R_{a'}} \otimes \phi_b). \tag{30}$$

Composition with $M : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$,

$$\begin{aligned} M : &= (S \otimes id \otimes S \otimes id)(\Delta \otimes \Delta)\Delta \\ &= (S \otimes id \otimes S \otimes id)(\Delta \otimes id \otimes id)(id \otimes \Delta)\Delta \\ &= (S \otimes id \otimes S \otimes id)(id \otimes \Delta \otimes id)(id \otimes \Delta)\Delta \end{aligned} \tag{31}$$

gives

$$U[M(t)] = \phi_{R_a}[S(t_{(1)})]\phi_{a'}[t_{(2)}]\phi_{R_{a'}}[S(t_{(3)})]\phi_b[t_{(4)}],$$

where coassociativity of $\mathcal{H}_R$ allows to use Sweedler's notation throughout. The above is

$$\phi_{R_a}[S(t_{(1)})]m[(\phi_{a'} \otimes \phi_{R_{a'}})(id \otimes S)\Delta(t_{(2)})]\phi_b(t_{(3)}). \tag{32}$$

As

$$\begin{aligned} m[(\phi_{a'} \otimes \phi_{R_{a'}})(id \otimes S)\Delta(t)] &= \phi_{a'}(m[(id \otimes S)\Delta(t)]) \\ &= \phi_{a'}(\bar{e}(t)) = \phi_{a'}(0) = 0, \end{aligned}$$

$\forall t \neq 1$, this has contributions only for $t^{(2)} = e$, in which case we obtain

$$m[(\phi_{R_a} \otimes \phi_b)(S \otimes id)\Delta(t)], \tag{33}$$

as desired. We used $S_{R_{a'}}(\phi_b) = \phi_{a'} \circ S$, which we prove later, see (45).

Nothing in this derivation prevents us to generalize to arbitrary rooted trees, extending from $\mathcal{H}_{Chen}$ to $\mathcal{H}$. We thus define, for a decorated rooted tree $t \in \mathcal{H}$ with n vertices,

$$G^t_{b,\infty} = -\int_b^\infty f_{i_n}(x) \prod_j G^{[t']_j}_{x,\infty} dx, \tag{34}$$

where the product is over the decorated branches of the decorated tree t, $B_-(t) = \prod_j t'_j$ and $f_{i_n}(x)$ is the label attached to the root. We still write $\phi_b(t)$ for $G^t_{b,\infty}$, but stress that $\phi_b(t) : \mathcal{H} \to V$ now gives parametrized representation for the full Hopf algebra $\mathcal{H}$ of decorated rooted trees.

We also define, $\forall t \neq e$, the functions $S_{R_a}(\phi_b)(t)$ and $\Gamma_{a,b}(t)$ without any change:

$$S_{R_a}(\phi_b)(t) = -R_a[\phi_b(t) + m[(S_{R_a} \otimes id)(\phi_b \otimes \phi_b)P_2\Delta(t)]] \tag{35}$$

and

$$\begin{aligned}
\Gamma_{a,b}(t) &= \sum S_{R_a}(\phi_b(t_{(1)}))\phi_b(t_{(2)}) && (36)\\
&= m[(S_{R_a} \otimes id)(\phi_b \otimes \phi_b)\Delta(t)] && (37)\\
&= m[(\phi_{R_a} \otimes \phi_b)(S \otimes id)\Delta(t)]. && (38)
\end{aligned}$$

Then, in a straightforward generalization one concludes from the above derivation

Lemma 1

$$\begin{aligned}
\Gamma_{a,b}(t) &= [S_{R_a}(\phi_b) \star \phi_b](t) = [\phi_{R_a} \circ S \star \phi_b](t)\\
&= \sum \Gamma_{a,a'}(t_{(1)})\Gamma_{a',b}(t_{(2)}). && (39)
\end{aligned}$$

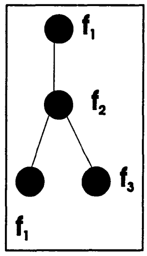

Figure 2: An example.

This lemma holds for any scalar function of rooted trees (with generalizations to matrix functions worked out in [8]) and hence applies to full QFT [5] as well.

Example. Let t be the decorated rooted tree of Fig.(2). Then, $G^t_{b,\infty}$ is given as

$$G^t_{b,\infty} = \int_b^\infty f_1(x_1) \int_{x_1}^\infty f_2(x_2) \int_{x_2}^\infty f_1(x_3)dx_3 \int_{x_2}^\infty f_3(x_4)dx_4dx_2dx_1. \tag{40}$$

Accordingly, $\Gamma_{a,b}(t)$ becomes

$$\Gamma_{a,b}(t)=\underbrace{\int_a^b f_1(x_1) \underbrace{\int_a^{x_1} f_2(x_2) \underbrace{\int_a^{x_2} f_1(x_3)dx_3} \underbrace{\int_a^{x_2} f_3(x_4)dx_4}\, dx_2}\, dx_1}, \tag{41}$$

as the reader should check. The underbracings indicate the tree structure of the nested and disjoint subintegrations, which is also exemplified in Fig.(2).

A few remarks. We obtain a natural generalization of the iterated integral. Actually, due to the fact that iterated integrals obey the shuffle product, it is not yet a generalization, as any bare integral representing a tree with side branchings is a linear combination of integrals representing trees in $\mathcal{H}_{Chen}$. Hence, at this stage, the generalization is merely a convenient notation. But there are more generalizations lying ahead, considering representations of $\mathcal{H}$ which extend the notions of iterated integrals truely, still obeying the convolution which implies Chen's lemma for ordinary iterated integrals. In particular, bare Green functions as typically derived from Feynman rules in the perturbative approach to a local QFT represent rooted trees in a manner such that trees with sidebranchings cannot be reexpressed in terms of trees without sidebranchings, as a shuffle product is absent in such circumstances.[2] Nevertheless, the derivation of the convolution of renormalized functions is a mere application of the coassociativity of $\mathcal{H}$ itself, and hence applies whenever appropriate representations of the Hopf algebra are available.

It will be an interesting exercise in the future to understand the monodromy of Green functions from this approach in the same manner as one can understand the monodromy of the polylogarithm from the study of the renormalized solution of the K-Z equation. Green functions in QFT are a more general class of functions than polylogs. Nevertheless, at lower loop orders, they are intimately related which might well be understood one day as testimony to the fact that Green functions in pQFT realize in a wider set-up algebraic structures which polylogs strictly obey.

Another generalization lies in the possibility to consider tree-indexed parameters in the integral. This will turn out a convenient means to parametrize the freedom which we have in the renormalization approach. To this idea we come back soon.

It is an interesting question which information about a manifold M such generalizations can provide, using as functions f_i the pull-back of appropriate forms ω_i via paths on that manifold, or, vice versa, how one can construct manifolds providing, eventually, iterated integrals which evaluate to the same renormalized Green functions as a QFT. This amounts to setting up and solving appropriate systems of equations, making use of the recursive properties of the Hopf algebra. Apart from a few further remarks along these lines in the next section the reader will find examples in [8].

Before we consider further generalizations by tree-indexed scales, we come to some interesting structures which can be readily observed at this level.

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[2]Though, as reported elsewhere [8], some remainders of it are still visible in QFT.

3 Multiplicativity of renormalization and consequences

So far, we observed how a change in the renormalization point is expressed by the generalized form of Chen's Lemma. This gives a very nice handle on the renormalization group (see below), and relates it to quite general algebraic considerations. While in this paper we will only outline the basic concepts, concrete applications will be worked out in future work. Also, in [5], the reader already finds applications which prove the usefulness of the reduction of renormalization concepts to the Hopf algebraic set-up.

3.1 Multiplicativity

Remarkable features appear when one engulfes in a detailed study of the properties under a change of renormalization schemes. To this end, let us come back to the map $\phi_b : \mathcal{H} \to V$. Clearly, $\forall t \neq e$,

$$\begin{aligned} 0 = \phi_b(0) = \phi_b(\bar{e}(t)) &= \phi_b(m[(S \otimes id)\Delta(t)]) \\ &= m[(\phi_b \otimes \phi_b)(S \otimes id)\Delta(t)]. \end{aligned} \tag{42}$$

Compare the expression on the rhs with the expression for $\Gamma_{a,b}(t)$

$$\Gamma_{a,b}(t) = m[(\phi_{R_a} \otimes \phi_b)(S \otimes id)\Delta(t)]. \tag{43}$$

Hence, this expression is non-vanishing only because of $S_{R_a} \circ \phi_b \circ S \equiv \phi_{R_a} \neq \phi_b$, hence, essentially only if $a \neq b$.[3] There is a map $\Delta_b : \mathcal{H} \to V \otimes V$ induced by ϕ_b,

$$\Delta_b = (\phi_b \otimes \phi_b) \circ \Delta, \tag{44}$$

and an induced map $S_b : \mathcal{H} \to V$, $S_b = \phi_b \circ S$. It is more interesting to consider the map $\mathcal{R} : \mathcal{H}^\star \otimes V \to \mathcal{H}^\star \otimes V$ given by $\mathcal{R}(\phi) = S_R[\phi \circ S]$ defined for any $\phi : \mathcal{H} \to V$ (hence, for any $\phi \in \mathcal{H}^\star \otimes V$). We will now show that $\mathcal{R}(\mathcal{R}(\phi)) = \mathcal{R}(\phi)$, which is the natural extension of $R^2 = R$. This will allow us to define beautiful cohomological properties for renormalization. We start with the consideration of renormalization schemes

[3] One has $\phi_{R_b} = \phi_b$, as one immediately checks.

which merely vary external parameters. In the following, the reader should have in mind that a, b are to be considered as representatives of appropriate sets of external parameters (masses, external momenta in Green functions) which parametrize analytic expressions representing elements of $\mathcal{H}$.

It suffices to show

$$S_{R_a}(\phi_b)(t) = \phi_a(S(t)), \tag{45}$$

which one readily proves by induction on the number of vertices of t:

$$\begin{aligned} S_{R_a}(\phi_b)(t) &= -R_a[\phi_b(t) + m[(S_{R_a} \otimes id)(\phi_b \otimes \phi_b)P_2\Delta(t)]] & (46)\\ &= -\phi_a(t) - R_a[m[(\phi_a \otimes \phi_b)(S \otimes id)P_2\Delta(t)]] & (47)\\ &= \phi_a\left(-t - m[(S \otimes id)P_2\Delta(t)]\right) & (48)\\ &= \phi_a(S(t)). & (49)\end{aligned}$$

In the second line, we used that

$$R_a[\phi_a(t)\phi_b(t')] = R_a[\phi_a(t)]R_a[\phi_b(t')],\ \forall t, t' \in \mathcal{H}, \tag{50}$$

an equation which is fulfilled by R_a, but not by general renormalization maps.

If we regard a renormalization map $R : V \to V$ as simply a map from V (considered as a vector space) to V, then, in general, $R[xy] \neq R[x]R[y]$, $\forall x, y \in V$.

A good example is a minimal subtraction scheme, which we will discuss in some detail below. We can define its renormalization map R_{MS} by a projection to the pole part: if

$$1 \neq v = \sum_{i:=0}^{\infty} c_{-k+i}\epsilon^{-k+i} \in \mathbf{R}[\epsilon^{-1}, [\epsilon]] \tag{51}$$

for some positive integer k, then

$$R_{MS}(v) = \sum_{i:=0}^{k-1} c_{-k+i}\epsilon^{-k+i}. \tag{52}$$

Clearly, $R_{MS}[xy] \neq R_{MS}[x]R_{MS}[y]$. Nevertheless, R_{MS} fulfils the multiplicativity constraints (m.c.'s) which we formulate for an arbitrary renormalization map R as

$$R\left[\prod_{i=1}^{r} R[x_i]\right] = \prod_{i=1}^{r} R[x_i] \tag{53}$$

and

$$R\left[\prod_{i=1}^{r} (x_i - R[x_i])\right] = 0, \tag{54}$$

both valid for some positive integer $r > 1$, and arbitrary $x_i \in V$. Note that, setting $r = 2$ and $x_2 = 1$, the first constraint implies $R[R[x_1]] = R[x_1]$ and hence $R[x_1 - R[x_1]] = 0$. The second constraint establishes the same property also for products of such differences.

Those constraints can be concluded from a single condition:[4]

$$R[xy] - R[R[x]y] - R[xR[y]] + R[x]R[y] = 0, \tag{55}$$

which implies the m.c.'s.

We call these constraints the multiplicativity constraints, as one can show that for maps R in accordance with these constraints one has

Proposition 2

$$S_R\left[\prod_i \phi(t_i)\right] = \prod_i S_R[\phi(t_i)], \ \forall \phi \in \mathcal{H} \otimes V. \tag{56}$$

We can prove this statement under fairly general circumstances:[5] Let $\mathcal{H}$ be a commutative, graded Hopf algebra with coproduct Δ, antipode S, multiplication m, unit e and counit $\bar{e}$, over some number field $\mathcal{F}$, and such that the subalgebra $\mathcal{H}_0$ of elements of degree zero is reduced to scalars, so that $\mathcal{H}_0$ is the kernel of $(id - E \circ \bar{e})$. Let E be the standard inclusion of $\mathcal{F}$ in $\mathcal{H}$, $E : \mathcal{F} \to \mathcal{F}e \in \mathcal{H}$. Note that $S^2 = id$.

Let a representation $\phi : \mathcal{H} \to V$ be given. This includes the case $\phi = id$, $V = \mathcal{H}$ itself. In the following, we discuss only this case, the

[4] I thank Christian Brouder for pointing my attention to this fact.

[5] This result has far reaching consequences showing the conceptual significance of renormalization by its relation to the Riemann–Hilbert problem [10].

changes necessary for the general case are obvious and demand only the insertion of the map $\phi : \mathcal{H} \rightarrow V$ at appropriate places. Also, the prove goes through for the non-commutative case, delivering the expected homomorphism property $S_R(XY) = S_R(Y)S_R(X)$.

Let then R be a map $R : \mathcal{H} \rightarrow \mathcal{H}$ which fulfils the m.c.'s (53,54). It need not be an algebra endomorphism, $R(xy) \neq R(x)R(y)$. But we demand $R(e) = e$. Define $P_2 : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ by

$$P_2 = (id - E \circ \bar{e}) \otimes (id - E \circ \bar{e}), \tag{57}$$

so that we can write

$$\Delta(X) = P_2(\Delta(X)) + e \otimes X + X \otimes e - [E \circ \bar{e} \otimes E \circ \bar{e}]\Delta(X). \tag{58}$$

Then, the antipode can be written in the form

$$S(X) = -X - m[(S \otimes id)P_2\Delta(X)] + 2e\bar{e}(X), \tag{59}$$

using $m[(S \otimes id)\Delta(X)] = E \circ \bar{e}(X)$ and (58). For any map R as above, we define

$$S_R(X) = -R[X + m[(S_R \otimes id)P_2\Delta(X)]] + 2e\bar{e}(X). \tag{60}$$

This definition works recursively, as S_R on the rhs is applied to elements of lower degree than X. We also set $S_R(e) = e$. We now prove Prop.(2): It suffices to prove the assertion for elements which fulfil $\bar{e}(X) = 0$. Otherwise, one decomposes $X = (X - E \circ \bar{e}(X)) + E \circ \bar{e}(X)$.

So let X, Y be two elements which are annihilated by the counit. Then,

$$S_R(XY) = -R[XY] - R[U], \tag{61}$$

where $U = m[(S_R \otimes id)P_2\Delta(XY)]$, hence

$$\begin{aligned} U \;=\; & S_R[X]Y + S_R[Y]X + S_R[XY']Y'' + S_R[Y']XY'' + S_R[X'Y]X'' \\ & + S_R[X']X''Y + S_R[X'Y']X''Y'', \end{aligned} \tag{62}$$

abbreviating $P_2(\Delta(X)) = X' \otimes X''$, (omitting the summation sign). Now, as $\mathcal{H}$ is a graded Hopf algebra by assumption, we can proceed by induction on that grading. For the start of the induction we take the grade $n = 1$, where the assertion is immediate.

Assume that the assertion holds for products XY of grade n, we prove that it holds for grade $n+1$.

Thus, by assumption we can write U as

$$U = S_R[X]Y + S_R[Y]X + S_R[X]S_R[Y']Y'' + S_R[Y']XY'' + S_R[Y]S_R[X']X'' + S_R[X']X''Y + S_R[X']S_R[Y']X''Y''. \quad (63)$$

Now, we use the fact that R fulfills the multiplicativity constraints to write

$$R[XY] = R[X]R[Y] + R[R[X]\tilde{Y}] + R[R[Y]\tilde{X}] \quad (64)$$
$$R[U_XU_Y] = R[U_X]R[U_Y] + R[R[U_X]\tilde{U_Y}] + R[R[U_Y]\tilde{U_X}] \quad (65)$$

where $\tilde{Z} := Z - R[Z], \forall Z \in \mathcal{H}$ and

$$U_X := S_R(X')X'' = m[(S_R \otimes id)P_2\Delta(X)] \quad (66)$$

and similarly for U_Y. Now use $S_R(X) = -R[X] - R[U_X]$ which enables us to completely decompose U and $R[XY]$ in terms of $R[X], R[Y]$, $R[U_X], R[U_Y]$. Using (64,65) one finds

$$-R[XY] - R[U] = (R[X] + R[U_X])(R[Y] + R[U_Y]) = S_R(X)S_R(Y), \quad (67)$$

as desired.

Now, renormalization maps all serve one and the same purpose: to eliminate the undesired divergences in the theory. Typically, they can be considered as transformations which do not alter the behaviour at large internal scales (internal refererring here to scales which are to be integrated out) such that one can establish well-defined ratios like the functions $\Gamma_{a,b}(t)$ defined before. In such ratios the dependence on large internal scales drops out and hence we find finite results for renormalized Green functions. General renormalization maps R useful in applications in QFT can be simply considered as maps which fulfil the multiplicativity constraints, but *not* necessarily the algebra homomorphism property $R[xy] = R[x]R[y]$.

Realizing that one can label contributions to Green functions of pQFT by decorated rooted trees in the same manner as we did so far with functions $G^t_{b,\infty}$, we consider quite generally maps ϕ from $\mathcal{H}$ to some appropriate space V.

Let us then introduce, for any such $\phi : \mathcal{H} \to V$, from which we demand nothing more than the algebra homomorphism property $\phi(ab) = \phi(a)\phi(b)$,

$$\phi_R : \mathcal{H} \to V,\ \phi_R(t) = S_R[\phi](S(t)) \tag{68}$$

where R is any map which fulfils (55) and hence the multiplicativity constraints (53,54). $S_R : \mathcal{H} \to V$ is still defined as

$$S_R(\phi) = -R[\phi + m((S_R \otimes id)(\phi \otimes \phi)P_2\Delta]. \tag{69}$$

Then, as before, let $\mathcal{R}$ be the corresponding map

$$\mathcal{R} : \mathcal{H}^\star \otimes V \to \mathcal{H}^\star \otimes V,\ \mathcal{R}(\phi) = \phi_R = S_R[\phi \circ S]. \tag{70}$$

All the maps ϕ_R are algebra homomorphisms, due to (56).

Then, we define $\Gamma_{R,\phi} : \mathcal{H} \to V$ by

$$\Gamma_{R,\phi}(t) = m[(\phi_R \otimes \phi)(S \otimes id)\Delta(t)]. \tag{71}$$

For example, setting $\phi = \phi_b$, $R = R_a$ we recover the previous definition. We clearly have

$$\Gamma_{R,\phi}(t) = [\phi_R \circ S \star \phi](t). \tag{72}$$

Note that $\Gamma_{R,\phi}$ calculates the ratio of the two representations ϕ_R, ϕ with respect to the convolution product, forming the ratio with the help of the antipode, as it should be. In general, one can define, for any two representations $\phi_u, \phi_v \in \mathcal{H}^\star \otimes V$,

$$\Gamma_{u,v}(t) = [\phi_u \circ S \star \phi_v](t). \tag{73}$$

Then, the generalized form of Chen's Lemma takes the form

$$\Gamma_{u,v}(t) = [\Gamma_{u,s} \star \Gamma_{s,v}]\,(t), \tag{74}$$

where u, s, v are labels indexing different representations $\phi_u, \phi_v, \phi_s : \mathcal{H} \to V$. ϕ_{R_a}, ϕ_b were examples, as are ϕ_R, ϕ.

Let us show that $\mathcal{R} \circ \mathcal{R}(\phi) = \mathcal{R}(\phi)$,

$$\begin{aligned}
\mathcal{R} \circ \mathcal{R}(\phi) &= S_R(S_R(\phi)) \\
&= -R[S_R(\phi) + m((S_R \otimes id)(S_R(\phi) \otimes S_R(\phi))P_2\Delta] \\
&= -R[S_R(\phi) + m((S_R \otimes id)(id \otimes S_R)(\phi \otimes \phi)(S \otimes id)P_2\Delta)] \\
&= -R[S_R(\phi) + S_R[m((\phi \otimes \phi)(S \otimes id)P_2\Delta)]] \\
&= R[S_R[-\phi - m((\phi \otimes \phi)(S \otimes id)P_2\Delta)]] \\
&= R[S_R[\phi \circ S]] = S_R(\phi \circ S) = \mathcal{R}(\phi).
\end{aligned}$$

This proof works by induction on the number of vertices. From the second to the third line we used the assertion for lesser than n vertices, by employing $S_R \circ S_R(\phi) = \mathcal{R} \circ \mathcal{R}(\phi) = \mathcal{R}(\phi) = S_R(\phi \circ S)$. In the last line we utilized that S_R maps to the range of R and $R^2 = R$.

Hence, renormalization maps R which fulfill the multiplicativity constraints fulfill $\mathcal{R}^2(\phi) = \mathcal{R}(\phi)$. Further, if we have a renormalization scheme R' and representations $\phi', \phi \in \mathcal{H}^\star \otimes V$ such that $S_{R'}(\phi') = \phi \circ S$ (that is, the natural generalization of (45) holds), then Lemma 1 holds in the form

$$\Gamma_{R,\phi'} = \Gamma_{R,\phi} \star \Gamma_{R',\phi'}, \tag{75}$$

where R can be any renormalization scheme. Here, $\phi'_{R'} := S_{R'}(\phi' \circ S)$, and the condition $S_{R'}(\phi') = \phi \circ S$ essentially guarantees that we renormalize the second term on the rhs of (75) at a renormalization point at which the unrenormalized functions ϕ in the first term on the rhs are evaluated. Hence, the lhs is a concatenation of two renormalized Green function on the rhs, the first evaluating bare Green functions at parameters which we use for renormalization of the second, which shifts the bare function from ϕ to ϕ'. The infinitesimal version of this reparametrization can be regarded as the generator of the flow of the renormalization group.

Indeed, in (75) we see that the rhs depends on the intermediate representation ϕ, which does not appear on the lhs. If we regard ϕ as charaterized by an appropriate (set of) parameter(s) b, and ϕ' characterized by a (set of) parameter(s) b', we can apply a differentiation with respect to b to find

$$0 = \frac{d}{db}\left[\Gamma_{R,\phi} \star \Gamma_{R',\phi'}\right] = \left(\frac{\partial}{\partial b}\Gamma_{R,\phi}\right) \star \Gamma_{R',\phi'} + \Gamma_{R,\phi} \star \left(\frac{\partial}{\partial b}\Gamma_{R',\phi'}\right), \tag{76}$$

which is a proto-type renormalization group equation. It expresses as a differential equation the independence of an intermediate scale. Note that in the limit $b \to b'$ the finite ratio $\Gamma_{R',\phi'}$ becomes an infinitesimal quantity. Note further that the dependence on b of the second term on the rhs is given by the fact that $S_{R'}(\phi') = \phi \circ S$. The exercise to cast the renormalization group explicitly in this language is a purely notational one, taking into account the dependence on parameters like charges, masses etc and hence establishing a coupled systems of such equations, which we postpone to future work. Various viewpoints about renormalization group equations ranging from standard BPHZ approaches to the Wilson viewpoint can be obtained from (75,76) depending on which parameters for an intermediate scale one chooses, with dimensionful parameters of bare Green functions or physical cut-offs being some obvious choices.

3.2 Cohomological properties of renormalization

Let us consider the following problem. Given is a perturbative QFT, defined by Feynman rules. This defines a series in graphs graded by the number of vertices. The graphs translate to unique analytical expressions, which decompose into Feynman integrands and integrations, determined by the closed loops in the graph. Powercounting establish a well-defined subset of superficially divergent subgraphs, and eventually, we realize that each of these graphs represents an element of $\mathcal{H}$ [2, 3, 5].

The analytic expressions are parametrized by external momenta and masses, which can be regarded as complex parameters generalizing the external scale b of the iterated integral. For a given Feynman graph Γ, let for now b_Γ represent this set of parameters. The integration is over internal loop momenta along propagators (edges) (or over internal vertices, in x-space) and diverges (in momentum space) when the internal momenta get large. Let us specify a renormalization scheme by saying that for any graph Γ we have defined a set of conditions on the parameters b_Γ, for example conditions that the square of external momenta equals some mass square. Let μ_Γ be the set of parameters b_Γ specified in accordance with these conditions. These parameters are provided by the integrand constructed according to the Feynman rules.[6]

The renormalized Green function established by this set-up can be calculated as

$$\Gamma_{R_{\mu_\Gamma},\phi_{b_\Gamma}}(t_\Gamma) = [S_{R_{\mu_\Gamma}}(\phi_{b_\Gamma}) \star \phi_{b_\Gamma}](t_\Gamma), \tag{77}$$

where $t_\Gamma \in \mathcal{H}$ is obtained from Γ, and ϕ_{b_Γ} maps it to an analytic expressions according to the Feynman rules and all other notation is self-evident.

[6] There is an analogous story in x-space to be developped elsewhere.

This can be written in the form, in an obvious shorthand notation,

$$\Gamma_{R,\phi} = m \circ (\mathcal{R}_\mu \otimes id)(\phi_b \otimes \phi_b) \circ (S \otimes id)\Delta, \tag{78}$$

which shows that we fail by the deviation of $\mathcal{R}$ from the identity (in $\mathcal{H}^\star \otimes V$) to get a trivial result. The interesting operator in the above is clearly $\mathcal{R}_\mu \otimes id$.

Let us now concatenate the renormalization step n times. Hence, we assume that we have given a renormalized Green function Γ_R, ϕ_{b_0} as above, where the notation stresses that we use a set of parameters $b_{0\Gamma}$ for the bare function, and some arbitrary renormalization R.

Let us now vary these external parameters through n steps until they reach a bare Green function ϕ_{b_n}, which uses other values for external parameters. In each step, we will use a renormalized Green function which subtracts $\phi_{b_i}(t_\Gamma)$ at $\mu_\Gamma = b_{i-1,\Gamma}$, thus $\phi_{b_i R_{b_{i-1}}} = \phi_{b_{i-1}}$. Hence, we achieve a concatenation of renormalizations where each step fullfils (75).

Thus consider a sequence of renormalizations sending $\phi_{b_0} \to \phi_{b_n}$ by an intermediate sequence of renormalizations $b_0 \to b_1 \to \ldots \to b_n$, using

$$\begin{aligned} \phi_{b_{i-1}} \star \Gamma_{R_{b_{i-1}}, \phi_{b_i}} &= \phi_{b_{i-1}} \star \mathcal{R}_{b_{i-1}}(\phi_{b_i} \circ S) \star \phi_{b_i} \\ &= \phi_{b_{i-1}} \star \phi_{b_{i-1}} \circ S \star \phi_{b_i} \\ &= \phi_{b_{i-1}}[id \star S] \star \phi_{b_i} \\ &= \phi_{b_{i-1}}[\bar{e}] \star \phi_{b_i} \\ &= \phi_{b_i}. \end{aligned} \tag{79}$$

Let us introduce

$$d_R : \mathcal{H}^\star \otimes V \to (\mathcal{H}^\star \otimes V)^{\otimes 2} \tag{80}$$

by

$$d_R(\phi_{b_i}) = \mathcal{R}_{b_{i-1}}(\phi_{b_i}) \otimes \phi_{b_i}, \tag{81}$$

so that we obtain

$$\begin{aligned} \phi_{b_n} = {} & M[[\phi_{b_1} \otimes d_{R_{b_1}}(\phi_{b_2}) \otimes d_{R_{b_2}}(\phi_{b_3}) \otimes \ldots \otimes d_{R_{b_{n-1}}}(\phi_{b_n})] \\ & \circ (id \otimes [S \otimes id]^{\otimes(n-1)} \Delta^{2n-1})], \end{aligned} \tag{82}$$

where M is a concatenation of $2n$ multiplication maps, $M : V^{\otimes 2n+1} \to V$, and $\Delta^{2n+1} : \mathcal{H} \to \mathcal{H}^{\otimes 2n+1}$ is the obvious map in $\mathcal{H}$ sending $\mathcal{H} \to \mathcal{H}^{\otimes 2n+1}$ using the coproduct, unique due to coassociativity of the latter.

This motivates to define d_R for elements $(\mathcal{H}^\star \otimes V)^{\otimes n}$.

$$d_R(\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_n) = \sum_{i=1}^{n} (-1)^{i+1} \phi_1 \otimes \ldots \otimes d(\phi_i) \otimes \ldots \otimes \phi_n, \tag{83}$$

where it is understood that $d(\phi_i) = \mathcal{R}_{i-1}(\phi_i) \otimes \phi_i$, $i > 1$ and $d(\phi_1) = d_R(\phi_1) = \mathcal{R}(\phi_1) \otimes \phi_1$. Hence, at the i-th entry ($i > 1$), $\mathcal{R}$ replaces ϕ_i by the element $\mathcal{R}_{i-1}(\phi) \otimes \phi_i = \phi_{i-1} \otimes \phi_i$. At the first entry, we obtain $\mathcal{R}(\phi_1)$. It is convenient to introduce $\phi_0 := \mathcal{R}(\phi_1)$, with $\mathcal{R}(\phi_0) = \phi_0$.

Then, one immediately checks, using $\mathcal{R}^2 = \mathcal{R}$,

$$d_R^2 = 0, \tag{84}$$

for example:

$$d_R(d_R(\phi_1)) = d_R(\phi_0 \otimes \phi_1) = \phi_0 \otimes \phi_0 \otimes \phi_1 - \phi_0 \otimes \mathcal{R}_{\prime}(\phi_1) \otimes \phi_1 = 0. \tag{85}$$

In this language, we have

$$\phi_{b_n} = M[[\phi_{b_1} \star d(\phi_2)(S \otimes id) \star \ldots \star d(\phi_n)(S \otimes id)]\Delta^{2n+1}] \tag{86}$$

and for the renormalized Green function

$$d_R(\phi_n) = M[d(\phi_1)(S \otimes id) \star (d\phi_2)(S \otimes id) \star \ldots \star (d\phi_n)(S \otimes id)\Delta^{2n+2}]. \tag{87}$$

In short, taking into account that the actions of $M, \Delta^{\cdots}, (S \otimes id)$ are obvious:

$$\phi_n = \phi_1 d\phi_2 \ldots d\phi_n \tag{88}$$

and

$$d\phi_n = d\phi_1 \ldots d\phi_n. \tag{89}$$

We recommend that the reader tries these formula out on several simple examples and marvels at their obvious cohomological relevance especially in comparison with [12].

We realize that the change of scales, so typically a step in the whole of physics, naturally carries cohomological structure which gives hope to be able to cast locality in a well-defined mathematical framework in the future.

Much more can and should be said about these aspects. Here, we have to refer the reader to future work [11].

3.3 Tree-indexed scales and the equivalence of schemes

Let us come back to Chen's Lemma and to iterated integrals. There are further generalizations lying ahead. So far, we used iterated integrals as quantities which are naturally indexed by rooted trees, and to which the previous considerations apply. The rooted trees determined how the various differential forms $f_i(x)dx$ are combined under the (indefinite) integral operator, but outer boundaries where kept constant throughout.

3.3.1 Tree indexed scales

A generalization which turns out to be quite useful in practice is to let even these boundaries be indexed by decorated rooted trees. Hence, we redefine

$$-G_{b,\infty}(t) = \int_{b_t}^{\infty} f_{i_n}(x) \prod G_{x,\infty}^{t'} dx \tag{90}$$

which is defined as before, only that we now label the lower outer boundary by a decorated rooted tree t. We understand that the map R_a maps lower boundaries b_t to a_t, and that the coproduct action extends to this label. Similar considerations apply to full-fledged Green functions of pQFT, where one can utilize the presence of scale dependent parameters to make them tree-dependent in the same manner. This idea will be pursued in the next section, and in [8].

If $t_2(f_1, f_2)$ is the decorated rooted tree of Fig.(1) with coproduct

$$\Delta(t_2(f_1, f_2)) = t_2(f_1, f_2) \otimes e + e \otimes t_2(f_1, f_2) + t_1(f_1) \otimes t_1(f_2) \tag{91}$$

we formally obtain (abbreviating $t_2(f_1, f_2) = t_{2_{12}}, t_1(f_1) = t_{1_1}, t_1(f_2) = t_{1_2}$),

$$S_{R_a}(\phi_b)(t_2(f_1, f_2)) = \left[-\int_{a_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} + \int_{a_{t_{1_2}}}^{\infty} \int_{t_{1_1}}^{\infty} \right] f_2(x) f_1(y) dy dx. \quad (92)$$

and

$$\Gamma_{a,b}(t_2(f_1, f_2)) \quad (93)$$
$$= \left[\int_{b_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} - \int_{b_{t_{1_2}}}^{\infty} \int_{a_{t_{1-1}}}^{\infty} + \int_{a_{t_{1_2}}}^{\infty} \int_{a_{t_{1_1}}}^{\infty} - \int_{a_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} \right] f_2(x) f_1(y) dy dx.$$

In this notation, a, b are to be regarded as representing actually a whole set of constants a_t, b_t, parametrizing the relevant scales for the decorated tree considered.

Still, (39) applies and describes what happens if we change the renormalization point. $\Gamma_{a,b} = \Gamma_{a,s} \star \Gamma_{s,b}$ now becomes

$$\left[\int_{b_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} - \int_{b_{t_{1_2}}}^{\infty} \int_{a_{t_{1_1}}}^{\infty} + \int_{a_{t_{1_2}}}^{\infty} \int_{a_{t_{1_1}}}^{\infty} - \int_{a_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} \right] f_2(x) f_1(y) dy dx$$
$$= \left[\int_{s_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} - \int_{s_{t_{1_2}}}^{\infty} \int_{a_{t_{1_1}}}^{\infty} + \int_{a_{t_{1_2}}}^{\infty} \int_{a_{t_{1_1}}}^{\infty} - \int_{a_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} \right] f_2(x) f_1(y) dy dx$$
$$+ \left[\int_{b_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} - \int_{b_{t_{1_2}}}^{\infty} \int_{s_{t_{1_1}}}^{\infty} + \int_{s_{t_{1_2}}}^{\infty} \int_{s_{t_{1_1}}}^{\infty} - \int_{s_{t_{2_{12}}}}^{\infty} \int_{x}^{\infty} \right] f_2(x) f_1(y) dy dx$$
$$+ \left[\int_{s_{t_{1_1}}}^{\infty} - \int_{a_{t_{1_1}}}^{\infty} \right] f_1(y) dy \left[\int_{b_{t_{1_2}}}^{\infty} - \int_{s_{t_{1_2}}}^{\infty} \right] f_2(x) dx,$$

which is evidently true, as the reader should check. It is instructive to see the Lemma (1) in action for this simple example.

Finiteness of $\Gamma_{a,b}$ now imposes conditions on the tree-indexed parameters, a fact which we will utilize in the next section.

3.3.2 Equivalence of schemes

Conceptually, the presence of tree-indexed parameters allows to describe different renormalization schemes on a similar footing. The idea is the following. Let us compare for example a BPHZ on-shell scheme in comparison with minimal subtracted dimensional renormalization. In the former case, one effectively subtracts at the level of integrands. Hence, one has an integrand which gives rise to a non-existent measure with respect to the loop integrations. The integrand is parametrized by several constants (masses, external momenta), which essentially play the role of the boundaries in our iterated integrals. A renormalization scheme in the BPHZ spirit would map, upon applying the antipode $S_{R_{BPHZ}}$, such an integrand to another one, for which these parameters fulfill certain conditions (on-shell, for example). The structure of $S_{R_{BPHZ}}$ achieves that these counterterms are local [2, 6, 3].

Then, the map $\Gamma_{R_{BPHZ},\phi}(t)$ so-constructed delivers a subtracted integrand which actually establishes a well-defined measure with respect to all loop integrations. Here, t is the decorated rooted tree assigned to the integrand according to powercounting [2, 3, 6, 5]. Hence, BPHZ-type schemes avoid the use of regularization altogether.

On the other hand, in dimensional renormalization using minimal subtraction (MS scheme), one introduces regularization and evaluates the bare Green-functions first, obtaining Laurent series in the regularization parameters. The antipode achieves a subtraction of these poles which respects locality, and similarly one constructs the MS-renormalized $\Gamma_{R_{MS},\phi}(t)$ using $S_{R_{MS}}$.

In the next section we will use the idea to have tree-indexed scales to show that we can regard a MS-scheme as a BPHZ type scheme on the expense of having to introduce tree-dependent scales.

4 Applications

To keep the amount of notation simple, we will consider representations of $\mathcal{H}$ defined as follows. Assume that we are given a set of functions $B_k(x)$ which are Laurent series in x with a first-order pole. Using vertex weights and the corresponding notation as defined in the beginning of section five below we can define a function

$$G_z(t;x) = \prod_{v\in T^{[0]}} B_{w(v)}(x)z^{-nx} = B_t(x)z^{-nx}, \tag{94}$$

where n is the number of vertices of t and z is to be regarded as the scale parametrizing the representation and x is the regularization parameter. We also wrote $\prod_{v\in T^{[0]}} B_{w(v)}(x) = B_t(x)$. Quite a number of interesting applications can be brought to this form [5, 8]. Typically, whenever we iterate a Feynman diagram in terms of itself as described by a rooted tree, one finds such representations. Many examples are given in [5]. The case of general Feynman diagrams is obtained by finding a proper notation for the case of different decorations, and by taking into account a proper form-factor decomposition. We refer the reader to [8] for further applications, extending to such cases.

Hence, in the notation of the previous section, we set $\phi_z(t) = G_z(t;x)$. Then, we define the MS renormalization scheme by setting

$$R_{MS} \circ \phi_z = \langle \phi_1 \rangle, \tag{95}$$

where angle brackets denote projection on the pole part of the Laurent series in x inside the brackets. Let then the counterterm defined by $S_{R_{MS}}(\phi_z)$ and the renormalized Green function by $\Gamma_{MS}(\phi_z)(t) = [S_{R_{MS}}(\phi_z) \star \phi_z](t)$, as usual. The reader should have no difficulties confirming that for t_2, the rooted tree with two vertices,

$$S_{R_{MS}}(\phi_z)(t_2) = -\langle B_2 B_1 \rangle + \langle \langle B_1 \rangle B_1 \rangle, \tag{96}$$

and

$$\Gamma_{R_{MS},\phi_z}(t_2) = B_2 B_1 z^{-2x} - \langle B_1 \rangle B_1 z^{-x} - \langle B_2 B_1 \rangle + \langle \langle B_1 \rangle B_1 \rangle. \tag{97}$$

Let us compare such an approach with an on-shell approach. Using the same bare functions, we define the on-shell renormalization map R_μ as

$$R_\mu \circ \phi_z = \phi_\mu. \tag{98}$$

If the $G_z(t;x)$ are provided by integrals whose integration is regularized by x where z is a parameter of the integrand, then the renormalization map just sets this external parameter to the value μ.

$\Gamma_{R_\mu,\phi_z}(t) = [S_{R_\mu}[\phi_z] \star \phi_z](t)$ is a function which has a subtracted integrand such that typically the limit $x \to 0$ exists at the level of the integrand. It becomes a Taylor series in $\log(z/\mu)$.

Let us now cast the MS renormalized Green function in this form on the expense of introducing tree-dependent scales μ_t.

Hence, we redefine $R_\mu(\phi)(t) = \phi_{\mu_t}(t)$, and get, still spelling out the example $t = t_2$,

$$S_{R_\mu}(\phi_z)(t_2) = -B_2 B_1 \mu_{t_2}^{-2x} + B_1 B_1 \mu_{t_1}^{-2x}. \tag{99}$$

We remind ourselves that $S(t_2) = -t_2 + t_1 t_1$.

It is easy to work out the general case. To see this, we look at the trees with up to three vertices. The antipode $Z_t := S_{R_{MS}}(\phi_z)(t)$ in MS reads

$$\begin{aligned}
Z_{t_1} &= -\langle B_1\rangle, && (100)\\
Z_{t_2} &= -\langle B_2 B_1\rangle + \langle\langle B_1\rangle B_1\rangle, && (101)\\
Z_{t_{3_1}} &= -\langle B_3 B_2 B_1\rangle + \langle\langle B_1\rangle B_2 B_1\rangle + \langle\langle B_2 B_1\rangle B_1\rangle \\
&\quad -\langle\langle\langle B_1\rangle B_1\rangle B_1\rangle, && (102)\\
Z_{t_{3_2}} &= -\langle B_3 B_1^2\rangle + 2\langle\langle B_1\rangle B_2 B_1\rangle \\
&\quad -\langle\langle\langle B_1\rangle B_1\rangle B_1\rangle. && (103)
\end{aligned}$$

In general, one finds

$$Z_t = \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_C} \left\langle \left[\prod_i \langle B_{t_i}\rangle\right] B_{t_R} \right\rangle. \tag{104}$$

The antipode Z_t^μ in a subtraction scheme using tree-indexed parameter sets μ_t reads

$$Z_t^\mu = \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_C} \left[\prod_i B_{t_i} \mu_{t_i}^{-\#(t_i)x}\right] B_{t_R} \mu_{t_R}^{-\#(t_R)x}. \tag{105}$$

We remind the reader that $\#(t)$ equals the number of vertices of t.

Equating $Z_t = Z_t^\mu$ determines μ_t recursively

$$\mu_t = \exp\left[\left(\frac{-1}{xt^!}\right) \log(B_t'/B_t)\right], \tag{106}$$

where

$$B' := \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_C} \left[\prod_i S_{R_{MS}}(\phi_z)(t_i)\right] S_{R_{MS}}(\phi_z)(t_R). \tag{107}$$

One also confirms that now

$$\Gamma_{R_{MS},\phi_z}(t) = \Gamma_{R_\mu,\phi_z}(t) \tag{108}$$

holds. Note that we can discard the use of a regularization scheme in Γ_{R_{MS},ϕ_z} as we observe that the scales μ_t are functions which exist in the limit $x \to 0$, which one confirms by using $B'_t/B_t = 1 + \mathcal{O}(x)$ in (106).

Conceptually, this eliminates any difference between a BPHZ type scheme and regularization followed by a minimal subtraction. Each divergent subgraph can be subtracted at its own scale, such that a subtraction with such tree-dependent sets of parameters equals the result of the use of a MS scheme. Actually, there remains an argument in favour of minimal subtraction: it incorporates from the beginning the wisdom that it is only logarithmic divergence which counts. Any integrand providing a different degree of divergence can be cast in the form of a log divergent integrand, (multiplied by a polynomial in external parameters) plus scale-independent terms.[7] The latter do not contribute anyhow after renormalization, and are economically eliminated in dimensional renormalization from the beginning.

We know that B_t is of order $\mathcal{O}(1/x^{\#(t)})$, as is $B'(t)$. One immediately proves that the difference is of lower order

$$[B_t - B'_t] \sim \mathcal{O}(1/x^{\#(t)-1}). \tag{109}$$

This is a direct consequence of the fact that the antipode S of $\mathcal{H}$ fulfills $S^2 = 1$. Indeed, to leading order in $1/x$ we have

$$\begin{aligned} B'_t &= \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_C} \left\langle \left[\prod_i \langle Z_{t_i} \rangle \right] Z_{t_R} \right\rangle & (110) \\ &= S_{R_{MS}}[Z_t] & (111) \\ &= Z_{S(t)} = \phi_z(S^2(t)) = B_t, & (112) \end{aligned}$$

where we used that to leading order $\langle\langle A\rangle B\rangle = \langle A\rangle\langle B\rangle$ for arbitrary expressions A, B so that the first line to leading order agrees with (107), and we used (45). As these properties are true not only for minimal subtraction but for any renormalization scheme, we conclude that in any renormalization scheme the leading pole part is the same, a property well-known to the practitioner. It is nice to see it traced back to the fact that $S^2 = 1$.

[7] An easy example: $\int_1^\infty \frac{x\,dx}{x+c} = -c\int_1^\infty \frac{dx}{x+c} + \int_1^\infty dx$.

Once more, it is instructive to write down the first couple of cases for B'_t.

$$\begin{aligned}
B'_{t_1} &= \langle B_{t_1}\rangle && (113)\\
B'_{t_2} &= \langle B_{t_2}\rangle - \langle\langle B_{t_1}\rangle B_{t_1}\rangle + \langle B_{t_1}\rangle\langle B_{t_1}\rangle && (114)\\
B'_{t_{3_1}} &= \langle B_{t_{3_1}}\rangle - \langle\langle B_{t_1}\rangle B_{t_2}\rangle - \langle\langle B_{t_2}\rangle B_{t_1}\rangle \\
&\quad + \langle\langle\langle B_{t_1}\rangle B_{t_1}\rangle B_{t_1}\rangle - 2\langle B_{t_1}\rangle\langle B_{t_2}\rangle \\
&\quad + 2\langle B_{t_1}\rangle\langle\langle B_{t_1}\rangle B_{t_1}\rangle - \langle B_{t_1}\rangle\langle B_{t_1}\rangle\langle B_{t_1}\rangle && (115)\\
B'_{t_{3_2}} &= \langle B_{t_{3_2}}\rangle - 2\langle\langle B_{t_1}\rangle B_{t_2}\rangle \\
&\quad + \langle\langle B_{t_1}\rangle\langle B_{t_1}\rangle B_{t_1}\rangle - 2\langle B_{t_1}\rangle\langle B_{t_2}\rangle \\
&\quad + 2\langle B_{t_1}\rangle\langle\langle B_{t_1}\rangle B_{t_1}\rangle - \langle B_{t_1}\rangle\langle B_{t_1}\rangle\langle B_{t_1}\rangle. && (116)
\end{aligned}$$

It is not a big surprise to see that a change of renormalization scheme can produce a lot of junk at subleading orders.

A final remark concerns momentum schemes (schemes which subtract at specified values of external momenta), hence schemes which fulfil (45). We know already that the convolution product holds for arbitrary renormalization schemes. The structure of the convolution product indicates how we translate a renormalized Green function, $\Gamma_{R,\phi_b}(t)$, determined by a scheme R and parameter(s) b, to $\Gamma_{R,\phi_{b'}}(t)$. Both utilize the same freely chosen renormalization scheme R giving an operator d_R, $d_R^2 = 0$. The transition $b \to b'$ uses the convolution by reparametrizations of the external parameter(s) b, hence a convolution using momentum schemes. They thus typically provide the general mediator for renormalized Green functions, as was exhibited in [5, 8].

5 Tree factorials and CM weights

In this section we want to prove some basic results concerning tree-factorials and Connes Moscovici weights. Both entities are combinatorial in nature. For simplification, we work in the undecorated Hopf algebra. Similar identities were derived by Butcher in his work on numerical integration methods [14, 13]. As our derivation is different we still give it in some detail.

Let $T^{[0]}$ be the set of vertices of the rooted tree T. For any vertex v of T, let $t_v = P^c(T)$, where c is the single cut which removes the edge

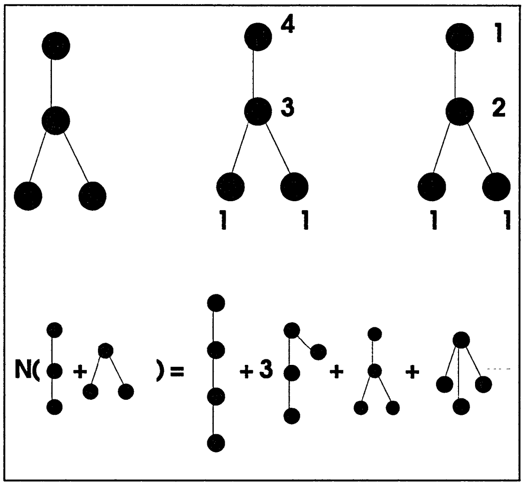

Figure 3: We define vertex weights and tree factorials, symmetry factors, CM-weights and feet of an undecorated rooted tree. For the tree t given at the left in the upper row we indicate the vertex weights at each vertex in the middle and the vertex symmetries at the right. The second row considers natural growth of $\delta_3 = t_{3_1} + t_{3_2}$. It obtains on the rhs of the equation four trees with CM-weights $1, 3, 1, 1$. We also indicate the feet of these four trees.

incoming to v. If v is the root, we set $t_v = t$. Also, $\#(t_v)$ is the number of vertices of the monomial t_v. Then, we define the tree factorial by

$$T^! = \prod_{v \in T^{[0]}} \#(t_v) \equiv \prod_{v \in T^{[0]}} w(v), \tag{117}$$

which also defines the vertex weights $w(v)$. Fig.(3) gives instructive examples. Finally, we set $e^! = 1$.

The next definition concerns the symmetry factor S_T of a tree T. For any $v \in T^{[0]}$ consider $B_-(t_v)$. This is a monomial in rooted trees, hence a product of branches

$$B_-(t_v) = \prod_{i=1}^{f(v)} t_{v,i}. \tag{118}$$

In general, some of these branches can be the same rooted tree, and hence these products can be written as products $\prod'$ over different rooted trees with integer powers:

$$\prod_{i=1}^{f(v)} t_{v,i} = {\prod'}_j [t_{v,j}]^{n_j}. \tag{119}$$

We associate to the vertex v its vertex symmetry, built from the factorials of the multiplicities n_j with which a tree $t_{v,j}$ originates from v,

$$S_{T,v} = {\prod'}_j n_j! \tag{120}$$

and define

$$S_T = \prod_{v \in T^{[0]}} S_{T,v}. \tag{121}$$

Fig.(3) gives examples for these notions, which were already used in [5]. In that paper, the symmetry factor S_T was dubbed $\Pi(T)$ and the vertex symmetry of the root $S_{T,r}$ was simply denoted as $\pi(T)$.[8]

We now define the Connes Moscovici weights $CM(T)$ for a tree T with n vertices as

$$CM(T) := \langle Z_T, \delta_n \rangle, \tag{122}$$

where $\langle \cdot, \cdot \rangle$ is the pairing of [3], $\langle Z_T, T' \rangle = \delta_{T,T'}$, and $\delta_n = N^n(e)$ is the n-fold application of natural growth to e, delivering the generators of the commutative part of the CM-Hopf subalgebra of $\mathcal{H}$, obtained by natural growth N applied n times, cf. Fig(3).

Two further definitions are useful. The first concerns the feet of a rooted tree T. It is a subset $\mathcal{F}(T^{[0]})$ of $T^{[0]}$ provided by those vertices which have fertility zero, hence no outgoing edges. Then, $\mathcal{F}(T)$ is the set of trees consisting of all trees which have one foot removed. The cardinality of this set equals the number of feet of T.[9]

Also, we let $\mathcal{N}(T)$ be the set of those trees which are generated from T by natural growth $N(T)$

$$T' \in \mathcal{N}(T) \Leftrightarrow \langle Z_{T'}, N(T) \rangle \neq 0, \tag{123}$$

[8]For a tree T with root vertex r one can consider $S_{T,r}$ to define what could be dubbed the Moebius function $\mu(T)$ of T: $\mu(T) = 0$ iff the branches at r are not square-free (hence if some of the powers $n_i > 1$) and $\mu(T) = 1$ if $f(r)$ is even, and $\mu(T) = -1$ if $f(r)$ is odd.

[9]What we call feet here is often called leaves in the literature on graphs.

and counting multiplicities appropriately. See Fig.(3) to get acquainted with all these notions.

We want to derive the following three results. First,

$$\frac{n}{T^!} = \sum_{t\in\mathcal{F}(T)} \frac{1}{t^!}, \tag{124}$$

which leads to the second result,

$$CM(t) = \frac{n!}{t^! S_t}, \forall t \in \mathcal{T}^{[n]}, \tag{125}$$

where the tree-factorials and Connes-Moscovici weights are defined as in [5]. Here, $\mathcal{T}^{[n]}$ is the set of trees with n vertices. Finally,

$$\sum_{T\in\mathcal{T}^{[n]}} \frac{CM(T)}{T^!} = \frac{(n-1)!}{2^{(n-1)}}. \tag{126}$$

We first note that (124) obviously holds for trees $t = B_+^n(e)$, where it reduces to the familiar $n/n! = 1/(n-1)!$. Hence, the tree factorial is another example of the replacement *integers to rooted trees.*

The next observation is

$$\frac{n}{t^!} = \prod_{j=1}^{k} \frac{1}{t_j^!}, \tag{127}$$

where $B_-(t) = \prod_{j=1}^k t_j$. This identity is a mere way of writing the definition of the tree factorial, using that any tree factorial factorizes n for a tree with n vertices.

Instead of proving (124), we prove

$$\frac{n_1 + \ldots + n_k}{T_1^! T_2^! \ldots T_k^!} = \sum_{t'\in\mathcal{F}(B_+(t_1 \ldots t_k))} B_-(t')^!, \tag{128}$$

where we define

$$B_-(t')^! = \frac{T_i^!}{T_1^! \ldots T_k^!} \frac{1}{(T_i')^!} \tag{129}$$

and T_i' is defined by $B_+(T_1 \ldots T_i' \ldots T_k) = t'$. Thus, we use that the feet must have been attached to some branch T_i of $B_+(T_1 \ldots T_k)$ which gives $T_i' \in \mathcal{F}(T_i)$.

Let us first show that (124) is a consequence of (128). To see this, it suffices to note that

$$\mathcal{F}(B_+(t)) = \{B_+(t') \mid t' \in \mathcal{F}(t)\}, \tag{130}$$

as obviously root and feet are different ends of a rooted tree. Fig.(3) gives an instructive example. Hence, setting $k = 1$ in (128),

$$\frac{n}{t^!} = \sum_{t' \in \mathcal{F}(B_+(t))} B_-(t')^! = \sum_{t' \in \mathcal{F}(t)} \frac{1}{t'^!}, \tag{131}$$

which proves (124).

It remains to prove (128). We have, using induction on the total number of vertices in $T_1 \ldots T_n$ and the fact that a single T_i has lesser vertices than the product,

$$\frac{n_1 + \ldots + n_k}{T_1^! \ldots T_k^!} = \sum_{i=1}^{k} \frac{n_i}{T_i^!} \frac{T_i^!}{T_1^! \ldots T_k^!} \tag{132}$$

$$= \sum_{i=1}^{k} \sum_{t_i' \in \mathcal{F}(T_i)} \frac{1}{(t_i')^!} \frac{1}{T_1^! \ldots \overset{\wedge}{T_i^!} \ldots T_k^!} \tag{133}$$

$$= \sum_{t' \in \mathcal{F}(B_+(T_1 \ldots T_k))} [B_-(t')]^!. \ \square \tag{134}$$

In the above, the $\wedge$ means to omit the corresponding tree factorial.

We are now in the position to prove (125). We assume it holds for trees with $n - 1$ vertices and simply reduce it to (124) to show that it holds for n vertices.

$$S_t CM(t) = \frac{n!}{t^!} \tag{135}$$

$$= \sum_{t' \in \mathcal{F}(t)} CM(t') s_{t'}$$

$$= (n-1)! \sum_{t' \in \mathcal{F}(t)} \frac{S_{t'}}{S_{t'} t'^!} \tag{136}$$

$$= (n-1)! \sum_{t' \in \mathcal{F}(t)} \frac{1}{t'^!}, \tag{137}$$

which is the desired result. □
It remains to prove (126). By the definition of the CM-weights as counting multiplicities of trees under natural growth we can write

$$\sum_{t\in\mathcal{T}^{[n]}}\frac{CM(t)}{t^!}=\sum_{t'\in\mathcal{T}^{[n-1]}}\frac{CM(t')}{(t')^!}\sum_{t''\in\mathcal{N}(t')}\frac{(t')^!}{(t'')^!}. \tag{138}$$

Hence we must show

$$\sum_{t''\in\mathcal{N}(t')}\frac{(t')^!}{(t'')^!}=\frac{n-1}{2}. \tag{139}$$

Then, (126) follows immediately by induction:

$$\sum_{t\in\mathcal{T}^{[n]}}\frac{CM(t)}{t^!}=\sum_{t'\in\mathcal{T}^{[n-1]}}\frac{CM(t')}{(t')^!}\frac{n-1}{2} \tag{140}$$

$$\Rightarrow\sum_{t\in\mathcal{T}^{[n]}}\frac{CM(t)}{t^!}=\frac{(n-2)!}{2^{n-2}}\frac{n-1}{2}=\frac{(n-1)!}{2^{n-1}}. \tag{141}$$

To derive (139) we write

$$\sum_{t''\in\mathcal{N}(t')}\frac{(t')^!}{(t'')^!}=\frac{n-1}{n}\sum_{t''\in\mathcal{N}(t')}\frac{\prod(br(t'))^!}{\prod(br(t''))^!}, \tag{142}$$

using the definition of the tree factorial via a product of the factorials of the branches.

Natural growth either happens at one of the branches of t', or at the root of t'. In the latter case, the above sum gives a contribution $\frac{n-1}{n}$. In the former case, assume natural growth happens at the branch t'_i of t'. Then,

$$\frac{\prod(br(t'))^!}{\prod(br(t''))^!}=\frac{(t'_i)^!}{(t''_i)^!}, \tag{143}$$

where $t''_i\in\mathcal{N}(t'_i)$. Hence, we can use induction in the above sum, as branches of a tree have lesser vertices than the tree itself. Thus, applying (139) for branches:

$$\frac{n-1}{n}\sum_{t''\in\mathcal{N}(t')}\frac{\prod(br(t'))^!}{\prod(br(t''))^!} \tag{144}$$

$$=\frac{n-1}{n}\left(1+\sum_{i=1}^{f(t')}\frac{\#(t'_i)}{2}\right)=\frac{n-1}{n}\left(1+\frac{\#(t')-1}{2}\right)=\frac{n-1}{2}. \tag{145}$$

We used that the sum over all branches of t' delivers $n-2$ vertices, as t' has $n-1$ vertices. Thus, Eq.(46) of [5] is proven.

6 Operator Product Expansion

In a way, the most general problem one faces in QFT can be stated as the problem of finding the limit of matrix elements

$$\begin{aligned}&\lim_{y\to x}\langle 0\mid O_a(x)O_b(y)P(x_1,\ldots)\mid 0\rangle\\&=\sum_k f_k(x-y)\langle 0\mid O_k((x+y)/2)P(x_1,\ldots)\mid 0\rangle,\end{aligned}\tag{146}$$

where O_a, O_b, O_k are operators and $P(x_1,\ldots)$ is a polynomial in field operators at points $x_1,\ldots$, and we sum over all operators O_k with appropriate quantum numbers and Wilson coefficients f_k. This is the problem of the operator product expansion. The Wilson coefficients will behave as

$$f_k(x-y)\sim(x-y)^{-(d_a+d_b-d_k)}(\text{polynomial in}\log(x-y)),\tag{147}$$

where d_a, d_b, d_k are the dimensions of the operators O_a, O_b, O_k.

Viewed in momentum space, it becomes an renormalization problem: in the desired limit, we will get a series of contributions which will be plagued by UV divergences when we integrate momenta. Hence, we can proceed as before and sort the resulting contributions by their tree-structure, followed by a renormalization based on the antipode of the resulting trees. The convolution product extends to the coefficient functions f_k and one thus finds that the convolution product of iterated integral is a simple representation of the convolution which describes the change of scales in OPEs. This motivated the title of this paper.

The relation between a convolution of the form (75) and the OPE is most clearly understood if one considers the OPE as a problem of the change of renormalization conditions, hence a problem of the change of scales, in our terminology. It is the standard practice in OPE's to find the desired expansion by doing the necessary subtractions for the case $x=y$, which means to find a larger set of appropriate forests [15, 16].

Let us essentially restrict to ϕ^4 theory and consider the problem

$$\lim_{x\to y}\phi(x)\phi(y) = f(x-y)\phi^2((x+y)/2) + \mathcal{O}((x-y)^2). \tag{148}$$

One starts with the renormalized function

$$\int d^D q e^{-iq(x-y)}\Gamma(q,\{p_k\}) \tag{149}$$

associated to the vev (146). Here, $\{p_k\}$ indicates the external momenta associated to the points x_i in (146) after Fourier transform, and q is the momentum according to the Fourier transform of $x-y$. $\Gamma(q,\{p_k\})$ is a renormalized amputated $(n+2)$-point Green function in momentum space (the grey blob in Fig.(4)), dressed with two extra propagators to connect it to x and y.

The corresponding Feynman integrand can be either read as belonging to a $(n+2)$-point Green function $G^{[n+2]}$ or to a n-point Green function with inserted operator ϕ^2, $G^{[n]}_{\phi^2}$. The elements of $\mathcal{H}$ which we shall associate to either case will be different.

In the former case, before the limit $x\to y$ is taken, the renormalized Green function $\Gamma_{G^{[n+2]}}$ is a well-defined finite ratio for a fixed renormalization scheme R. Now, we want to change this ratio in accord with a new renormalization condition which demands that the amputated Greens function vanishes for $\sqrt(q^2)\to\infty$, so that the momentum integral over q in (149) exists. Note that it would be log-divergent if the amputated Green's function has a finite value for $q^2\to\infty$.

This is a typical change achieved by a convolution product described in (75). The renormalized Green function associated with the operator insertion, $\Gamma_{G^{[n]}_{\phi^2}}$, is obtained by convoluting $\Gamma_{G^{[n+2]}}$ with the ratio which takes into account the change of renormalization conditions.

Clearly, as the convolution of two ratios is a ratio, we can describe the so obtained function as a sum over forests. The leading term $x=y$ is explicitely singled out in Fig.(4).

Again, a more detailed description of this approach is mainly a notational exercise and will be given elsewhere.

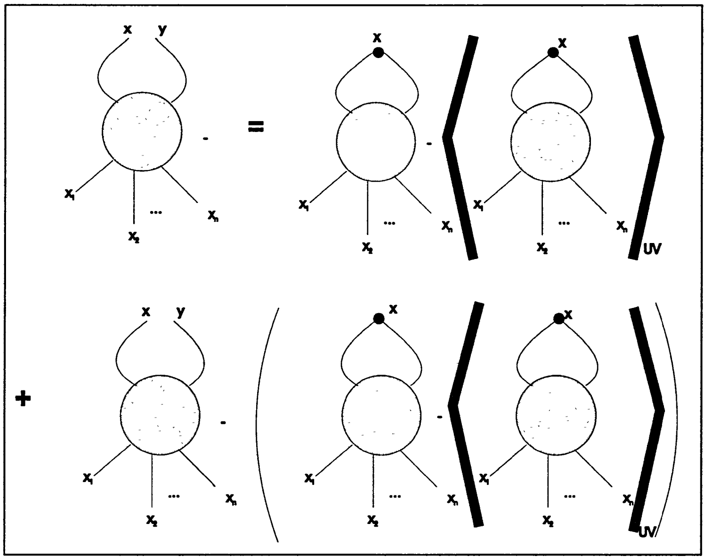

Figure 4: The operator product expansion amounts to a renormalization of a Green function such that the limit $x \to y$ is well defined. It amounts to a change in renormalization conditions such that the Fourier integral over q can be carried out for $y \to x$.

7 Comments, Conclusions

In this paper we emphasized the common algebraic structure of iterated integrals and renormalized Green functions in pQFT. Both fulfil a convolution law derived from the underlying Hopf algebra structure. The shuffle product of iterated integrals allows to restrict the Hopf algebra to $\mathcal{H}_{Chen}$, while Green functions represent the full Hopf algebra of decorated rooted trees. Typical properties of renormalization like RGE's and OPE's can be derived from this convolution law. The notational exercise to make this explicit will be presented in future work. The opposite exercise, expressing QFT Green functions as generalized iterated integrals, was undertaken in [8]. Here we presented only the conceptual foundations of renormalization theory. In the form presented here it inspired already new results exploring the connection to the Butcher group [13] which also emphasizes the richness of the concepts involved in the renormalization of a local QFT [9].

We argued that all renormalization schemes can be treated on the same footing, and gave explicit examples for the case of one-loop decorations. Two renormalization schemes play a distinguished role: on-shell schemes (momentum schemes in a massless theory) which serve as the general mediator for arbitrary changes of the renormalization point in any choosen scheme and the minimal subtraction scheme which annihilates scale independent quantities from the beginning.

A hope is that the analytic structure of Green functions will become coherent in a manner inspired by the study of the polylogarithm and multiple zeta values and the structures observed there, ranging from the K-Z equations and iterated integrals to Hopf algebras and, eventually, singular knot invariants and weight systems. The enormous progress in mathematics in recent years [17], so far being mainly related to topological QFT's, will hopefully enrich our understanding of QFT in four dimensions eventually. Recent results concerning counterterms of Feynman diagrams [18] as well as analytic structures of Green functions [19] justify some hope, when combined with the results of this paper.

One thing we have not considered here: non-trivial renormalization schemes establish algebraic structures on V which weaken the structure of a proper Hopf algebra [2]. As this paper is already quite long, we will describe the resulting weak Hopf algebra structure in more detail in forthcoming work [20].

Let us close this paper with one final observation which shows the urgent need to work out the connection between renormalization and NCG [3] in more detail. Using the results of section five we write for a rooted tree t with n vertices

$$\frac{1}{t^!} = \frac{S_t CM(t)}{n!}. \tag{150}$$

Plugging this into the model of section four (see [5] to find such models coming from realistic QFT's), we find

$$G_z(t;x) = B_t(x) z^{-nx} = \frac{1}{t^! x^n} F_t(x) z^{-nx} \tag{151}$$

where we factored out the pole parts such that $F(0) = 1$. Summing over all trees we obtain

$$\sum_n \sum_{t \in \mathcal{T}^{[n]}} G_z(t;x) = \sum_n \frac{1}{n!} F_{\delta_n}(x) [\frac{z^{-x}}{x}]^n, \tag{152}$$

where $F_{\delta_n}(x) = \sum_{t\in\mathcal{T}^{[n]}} CM(t)S_tF_t(x)$ is the natural representative of the natural grown δ_n, a nice result in the light of [3, 4], emphasizing that renormalization almost (by the deviation of $\mathcal{R}$ from *id*) inverts bare Green functions in the group assigned to $\mathcal{H}$ in the final section of [3], in fullagreement with [13] and [9].

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