

Symmetry breaking boundary conditions and WZW orbifolds

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Abstract

Symmetry breaking boundary conditions for WZW theories are discussed. We derive explicit formulae for the reflection coefficients in the presence of boundary conditions that preserve only an orbifold subalgebra with respect to an involutive automorphism of the chiral algebra. The characters and modular transformations of the corresponding orbifold theories are computed. Both inner and outer automorphisms are treated.

1 Introduction

Our understanding of conformally invariant boundary conditions for two-dimensional conformal field theories and of their classification has recently improved enormously. Such boundary conditions allow to study string perturbation theory in the background of certain solitonic solutions, so-called D-branes. They also possess applications in statistical mechanics, e.g., in the description of impurities and percolation problems.

For non-trivial conformal field theories, i.e., for string backgrounds that are not flat, conformal field theory techniques have been applied successfully in those situations where only finitely many primary fields occur. More precisely, these are cases where not only the bulk theory is rational, but also the part \mathfrak{A}^G of the bulk symmetry \mathfrak{A} that is not broken by the boundary conditions is still the chiral algebra of a rational theory. The situation is particularly manageable when \mathfrak{A}^G is actually an orbifold subalgebra, i.e., a subalgebra of \mathfrak{A} that is left pointwise fixed by some group G of automorphisms. For the case when G is a finite abelian group, an explicit description and classification of such boundary conditions has been established in [9, 10]. The crucial ingredients in those investigations are the representations of the modular group, both the one that is associated to the chiral conformal field theory based on \mathfrak{A} and the one associated to the orbifold chiral algebra \mathfrak{A}^G .

The purpose of the present note is to exploit these novel results in the special case when \mathfrak{A} is the chiral algebra of a WZW theory. In this case the theory of affine Lie algebras provides a powerful tool to compute the relevant modular matrices. There are, however, also other reasons to study this specific class of models. It has been conjectured [11] that the structure constants of the classifying algebra for the boundary conditions of a given automorphism type are given by the traces of the action of this automorphism on the spaces of chiral blocks. In the case of WZW theories, techniques are available (see e.g., [27, 3]) that allow to test this conjecture. Moreover, WZW theories correspond to strings propagating on group manifolds; thus they constitute the most directly accessible non-trivial generalization of strings propagating in a flat background. Boundary conditions of WZW theories therefore present a convenient testing ground for studying conjectures about the correspondence of geometric and algebraic formulations of boundary

conditions. While in the case of flat backgrounds one deals with the familiar Neumann and Dirichlet boundary conditions of free bosons (possibly supplemented with a background field strength), already in this modestly generalized situation the geometric interpretation of boundary conditions, in particular of those which break bulk symmetries, still remains to be clarified.

As already mentioned, the analysis of the boundary conditions of our interest requires in particular a rather detailed knowledge of the modular matrices of the orbifold theory. As a consequence, we first have to establish these data, which in itself constitutes a non-trivial result. In the technically more amenable case where the orbifold group G consists of inner automorphisms, such orbifolds have been studied earlier [19]. In the present paper, we will also determine the modular matrices for orbifolds by outer automorphisms. As it turns out, the twisted sector of such orbifold theories is provided by integrable irreducible representations of *twisted* affine Lie algebras. As a simplification, here we restrict ourselves to the case of orbifolds by involutions, i.e., to $G = \mathbb{Z}_2$. But as a matter of fact our results on WZW orbifolds are already sufficient to extract also the reflection coefficients for more general symmetry breaking boundary conditions.

Once the modular matrices of the orbifold theory have been obtained, the results of [9, 10] allow to read off the corresponding symmetry breaking boundary conditions rather directly. In fact, we learn even more; by applying T-duality both in the bulk and to the boundary conditions we can also describe boundary conditions that preserve all symmetries of the bulk theory, but for a theory with non-trivial modular invariant in the bulk. Our results cover in particular the case of orbifolds by the charge conjugation automorphism. Thus, by T-duality, we are able to describe the boundary conditions that arise for the true diagonal torus partition function. (Typically, in the literature boundary conditions are considered for the charge conjugation modular invariant [5, 11, 10]; see, however, [22, 12].)

The paper is organized as follows. We start in section 2 by summarizing the general structure of \mathbb{Z}_2 -orbifolds, with emphasis on the building blocks of the orbifold characters and their modular transformations. Section 3 contains the technical core of the paper. We establish the techniques for dealing with arbitrary involutive automorphisms

of the chiral algebra of a WZW theory that leave the Virasoro element fixed, and compute the characters and modular matrices for the orbifold of the WZW theory by such an automorphism. In section 4 these results are combined with the information from [10] so as to determine the symmetry breaking boundary conditions of the WZW theory. Finally, an appendix collects some properties of twisted Theta functions.

2 \mathbb{Z}_2 -orbifolds

2.1 Elementary consistency conditions

We study the orbifold theory of an arbitrary rational conformal field theory by the \mathbb{Z}_2 -group that is generated by an order-two automorphism

$$\omega : \mathfrak{A} \rightarrow \mathfrak{A}, \quad \omega^2 = id, \quad (2.1)$$

of the chiral algebra \mathfrak{A} of the original theory. The chiral algebra of the orbifold theory is by definition the subalgebra $\mathfrak{A}^{\mathbb{Z}_2}$ of \mathfrak{A} that is left pointwise fixed by ω . To give rise to a consistent conformal field theory, this orbifold subalgebra must again possess a Virasoro element. Conformal invariance of the boundary conditions requires in addition that the Virasoro elements of \mathfrak{A} and $\mathfrak{A}^{\mathbb{Z}_2}$ actually coincide; thus we require that the automorphism ω leaves the Virasoro element fixed.

For any arbitrary automorphism $\tilde{\omega}$ of \mathfrak{A} and any \mathfrak{A} -representation R also $R \circ \tilde{\omega}$ is a representation of \mathfrak{A} . As a consequence, associated with $\tilde{\omega}$ there comes a bijection $\tilde{\omega}^*$ between irreducible representations of \mathfrak{A} that is defined by $R_\lambda \circ \tilde{\omega} \cong R_{\tilde{\omega}^*\lambda}$. Since the orbifold chiral algebra $\mathfrak{A}^\mathcal{O}$ only contains elements that are invariant under ω , there are no observables in the orbifold theory that could distinguish between R_λ and $R_{\tilde{\omega}^*\lambda}$. These representations, even when they are inequivalent representations of the original chiral algebra \mathfrak{A} , will give equivalent representations of $\mathfrak{A}^\mathcal{O}$. This identification is, however, not the only effect. Rather, when $R_{\tilde{\omega}^*\lambda} \cong R_\lambda$, then the automorphism $\tilde{\omega}$ is implemented by an automorphism of the \mathfrak{A} -module R_λ , and the invariant subspaces under this map constitute submodules for $\mathfrak{A}^\mathcal{O}$. As a consequence, one is faced with the task to split these \mathfrak{A} -modules into submodules over $\mathfrak{A}^\mathcal{O}$. This way we arrive at a certain set of irreducible $\mathfrak{A}^\mathcal{O}$ -modules, but typically we do not get all of the irreducible $\mathfrak{A}^\mathcal{O}$ -modules. The ones which

can be obtained by identifying and splitting \mathfrak{A} -modules constitute the untwisted sector of the orbifold theory, while all other representations of $\mathfrak{A}^{\mathcal{O}}$ are said to be in some twisted sector. We remark that by definition the untwisted sector is closed under operator products, but it is not closed under modular transformations. This very fact will enable us to determine the twisted sector.

Let us now specialize to the \mathbb{Z}_2 orbifold obtained for the automorphism (2.1). We first consider the untwisted sector. There are two types of fields, which are distinguished by the action of the orbifold group on the fields of the original theory they come from. More specifically, they differ in the size of the *stabilizer* subgroup

$$S_{\lambda} := \{\omega \in \mathbb{Z}_2 \mid \omega^* \lambda = \lambda\}. \quad (2.2)$$

When $S_{\mu} = \mathbb{Z}_2$, then we call the primary field μ *symmetric*, while for $S_{\mu} = \{id\}$, we call μ *non-symmetric*.¹ For non-symmetric μ , the two fields on the \mathbb{Z}_2 -orbit $\{\mu, \omega^* \mu\}$ are isomorphic modules of the orbifold chiral algebra. As a consequence, each non-symmetric orbit gives rise to a single primary field in the orbifold; we choose (arbitrarily) a representative λ for each such length-two orbit $\{\lambda, \omega^* \lambda\}$ and label the corresponding orbifold field as $(\lambda, 0, 0)$; its character is simply

$$\chi_{(\lambda, 0, 0)}^{\mathcal{O}}(\tau) = \chi_{\lambda}(\tau). \quad (2.3)$$

In contrast, each symmetric field λ of the original theory, satisfying $\omega^* \lambda = \lambda$, gets split, thus giving rise to two distinct fields in the untwisted sector of the orbifold; we label them as $(\lambda, \psi, 0)$ with $\psi \in \{\pm 1\}$. Their characters are to be obtained by a suitable projection, and accordingly can be written as

$$\chi_{(\lambda, \psi, 0)}^{\mathcal{O}}(\tau) = \frac{1}{2} (\chi_{\lambda}(\tau) + \psi \eta_{\lambda}^{-1} \chi_{\lambda}^{(0)}(2\tau)) \quad (2.4)$$

with certain phases η_{λ} . At point this is just a definition of the expression $\eta_{\lambda}^{-1} \chi_{\lambda}^{(0)}$ that is hereby introduced for every symmetric field; this definition implies in particular that

$$\chi_{\lambda}^{(0)}(\tau+2) = (T_{\lambda}^{(0)})^2 \chi_{\lambda}^{(0)}(\tau) \quad \text{with} \quad (T_{\lambda}^{(0)})^2 = T_{(\lambda, \psi, 0)}^{\mathcal{O}} = T_{\lambda}, \quad (2.5)$$

¹ In the case of permutation orbifolds, such fields have been called [4] *diagonal* and *off-diagonal*, respectively.

where T denotes the T-matrix of the original theory. Also, the explicit introduction of the phases η_λ is not really necessary, but this will prove to be convenient later on; we take the convention, though, to ascribe the value $\eta_\Omega = 1$ to the phase for the vacuum primary field Ω .

We can also immediately present some S-matrix elements of the orbifold theory, namely those for the S-transformation of the characters (2.3) coming from non-symmetric fields; they are expressible through the S-matrix S of the original theory as

$$\begin{aligned} S_{(\lambda,0,0),(\mu,0,0)}^{\mathcal{O}} &= S_{\lambda,\mu} + S_{\lambda,\omega^*\mu}, \\ S_{(\lambda,0,0),(\mu,\psi,0)}^{\mathcal{O}} &= S_{\lambda,\mu}, \\ S_{(\lambda,0,0),(\mu,\psi,1)}^{\mathcal{O}} &= 0. \end{aligned} \quad (2.6)$$

Here in the last line we introduced the notation $(\dot{\lambda}, \psi, 1)$ for the fields in the twisted sector of the orbifold that come from the symmetric field λ ; each symmetric field gives rise to two such fields, distinguished by the value of $\psi \in \{\pm 1\}$, while there are no twisted fields coming from non-symmetric fields λ .

In order for the result in the second line of (2.6) to be well-defined, i.e., independent on whether one takes the representative λ or $\omega^*\lambda$ of a non-symmetric orbit, we need $S_{\lambda,\mu} = S_{\omega^*\lambda,\mu}$ for every symmetric μ ; this is a condition that must be satisfied for every consistent orbifold action. Further, to have a consistent orbifold theory, $S^{\mathcal{O}}$ must be symmetric; this implies that $S_{\lambda,\mu} + S_{\lambda,\omega^*\mu} = S_{\mu,\lambda} + S_{\mu,\omega^*\lambda}$, which together with the symmetry of the original S-matrix S implies that the previous requirement generalizes to

$$S_{\lambda,\omega^*\mu} = S_{\omega^*\lambda,\mu} \quad (2.7)$$

for every μ . Thus for every consistent \mathbb{Z}_2 -orbifold action, the induced map on the primaries must have the property that (2.7) holds for all λ and μ . Moreover, in order for the orbifold to furnish a conformal field theory the automorphism ω of the chiral algebra must keep the Virasoro algebra fixed, which in turn implies that it maps the vacuum primary field Ω to itself, $\omega^*\Omega = \Omega$. These elementary results for the orbifold tell us in particular that

$$\sum_{\kappa} \frac{S_{\omega^*\lambda,\kappa} S_{\omega^*\mu,\kappa} S_{\omega^*\nu,\kappa}^*}{S_{\omega^*\Omega,\kappa}} = \sum_{\kappa} \frac{S_{\lambda,\omega^*\kappa} S_{\mu,\omega^*\kappa} S_{\nu,\omega^*\kappa}^*}{S_{\Omega,\omega^*\kappa}}, \quad (2.8)$$

which by the Verlinde formula means that the fusion rules of the original theory satisfy

$$N_{\omega^* \lambda, \omega^* \mu}^{\omega^* \nu} = N_{\lambda, \mu}^{\nu}. \quad (2.9)$$

Thus the map ω^* constitutes an *automorphism of the fusion rules*. One should, however, be aware of the fact that different automorphisms of the chiral algebra can give rise to one and the same automorphism of the fusion rules; to give an example, we will see that every inner automorphism of a WZW theory yields the identity map on the fusion rule algebra.

2.2 Characters in the twisted sector

To be able to analyze also the characters in the twisted sector, additional structure is needed. The characters in the twisted sector can be obtained by performing an S-transformation $\tau \mapsto -1/\tau$ on the functions $\chi^{(0)}$. The result can be written as a linear combination of character-like quantities. In all cases known to us each of those is, up to an over-all power of $q = \exp(2\pi i\tau)$, a power series in $q^{1/2}$, albeit not in q . Each such series arises from the sum of characters of two primary fields in the twisted sector, and the two characters will be obtained separately as the eigenfunctions of the T-operation $\tau \mapsto \tau+1$.

Therefore we demand that for every symmetric field λ there is another function

$$\chi_{\dot{\lambda}}^{(1)}(\tau), \quad (2.10)$$

such that the following relations are obeyed.

- The labels $\dot{\lambda}$ are in one-to-one correspondence with the labels λ . (Generically this one-to-one correspondence is, however, not canonical. In particular, in general one cannot dispense of using two different kinds of symbols for the labels of the functions $\chi^{(0)}$ and $\chi^{(1)}$.)
- The function $\chi_{\dot{\lambda}}^{(1)}$ has non-negative integral coefficients in its expansion in powers of $q = \exp(2\pi i\tau)$.
- The functions are eigenfunctions under T, i.e., we have

$$\chi_{\dot{\lambda}}^{(1)}(\tau+1) = T_{\dot{\lambda}}^{(1)} \chi_{\dot{\lambda}}^{(1)}(\tau) \quad (2.11)$$

for some suitable phases $T_{\dot{\lambda}}^{(1)}$.

■ The S-operation connects the functions $\chi_{\dot{\lambda}}^{(0)}$ and $\chi_{\dot{\mu}}^{(1)}$. More precisely, there are unitary matrices $S^{(0)}$ and $S^{(1)}$ such that

$$\chi_{\dot{\lambda}}^{(0)}(-\frac{1}{\tau}) = \sum_{\dot{\mu}} S_{\dot{\lambda}, \dot{\mu}}^{(0)} \chi_{\dot{\mu}}^{(1)}(\tau) \quad (2.12)$$

and

$$\chi_{\dot{\lambda}}^{(1)}(-\frac{1}{\tau}) = \sum_{\substack{\mu \\ \mu = \omega^* \mu}} S_{\dot{\lambda}, \mu}^{(1)} \chi_{\mu}^{(0)}(\tau). \quad (2.13)$$

This structure is present in all examples that are known to us. Note that even when the labels λ and $\dot{\lambda}$ take their values in the same set, the functions $\chi^{(0)}$ and $\chi^{(1)}$ are definitely not required to coincide. Thus even in this special situation there is no reason to require that the matrices $S^{(0)}$ and $S^{(1)}$ are symmetric, though this will be the case in specific examples.

Combining the conditions (2.11) and (2.12), we find the characters in the twisted sector as

$$\chi_{(\dot{\lambda}, \psi, 1)}^{\mathcal{O}}(\tau) = \frac{1}{2} \left(\chi_{\dot{\lambda}}^{(1)}(\frac{\tau}{2}) + \psi (T_{\dot{\lambda}}^{(1)})^{-1/2} \chi_{\dot{\lambda}}^{(1)}(\frac{\tau+1}{2}) \right). \quad (2.14)$$

They transform under T as

$$\chi_{(\dot{\lambda}, \psi, 1)}^{\mathcal{O}}(\tau+1) = \psi (T_{\dot{\lambda}}^{(1)})^{1/2} \chi_{(\dot{\lambda}, \psi, 1)}(\tau), \quad (2.15)$$

which determines the conformal weights (modulo integers) in the twisted sector,

$$T_{(\dot{\lambda}, \psi, 1)}^{\mathcal{O}} = \psi (T_{\dot{\lambda}}^{(1)})^{1/2}. \quad (2.16)$$

Note that the square root $(T_{\dot{\lambda}}^{(1)})^{1/2}$ is only defined up to a sign; it is to be understood as follows. For each value of $\dot{\lambda}$ we make an arbitrary choice of the square root, and keep this choice fixed once and forever. A different choice would lead to the opposite assignment of the label ψ . Once this choice has been made, fields with the same $\dot{\lambda}$, but different values of ψ , can be distinguished unambiguously, because their conformal weights differ by $1/2 \bmod \mathbb{Z}$.

We are now in a position to determine the rest of the S-matrix elements in the untwisted sector. We obtain

$$\begin{aligned} S_{(\lambda,\psi,0),(\mu,\psi',0)}^{\mathcal{O}} &= \frac{1}{2} S_{\lambda,\mu} , \\ S_{(\lambda,\psi,0),(\dot{\mu},\psi',1)}^{\mathcal{O}} &= \frac{1}{2} \psi \eta_{\lambda}^{-1} S_{\lambda,\dot{\mu}}^{(0)} , \\ S_{(\lambda,\psi,0),(\mu,0,0)}^{\mathcal{O}} &= \frac{1}{2} (S_{\lambda,\mu} + S_{\lambda,\omega^*\mu}) = S_{\lambda,\mu} . \end{aligned} \quad (2.17)$$

(The first and third lines do not rely on our assumptions about the functions $\chi^{(1)}$.)

For the modular S-transformation of the characters (2.14) we find

$$\chi_{(\lambda,\psi,1)}(-\frac{1}{\tau}) = \frac{1}{2} \sum_{\mu} \sum_{\psi'=\pm 1} \psi' (\eta_{\mu} S_{\lambda,\mu}^{(1)} \chi_{(\mu,\psi',0)}(\tau) + \psi P_{\lambda,\dot{\mu}} \chi_{(\dot{\mu},\psi',1)}(\tau)) , \quad (2.18)$$

where P is the matrix

$$P := (T^{(1)})^{1/2} S^{(1)} (T^{(0)})^2 S^{(0)} (T^{(1)})^{1/2} \quad (2.19)$$

(the symbol P is chosen in accordance with the notation for a similar matrix that appeared in a different context in [23]). By the unitarity of the various matrices appearing here, P is unitary as well. The precise form of the matrix P depends on our choice of square roots in the definition of $T^{\mathcal{O}}$. But the result for the matrix $S^{\mathcal{O}}$ is independent of our choice of square roots, because for a different choice we also must flip the labelling of characters, so that the expression $\psi\psi'P_{\lambda,\dot{\mu}}$ remains unchanged. From (2.18) we read off the remaining elements of the modular matrix $S^{\mathcal{O}}$ of the orbifold:

$$\begin{aligned} S_{(\dot{\lambda},\psi,1),(\mu,\psi',0)}^{\mathcal{O}} &= \frac{1}{2} \eta_{\mu} \psi' S_{\dot{\lambda},\mu}^{(1)} , \\ S_{(\dot{\lambda},\psi,1),(\dot{\mu},\psi',1)}^{\mathcal{O}} &= \frac{1}{2} \psi \psi' P_{\dot{\lambda},\dot{\mu}} , \\ S_{(\dot{\lambda},\psi,1),(\mu,0,0)}^{\mathcal{O}} &= 0 . \end{aligned} \quad (2.20)$$

Thus in particular we have managed to express, based on only very general assumptions, the matrix elements $S_{(\dot{\lambda},\psi,1),(\dot{\mu},\psi',1)}^{\mathcal{O}}$ in the twisted sector in terms of other $S^{\mathcal{O}}$ -elements and data of the original theory.

2.3 Consistency conditions

In order for the orbifold construction to furnish a consistent conformal field theory, the matrices $S^\mathcal{O}$ and $T^\mathcal{O}$ must possess the following properties:

- symmetry: $(S^\mathcal{O})^\dagger = S^\mathcal{O}$,
 - unitarity: $(S^\mathcal{O})^{-1} = (S^\mathcal{O})^*$,
 - conjugation: $C^\mathcal{O} := (S^\mathcal{O})^2$ is an order-two permutation,
 - $(S^\mathcal{O}T^\mathcal{O})^3 = (S^\mathcal{O})^2$.
- (2.21)

In addition, when inserting $S^\mathcal{O}$ into the Verlinde formula, we must obtain non-negative integral fusion coefficients.

Let us discuss the implications of the requirements (2.21) in some detail.

2.3.1 Symmetry of $S^\mathcal{O}$

To investigate the symmetry property of the matrix $S^\mathcal{O}$, we compare its entry $S_{(\lambda,\psi,0),(\dot{\mu},\psi',1)}^\mathcal{O}$ to $S_{(\dot{\mu},\psi',1),(\lambda,\psi,0)}^\mathcal{O}$. One sees that a necessary condition for symmetry of $S^\mathcal{O}$ is that

$$S_{\dot{\mu},\lambda}^{(1)} = \eta_\lambda^{-2} S_{\lambda,\dot{\mu}}^{(0)}. \quad (2.22)$$

This relation is compatible with the unitarity of the matrices because the numbers η_λ are phases. It also allows us to rewrite P as

$$P = (T^{(1)})^{1/2} (S^{(0)})^\dagger (\eta^{-1} T^{(0)})^2 S^{(0)} (T^{(1)})^{1/2}. \quad (2.23)$$

This is manifestly symmetric, which implies that (2.22) is also sufficient for $S^\mathcal{O}$ to be symmetric.

2.3.2 Unitarity of $S^\mathcal{O}$ and charge conjugation

The conditions that ensure that $S^\mathcal{O}$ is unitary and that the charge conjugation $C^\mathcal{O} = (S^\mathcal{O})^2$ is a permutation of order two are best studied together. For non-symmetric λ , straightforward calculation using

the unitarity of S shows that $(S^\mathcal{O}(S^\mathcal{O})^*)_{(\lambda,0,0),(\lambda',0,0)} = \delta_{(\lambda,0,0),(\lambda',0,0)}$ and $(S^\mathcal{O}(S^\mathcal{O})^*)_{(\lambda,0,0),(\lambda',\psi,g)} = 0$ are satisfied automatically. To determine the conjugation for non-symmetric λ , we just use that ω^* is a fusion rule automorphism and hence commutes with charge conjugation; it follows that charge-conjugate ω^* -orbits give charge-conjugate fields in the orbifold theory. (Thus when ω^* is itself charge conjugation, all fields $(\mu, 0, 0)$ of the orbifold theory are selfconjugate.)

For symmetric fields, the desired result $(S^\mathcal{O}(S^\mathcal{O})^*)_{(\lambda,\psi,0),(\lambda',\psi',0)} = \delta_{(\lambda,\psi,0),(\lambda',\psi',0)}$ is obtained if and only if $S^{(0)}$ and $S^{(1)}$ are related as

$$\sum_{\dot{\mu}} S_{\lambda,\dot{\mu}}^{(0)} (S_{\dot{\mu},\lambda'}^{(1)})^* = \eta_\lambda^2 \delta_{\lambda,\lambda'}, \quad (2.24)$$

or equivalently,

$$\sum_{\substack{\mu \\ \mu = \omega^* \mu}} S_{\dot{\lambda},\mu}^{(1)} \eta_\mu^2 (S_{\mu,\dot{\lambda}'}^{(0)})^* = \delta_{\dot{\lambda},\dot{\lambda}'}. \quad (2.25)$$

As for charge conjugation, we obtain

$$C_{(\lambda,\psi,0),(\lambda',\psi',0)}^\mathcal{O} = \frac{1}{2} (C_{\lambda,\lambda'} + \psi\psi' C_{\lambda,\lambda'}^{(0)}) \quad (2.26)$$

with

$$C_{\lambda,\lambda'}^{(0)} := \eta_\lambda^{-1} \eta_{\lambda'}^{-1} \sum_{\dot{\mu}} S_{\lambda,\dot{\mu}}^{(0)} S_{\dot{\mu},\lambda'}^{(0)} = \eta_\lambda^{-1} \eta_{\lambda'} (S^{(0)} S^{(1)})_{\lambda,\lambda'}. \quad (2.27)$$

Thus charge conjugation in the orbifold is consistent only if it is possible to choose sign factors

$$\epsilon_\lambda \in \{\pm 1\} \quad (2.28)$$

in such a manner that

$$C_{\lambda,\lambda'}^{(0)} = \epsilon_\lambda C_{\lambda,\lambda'}. \quad (2.29)$$

For the twisted sector, we find that $(S^\mathcal{O}(S^\mathcal{O})^*)_{(\dot{\lambda},\psi,1),(\dot{\lambda}',\psi',1)} = \delta_{(\dot{\lambda},\psi,1),(\dot{\lambda}',\psi',1)}$ follows from unitarity of the P -matrix (2.19), while the charge conjugation is

$$C_{(\dot{\lambda},\psi,1),(\dot{\lambda}',\psi',1)}^\mathcal{O} = \frac{1}{2} (C_{\dot{\lambda},\dot{\lambda}'}^{(1)} + \psi\psi' (P^2)_{\dot{\lambda},\dot{\lambda}'}) \quad (2.30)$$

with

$$C_{\dot{\lambda}, \dot{\lambda}'}^{(1)} := \sum_{\substack{\mu \\ \mu = \omega \star \mu}} S_{\dot{\lambda}, \mu}^{(1)} S_{\mu, \dot{\lambda}'}^{(0)}. \quad (2.31)$$

We conclude that $C^{(1)}$ must be a permutation of order two; it is convenient to use $C^{(1)}$ to define a conjugation on dotted indices, which we denote by a superscript '+'. Moreover, P^2 must be the same permutation, up to a sign that can depend on $\dot{\lambda}$:

$$(P^2)_{\dot{\lambda}, \dot{\lambda}'} = \dot{\epsilon}_{\dot{\lambda}} C_{\dot{\lambda}, \dot{\lambda}'}^{(1)}. \quad (2.32)$$

From (2.30) we then learn that

$$C_{(\dot{\lambda}, \psi, 1), (\dot{\lambda}', \psi', 1)}^{\mathcal{O}} = C_{\dot{\lambda}, \dot{\lambda}'}^{(1)} \delta_{\psi, \psi' \dot{\epsilon}_{\dot{\lambda}}}. \quad (2.33)$$

We can determine the signs $\dot{\epsilon}_{\dot{\lambda}}$ from the requirement that a consistent conjugation relates fields with identical conformal weight. The two conformal weights that have to coincide are $(T_{\dot{\lambda}}^{(1)})^{1/2} \psi$ and $(T_{\dot{\lambda}^+}^{(1)})^{1/2} \psi' = (T_{\dot{\lambda}^+}^{(1)})^{1/2} \psi \dot{\epsilon}_{\dot{\lambda}}$, so that

$$\dot{\epsilon}_{\dot{\lambda}} = (T_{\dot{\lambda}}^{(1)})^{1/2} / (T_{\dot{\lambda}^+}^{(1)})^{1/2}, \quad (2.34)$$

which is indeed a sign, and which allows us to write

$$P^2 = (T^{(1)})^{1/2} C^{(1)} (T^{(1)})^{-1/2}. \quad (2.35)$$

Thus the presence of the signs $\dot{\epsilon}_{\dot{\lambda}}$ is due to the fact that we choose the two square roots for $\dot{\lambda}$ and $\dot{\lambda}^+$ independently. As an easy consequence of the relation (2.34) we have

$$\dot{\epsilon}_{\dot{\lambda}^+} = \dot{\epsilon}_{\dot{\lambda}}, \quad (2.36)$$

which in turn implies that the conjugation has order two.

We can summarize our results about charge conjugation as

$$\begin{aligned} (\mu, 0, 0)^+ &= (\mu^+, 0, 0), \\ (\mu, \psi, 0)^+ &= (\mu^+, \epsilon_{\mu} \psi, 0), \\ (\dot{\mu}, \psi, 1)^+ &= (\dot{\mu}^+, \dot{\epsilon}_{\dot{\mu}} \psi, 1). \end{aligned} \quad (2.37)$$

As a side remark, let us also mention that in every conformal field theory, the requirements that $S^2 = C$ and $C^2 = \mathbb{1}$ imply that $S^{-1} = SC$, while the requirements that S is unitary and symmetric imply that $S^{-1} = S^*$; thus charge conjugation and complex conjugation of S-matrix elements are related as $S_{i+,j} = (S_{i,j})^*$. Let us check that this property is indeed satisfied for the orbifold. When non-symmetric fields are involved, the property is obeyed trivially. From the diagonal elements in the twisted sector we find the condition

$$P_{\dot{\mu},\dot{\mu}'}^* = \dot{\epsilon}_{\dot{\mu}} P_{\dot{\mu}^+, \dot{\mu}'} , \quad (2.38)$$

while the off-diagonal elements yield

$$(S_{\lambda,\dot{\mu}}^{(0)})^* = \eta_{\lambda}^{-2} S_{\lambda,\dot{\mu}^+}^{(0)} = S_{\dot{\mu}^+, \lambda}^{(1)} , \quad (2.39)$$

and

$$(S_{\lambda,\dot{\mu}}^{(0)})^* = \epsilon_{\lambda} \eta_{\lambda}^* \eta_{\lambda^+}^{-1} S_{\lambda^+, \dot{\mu}}^{(0)} . \quad (2.40)$$

By unitarity of $S^{(0)}$ and by the definition of the conjugation in the twisted sector, these relations are satisfied automatically.

2.3.3 The relation between $S^{\mathcal{O}}$ and $T^{\mathcal{O}}$

Finally we demand that the modular group relation $(S^{\mathcal{O}} T^{\mathcal{O}})^3 = (S^{\mathcal{O}})^2$ holds or, equivalently, that $T^{\mathcal{O}*} S^{\mathcal{O}} T^{\mathcal{O}*} = S^{\mathcal{O}} T^{\mathcal{O}} S^{\mathcal{O}} =: X^{\mathcal{O}}$. This is indeed the case; for $X_{(\lambda,0,0),(\dot{\lambda}',\psi',1)}^{\mathcal{O}}$ it is immediate (these matrix elements are zero), and for $X_{(\lambda,0,0),(\lambda',0,0)}^{\mathcal{O}}$ it follows directly from $(ST)^3 = S^2$. For $X_{(\dot{\lambda},\psi,1),(\dot{\lambda}',\psi',1)}^{\mathcal{O}}$, validity of the condition is checked by also using that $(T^{(0)})^2 = T$, while for $X_{(\lambda,0,0),(\lambda',\psi,0)}^{\mathcal{O}}$ and $X_{(\lambda,\psi,0),(\lambda',\psi',0)}^{\mathcal{O}}$, one has to employ the identity

$$\sum_{\psi=\pm 1} \sum_{\dot{\mu}} S_{\lambda,\dot{\mu}}^{(0)} T_{(\dot{\mu},\psi,1)}^{\mathcal{O}} S_{\dot{\mu},\lambda'}^{(1)} = 0 , \quad (2.41)$$

which is a consequence of $T_{(\dot{\lambda},-\psi,1)}^{\mathcal{O}} = -T_{(\dot{\lambda},\psi,1)}^{\mathcal{O}}$. Finally, for ‘mixed’ matrix elements involving both untwisted symmetric and twisted fields, one has to compare

$$(T^{\mathcal{O}*} S^{\mathcal{O}} T^{\mathcal{O}*})_{(\lambda,\psi,0),(\dot{\lambda}',\psi',1)} = \frac{1}{2} \psi \psi' \eta_{\lambda}^{-1} T_{\lambda}^* S_{\lambda,\dot{\lambda}'}^{(0)} (T_{\dot{\lambda}'}^{(1)})^{-1/2} \quad (2.42)$$

to

$$(S^{\mathcal{O}} T^{\mathcal{O}} S^{\mathcal{O}})_{(\lambda, \psi, 0), (\dot{\lambda}', \psi', 1)} = \frac{1}{2} \psi \psi' \eta_{\lambda}^{-1} \sum_{\dot{\mu}} S_{\lambda, \dot{\mu}}^{(0)} (T_{\dot{\mu}}^{(1)})^{1/2} P_{\dot{\mu}, \dot{\lambda}'} . \quad (2.43)$$

Equality holds if and only if

$$S^{(0)} (T^{(1)})^{1/2} P = T^{-1} S^{(0)} (T^{(1)})^{-1/2} ; \quad (2.44)$$

using the definition of P , this is in turn equivalent to the identity (2.35). We conclude that the requirement $(S^{\mathcal{O}} T^{\mathcal{O}})^3 = (S^{\mathcal{O}})^2$ does not lead to any new constraints on $S^{\mathcal{O}}$ or $T^{\mathcal{O}}$.

2.3.4 Collection of the results

We summarize the findings above by the statement that the consistency conditions (2.21) for the matrices $S^{\mathcal{O}}$ and $T^{\mathcal{O}}$ are equivalent to the following set of properties of the matrices $S^{(0)}$, $S^{(1)}$ and $T^{(1)}$:

$$\begin{aligned} (i) \quad & S^{(0)} \text{ is unitary ,} \\ (ii) \quad & S_{\dot{\lambda}, \dot{\lambda}'}^{(1)} = \eta_{\dot{\lambda}'}^{-2} S_{\dot{\lambda}', \dot{\lambda}}^{(0)} , \\ (iii) \quad & C_{\dot{\lambda}, \dot{\lambda}'}^{(0)} := \eta_{\dot{\lambda}}^{-1} \eta_{\dot{\lambda}'} (S^{(0)} S^{(1)})_{\dot{\lambda}, \dot{\lambda}'} = \epsilon_{\dot{\lambda}} C_{\dot{\lambda}, \dot{\lambda}'} , \\ (iv) \quad & C_{\dot{\lambda}, \dot{\lambda}'}^{(1)} := (S^{(1)} S^{(0)})_{\dot{\lambda}, \dot{\lambda}'} = \frac{(T_{\dot{\lambda}}^{(1)})^{1/2}}{(T_{\dot{\lambda}'}^{(1)})^{1/2}} (P^2)_{\dot{\lambda}, \dot{\lambda}'} . \end{aligned} \quad (2.45)$$

Here P is defined by (2.19); also, the numbers $\epsilon_{\dot{\lambda}}$ must all be ± 1 , and $C^{(1)}$ must be a permutation of order two. The requirement (iv) actually consists of two conditions, namely that $S^{(1)} S^{(0)}$ is a permutation of order two, and that up to signs (which are, however, uniquely determined) P^2 is that same permutation; also, (iv) is the only condition that constrains the conformal weights in the twisted sector.

We now draw further conclusions from the constraints (2.45). First, using unitarity of $S^{(0)}$ and $S^{(1)}$, we get from condition (iv) that

$$S_{\dot{\lambda}, \dot{\lambda}'}^{(0)} = (S_{\dot{\lambda}', \dot{\lambda}}^{(1)})^* \equiv \eta_{\dot{\lambda}}^2 (S_{\dot{\lambda}, \dot{\lambda}'}^{(0)})^* ; \quad (2.46)$$

this can be regarded as an analogue of simple current symmetries [25] of S-matrices. Second, combining conditions (ii) and (iii) with the

unitarity of $S^{(1)}$, we learn that

$$S_{\dot{\mu},\lambda^+}^{(1)} = \epsilon_\lambda \eta_{\lambda^+}^{-1} \eta_\lambda^{-1} \cdot (S_{\dot{\mu},\lambda}^{(1)})^*. \quad (2.47)$$

In particular, the relation

$$S_{\dot{\mu},\lambda^+}^{(1)} = (S_{\dot{\mu},\lambda}^{(1)})^* \quad (2.48)$$

is equivalent to having

$$\epsilon_\lambda = \eta_{\lambda^+} \eta_\lambda. \quad (2.49)$$

In this latter case it follows e.g., that for every selfconjugate λ the number η_λ^2 is a sign. (In particular, when ω^* is itself charge conjugation, so that we are dealing here only with selfconjugate λ , then $\eta_\lambda^2 = \pm 1$ in full generality.)

It is remarkable that only condition (iii) relates the matrices $S^{(0)}$ and S . It is not clear to us whether the special situation that all fields are symmetric (i.e. that $\omega^* = id$) implies that $S = S^{(0)}$.

2.4 Fusion rules

The fusion rules of the orbifold are expressible through $S^\mathcal{O}$ via the Verlinde formula. By direct computation we arrive at the following results. First, there is the expected twist selection rule

$$N_{(\dot{\lambda}_1, \tilde{\psi}_1, g_1), (\dot{\lambda}_2, \tilde{\psi}_2, g_2)}^{\mathcal{O}(\dot{\lambda}_3, \tilde{\psi}_3, g_3)} = 0 \quad \text{for } g_1 + g_2 + g_3 = 1 \bmod 2. \quad (2.50)$$

Here we introduced the convention that $\tilde{\psi}$ can take the values ± 1 for $g=1$, while for $g=0$ the allowed values are ± 1 for symmetric fields, but 0 for non-symmetric ones.

Also, in the untwisted sector we find linear combinations of the original fusion rule coefficients as long as non-symmetric fields are involved:

$$\begin{aligned} N_{(\lambda_1, 0, 0), (\lambda_2, 0, 0)}^{\mathcal{O}(\lambda_3, 0, 0)} &= N_{\lambda_1, \lambda_2}^{\lambda_3} + N_{\omega^* \lambda_1, \lambda_2}^{\lambda_3} + N_{\lambda_1, \omega^* \lambda_2}^{\lambda_3} + N_{\lambda_1, \lambda_2}^{\omega^* \lambda_3}, \\ N_{(\lambda_1, 0, 0), (\lambda_2, 0, 0)}^{\mathcal{O}(\lambda_3, \psi, 0)} &= N_{\lambda_1, \lambda_2}^{\lambda_3} + N_{\lambda_1, \omega^* \lambda_2}^{\lambda_3}, \\ N_{(\lambda_1, 0, 0), (\lambda_2, \psi, 0)}^{\mathcal{O}(\lambda_3, \psi, 0)} &= N_{\lambda_1, \lambda_2}^{\lambda_3}. \end{aligned} \quad (2.51)$$

In particular, in these cases the fusion rules of the orbifold theory are manifestly non-negative integers. Note that only fusion rule coefficients of the original theory appear; this is due to the fact that when at least one non-symmetric field is involved, then the twisted sector does not contribute to the Verlinde sum. A more interesting case is the remaining one in the untwisted sector, which is

$$N_{(\lambda_1, \psi_1, 0), (\lambda_2, \psi_2, 0), (\lambda_3, \psi_3, 0)}^{\mathcal{O}} = \frac{1}{2} (N_{\lambda_1, \lambda_2, \lambda_3} + \psi_1 \psi_2 \psi_3 (\eta_{\lambda_1} \eta_{\lambda_2} \eta_{\lambda_3})^{-1} N_{\lambda_1, \lambda_2, \lambda_3}^{(0)}) \quad (2.52)$$

with

$$N_{\lambda_1, \lambda_2, \lambda_3}^{(0)} := \sum_{\dot{\mu}} \frac{S_{\lambda_1, \dot{\mu}}^{(0)} S_{\lambda_2, \dot{\mu}}^{(0)} S_{\lambda_3, \dot{\mu}}^{(0)}}{S_{\Omega, \dot{\mu}}^{(0)}}. \quad (2.53)$$

These coefficients tell us how the chiral blocks of the original theory split under the orbifold action. In the case of \mathbb{Z}_2 -permutation orbifolds [4], this just amounts to symmetrization or antisymmetrization of the blocks, so that there is an immediate formula for the numbers $N^{(0)}$. In general, we encounter the following structure. An action of the orbifold group can not only be defined on the \mathfrak{A} -modules, but also on the spaces of chiral blocks of the \mathfrak{A} -theory. (Using the description of chiral blocks in terms of co-invariants, see e.g., [27, 3], this can be made explicit in the case of WZW theories.) The expression appearing in formula (2.53) is then the trace of this action or, to be more precise, since this action is only defined up to a sign, the difference of the contributions from the two invariant subspaces. Notice that the form of (2.53) is precisely the one that is familiar from the Verlinde formula. We would also like to point out that simple current symmetries can be implemented in a similar way on the spaces of chiral blocks [13]; in that case a formula for the traces has been conjectured that is of Verlinde form as well, but with the matrix $S^{(0)}$ in the denominator of (2.53) replaced by S [13].

For the fusion rules that involve two twist fields we find

$$N_{(\dot{\lambda}_1, \psi_1, 1), (\dot{\lambda}_2, \psi_2, 1), (\lambda_3, 0, 0)}^{\mathcal{O}} = \sum_{\substack{\mu \\ \mu = \omega^* \mu}} \eta_{\mu}^2 \frac{S_{\dot{\lambda}_1, \mu}^{(1)} S_{\dot{\lambda}_2, \mu}^{(1)} S_{\lambda_3, \mu}}{S_{\Omega, \mu}} \quad (2.54)$$

(independently of the values of ψ_1 and ψ_2) and

$$N_{(\dot{\lambda}_1, \psi_1, 1), (\dot{\lambda}_2, \psi_2, 1), (\lambda_3, \psi_3, 0)}^{\mathcal{O}} = \frac{1}{2} \left(\sum_{\mu} \eta_{\mu}^2 \frac{S_{\dot{\lambda}_1, \mu}^{(1)} S_{\dot{\lambda}_2, \mu}^{(1)} S_{\lambda_3, \mu}}{S_{\Omega, \mu}} + \psi_1 \psi_2 \psi_3 \sum_{\dot{\mu}} \eta_{\lambda_3}^{-1} \frac{P_{\dot{\lambda}_1, \dot{\mu}} P_{\dot{\lambda}_2, \dot{\mu}} S_{\lambda_3, \dot{\mu}}^{(0)}}{S_{\Omega, \dot{\mu}}^{(0)}} \right). \quad (2.55)$$

Note that upon summation over ψ_3 the second term in (2.55) cancels out, so that we have

$$\sum_{\psi_3=\pm 1} N_{(\dot{\lambda}_1, \psi_1, 1), (\dot{\lambda}_2, \psi_2, 1), (\lambda_3, \psi_3, 0)}^{\mathcal{O}} = \sum_{\substack{\mu \\ \mu=\omega^* \mu}} \eta_{\mu}^2 S_{\dot{\lambda}_1, \mu}^{(1)} S_{\dot{\lambda}_2, \mu}^{(1)} S_{\lambda_3, \mu} / S_{\Omega, \mu}, \quad (2.56)$$

which is of the same form as the expression (2.54) for $N_{(\dot{\lambda}_1, \psi_1, 1), (\dot{\lambda}_2, \psi_2, 1), (\lambda_3, 0, 0)}^{\mathcal{O}}$. In the last few equations we have presented the structure constants with three lower indices. Those with an upper index are obtained by either using a complex conjugate matrix $(S^{\mathcal{O}})^*$ in the Verlinde formula or, equivalently,² by raising the index with the help of the conjugation matrix $C^{\mathcal{O}}$ of the orbifold; e.g., one has

$$N_{(\dot{\lambda}_2, \psi_2, 1), (\lambda_3, 0, 0)}^{\mathcal{O}} (\dot{\lambda}_1, \psi_1, 1) = \sum_{\substack{\mu \\ \mu=\omega^* \mu}} \frac{(S_{\dot{\lambda}_1, \mu}^{(1)})^* S_{\dot{\lambda}_2, \mu}^{(1)} S_{\lambda_3, \mu}}{S_{\Omega, \mu}}. \quad (2.57)$$

It is readily checked that the orbifold field

$$J^{\mathcal{O}} := (0, -1, 0) \quad (2.58)$$

acts under fusion as

$$J^{\mathcal{O}} \star (\lambda, \epsilon, g) = (\lambda, -\epsilon, g) \quad (2.59)$$

and hence is a simple current of order two. It has integral conformal weight, and the extension of the orbifold theory by this simple current simply reproduces the original theory. Upon extension, orbifold fields coming from symmetric fields of the original theory form full orbits, whereas the fields coming from non-symmetric fields are fixed points which need to be resolved, thus giving rise to a pair of non-symmetric

² When deriving (2.57) this way from (2.54), in addition to the conjugation (2.33) one has to employ the identity (2.47).

fields. The virtue of this simple relationship is that it provides us with two distinct descriptions of one and the same situation – a pair of theories with respective chiral algebras \mathfrak{A} and $\mathfrak{A}^{\mathcal{O}} \subset \mathfrak{A}$. Comparison of the two descriptions often simplifies the analysis of this situation. This was instrumental in the investigation of boundary conditions in [9, 10], e.g., when discussing the relationship between boundary blocks (i.e., chiral blocks for one-point functions of bulk fields on the disk) and boundary states.

3 WZW orbifolds

In this section we apply the general results of section 2 to the case where the original theory is a WZW theory based on some untwisted affine Lie algebra $\hat{\mathfrak{g}}$. In this situation every automorphism of the full chiral algebra \mathfrak{A} , which for our present purposes is the semi-direct sum of $\hat{\mathfrak{g}}$ and the Virasoro algebra, is completely determined by its restriction ω to $\hat{\mathfrak{g}}$. Moreover, of particular interest in applications are those cases where this automorphism ω comes from an automorphism of the horizontal subalgebra \mathfrak{g} of $\hat{\mathfrak{g}} \equiv \mathfrak{g}^{(1)}$. In the case of interest to us here, i.e., orbifolds, this restriction is mandatory because the automorphism must act as the identity on the Virasoro algebra and hence must preserve \mathfrak{g} ; accordingly here we consider this particular kind of automorphisms of $\hat{\mathfrak{g}}$. Also, in order to make contact with the discussion in section 2 we require ω to have order two, i.e., generate a \mathbb{Z}_2 -group. For the case of *inner* automorphisms, WZW orbifolds with respect to more general finite orbifold groups have been discussed in [19]. In the present paper we use the techniques developed in [14, 8] which allow us to deal also with the case of outer automorphisms. There are two reasons to expect that in the case of an outer automorphism of $\hat{\mathfrak{g}} = \mathfrak{g}^{(1)}$ the twisted sector can be understood in terms of the *twisted* affine Lie algebra $\mathfrak{g}^{(2)}$. To compute the traces in the untwisted sector, we will use the theory of twining characters and orbit Lie algebras as introduced in [14], and in the case at hand the latter are twisted affine Lie algebras. Modular transformation of these traces yields again characters of twisted affine Lie algebras, although not necessarily of the same twisted affine Lie algebra that is relevant to the untwisted sector. The fact that the modules of twisted affine Lie algebras furnish the states of the twisted sector can, of course, also be understood from the fact that the latter

provide twisted representations of the chiral algebra \mathfrak{A} in the sense of [6].

3.1 Automorphisms

For simplicity, we denote the relevant automorphism of the finite-dimensional simple Lie algebra \mathfrak{g} again by ω . It is known [18, Prop. 8.1] that for every automorphism ω of \mathfrak{g} of finite order there is a suitable Cartan–Weyl basis of \mathfrak{g} in which it can be written as

$$\omega = \omega_{\circ} \circ \sigma_s, \quad (3.1)$$

where ω_{\circ} is a diagram automorphism, i.e., acts on the Chevalley generators of \mathfrak{g} as

$$\omega_{\circ}(E^i) = E^{\dot{\omega}i}, \quad \omega_{\circ}(F^i) = F^{\dot{\omega}i}, \quad \omega_{\circ}(H^i) = H^{\dot{\omega}i} \quad (3.2)$$

($i = 1, 2, \dots, r \equiv \text{rank } \hat{\mathfrak{g}}$) with some (possibly trivial) symmetry $\dot{\omega}$ of the Dynkin diagram of \mathfrak{g} , while σ_s is an inner automorphism

$$\sigma_s = \exp(2\pi i \text{ad}_{H_s}), \quad (3.3)$$

where $H_s \equiv (s, H)$ is some element of the Cartan subalgebra of \mathfrak{g} that satisfies $\omega(H_s) = H_s$. The latter property implies that the automorphisms ω_{\circ} and σ_s commute. It is not difficult to see that every automorphism of the form (3.1) extends uniquely to an automorphism of the untwisted affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}^{(1)}$, which acts like

$$\begin{aligned} \omega(E_n^i) &= (\zeta_{\omega_{\circ}})^n e^{2\pi i(s, \alpha^{(i)})} E_n^{\dot{\omega}i}, & \omega(F_n^i) &= (\zeta_{\omega_{\circ}})^n e^{-2\pi i(s, \alpha^{(i)})} F_n^{\dot{\omega}i}, \\ \omega(H_n^i) &= H_n^{\dot{\omega}i}, & \omega(K) &= K. \end{aligned} \quad (3.4)$$

Here $\alpha^{(i)}$ are the simple \mathfrak{g} -roots, K denotes the canonical central element of $\hat{\mathfrak{g}}$, and $\zeta_{\omega_{\circ}}$ is a sign defined by

$$\omega_{\circ}(E^{\theta}) = \zeta_{\omega_{\circ}} E^{\theta}, \quad (3.5)$$

where θ is the highest \mathfrak{g} -root. In particular, for $s=0$ the generators coming from the additional simple root $\alpha^{(0)}$ of $\hat{\mathfrak{g}}$ transform as

$$\omega_{\circ}(E^0) = \zeta_{\omega_{\circ}} E^0, \quad \omega_{\circ}(F^0) = \zeta_{\omega_{\circ}} F^0, \quad \omega_{\circ}(H^0) = H^0; \quad (3.6)$$

for $\zeta_{\omega_0} = +1$ this is nothing but the diagram automorphism of $\hat{\mathfrak{g}}$ that is obtained by the same prescription as in (3.2) when one extends $\dot{\omega}$ in the natural manner, i.e., as $\dot{\omega}0 = 0$. By direct calculation, one finds that³

$$\zeta_{\omega_0} = \begin{cases} -1 & \text{for } \mathfrak{g} = A_{2n}, \omega_0 = \omega_c, \\ 1 & \text{else.} \end{cases} \quad (3.7)$$

We identify the Cartan subalgebra with the weight space and correspondingly call s the *shift vector* that characterizes the inner automorphism σ_s . The map ω^* on the weight space only depends on ω_0 ; its action on the fundamental \mathfrak{g} -weights $\Lambda_{(i)}$ reads

$$\omega^* \Lambda_{(i)} = \Lambda_{(\dot{\omega}i)}. \quad (3.8)$$

In particular, for inner automorphisms ω the map ω^* is the identity. We refer to $\hat{\mathfrak{g}}$ -weights λ that satisfy $\omega^* \lambda = \lambda$ as *symmetric* weights; e.g., according to the statements above the shift vector s is symmetric,

$$\omega^* s = s. \quad (3.9)$$

Now the group of automorphisms of a finite-dimensional simple Lie algebra is a real compact Lie group; its factor group modulo inner automorphisms is isomorphic to the center $Z(G)$ of the universal covering group G whose Lie algebra is the compact real form of \mathfrak{g} . Moreover, every element of the center $Z(G)$ can be obtained by exponentiation of an element of the coweight lattice Q^* of \mathfrak{g} . As a consequence, the shift vector s is only defined up to a symmetric element of the coweight lattice. The inequivalent shift vectors are characterized in theorem 8.6 of [18].

For ω to have order two, $2s$ must be an element of the coweight lattice Q^* , and hence corresponds to a group element $\gamma_{2s} = e^{2\pi i \cdot 2s}$ in the center $Z(G)$. According to the results of [25, 26] it therefore determines uniquely a simple current of the WZW theory based on $\hat{\mathfrak{g}}$, which we denote by $J^{[s]}$. (But the shift vector s is *not*, in general, equal to $1/2$ the co-minimal fundamental weight that characterizes [15] the simple current $J^{[s]}$.) We also introduce the monodromy charge

$$Q_{J^{[s]}}(\lambda) := \Delta(\lambda) + \Delta(J^{[s]}) - \Delta(J^{[s]} \star \lambda) \bmod \mathbb{Z} \quad (3.10)$$

³ The automorphism obtained by setting ζ_{ω_0} to 1 in the exceptional A_{2n} -case does not leave the Virasoro algebra invariant; it maps L_n to $(-1)^n L_n$.

of λ with respect to the simple current $J^{[s]}$. When multiplied with the order of $J^{[s]}$, $Q_{J^{[s]}}(\lambda)$ equals minus the conjugacy class of λ with respect to that element $\gamma_{2s} \in Z(G)$ that is obtained as the exponential of the element $2s$ of the coweight lattice. Thus the exponentiated monodromy charge is given by

$$\exp(2\pi i Q_{J^{[s]}}(\lambda)) = (\eta_\lambda^{[s]})^{-2} \quad (3.11)$$

with

$$\eta_\lambda^{[s]} := \exp(2\pi i(s, \lambda)). \quad (3.12)$$

It follows [25] in particular that

$$S_{\lambda, \mu} \eta_\mu^{[s]2} = S_{J^{[s]} \star \lambda, \mu} \quad (3.13)$$

for all λ, μ .

The (conjugacy classes of) order-two automorphisms ω of a complex finite-dimensional simple Lie algebra are uniquely characterized by their fixed point algebra \mathfrak{g}^ω , which is the subalgebra of \mathfrak{g} that is left pointwise fixed under ω . Equivalently, these automorphisms are in one-to-one correspondence to the real forms of these complex Lie algebras. A complete list of the latter can e.g., be found in table II on p. 514 of [17]. It is straightforward (though lengthy) to determine the corresponding diagram automorphisms and shift vectors. We summarize the pertinent results in table 1. In table 1 we use the following notation. Except for $\mathfrak{g} = D_4$, ω_c is the unique non-trivial diagram automorphism of the relevant Lie algebra, while for $\mathfrak{g} = D_4$ it is the diagram automorphism for which ω exchanges the two spinor nodes of the Dynkin diagram. In all cases where the group of simple currents of the theory is cyclic, J stands for the generator of this group that has highest weight $k\Lambda_{(1)}$, except for $\mathfrak{g} = C_n$, where the weight is $k\Lambda_{(n)}$; for D_n we use the notation J_v for the vector current, of highest weight $k\Lambda_{(1)}$, which has order two and whose monodromy charge distinguishes spinor and tensor representations, while J_s is the spinor current, of highest weight $k\Lambda_{(n)}$.

By comparing the results for the numbers (s, s) with the values [15] of the conformal dimensions $\Delta(J)$ of simple currents, one finds that the identity

$$\Delta(J^{[s]})/k = 2(s, s) \bmod \mathbb{Z} \quad (3.14)$$

Table 1: Order-2 automorphisms of finite-dimensional simple Lie algebras.

\mathfrak{g}	ω_o	s	(s, s)	$J^{[s]}$	\mathfrak{g}^ω
A_1	id	$\frac{1}{2} \Lambda_{(1)}$	$1/8$	J	$\mathfrak{u}(1)$
$A_n, n \geq 1$	id	$\frac{1}{2} \Lambda_{(\ell)}, \ell = 2, \dots, [\frac{n+1}{2}]$	$\ell(n+1-\ell)/4(n+1)$	$J^{-\ell}$	$A_{\ell-1} \oplus A_{n-\ell} \oplus \mathfrak{u}(1)$
A_{2n}	ω_c	0	0	$\mathbf{1}$	B_n
A_{2n+1}	ω_c ω_c	0 $\frac{1}{2} \Lambda_{(n+1)}$	0 $(n+1)/8$	$\mathbf{1}$ J^{n+1}	C_{n+1} D_{n+1}
B_n	id	$\frac{1}{2} \Lambda_{(\ell)}, \ell = 2, \dots, n-1$	$\ell/4$	J^ℓ	$D_\ell \oplus B_{n-\ell}$
C_n	id id	$\Lambda_{(\ell)}, \ell = 1, \dots, [n/2]$ $\frac{1}{2} \Lambda_{(n)}$	$\ell/2$ $n/8$	$\mathbf{1}$ J	$C_\ell \oplus C_{n-\ell}$ $A_{n-1} \oplus \mathfrak{u}(1)$
D_n	id id ω_c ω_c	$\frac{1}{2} \Lambda_{(\ell)}, \ell = 2, \dots, [\frac{n+1}{2}]$ $\frac{1}{2} \Lambda_{(n)}$ 0 $\frac{1}{2} \Lambda_{(\ell)}, \ell = 1, \dots, [\frac{n-1}{2}]$	$\ell/4$ $n/16$ 0 $\ell/4$	J_v^ℓ J_s $\mathbf{1}$ J_v^ℓ	$D_\ell \oplus D_{n-\ell}$ $A_{n-1} \oplus \mathfrak{u}(1)$ B_{n-1} $B_\ell \oplus B_{n-\ell-1}$
E_6	id id ω_c ω_c	$\frac{1}{2} \Lambda_{(1)}$ $\frac{1}{2} \Lambda_{(6)}$ 0 $\frac{1}{2} \Lambda_{(6)}$	$1/3$ $1/2$ 0 $1/2$	J $\mathbf{1}$ $\mathbf{1}$ $\mathbf{1}$	$D_5 \oplus \mathfrak{u}(1)$ $A_5 \oplus \mathfrak{u}(1)$ F_4 C_4
E_7	id id id	$\frac{1}{2} \Lambda_{(5)}$ $\frac{1}{2} \Lambda_{(6)}$ $\frac{1}{2} \Lambda_{(7)}$	1 $3/8$ $7/8$	$\mathbf{1}$ J J	$D_6 \oplus \mathfrak{u}(1)$ $E_6 \oplus \mathfrak{u}(1)$ A_7
E_8	id id	$\frac{1}{2} \Lambda_{(1)}$ $\frac{1}{2} \Lambda_{(7)}$	$1/2$ 1	$\mathbf{1}$ $\mathbf{1}$	$E_7 \oplus \mathfrak{u}(1)$ D_8
F_4	id id	$\frac{1}{2} \Lambda_{(1)}$ $\Lambda_{(4)}$	$1/2$ 1	$\mathbf{1}$ $\mathbf{1}$	$C_3 \oplus A_1$ B_4
G_2	id	$\frac{1}{2} \Lambda_{(1)}$	$1/2$	$\mathbf{1}$	$A_1 \oplus A_1$

holds for all entries in table 1. Together with the result (3.11) and the identity

$$Q_J(J \star \lambda) + Q_J(\lambda) = 2 \Delta(J) \bmod \mathbb{Z}, \quad (3.15)$$

this implies that

$$(\eta_\lambda^{[s]} \eta_{J^{[s]} \star \lambda}^{[s]})^2 = e^{-4\pi i \Delta(J^{[s]})} = e^{-8\pi i k(s,s)} \quad (3.16)$$

for every λ .

3.2 Implementation of automorphisms

To be able to determine the characters of the orbifold, we have to know the action of the automorphism ω on the primary fields. To this end we implement ω by suitable maps $\mathcal{T}_\lambda^\omega$ on irreducible highest weight modules \mathcal{H}_λ of the affine Lie algebra $\hat{\mathfrak{g}}$.⁴ They must satisfy the twisted intertwining property $\mathcal{T}_\lambda^\omega \circ x = \omega(x) \circ \mathcal{T}_\lambda^\omega$ for all $x \in \hat{\mathfrak{g}}$. This requirement determines them up to a scalar factor. One possible implementation is

$$\tilde{\mathcal{T}}_\lambda^\omega := \exp(2\pi i(s, H)) \mathcal{T}_\lambda^{\omega_0}. \quad (3.17)$$

Here $\mathcal{T}_\lambda^{\omega_0}$ is the preferred implementation of the diagram automorphism, i.e., the one that acts as the identity map on the highest weight space of \mathcal{H}_λ .⁵ The inner part, $\exp(2\pi i(s, H))$, can be regarded as the representation matrix $R_\lambda(\gamma^{[s]})$ of an element $\gamma^{[s]}$ of the simply connected, connected, compact Lie group whose Lie algebra is the compact real form of \mathfrak{g} . Unlike ω itself, the twisted intertwiners (3.17) do not square to the identity map. Rather $(\tilde{\mathcal{T}}_\lambda^\omega)^2$ acts on \mathcal{H}_λ as a multiple $(\eta_\lambda^{[s]})^2 id$ of the identity, where $\eta_\lambda^{[s]}$ is the number introduced in (3.12). An implementation which does have order two is thus given by

$$\mathcal{T}_\lambda^\omega := (\eta_\lambda^{[s]})^{-1} \tilde{\mathcal{T}}_\lambda^\omega. \quad (3.18)$$

It should be realized that the requirement that the map $\mathcal{T}_\lambda^\omega$ has order two does not fix it uniquely, but leaves an over-all sign undetermined;

⁴ By a slight abuse of notation, we use the symbol λ both for a highest weight of \mathfrak{g} and for a highest weight of $\hat{\mathfrak{g}}$. This is justified by the fact that (as in any application to conformal field theory) we work at a fixed non-negative integral value of the level of the $\hat{\mathfrak{g}}$ -modules, so that the \mathfrak{g} -weight uniquely determines its affine extension.

⁵ This implementation has been used in [14, 8].

in the prescription (3.18) we have fixed this ambiguity by requiring that the map acts as $+id$ on the highest weight vector.

Recall that $\eta_\lambda^{[s]2}$ is the monodromy charge with respect to the simple current $J^{[s]}$; thus the number $\eta_{\lambda^+}^{[s]2} = \exp(2\pi i(2s, \lambda^+)) = \exp(2\pi i(2s^+, \lambda))$ is the monodromy charge with respect to the inverse of that simple current, so the product of these two numbers is 1. It follows that

$$\epsilon_\lambda^{[s]} := \eta_\lambda^{[s]} \eta_{\lambda^+}^{[s]} \quad (3.19)$$

is equal to ± 1 .

Our goal is now to compute the characters of the primary fields in the untwisted sector of the orbifold. While the identification of pairs of non-symmetric fields does not pose any problem, in the symmetric case we need to get a handle on the eigenspaces of the action of $\mathcal{T}_\lambda^\omega$ on the modules having symmetric highest weight. Using the projectors

$$\mathbf{1} + \psi \mathcal{T}_\lambda^\omega \quad (3.20)$$

on the eigenspaces, where ψ takes the values ± 1 , the characters of the orbifold primaries in the untwisted sector are

$$\begin{aligned} \chi_{(\lambda, \psi, 0)}(\tau, z) &= \frac{1}{2} \operatorname{Tr}_{\mathcal{H}_\lambda} (\mathbf{1} + \psi \mathcal{T}_\lambda^\omega) q^{L_0 - c/24} e^{2\pi i(z, H)} \\ &= \frac{1}{2} \chi_\lambda(\tau, z) + \frac{1}{2} (\eta_\lambda^{[s]})^{-1} \psi \chi_\lambda^\omega(\tau, z). \end{aligned} \quad (3.21)$$

Here we introduced the expression

$$\chi_\lambda^\omega(\tau, z) := \operatorname{Tr}_{\mathcal{H}_\lambda} \tilde{\mathcal{T}}_\lambda^\omega q^{L_0 - c/24} e^{2\pi i(z, H)} \quad (3.22)$$

for every automorphism ω of $\hat{\mathfrak{g}}$; these quantities χ_λ^ω are known [14] as *twining characters* for the automorphism ω . They can be regarded as a character valued indices.

In the case of diagram automorphisms, the twining characters have been computed in [14, 8]. It has been proven that, upon correctly adjusting its arguments, the twining character coincides with the ordinary character of some other Lie algebra, which is called the orbit Lie algebra. One key ingredient in the proof of this assertion was the observation that the subgroup of the Weyl group of $\hat{\mathfrak{g}}$ that commutes with the action of the automorphism on the weight space is isomorphic

to the Weyl group of the orbit Lie algebra. The orbit Lie algebra is a Kac–Moody algebra of the same type as the original algebra. Thus in the case at hand, it is still an affine Lie algebra, albeit no longer necessarily an untwisted one; indeed it is a *twisted* affine Lie algebra precisely if the automorphism ω of $\hat{\mathfrak{g}}$ is outer. The appearance of twisted affine Lie algebras for those twining characters will imply, after modular transformation, that the twisted sector of the orbifold theory is controlled by the representation theory of a twisted affine Lie algebra. This does not come as a surprise, since twisted affine Lie algebras provide twisted representations for untwisted affine Lie algebras. For inner automorphisms the orbit Lie algebra coincides with $\hat{\mathfrak{g}}$, so in particular its horizontal subalgebra is just the horizontal subalgebra \mathfrak{g} of $\hat{\mathfrak{g}}$. For the outer automorphisms the horizontal subalgebra of the orbit Lie algebra is C_n for $\mathfrak{g} = A_{2n}$, B_{n+1} for $\mathfrak{g} = A_{2n+1}$, C_{n-1} for $\mathfrak{g} = D_n$ and F_4 for $\mathfrak{g} = E_6$.

3.3 Weyl groups and lattices

As we will see below, for orbifolds with respect to inner automorphisms, the Weyl group \hat{W} of $\hat{\mathfrak{g}}$ plays an analogous role for the twining characters as it does for the characters of the original WZW theory. In contrast, in the outer case, where the map ω^* on the weight space is non-trivial, this role is taken over by the commutant

$$\hat{W}_{\omega^*} := \{\hat{w} \in \hat{W} \mid \hat{w} \circ \omega^* = \omega^* \circ \hat{w}\} \quad (3.23)$$

of ω^* in \hat{W} . This subgroup \hat{W}_{ω^*} can be described more explicitly as follows. The Weyl group \hat{W} has the structure $\hat{W} = W \ltimes L^\vee$ of a semi-direct product of the Weyl group W of the horizontal subalgebra \mathfrak{g} with the coroot lattice L^\vee of \mathfrak{g} . Thus every element $\hat{w} \in \hat{W}$ can be uniquely written as

$$\hat{w} = w \circ t_\beta \quad (3.24)$$

for some $\beta \in L^\vee$, where t_β stands for translation by β ; conversely, all these maps are in the Weyl group \hat{W} . Now since ω is also an automorphism of \mathfrak{g} , the product $w_\omega := (\omega^*)^{-1} w \omega^*$ is again an element of W . Moreover, ω^* restricts to an automorphism of the coroot lattice L^\vee , and hence

$$\hat{w} \circ \omega^* = w \circ t_\beta \circ \omega^* = \omega^* \circ w_\omega \circ t_{(\omega^*)^{-1}\beta}. \quad (3.25)$$

As the decomposition (3.24) is unique, this implies that ω^* commutes with \hat{w} if and only if $w_\omega = w$ and $\omega^*\beta = \beta$. It follows that \hat{W}_{ω_o} is the semi-direct product

$$\hat{W}_{\omega_o} = W_{\omega_o} \ltimes L_{\omega_o}^\vee \quad (3.26)$$

of the symmetric part $L_{\omega_o}^\vee$ of the coroot lattice of \mathfrak{g} with the group

$$W_{\omega_o} := \{w \in W \mid w \circ \omega^* = \omega^* \circ w\}. \quad (3.27)$$

It is known [14, 8, 21] that W_{ω_o} is a Coxeter group; we will denote by ε_{ω_o} its sign function. In fact, W_{ω_o} is nothing but the Weyl group of the orbit Lie algebra [14, 8] of the finite-dimensional simple Lie algebra \mathfrak{g} .

A general element of $L_{\omega_o}^\vee$ is of the form⁶

$$\sum_{i=1}^r n_i \alpha^{(i)} \quad \text{with } n_i \in \mathbb{Z} \text{ and } n_{\dot{\omega}i} = n_i \text{ for all } i. \quad (3.28)$$

In other words, the basis elements of $L_{\omega_o}^\vee$ are associated to orbits of $\dot{\omega}$, and for length-1 orbits the basis element is just $\alpha^{(i)}$, while for length-2 orbits it is $\alpha^{(i)} + \alpha^{(\dot{\omega}i)}$. By computing the inner products between these basis vectors, we find that these lattices are just scaled root lattices, as listed in the following table.

\mathfrak{g}	\mathfrak{g}^{ω_o}	$L_{\omega_o}^\vee$	$\dot{L}_{\omega_o}^\vee$
A_{2n}	C_n	$\sqrt{2} Q(B_n)$	$\frac{1}{\sqrt{2}} Q(B_n)$
A_{2n-1}	B_n	$\sqrt{2} Q(B_n)$	$Q(C_n)$
D_{n+1}	C_n	$\sqrt{2} Q(C_n) \equiv Q(D_n)$	$Q(B_n)$
E_6	F_4	$\sqrt{2} Q(F_4)$	$Q(F_4)$

(3.29)

Table (3.29) also provides another set of lattices, denoted by $\dot{L}_{\omega_o}^\vee$; they are constructed in the following manner. The lattice $L_{\omega_o}^\vee$ is the ‘symmetric’ (i.e., pointwise fixed) sublattice of the coroot lattice L^\vee . We will also need the symmetric sublattice of $L^{\vee*}$, which we denote

⁶ In all cases with non-trivial ω_o , \mathfrak{g} is simply laced, so that the root lattice and coroot lattice coincide.

by $\dot{L}_{\omega_0}^{\vee*}$. When ω_0 is non-trivial, then unlike L^\vee and $L^{\vee*}$ themselves, these are no longer dual to each other. Rather, the dual lattice $L_{\omega_0}^{\vee*}$ of $L_{\omega_0}^\vee$ is a lattice that contains the symmetric weight lattice $\dot{L}_{\omega_0}^{\vee*}$ as a sublattice, while the dual lattice $(\dot{L}_{\omega_0}^{\vee*})^* =: \dot{L}_{\omega_0}^\vee$ contains the symmetric coroot lattice $L_{\omega_0}^\vee$; thus

$$L_{\omega_0}^\vee \subseteq \dot{L}_{\omega_0}^\vee, \quad \dot{L}_{\omega_0}^{\vee*} \subseteq L_{\omega_0}^{\vee*}. \quad (3.30)$$

All these lattices have the same rank, which we denote by r_{ω_0} . Denoting the length of the $\dot{\omega}$ -orbit through i , which is either 1 or 2, by ℓ_i , we have explicitly

$$\begin{aligned} L_{\omega_0}^\vee &= \{ \sum_{i=1}^r n_i \alpha^{(i)\vee} \mid n_{\dot{\omega}i} = n_i \in \mathbb{Z} \}, \\ \dot{L}_{\omega_0}^\vee &= \{ \sum_{i=1}^r n_i \alpha^{(i)\vee} \mid \ell_i n_i \in \mathbb{Z}, n_{\dot{\omega}i} = n_i \}, \\ L_{\omega_0}^{\vee*} &= \{ \sum_{i=1}^r \lambda^i \Lambda_{(i)} \mid \ell_i \lambda^i \in \mathbb{Z}, \lambda^{\dot{\omega}i} = \lambda^i \}, \\ \dot{L}_{\omega_0}^{\vee*} &= \{ \sum_{i=1}^r \lambda^i \Lambda_{(i)} \mid \lambda^{\dot{\omega}i} = \lambda^i \in \mathbb{Z} \}. \end{aligned} \quad (3.31)$$

The inclusions (3.30) are both of finite index. In fact, the indices are equal. Namely, there are the isomorphisms

$$\dot{L}_{\omega_0}^{\vee*} / L_{\omega_0}^\vee \cong (L_{\omega_0}^{\vee*} / \dot{L}_{\omega_0}^\vee)^* \cong L_{\omega_0}^{\vee*} / \dot{L}_{\omega_0}^\vee \quad (3.32)$$

of finite abelian groups (the first isomorphism is canonical, while the second is a non-canonical isomorphism between a finite abelian group and its dual group), which implies in particular that

$$|\dot{L}_{\omega_0}^\vee / L_{\omega_0}^\vee| = |L_{\omega_0}^{\vee*} / \dot{L}_{\omega_0}^{\vee*}|. \quad (3.33)$$

As a sublattice of the coroot lattice L^\vee , which is an even lattice, $L_{\omega_0}^\vee$ is even as well. Moreover, except for $\mathfrak{g} = A_{2n}$ with outer automorphism, the lattice $\sqrt{2}\dot{L}_{\omega_0}^\vee \supseteq \sqrt{2}L_{\omega_0}^\vee$ is also an even lattice. This can be read off table (3.29) above, where the lattices $\dot{L}_{\omega_0}^\vee$ are listed, but also follows from the following general argument. First, every $\dot{\beta} \in \dot{L}_{\omega_0}^\vee$ can be written as

$$\dot{\beta} = \sum_{i=1}^r \frac{m_i}{\ell_i} \alpha^{(i)\vee} = \sum_{i=\dot{\omega}i} m_i \alpha^{(i)\vee} + \frac{1}{2} \sum_{i < \dot{\omega}i} m_i (\alpha^{(i)\vee} + \alpha^{(\dot{\omega}i)\vee}) \quad (3.34)$$

with $m_i \in \mathbb{Z}$ for all $i = 1, 2, \dots, r$. Using the symmetry property $A^{\dot{\omega}i, \dot{\omega}j} = A^{i,j}$ of the Cartan matrix A of \mathfrak{g} with respect to $\dot{\omega}$, it follows that

$$\begin{aligned} (\dot{\beta}, \dot{\beta}') &= \frac{1}{2} \sum_{\substack{i,j \\ i < \dot{\omega}i, j < \dot{\omega}j}} m_i m'_j (\alpha^{(i)\vee}, \alpha^{(j)\vee} + \alpha^{(\dot{\omega}j)\vee}) \\ &\quad + \sum_{\substack{i,j \\ i = \dot{\omega}i \text{ or } j = \dot{\omega}j}} m_i m'_j (\alpha^{(i)\vee}, \alpha^{(j)\vee}) \in \frac{1}{2} \mathbb{Z}. \end{aligned} \quad (3.35)$$

Moreover, specializing to $\dot{\beta}' = \dot{\beta}$ and using the explicit form of the Cartan matrix, one verifies that

$$(\dot{\beta}, \dot{\beta}) \in \begin{cases} \frac{1}{2} \mathbb{Z} & \text{for } \mathfrak{g} = A_{2n}, \omega_o = \omega_c, \\ \mathbb{Z} & \text{else.} \end{cases} \quad (3.36)$$

To relate vectors in the various lattices, we introduce the linear map

$$f_{\omega_o} : \lambda = \sum_i \lambda^i \Lambda_{(i)} \mapsto f_{\omega_o}(\lambda) := \sum_i \frac{1}{\ell_i} \lambda^i \Lambda_{(i)} \quad (3.37)$$

on the weight space of \mathfrak{g} ; this map restricts to bijections

$$f_{\omega_o}(L_{\omega_o}^\vee) = \dot{L}_{\omega_o}^\vee, \quad f_{\omega_o}(\dot{L}_{\omega_o}^{\vee*}) = L_{\omega_o}^{\vee*} \quad (3.38)$$

(they are *not* isomorphisms of lattices; e.g., they are not isometries). Since the elements of the Weyl group W of \mathfrak{g} preserve the weight lattice $L^{\vee*}$ as well as the coroot lattice L^\vee , it follows that the elements of W_{ω_o} preserve both $\dot{L}_{\omega_o}^{\vee*}$ and $L_{\omega_o}^\vee$. Moreover, as elements of W , they are manifestly isometries, and hence they also preserve the dual lattices $\dot{L}_{\omega_o}^\vee$ and $L_{\omega_o}^{\vee*}$. In short, for all the lattices (3.31) the elements of W_{ω_o} are lattice-preserving isometries. Further, the action of \hat{W}_{ω_o} , and hence also of $W_{\omega_o} \ltimes hL_{\omega_o}^\vee$ for any $h \in \mathbb{Z}_{>0}$ – and similarly also the action of $W_{\omega_o} \ltimes h\dot{L}_{\omega_o}^\vee$ – partitions the symmetric subspace of the weight space of \mathfrak{g} into chambers. In order to account for this feature, we introduce the following notation. The orbit spaces for the two actions are denoted by

$$P_h^{\omega_o+} := \dot{L}_{\omega_o}^{\vee*} / (W_{\omega_o} \ltimes hL_{\omega_o}^\vee) \quad \text{and} \quad \dot{P}_h^{\omega_o+} := L_{\omega_o}^{\vee*} / (W_{\omega_o} \ltimes h\dot{L}_{\omega_o}^\vee), \quad (3.39)$$

respectively. In both cases the subset of orbits on which the action is free plays a particularly important role. We denote them by

$$P_k^{\omega_o++} := (P_{k+\mathfrak{g}^\vee}^{\omega_o+})^\circ \quad \text{and} \quad \dot{P}_k^{\omega_o++} := (\dot{P}_{k+\mathfrak{g}^\vee}^{\omega_o+})^\circ, \quad (3.40)$$

where for later convenience we have shifted the index by the dual Coxeter number g^\vee of $\hat{\mathfrak{g}}$.

In the case of undotted weights, natural representatives for $P_h^{\omega_\circ+}$ and $P_k^{\omega_\circ++}$ exist:

$$\begin{aligned} P_h^{\omega_\circ+} &= \{ \lambda = \sum_{i=1}^r \lambda^i \Lambda_{(i)} \mid \lambda^{\dot{\omega}^i} = \lambda^i \in \mathbb{Z}_{\geq 0}, (\lambda, \theta^\vee) \leq h \}, \\ P_k^{\omega_\circ++} &= \{ \lambda = \sum_{i=1}^r \lambda^i \Lambda_{(i)} \mid \lambda^{\dot{\omega}^i} = \lambda^i \in \mathbb{Z}_{>0}, (\lambda, \theta^\vee) < k + g^\vee \}. \end{aligned} \quad (3.41)$$

Notice that in general it is *not* true that $\dot{P}_h^{\omega_\circ++}$ coincides with $f_{\omega_\circ}(P_h^{\omega_\circ++})$ (the latter does hold, however, when the map f_{ω_\circ} commutes with the action of W_{ω_\circ} which, besides the trivial case where ω is inner, happens if and only if $\mathfrak{g} = A_{2n}$). But we will see later (see formula (3.56) below) that the set $\dot{P}_h^{\omega_\circ++}$ is closely related to the set of integrable weights of twisted affine Lie algebras.

It should also be noted that from their definition it is not obvious (except when f_{ω_\circ} commutes with W_{ω_\circ}) whether the cardinalities of the two sets $P_k^{\omega_\circ+}$ and $\dot{P}_k^{\omega_\circ+}$ are the same. But it follows from the unitarity of the matrix S^{ω_\circ} that will be introduced in (3.54) below that this equality indeed holds true:

$$|\dot{P}_k^{\omega_\circ+}| = |P_k^{\omega_\circ+}| \quad (3.42)$$

for all horizontal algebras \mathfrak{g} , all automorphisms ω and all levels k .

3.4 Twisted characters

We are now ready to introduce the relevant *twisted characters* that appear as constituents of the orbifold characters. They are functions which generalize the ordinary characters of affine Lie algebras, and accordingly they depend on two arguments, a number τ in the upper complex half plane and an element z of the weight space $L^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ of \mathfrak{g} , as well as on two parameters, a non-negative integer h and an element of the relevant dual lattice, that is, $\lambda \in \dot{L}_{\omega_\circ}^{\vee*}$ and $\bar{\lambda} \in L_{\omega_\circ}^{\vee*}$, respectively. In addition they depend on two further parameters, the *twist parameters* $s_1, s_2 \in L^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. The ordinary $\hat{\mathfrak{g}}$ -characters are recovered when $s_1 = 0 = s_2$ and when the relevant lattice is the ordinary coroot lattice L^\vee of \mathfrak{g} .

The ordinary characters χ_λ can be written as quotients of functions $\Xi_{\lambda+\rho, k+g^\vee}$ and Ξ_{ρ, g^\vee} , which in turn are odd Weyl sums over Theta functions for the lattice L^\vee [18]. Similarly, the twisted characters are given by

$$\chi_\lambda^{\omega_\circ}[s_1, s_2](\tau, z) := \Xi_{\lambda+\rho, k+g^\vee}^{\omega_\circ}[s_1, s_2](\tau, z) / \Xi_{\rho, g^\vee}^{\omega_\circ}[s_1, s_2](\tau, z), \quad (3.43)$$

and

$$\dot{\chi}_\lambda^{\omega_\circ}[s_1, s_2](\tau, z) := \dot{\Xi}_{\lambda+\dot{\rho}, k+g^\vee}^{\omega_\circ}[s_1, s_2](\tau, z) / \dot{\Xi}_{\dot{\rho}, g^\vee}^{\omega_\circ}[s_1, s_2](\tau, z), \quad (3.44)$$

respectively. The functions Ξ^{ω_\circ} and $\dot{\Xi}^{\omega_\circ}$ are the antisymmetrized (with respect to the sign function $\varepsilon_{\omega_\circ}$ of W_{ω_\circ}) sums

$$\begin{aligned} \Xi_{\lambda, h}^{\omega_\circ}[s_1, s_2](\tau, z) &:= \sum_{w \in W_{\omega_\circ}} \varepsilon_{\omega_\circ}(w) \Theta_{w(\lambda), h}[s_1, s_2](\tau, z), \\ \dot{\Xi}_{\dot{\lambda}, h}^{\omega_\circ}[s_1, s_2](\tau, z) &:= \sum_{w \in W_{\omega_\circ}} \varepsilon_{\omega_\circ}(w) \dot{\Theta}_{w(\dot{\lambda}), h}[s_1, s_2](\tau, z) \end{aligned} \quad (3.45)$$

over the $\mathfrak{g}_{\omega_\circ}$ -Weyl group W_{ω_\circ} , where in turn $\Theta_{\mu, h}$ and $\dot{\Theta}_{\dot{\mu}, h}$ are *twisted Theta functions* associated to the lattices $L_{\omega_\circ}^\vee$ and $\dot{L}_{\omega_\circ}^\vee$, respectively, at level h , i.e.,

$$\begin{aligned} \Theta_{\lambda, h}[s_1, s_2](\tau, z) &:= \sum_{\beta \in L_{\omega_\circ}^\vee} e^{2\pi i \tau (\lambda + h s_1 + h \beta, \lambda + h s_1 + h \beta) / 2h} e^{2\pi i (z + s_2, \lambda + h s_1 + h \beta)}, \\ \dot{\Theta}_{\dot{\lambda}, h}[s_1, s_2](\tau, z) &:= \sum_{\dot{\beta} \in \dot{L}_{\omega_\circ}^\vee} e^{2\pi i \tau (\dot{\lambda} + h s_1 + h \dot{\beta}, \dot{\lambda} + h s_1 + h \dot{\beta}) / 2h} e^{2\pi i (z + s_2, \dot{\lambda} + h s_1 + h \dot{\beta})}. \end{aligned} \quad (3.46)$$

Various properties of the functions (3.46) and (3.45) are listed in appendix A. They give rise to the following properties of the twisted characters (3.43) and (3.44).

■ We can restrict our attention to a fundamental domain of the action of $\hat{W}_{\omega_\circ} = W_{\omega_\circ} \ltimes h L_{\omega_\circ}^\vee$ on $\dot{L}_{\omega_\circ}^{\vee*}$, respectively of $W_{\omega_\circ} \ltimes h \dot{L}_{\omega_\circ}^\vee$ on $L_{\omega_\circ}^{\vee*}$. Thus we need to consider only characters with

$$\lambda \in P_k^{\omega_\circ+} \quad \text{and} \quad \dot{\lambda} \in \dot{P}_k^{\omega_\circ+}, \quad (3.47)$$

respectively.

■ From the result (A.28) we deduce

$$\begin{aligned}\chi_{\lambda}^{\omega_{\circ}}[s_1, s_2 + \mu](\tau, z) &= e^{2\pi i(\lambda, \mu)} e^{2\pi i k(s_1, \mu)} \chi_{\lambda}^{\omega_{\circ}}[s_1, s_2](\tau, z) \quad \text{for } \mu \in \dot{L}_{\omega_{\circ}}^{\vee*} \cap Q^*, \\ \dot{\chi}_{\lambda}^{\omega_{\circ}}[s_1, s_2 + \dot{\mu}](\tau, z) &= e^{2\pi i(\dot{\lambda}, \dot{\mu})} e^{2\pi i k(s_1, \dot{\mu})} \dot{\chi}_{\lambda}^{\omega_{\circ}}[s_1, s_2](\tau, z) \quad \text{for } \dot{\mu} \in L_{\omega_{\circ}}^{\vee*} \cap Q^*,\end{aligned}\tag{3.48}$$

where Q^* is the coweight lattice of \mathfrak{g} .

■ The behavior under the T-transformation follows from the transformation properties (A.20) of the Ξ -functions. For the functions $\chi^{\omega_{\circ}}$ it reads

$$\chi_{\lambda}^{\omega_{\circ}}[s_1, s_2](\tau + 1, z) = e^{-i\pi k(s_1, s_1)} T_{\lambda} \chi_{\lambda}^{\omega_{\circ}}[s_1, s_1 + s_2](\tau, z) \tag{3.49}$$

with

$$\begin{aligned}T_{\lambda} &:= \exp\left(2\pi i[(\lambda + \rho, \lambda + \rho)/2(k + g^{\vee}) - (\rho, \rho)/2g^{\vee}]\right) \\ &= \exp(2\pi i(\Delta_{\lambda} - c/24)).\end{aligned}\tag{3.50}$$

Here it is assumed that the condition (A.7) is satisfied, i.e., that $(s_1, \beta) \in \mathbb{Z}$ for all $\beta \in L_{\omega_{\circ}}^{\vee}$, or in short, that

$$s_1 \in L_{\omega_{\circ}}^{\vee*}; \tag{3.51}$$

this is indeed the case for all vectors s that appear in table 1, which will be the situation we are actually interested in.

■ In the cases where $\dot{L}_{\omega_{\circ}}^{\vee}$ differs from $L_{\omega_{\circ}}^{\vee}$, it is not an even lattice; as a consequence the $\dot{\chi}$ -characters do not transform nicely under the T-transformation. But they still do so under T^2 , namely

$$\dot{\chi}_{\lambda}^{\omega_{\circ}}[s_1, s_2](\tau + 2, z) = e^{-2i\pi k(s_1, s_1)} (T_{\lambda}^{\omega_{\circ}})^2 \dot{\chi}_{\lambda}^{\omega_{\circ}}[s_1, 2s_1 + s_2](\tau, z) \tag{3.52}$$

with

$$T_{\lambda}^{\omega_{\circ}} := \exp\left(2\pi i[(\dot{\lambda} + \dot{\rho}, \dot{\lambda} + \dot{\rho})/2(k + g^{\vee}) - (\dot{\rho}, \dot{\rho})/2g^{\vee}]\right). \tag{3.53}$$

Here again it is assumed that condition (A.7) is satisfied for s_1 , except for the case of $\mathfrak{g} = A_{2n}$, where the stronger restriction (A.11) applies.

■ For the behavior under the S-transformation we use the formulae

(A.21) and (A.22) as well as (A.27). Introducing the matrix S^{ω_\circ} with entries

$$S^{\omega_\circ}_{\lambda, \dot{\mu}} := |\dot{L}^{\vee*}_{\omega_\circ} / (k + \mathfrak{g}^\vee) L^{\vee}_{\omega_\circ}|^{-1/2} i^{(d_{\omega_\circ} - r_{\omega_\circ})/2} \sum_{w \in W_{\omega_\circ}} \varepsilon_{\omega_\circ}(w) \cdot e^{-2\pi i (w(\lambda + \rho), \dot{\mu} + \dot{\rho}) / (k + \mathfrak{g}^\vee)}, \quad (3.54)$$

we obtain

$$\begin{aligned} \chi^{\omega_\circ}_\lambda[s_1, s_2](-\tfrac{1}{\tau}, \tfrac{z}{\tau}) &= e^{2\pi i k(s_1, s_2)} \sum_{\dot{\mu} \in \dot{P}_k^{\omega_\circ+}} S^{\omega_\circ}_{\lambda, \dot{\mu}} \dot{\chi}^{\omega_\circ}_\mu[s_2, -s_1](\tau, z), \\ \dot{\chi}^{\omega_\circ}_\mu[s_1, s_2](-\tfrac{1}{\tau}, \tfrac{z}{\tau}) &= e^{2\pi i k(s_1, s_2)} \sum_{\lambda \in P_k^{\omega_\circ+}} S^{\omega_\circ}_{\lambda, \dot{\mu}} \chi^{\omega_\circ}_\lambda[s_2, -s_1](\tau, z). \end{aligned} \quad (3.55)$$

■ Finally we point out that there is a close connection between the functions $\chi^{\omega_\circ}_\lambda[0, 0](\tau)$ and the characters $\chi'_{\mu'}$ of the twisted affine Lie algebra $\mathfrak{g}^{(2)}$. This is already apparent from the fact that the lattices $\dot{L}^{\vee}_{\omega_\circ}$ as listed in table (3.29) are precisely $2^{-1/2}$ times the lattices that appear in the Weyl group of these twisted affine Lie algebras (see remark 6.7. of [18]). Closer inspection shows that indeed we have

$$\chi'_{\mu'}(\tau, z) = \dot{\chi}^{\omega_\circ}_{\mu'/\sqrt{2}}[0, 0](2\tau, \sqrt{2}z). \quad (3.56)$$

In particular, we learn that the coefficients that appear in the expansion of $\dot{\chi}^{\omega_\circ}_{\mu'/\sqrt{2}}[0, 0]$ in powers of q are non-negative integers.

3.5 The orbifold data

We are now in a position to compare what we have learned about the twisted characters $\chi^{\omega_\circ}_\mu[s_1, s_2]$ and $\dot{\chi}^{\omega_\circ}_\mu[s_1, s_2]$ – in particular their modular properties – with the general results about orbifolds that were presented in section 2. This way we can identify the data that characterize the WZW orbifold, i.e., the functions $\chi^{(0)}_\mu$ and $\chi^{(1)}_\mu$ and the associated modular matrices.

First of all, the labels λ of symmetric fields as well as the labels $\dot{\lambda}$ in the twisted sector are precisely as defined above; in particular, λ takes values in the set (3.41), i.e., is a symmetric \mathfrak{g} -weight that is integrable

for $\hat{\mathfrak{g}}$ at level k . Further, let us choose the convention for the phases η_λ as introduced in (2.4) so as to match the phases $\eta_\lambda^{[s]}$ appearing in (3.12), i.e.,

$$\eta_\lambda := \eta_\lambda^{[s]} \equiv \exp(2\pi i(s, \lambda)) \quad (3.57)$$

for every symmetric weight λ . Also, for the moment we exclude the exceptional case of outer automorphisms $\omega_o = \omega_c$ of A_{2n} , which will be treated afterwards. Then the functions $\chi^{(0)}$ that describe the projection of symmetric fields in the untwisted sector are given by

$$\chi_\lambda^{(0)}(2\tau) = \chi_\lambda^{\omega_o}[0, s](\tau) \quad (3.58)$$

while for the twisted sector, we have to set

$$\chi_\lambda^{(1)}(\frac{\tau}{2}) = \dot{\chi}_\lambda^{\omega_o}[s, 0](\tau), \quad (3.59)$$

with ω_o the diagram automorphism and s the shift vector as listed in table 1. To be precise, the latter identification is unique up to possibly a phase that could, however, be absorbed in the definition of $S^{(1)}$. With the prescription given here, it follows from our general discussion that the coefficients of an expansion in q of the expression (3.58) are non-negative integers.

In the exceptional A_{2n} case, where $\omega_o = \omega_c$ and $s=0$, we must in addition account for the minus sign (3.7) that is present in the automorphism (3.4) of the affine Lie algebra $A_{2n}^{(1)}$. This sign amounts to a relative minus sign between contributions from even and odd grades, hence we can immediately conclude that in place of (3.58) we now have

$$\chi_\lambda^{(0)}(2\tau) = T_\lambda^{-1/2} \chi_\lambda^{\omega_c}[0, 0](\tau + \frac{1}{2}). \quad (3.60)$$

For obtaining the analogue of (3.59), we need to express the S-transformed character $\chi_\lambda^{(0)}(-\frac{2}{\tau})$ as a linear combination of characters in the twisted sector, i.e., as a linear combination of as many power series in $q^{1/2}$ as there are symmetric fields in the theory. When addressing this task directly via the expression $\chi_\lambda^{\omega_c}[0, 0](-\frac{1}{\tau} + \frac{1}{2})$, the calculations turn out to become clumsy. However, the following observation can be employed to rewrite χ^{ω_c} in a much more amenable manner. Namely, the C_n -modules that appear at the even and odd grades, respectively, of the integrable highest weight modules of the twisted affine Lie algebra

$A_{2n}^{(2)}$ are distinguished by their C_n conjugacy class. It follows that a shift $\tau \mapsto \tau + 1/2$ in the argument of the $A_{2n}^{(2)}$ -characters, and hence also of the twining characters χ^{ω_c} , can be undone by inserting the central group element corresponding to the conjugacy class into the trace. In the case of $A_{2n}^{(2)}$ -characters, this group element is given by $\exp(2\pi i H_{\tilde{s}_o})$ with \tilde{s}_o the C_n -weight $\tilde{s}_o = \Lambda_{(n)}$. Translated to the twining characters, this becomes $\exp(2\pi i H_{s_o})$ with A_{2n} -weight⁷

$$s_o := \frac{1}{4} (\Lambda_{(n)} + \Lambda_{(n+1)}). \quad (3.61)$$

Thus we can rewrite (3.60) as

$$\chi_\lambda^{(0)}(2\tau) = \chi_\lambda^{\omega_c}[0, s_o](\tau). \quad (3.62)$$

This is of the same form as the generic result (3.58); accordingly, in place of (3.59) we now have

$$\chi_\lambda^{(1)}(\frac{\tau}{2}) = \dot{\chi}_\lambda^{\omega_c}[s_o, 0](\tau). \quad (3.63)$$

To determine the matrices $S^{(0)}$ and $S^{(1)}$, we calculate

$$\begin{aligned} \chi_\lambda^{(0)}(-\frac{1}{\tau}) &= \chi_\lambda^{\omega_o}[0, s](-\frac{1}{2\tau}) \\ &= \sum_{\dot{\mu} \in \dot{P}_k^{\omega_o+}} S_{\lambda, \dot{\mu}}^{\omega_o} \dot{\chi}_{\dot{\mu}}^{\omega_o}[s, 0](2\tau) = \sum_{\dot{\mu} \in \dot{P}_k^{\omega_o+}} S_{\lambda, \dot{\mu}}^{\omega_o} \chi_{\dot{\mu}}^{(1)}(\tau), \\ \chi_{\dot{\mu}}^{(1)}(-\frac{1}{\tau}) &= \dot{\chi}_{\dot{\mu}}^{\omega_o}[s, 0](-\frac{2}{\tau}) = \sum_{\lambda \in P_k^+} S_{\lambda, \dot{\mu}}^{\omega_o} \chi_\lambda^{\omega_o}[0, -s](\frac{\tau}{2}) \\ &= \sum_{\lambda \in P_k^+} S_{\lambda, \dot{\mu}}^{\omega_o} \eta_\lambda^{-2} \chi_\lambda^{\omega_o}[0, s](\frac{\tau}{2}) = \sum_{\lambda \in P_k^+} S_{\lambda, \dot{\mu}}^{\omega_o} \eta_\lambda^{-2} \chi_\lambda^{(0)}(\tau), \end{aligned} \quad (3.64)$$

while $T^{(1)}$ is determined by employing (3.53) and (3.48),

$$\begin{aligned} \chi_\lambda^{(1)}(\tau+1) &= \dot{\chi}_\lambda^{\omega_o}[s, 0](2\tau+2) = e^{-2\pi i k(s, s)} (T_\lambda^{\omega_o})^2 \dot{\chi}_\lambda^{\omega_o}[s, 2s](2\tau) \\ &= e^{-2\pi i k(s, s)} e^{2\pi i (\dot{\lambda}, 2s)} e^{2\pi i k(s, 2s)} (T_\lambda^{\omega_o})^2 \dot{\chi}_\lambda^{\omega_o}[s, 0](2\tau) \\ &= e^{2\pi i k(s, s)} e^{2\pi i (\dot{\lambda}, 2s)} (T_\lambda^{\omega_o})^2 \chi_\lambda^{(1)}(\tau). \end{aligned} \quad (3.65)$$

⁷ The factor of 4 arises because the identification between weight space and Cartan subalgebra of C_n that is induced from the corresponding identification for A_{2n} differs by this factor from the standard one.

In the case of the outer automorphism of A_{2n} we must in addition include the appropriate shift vector s_o that was introduced in (3.61) above. It is gratifying that according to the result (A.11) this shift is precisely what is needed in order for $\chi^{(1)}$ to transform nicely under the T-operation. Furthermore, in this exceptional case we can also use the simple relationship $\dot{\lambda} = \lambda/2$. We then find

$$\begin{aligned} S_{\lambda, \dot{\mu}}^{(0)} &= S_{\lambda, \dot{\mu}}^{\omega_o}, \\ S_{\dot{\mu}, \lambda}^{(1)} &= \eta_{\lambda}^{-2} S_{\lambda, \dot{\mu}}^{\omega_o}, \\ T_{\dot{\lambda}}^{(1)} &= \begin{cases} e^{2\pi i k(s_o, s_o)} e^{2\pi i(\lambda, s_o)} (T_{\lambda})^{1/2} & \text{for } \mathfrak{g} = A_{2n}, \omega_o = \omega_c, \\ e^{2\pi i k(s, s)} e^{2\pi i(\dot{\lambda}, 2s)} (T_{\dot{\lambda}}^{\omega_o})^2 & \text{else.} \end{cases} \end{aligned} \quad (3.66)$$

Let us check that the consistency conditions (i) through (iv) of \mathbb{Z}_2 -orbifolds that we derived in section 2 are indeed satisfied by these matrices.

- Concerning (i) we remark that the unitarity of S^{ω_o} follows by precisely the same arguments [18] as for the Kac–Peterson S-matrix of affine Lie algebras.
- The relation (ii) between $S^{(0)}$ and $S^{(1)}$ is manifest in (3.66).
- To address property (iii) we start by computing the matrix elements $(S^{\omega_o} (S^{\omega_o})^t)_{\lambda, \lambda'}$. This is non-zero if and only if there is an element $w \in W_{\omega_o}$ such that $w(\lambda') = -\lambda$. This can only be the longest element $w_{\max}^{\omega_o}$ of the Weyl group of the (horizontal) orbit Lie algebra. Therefore we need $\lambda^+ = \lambda'$, where the conjugation is to be taken in the orbit theory. The sign introduced by $w_{\max}^{\omega_o}$ cancels against the prefactors i for the same reason they do so in the Kac–Peterson formula of the untwisted affine Lie algebra that is based on the horizontal orbit Lie algebra. (The latter is *not* the affine orbit theory, which would be a twisted affine Lie algebra; here only properties of the Weyl groups of the horizontal subalgebras matter.) Thus

$$S^{\omega_o} (S^{\omega_o})^t = C^{\omega_o}, \quad (3.67)$$

where C^{ω_o} is the conjugation matrix of the orbit Lie algebra. C^{ω_o} is in particular a permutation of order two; inspection shows that C^{ω_o} simply coincides with the restriction of C to symmetric fields, which in turn implies that

$$C_{\lambda, \mu}^{(0)} = \eta_{\lambda}^{-2} C_{\lambda, \mu}. \quad (3.68)$$

(In fact, only when ω is inner and charge conjugation outer, then C^{ω_0} is non-trivial, while in all other cases C^{ω_0} is the identity permutation.) Validity of property (iii) thus follows from the fact that according to relation (3.19), η_λ^2 is a sign. The notation for that sign in (3.19) was chosen with hindsight; indeed, we simply have

$$\epsilon_\lambda = \epsilon_\lambda^{[s]} \quad (3.69)$$

for all symmetric weights λ .

■ To verify the first relation of condition (iv), we insert the explicit form (3.54) of S^{ω_0} to deduce

$$\begin{aligned} (S^{(1)}S^{(0)})_{\dot{\lambda}, \dot{\lambda}'} &= (-1)^{(d_{\omega_0} - r_{\omega_0})/2} \frac{1}{|W_{\omega_0}|} e^{2\pi i(2s, \rho)} \\ &\quad \sum_{w, w' \in W_{\omega_0}} \varepsilon_{\omega_0}(ww') \delta_{w(\dot{\lambda} + \dot{\rho}) + w'(\dot{\lambda}' + \dot{\rho} + 2(k + g^\vee)s), 0 \bmod (k + g^\vee) \dot{L}_{\omega_0}^\vee}. \end{aligned} \quad (3.70)$$

Now it follows immediately from the semi-direct product structure of W_{ω_0} that for every $w \in W_{\omega_0}$ and every $\dot{\lambda} \in \dot{P}_k^{\omega_0+}$ there exists a unique vector $\dot{\beta} \in (k + g^\vee) \dot{L}_{\omega_0}^\vee$ and a unique Weyl group element $w' \in W_{\omega_0}$, as well as a unique $\dot{\mu} \in \dot{P}_k^{\omega_0+}$ such that

$$w(\dot{\lambda} + \dot{\rho}) = -w'(\dot{\mu} + \dot{\rho} + 2(k + g^\vee)s + \dot{\beta}). \quad (3.71)$$

Denoting the weight $\dot{\mu}$ appearing here by $\dot{\lambda}^+$, it follows that

$$\begin{aligned} (S^{(1)}S^{(0)})_{\dot{\lambda}, \dot{\lambda}'} &= (-1)^{(d_{\omega_0} - r_{\omega_0})/2} \frac{1}{|W_{\omega_0}|} e^{2\pi i(2s, \rho)} \sum_{w \in W_{\omega_0}} \varepsilon_{\omega_0}(ww') \delta_{\dot{\lambda}, \dot{\lambda}^+} \\ &= \pm \delta_{\dot{\lambda}, \dot{\lambda}^+}. \end{aligned} \quad (3.72)$$

Moreover, by noticing that the square of the S-transformation acts on the arguments as $S^2: \tau \mapsto \tau, z \mapsto -z$ and considering characters at $z=0$, the sign can be seen to be +1. Thus indeed $C^{(1)} := S^{(1)}S^{(0)}$ is a permutation of order two.

■ Finally, the second relation of (iv) follows from the fact that the element TST^2STST^2S of $\text{PSL}(2, \mathbb{Z})$ equals the element S^2 , so that

$$\begin{aligned} \sum_{\dot{\lambda}'} (\dot{\epsilon}P^2)_{\dot{\lambda}, \dot{\lambda}'} \dot{\chi}_{\dot{\lambda}'}(\tau, z) &= (TST^2STST^2S) \cdot \dot{\chi}_{\dot{\lambda}}(\tau, z) \\ &= \dot{\chi}_{\dot{\lambda}}(\tau, -z) = \sum_{\dot{\lambda}'} C_{\dot{\lambda}, \dot{\lambda}'}^{(1)} \dot{\chi}_{\dot{\lambda}'}(\tau, z). \end{aligned} \quad (3.73)$$

By the linear independence of the characters, this implies that $\epsilon P^2 = C^{(1)}$, as required.

The results obtained above can be checked most directly in those cases where a different formulation of the orbifold conformal field theory is available. A class of examples where this is the case is provided by certain conformal embeddings [24, 1]. One finds one infinite series, the embedding of $\mathfrak{so}(n)$ at level 2 in $\mathfrak{su}(n)$ at level 1 ($n \geq 3$), and one isolated case, namely C_4 at level 1, which is a special subalgebra of E_6 at level 1.

The simplest among these is the first member of the infinite series, i.e., the embedding of $(A_1)_4$ in $(A_2)_1$, which corresponds to the D-type modular invariant of the A_1 -WZW theory at level 4. In this case, the automorphism is the diagram automorphism of A_2 , which also coincides with charge conjugation. There is a single symmetric primary field, with highest weight $\Omega = \Lambda_{(0)} \equiv (0,0)$. Its character can be written as the sum of two characters of A_1 at level 4, $\chi_{(0,0)}(\tau) = \chi_0^{\text{su}(2)}(\tau) + \chi_4^{\text{su}(2)}(\tau)$. The orbifold chiral algebra is A_1 at level 4; thus the twining character must be the difference of two A_1 -characters. Indeed, one can verify explicitly the identity $\chi^{(0)}(2\tau) = \chi_0^{\text{su}(2)}(\tau) - \chi_4^{\text{su}(2)}(\tau)$. Performing an S-transformation of the A_1 -characters, we then find $\chi^{(1)}(\frac{\tau}{2}) = \chi_1^{\text{su}(2)}(\tau) + \chi_3^{\text{su}(2)}(\tau)$. We have checked explicitly this character identity, too.

Two other special situations concern inner automorphisms and charge conjugation. They will be dealt with separately in the next two subsections.

3.6 Inner automorphisms

In the case where the automorphism $\omega = \sigma_s$ is inner, many of our results simplify. Let us describe some of the simplifications. First note that in the inner case the rank of the fixed point algebra equals the rank of $\hat{\mathfrak{g}}$, $\text{rank } \mathfrak{g}^{\sigma_s} = \text{rank } \mathfrak{g}$. More importantly, the fact that $\omega_o = id$ immediately implies that in lattice sums we just deal with the ordinary coroot lattice, and the relevant lattice symmetries are just given by the ordinary Weyl group, i.e., we have

$$L_{id}^\vee = \dot{L}_{id}^\vee = L^\vee, \quad L_{id}^{\vee*} = \dot{L}_{id}^{\vee*} = L^{\vee*}, \quad W_{id} = W. \quad (3.74)$$

It follows in particular that the entries of $T^{(1)}$ now read

$$T_{\dot{\lambda}}^{(1)} = \left(e^{\pi i k(s,s)} e^{2\pi i(\lambda,s)} T_{\lambda} \right)^2, \quad (3.75)$$

where of course $\dot{\lambda} \equiv \lambda$. Similarly, the matrix elements of $S^{\omega_{\circ}}$ coincide with the corresponding elements of the ordinary S-matrix of the original WZW theory, $S_{\lambda,\mu}^{\omega_{\circ}} = S_{\lambda,\mu}$, and indeed (3.54) then is nothing but the Kac–Peterson formula [18] for S .

This result implies in particular that in the case of inner automorphisms the number $N_{\lambda_1, \lambda_2, \lambda_3}^{(0)}$ that was identified with the difference of the dimensions of the invariant subspaces under the action of the automorphism on the chiral blocks is in fact equal to the fusion rules, i.e., $N_{\lambda_1, \lambda_2, \lambda_3}^{(0)} = N_{\lambda_1, \lambda_2, \lambda_3}$. Indeed, using the explicit description of chiral blocks as co-invariants (see e.g., [27, 3]), one verifies that *inner* automorphisms of $\hat{\mathfrak{g}}$ act on the chiral blocks as multiples of the identity.

The functions $\chi^{(0)}$ and $\chi^{(1)}$ are both just shifted versions of ordinary characters,

$$\chi_{\dot{\lambda}}^{(0)}(2\tau) = \chi_{\lambda}[0, s](\tau), \quad \chi_{\dot{\lambda}}^{(1)}\left(\frac{\tau}{2}\right) = \chi_{\lambda}[s, 0](\tau). \quad (3.76)$$

Thus the orbifold characters coming from symmetric fields read

$$\begin{aligned} \chi_{(\lambda, \psi, 0)}(\tau, z) &= \frac{1}{2} \left(\chi_{\lambda}[0, 0](\tau, z) + \psi \eta_{\dot{\lambda}}^{[s]-1} \chi_{\lambda}[0, s](\tau, z) \right), \\ \chi_{(\lambda, \psi, 1)}(\tau, z) &= \frac{1}{2} \left(\chi_{\lambda}[s, 0](\tau, z) + \psi \eta_{\dot{\lambda}}^{[s]-1} e^{-2\pi i k(s,s)} \chi_{\lambda}[s, s](\tau, z) \right). \end{aligned} \quad (3.77)$$

Let us also mention that in the inner case it can be seen rather directly that the shifted characters $\chi_{\lambda}[s, 0]$ are the correct quantities for the twisted sector. Namely, there exists a continuous family $\{\sigma_v\}$ of shift automorphisms of the semi-direct sum of the affine Lie algebra $\hat{\mathfrak{g}}$ and the Virasoro algebra [7, 20, 16], depending on a vector v in the Cartan subalgebra of \mathfrak{g} and acting as

$$\begin{aligned} \sigma_v(H_n^i) &= H_n^i + v^i K \delta_{n,0}, & \sigma_v(K) &= K, \\ \sigma_v(E_n^{\alpha}) &= E_{n+(\alpha, v)}^{\alpha}, \\ \sigma_v(L_n) &= L_n + (v, H_n) + \frac{1}{2}(v, v) K \delta_{n,0}. \end{aligned} \quad (3.78)$$

Noticing that

$$\chi_{\lambda}[s, 0](\tau, z) = e^{2\pi i k(z,s)} e^{2\pi i \tau k(s,s)/2} \chi_{\lambda}[0, 0](\tau, z + \tau s), \quad (3.79)$$

this implies that

$$\chi_\lambda[s, 0](\tau, z) = \text{Tr}_{\mathcal{H}_\lambda} e^{2\pi i \tau (\sigma_s(L_0))} e^{2\pi i (z, \sigma_s(H_0))} . \quad (3.80)$$

In words, the shifted character with shift $[s, 0]$ is precisely the character for the σ_s -twisted action of the affine Lie algebra. This confirms in particular once more that the coefficients in the expansion of $\chi_\lambda[s, 0](\tau, z=0)$ in powers of q are non-negative integers.

Charge conjugation in the untwisted sector is given by

$$C_{\lambda, \lambda'}^{(0)} = \eta_\lambda^{[s]-2} C_{\lambda, \lambda'} , \quad (3.81)$$

while with the help of the simple current symmetry (3.13) of the S -matrix, we find that the charge conjugation in the twisted sector reads

$$C_{\lambda, \lambda'}^{(1)} = C_{\lambda, J^{[s]} \star \lambda'} . \quad (3.82)$$

Finally we mention that in the case of inner automorphisms the matrix P as defined in formula (2.19) differs from S just by phases. Namely, using (3.75) we have

$$P = e^{2\pi i k(s, s)} \eta^{[s]} T S (\eta^{[s]})^{-2} T S T \eta^{[s]} . \quad (3.83)$$

Using $(ST)^3 = S^2$ and again the simple current relation (3.13), it follows that

$$P_{\lambda, \lambda'} = (\eta_\lambda^{[s]} \eta_{\lambda'}^{[s]})^{-1} e^{-2\pi i k(s, s)} S_{\lambda, \lambda'} . \quad (3.84)$$

Let us also recall that the automorphism ω determines the shift vector s only up to an element β of the coweight lattice. Also, every element β of the coweight lattice determines a conjugacy class, which in turn specifies a simple current J_β . It follows from the explicit character formula that

$$\chi_\lambda[s_1 + \beta, s_2](\tau, z) = \chi_{J_\beta \star \lambda}[s_1, s_2](\tau, z) .$$

This means that one can associate only *orbits* of primary fields with respect to the whole group of simple currents of the original theory in a natural manner to a primary field in the twisted sector of the orbifold.

3.7 Charge conjugation

An order-two automorphism that is present in any arbitrary conformal field theory, and hence is of particular interest, is *charge conjugation*. In the WZW case this comes from the charge conjugation automorphism of the relevant finite-dimensional simple Lie algebra \mathfrak{g} . This involution of \mathfrak{g} can be universally described as the automorphism that acts on the generating elements of a suitable Chevalley basis of \mathfrak{g} as

$$\omega(E^{\alpha^{(i)}}) := E^{-\alpha^{(i)}}, \quad \omega(E^{-\alpha^{(i)}}) := E^{\alpha^{(i)}}, \quad \omega(H^i) := -H^i.$$

In order to conform with the general treatment above, here we take the charge conjugation with respect to a different basis of \mathfrak{g} , such that it is of the specific form (3.1). Data for these special automorphisms are listed in the following table:

\mathfrak{g}	ω_o	s	(s, s)	$J^{[s]}$	\mathfrak{g}^ω
A_1	id	$\frac{1}{2} \Lambda_{(1)}$	$1/8$	J	$\mathfrak{u}(1)$
A_{2n}	ω_c	0	0	$\mathbf{1}$	B_n
A_{2n+1}	ω_c	$\frac{1}{2} \Lambda_{(n+1)}$	$(n+1)/8$	J^{n+1}	D_{n+1}
B_{2n}	id	$\frac{1}{2} \Lambda_{(n)}$	$n/4$	J^n	$B_n \oplus D_n$
B_{2n+1}	id	$\frac{1}{2} \Lambda_{(n+1)}$	$(n+1)/4$	J^{n+1}	$D_{n+1} \oplus B_n$
C_n	id	$\frac{1}{2} \Lambda_{(n)}$	$n/8$	J	$A_{n-1} \oplus \mathfrak{u}(1)$
D_{2n}	id	$\frac{1}{2} \Lambda_{(n)}$	$n/4$	J_v^n	$D_n \oplus D_n$
D_{2n+1}	ω_c	$\frac{1}{2} \Lambda_{(n)}$	$n/4$	J_v^n	$B_n \oplus B_n$
E_6	ω_c	$\frac{1}{2} \Lambda_{(6)}$	$1/2$	$\mathbf{1}$	C_4
E_7	id	$\frac{1}{2} \Lambda_{(7)}$	$7/8$	J	A_7
E_8	id	$\frac{1}{2} \Lambda_{(7)}$	1	$\mathbf{1}$	D_8
F_4	id	$\frac{1}{2} \Lambda_{(1)}$	$1/2$	$\mathbf{1}$	$C_3 \oplus A_1$
G_2	id	$\frac{1}{2} \Lambda_{(1)}$	$1/2$	$\mathbf{1}$	$A_1 \oplus A_1$

(3.85)

In the charge conjugation case the relevant real form of \mathfrak{g} is the normal (or split, maximally non-compact) real form, and we have $\dim \mathfrak{g}^\omega = \frac{1}{2} (\dim \mathfrak{g} - \text{rank } \mathfrak{g})$. A quantity that we encounter in this case is the

Frobenius–Schur indicator of an irreducible representation of \mathfrak{g} , respectively [2] of a primary field of the WZW theory. By definition, this is the number ϵ_λ that takes the value $\epsilon_\lambda = 0$ for non-selfconjugate irreducible representations, while in the selfconjugate case it distinguishes between orthogonal (real) irreducible representations, for which $\epsilon_\lambda = 1$, and symplectic (pseudo-real) irreducible representations, which have $\epsilon_\lambda = -1$. We observe that in all cases where symplectic irreducible representations occur, i.e., for \mathfrak{g} one of

$$A_{4\ell+1}, \quad B_{4\ell+1}, \quad B_{4\ell+2}, \quad C_r, \quad D_{4\ell+2}, \quad E_7, \quad (3.86)$$

the value of the Frobenius–Schur indicator is given by

$$\epsilon_\lambda = (-1)^{c_\lambda}, \quad (3.87)$$

where c_λ is the integer

$$c_\lambda := 2 \cdot [(2s, \lambda) \bmod \mathbb{Z}]. \quad (3.88)$$

Inspection shows that for all algebras \mathfrak{g} for which charge conjugation is inner (so that all representations are selfconjugate), c_λ coincides with the conjugacy class of the \mathfrak{g} -weight λ , which in turn is twice the monodromy charge of λ with respect to the simple current $J^{[s]}$,

$$c_\lambda = 2 [\Delta(\lambda) + \Delta(J^{[s]}) - \Delta(J^{[s]\star}\lambda) \bmod \mathbb{Z}]. \quad (3.89)$$

Note that in all these cases $J^{[s]}$ has order 2, i.e., $(J^{[s]})^2 = \mathbf{1}$, and $c_{J^{[s]\star}\lambda} = c_\lambda$.

In particular we see that the quantity ϵ_λ that was introduced in section 2 is in this case just the exponentiated conjugacy class; it therefore coincides with the Frobenius–Schur indicator. On the other hand, the general results of section 2 imply that each of the two fields $(\lambda, +1, 0)$ and $(\lambda, -1, 0)$ is self-conjugate if $\epsilon_\lambda = 1$, while they are each other's conjugate if $\epsilon_\lambda = -1$. This is intuitively clear, since in the latter case the module of the original theory is symplectic and should be split by charge conjugation into two modules of the orbifold chiral algebra with identical Virasoro-specialized character.

4 Boundary conditions

In [9, 10] a general prescription has been obtained by which one can determine the set of conformally invariant boundary conditions that

preserve a given subalgebra \mathfrak{A}^G of the chiral algebra \mathfrak{A} , for the case when \mathfrak{A}^G is the fixed point subalgebra with respect to a finite abelian orbifold group G . In the particular case where the orbifold group is just $G = \mathbb{Z}_2$, many of the results of [9, 10] simplify enormously. Let us present some of those results in the form in which they arise in this specific situation.

The conformally invariant boundary conditions preserving $\mathfrak{A}^{\mathbb{Z}_2} =: \mathfrak{A}^\omega$ are in one-to-one correspondence with the one-dimensional irreducible representations of a certain finite-dimensional semisimple associative commutative algebra $\mathcal{C}(\mathfrak{A}^\omega)$, called the *classifying algebra*. This algebra has a distinguished basis whose elements are in one-to-one correspondence with the chiral blocks for the one-point correlation functions of bulk fields on the disk. Let us start by describing this basis in detail for the case of interest to us. According to the results of [9, 10], it looks as follows. Each orbifold field $(\lambda, 0, 0)$ that comes from a pair of non-symmetric fields of the original theory gives rise to two basis elements, which we label as

$$\tilde{\Phi}_\lambda \quad \text{and} \quad \tilde{\Phi}_{\omega^*\lambda} \quad (\omega^*\lambda \neq \lambda), \quad (4.1)$$

while each untwisted orbifold field $(\lambda, \psi, 0)$ that comes from a symmetric field yields a single basis element, which we simply denote by

$$\tilde{\Phi}_{(\lambda, \psi)} \quad \text{with} \quad \psi \in \{\pm 1\} \quad (\omega^*\lambda = \lambda). \quad (4.2)$$

Fields in twisted sectors of the orbifold, on the other hand, do not correspond to any element of $\mathcal{C}(\mathfrak{A}^\omega)$.

As already mentioned, to each element $\tilde{\Phi}$ in the basis of the classifying algebra there corresponds a boundary block (i.e., a chiral block for the one-point functions of bulk fields on the disk); we denote this chiral block by $\tilde{\beta}$. For non-symmetric fields λ , $\tilde{\beta}_\lambda$ is the ordinary boundary block (also known as Ishibashi state) of the original WZW theory, while for symmetric λ we get two distinct boundary blocks $\tilde{\beta}_{(\lambda, \psi)}$. The ‘regularized scalar products’ of these boundary blocks are given by the

characters of the orbifold theory:

$$\begin{aligned}
 \langle \tilde{\beta}_\lambda | q^{L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - c/12} | \tilde{\beta}_\mu \rangle &= \delta_{\lambda, \mu} \chi_\lambda(2\tau), \\
 \langle \tilde{\beta}_\lambda | q^{L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - c/12} | \tilde{\beta}_{(\mu, \psi)} \rangle &= 0, \\
 \langle \tilde{\beta}_{(\lambda, \psi)} | q^{L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - c/12} | \tilde{\beta}_{(\mu, \psi')} \rangle &= \delta_{\lambda, \mu} \delta_{\psi, \psi'} \cdot \frac{1}{2} (\chi_\lambda(2\tau) + \psi \chi_\lambda^\omega(2\tau)).
 \end{aligned} \tag{4.3}$$

Here χ_λ^ω is the *twining character* for the automorphism ω ; decomposing this automorphism as $\omega = \omega_\circ \circ \sigma_s$ into its diagram and inner parts (see formula (3.1)), χ_λ^ω is given by the expression (3.43) with $s_1 = 0$ and $s_2 = s$.

Next we list the boundary conditions that preserve the orbifold chiral algebra. According to [9, 10] they can be labelled by orbits with respect to the simple current $(\Omega, -1, 0)$ of the orbifold, including multiplicities that take into account how this simple current acts by the fusion product. In the untwisted sector, the full orbits of this action are $\{(\mu, 1, 0), (\mu, -1, 0)\}$, and each such orbit gives rise to a single boundary condition, which we label by μ . The fixed points are $\{(\mu, 0, 0)\}$; each of them provides us with two distinct boundary conditions, which we label by μ and $\omega^* \mu$. The twisted sector supplies us with additional boundary conditions. All orbits in the twisted sector have length two, i.e., they are of the form $\{(\dot{\mu}, 1, 1), (\dot{\mu}, -1, 1)\}$; accordingly each of them amounts to a single boundary condition, which we label by $\dot{\mu}$. Thus altogether the list of boundary conditions reads

$$\begin{aligned}
 &\mu \quad \text{for } \{(\mu, 1, 0), (\mu, -1, 0)\}, \\
 &\left. \begin{array}{l} \mu \\ \omega^* \mu \end{array} \right\} \quad \text{for } \{(\mu, 0, 0)\} \\
 &\dot{\mu} \quad \text{for } \{(\dot{\mu}, 1, 1), (\dot{\mu}, -1, 1)\}.
 \end{aligned} \tag{4.4}$$

The structure constants are most conveniently expressed in terms of a certain matrix \tilde{S} which, roughly, connects the boundary blocks to the boundary conditions. This matrix has two distinct types of labels; the row index refers to boundary blocks, while the column index refers to boundary conditions. In the case at hand, we obtain

$$\left. \begin{array}{l} \tilde{S}_{(\lambda, \psi), \mu} = S_{\lambda, \mu} \\ \tilde{S}_{(\lambda, \psi), \dot{\mu}} = \psi \eta_\lambda^{-1} S_{\lambda, \dot{\mu}}^{(0)} \end{array} \right\} \text{ for } \omega^* \lambda = \lambda, \quad \left. \begin{array}{l} \tilde{S}_{\lambda, \mu} = S_{\lambda, \mu} \\ \tilde{S}_{\lambda, \dot{\mu}} = 0 \end{array} \right\} \text{ for } \omega^* \lambda \neq \lambda. \tag{4.5}$$

Here $S^{(0)}$ is determined by the formulae (3.66) and (3.54) of section 3; it depends only on the class of the automorphism ω modulo inner automorphisms and reads explicitly

$$S_{\lambda, \dot{\mu}}^{(0)} := \mathcal{N}_{\omega_o} i^{(d_{\omega_o} - r_{\omega_o})/2} \sum_{w \in W_{\omega_o}} \varepsilon_{\omega_o}(w) e^{-2\pi i (w(\lambda + \rho), \dot{\mu} + \dot{\rho}) / (k + g^\vee)}. \quad (4.6)$$

Here W_{ω_o} is the Weyl group of the horizontal subalgebra of the orbit Lie algebra and ε_{ω_o} is its sign function; d_{ω_o} and r_{ω_o} are the dimension and the rank of this horizontal Lie algebra. We also recall that for inner automorphisms the horizontal subalgebra of the orbit Lie algebra coincides with the horizontal subalgebra \mathfrak{g} of $\hat{\mathfrak{g}}$; for the outer automorphisms it is given by C_n for $\mathfrak{g} = A_{2n}$, B_{n+1} for $\mathfrak{g} = A_{2n+1}$, C_{n-1} for $\mathfrak{g} = D_n$ and by F_4 for $\mathfrak{g} = E_6$. Finally \mathcal{N}_{ω_o} is the real positive number which is determined by requiring $S^{(0)}$ to be unitary.

Having obtained the matrix \tilde{S} , the structure constants of the classifying algebra $\mathcal{C}(\mathfrak{A}^\omega)$ are computed by the Verlinde-like formula

$$\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2)}^{(\lambda_3, \psi_3)} = \sum_m \frac{\tilde{S}_{(\lambda_1, \psi_1), m} \tilde{S}_{(\lambda_2, \psi_2), m} \tilde{S}_{(\lambda_3, \psi_3), m}^{-1}}{\tilde{S}_{(\Omega, 1), m}}, \quad (4.7)$$

where the symbol m stands either for an orbit $m = \mu$ (with multiplicities) in the untwisted sector or an orbit $m = \dot{\mu}$ of primary fields in the twisted sector.

As long as one of the indices refers to a non-symmetric field, this formula yields ordinary fusion rule coefficients (as expected from general arguments [9]):

$$\tilde{N}_{\lambda_1, \lambda_2}^{\lambda_3} = N_{\lambda_1, \lambda_2}^{\lambda_3}, \quad \tilde{N}_{\lambda_1, \lambda_2}^{(\lambda_3, \psi_3)} = N_{\lambda_1, \lambda_2}^{\lambda_3}, \quad \tilde{N}_{\lambda_1, (\lambda_2, \psi_2)}^{(\lambda_3, \psi_3)} = N_{\lambda_1, \lambda_2}^{\lambda_3}.$$

For the case of three symmetric fields we have instead

$$\begin{aligned} \tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)} &= N_{\lambda_1, \lambda_2, \lambda_3} + \frac{\psi_1 \psi_2 \psi_3}{\eta_{\lambda_1} \eta_{\lambda_2} \eta_{\lambda_3}} \\ N_{\lambda_1, \lambda_2, \lambda_3}^{(0)} &= N_{(\lambda_1, \psi_1, 0), (\lambda_2, \psi_2, 0), (\lambda_3, \psi_3, 0)}^{\mathcal{O}}. \end{aligned}$$

We are now able to display explicitly the *reflection coefficients* $R_{(\lambda, \psi), \Omega}^m$ and $R_{\lambda, \Omega}^m$, which are the operator product coefficients that describe how a bulk field excites a boundary vacuum field $\Psi_\Omega^{m, m}$ when it

approaches the boundary, according to

$$\phi_{l,l^+}(re^{i\sigma}) \sim \sum_{l'} \sum_m (r^2-1)^{-2\Delta_l+\Delta_{l'}} R_{l,l'}^m \Psi_{l'}^{m,m}(e^{i\sigma}) \quad \text{for } r \rightarrow 1. \quad (4.8)$$

The reflection coefficients for the boundary condition $m = \mu$ read

$$R_{(\lambda,\psi),\Omega}^\mu = \frac{S_{\lambda,\mu}}{S_{\Omega,\mu}}, \quad R_{\lambda,\Omega}^\mu = \frac{S_{\lambda,\mu}}{S_{\Omega,\mu}}, \quad (4.9)$$

while for boundary conditions of type $m = \dot{\mu}$ they are

$$R_{(\lambda,\psi),\Omega}^{\dot{\mu}} = \frac{S_{(\lambda,\psi,0),(\dot{\mu},\varphi,1)}^{\mathcal{O}}}{S_{(\Omega,1,0),(\dot{\mu},\varphi,1)}^{\mathcal{O}}} = \psi \eta_\lambda^{-1} \frac{S_{\lambda,\dot{\mu}}^{(0)}}{S_{\Omega,\dot{\mu}}^{(0)}}, \quad R_{\lambda,\Omega}^{\dot{\mu}} = 0. \quad (4.10)$$

As usual the the *boundary state* $|B^m\rangle$ that is associated to a boundary condition m is a linear combination of boundary blocks $\tilde{\beta}_{(\lambda,\psi)}$ and $\tilde{\beta}_\lambda$, with coefficients given by the reflection coefficients:

$$\begin{aligned} |B^m\rangle &= \sum_{(\lambda,\psi)} C^m R_{(\lambda,\psi),\Omega}^m \tilde{\beta}_{(\lambda,\psi)} + \sum_\lambda C^m R_{\lambda,\Omega}^m \tilde{\beta}_\lambda \\ &= \sum_{(\lambda,\psi)} \tilde{S}_{(\lambda,\psi),m} \tilde{\beta}_{(\lambda,\psi)} + \sum_\lambda \tilde{S}_{\lambda,m} \tilde{\beta}_\lambda. \end{aligned} \quad (4.11)$$

Here the normalization $C^m := \tilde{S}_{(\Omega,1),m}$ ensures the correct normalization of the vacuum boundary field.

The boundary conditions come in two sets, corresponding to two automorphism types [11]. We first comment on the ones labelled by $m = \mu$. In this case the reflection coefficients are precisely the (generalized) quantum dimensions of the original theory. Indeed, these boundary conditions do not only preserve the orbifold subalgebra \mathfrak{A}^ω , but even the full chiral algebra \mathfrak{A} . It is well known [5] that such boundary conditions are governed by the fusion rules of the \mathfrak{A} -theory. In our description this behavior is recovered as follows. The subalgebra $\mathcal{C}_+(\mathfrak{A}^\omega)$ of $\mathcal{C}(\mathfrak{A}^\omega)$ that is spanned by the basis elements $\tilde{\Phi}_\lambda$ for the non-symmetric fields and by the sums $\tilde{\Phi}_{(\lambda,1)} + \tilde{\Phi}_{(\lambda,-1)}$ for the symmetric fields is an ideal of $\mathcal{C}(\mathfrak{A}^\omega)$, and it is in fact isomorphic to the fusion rule algebra of the WZW theory. Its irreducible representations are

labelled by orbits (with multiplicities) of fields in the untwisted sector, which in turn correspond to primary fields in the original WZW theory. Since according to the results of [9, 10] the boundary conditions that preserve the full bulk symmetry are precisely those that come from the untwisted sector of the orbifold, this is not surprising at all.

The second automorphism type of boundary conditions is provided by those which correspond to orbits in the twisted sector; they are thus labelled with dotted indices, $m = \dot{\mu}$. They do not preserve the full bulk symmetry, but only \mathfrak{A}^ω . We get as many symmetry breaking boundary conditions as there are symmetric primary fields in the original theory. The corresponding boundary states only involve the boundary blocks $\tilde{\beta}_{(\lambda, \psi)}$ of symmetric fields; moreover, due to the factor of ψ in (4.10) only the combinations $\tilde{\beta}_{(\lambda, 1)} - \tilde{\beta}_{(\lambda, -1)}$ appear. The latter are just twisted boundary blocks and reflect the breaking of the bulk symmetries. These boundary conditions are described by the irreducible representations of a complementary ideal $\mathcal{C}_-(\mathfrak{A}^\omega)$ of the classifying algebra, namely the one which is spanned by the differences $\tilde{\Phi}_{(\lambda, 1)} - \tilde{\Phi}_{(\lambda, -1)}$ of basis elements of $\mathcal{C}(\mathfrak{A}^\omega)$.

According to the general results of [1, 2] there exists a conjugation on the space of boundary conditions. When restricted to the boundary conditions that preserve the full bulk symmetry, this is nothing but the charge conjugation of the original theory. For the other type of boundary conditions the non-vanishing elements in the conjugation are

$$C_{\dot{\mu}, \dot{\mu}'}^{\mathcal{B}} \equiv \frac{1}{2} \sum_{\lambda, \psi} \tilde{S}_{(\lambda, \psi), \dot{\mu}} \tilde{S}_{(\lambda, \psi), \dot{\mu}'} = C_{\dot{\mu}, \dot{\mu}'}^{(1)}.$$

The derivation of this result uses only purely chiral information, namely the relation between $S^{(0)}$ and $S^{(1)}$ and the knowledge that $C^{(1)}$ is a permutation of order two.

Let us finally remark that for the determination of boundary conditions, we only need to know the S-matrix elements that involve at least one primary field from the untwisted sector. As a consequence, while the determination of the WZW orbifolds becomes technically more involved for groups G other than \mathbb{Z}_2 , the generalization of our results to boundary conditions that preserve any orbifold algebra under an arbitrary abelian group of automorphisms is straightforward.

A Appendix

Here we collect the pertinent properties of the twisted Theta functions (3.46) and their Weyl sums (3.45) that are needed in the main text. These results are actually of a more general validity than is needed for our present purposes. Namely, they hold for any pair of lattices L and \dot{L} of rank r that satisfy the following conditions. First, L must be a sublattice of \dot{L} , while \dot{L} in turn is a sublattice of $L/2$, i.e.,

$$L \subseteq \dot{L} \subseteq \frac{1}{2} L; \quad (\text{A.1})$$

this also implies that $\dot{L}^* \subseteq L^* \subseteq \dot{L}^*/2$. Second, both L and $\sqrt{2}\dot{L}$ are even lattices; when already \dot{L} is an even lattice, we put $l_{\dot{L}} := 1$, while otherwise we set $l_{\dot{L}} := 2$. Finally, we choose some subgroup \mathcal{W} of the group of isometries of these lattices. In the case of interest to us, $L = L_{\omega_o}^\vee$ and $\dot{L} = \dot{L}_{\omega_o}^\vee$ are associated to the coroot lattice of a finite-dimensional simple Lie algebra \mathfrak{g} in the manner described in subsection 3.3, so they have $r = r_{\omega_o}$, and $\mathcal{W} = W_{\omega_o}$ is the Weyl group of \mathfrak{g} .

In the case of inner automorphisms the required properties of the lattices are realized trivially, since we simply have $\dot{L}_{\omega_o}^\vee = L_{\omega_o}^\vee = L^\vee$, so that in particular $l_{\dot{L}} = 1$. For the case of outer automorphisms, on the other hand, we have $l_{\dot{L}} = 2$, except for $\mathfrak{g} = A_{2n}$. The latter case is in fact *not* covered by the setting described above, since the relevant lattice $\sqrt{2}\dot{L}_{\omega_o}^\vee = L_{\omega_o}^\vee/\sqrt{2} = Q(B_n)$ is then not even (see table (3.29) and formula (3.36)). Nevertheless this exceptional A_{2n} -case with outer automorphism can still be treated by essentially the same methods. Only a specific modification of the requirements to be imposed on shift vectors is necessary, see formula (A.11) below.

A.1 Twisted Theta functions

The twisted Theta functions of our interest are the lattice sums

$$\begin{aligned} \Theta_{\lambda, h}[s_1, s_2](\tau, z) &:= \sum_{\beta \in L} e^{2\pi i \tau (\lambda + h s_1 + h \beta, \lambda + h s_1 + h \beta) / 2h} e^{2\pi i (z + s_2, \lambda + h s_1 + h \beta)}, \\ \dot{\Theta}_{\dot{\lambda}, h}[s_1, s_2](\tau, z) &:= \sum_{\dot{\beta} \in \dot{L}} e^{2\pi i \tau (\dot{\lambda} + h s_1 + h \dot{\beta}, \dot{\lambda} + h s_1 + h \dot{\beta}) / 2h} e^{2\pi i (z + s_2, \dot{\lambda} + h s_1 + h \dot{\beta})}, \end{aligned} \quad (\text{A.2})$$

where τ is in the upper complex half plane, $z, s_1, s_2 \in L \otimes_{\mathbb{Z}} \mathbb{R}$, $h \in \mathbb{Z}_{>0}$, and $\lambda \in \dot{L}^*$, $\dot{\lambda} \in L^*$. Some properties of these functions are the following.

- $\Theta_{\lambda,h}[s_1, s_2]$ depends on λ only modulo hL . Also, it depends on λ and s_1 only via the combination $\lambda + hs_1$; but still it proves to be convenient to keep both parameters.
- For any automorphism (i.e., lattice preserving isometry) w of the lattice L one has

$$\Theta_{\lambda,h}[w(s_1), w(s_2)](\tau, z) = \Theta_{w^{-1}(\lambda),h}[s_1, s_2](\tau, w^{-1}(z)); \quad (\text{A.3})$$

in particular, the two functions coincide as functions of τ . (When L is the coroot lattice or root lattice of a finite-dimensional simple Lie algebra \mathfrak{g} , every automorphism is an element of the product of the Weyl group W of \mathfrak{g} with certain outer automorphisms.)

- Manifestly, s_1 is defined only modulo L and \dot{L} , respectively.
- Concerning shifts in s_2 , we have

$$\Theta_{\lambda,h}[s_1, s_2 + \mu](\tau, z) = e^{2\pi i(h(\mu, s_1) + (\mu, \lambda))} \Theta_{\lambda,h}[s_1, s_2](\tau, z) \quad (\text{A.4})$$

for every $\mu \in \dot{L}^*$. For $\mu = \beta \in L \supseteq L^*$ this reduces to

$$\Theta_{\lambda,h}[s_1, s_2 + \beta](\tau, z) = e^{2\pi i h(\beta, s_1)} \Theta_{\lambda,h}[s_1, s_2](\tau, z). \quad (\text{A.5})$$

- Next we study the T-transformation $T: \tau \mapsto \tau + 1, z \mapsto z$. Using the fact that $(\beta, \beta) \in 2\mathbb{Z}$ and $(\lambda, \beta) \in \mathbb{Z}$ for all $\beta \in L$ and all $\lambda \in \dot{L}^* \subseteq L^*$, we see that the twisted Theta functions get multiplied by a phase,

$$\Theta_{\lambda,h}[s_1, s_2](\tau + 1, z) = e^{2\pi i((\frac{\lambda, \lambda}{2h} - \frac{h}{2}(s_1, s_1))} \Theta_{\lambda,h}[s_1, s_1 + s_2](\tau, z), \quad (\text{A.6})$$

provided that

$$(s_1, \beta) \in \mathbb{Z} \quad \text{for all } \beta \in L. \quad (\text{A.7})$$

This nicely reflects the fact that the T-transformation is the element of the mapping class group of the torus that adds an a -cycle to the b -cycle of the torus.

- All these properties apply analogously to the twisted Theta functions $\dot{\Theta}_{\lambda,h}[s_1, s_2]$ that are defined with the lattice \dot{L} in place of L . However, in the case of the T-transformation we now have to take the l_L th power

in order for the twisted Theta function to acquire just a phase. More precisely, under the condition that

$$2(\dot{\lambda}, \dot{\beta}) \in \mathbb{Z} \quad (\text{A.8})$$

as well as

$$(\dot{\beta}, \dot{\beta}) + 2(s_1, \dot{\beta}) \in \mathbb{Z} \quad (\text{A.9})$$

for all $\dot{\beta} \in \dot{L}$ and all $\dot{\lambda} \in L^*$, it follows that

$$\dot{\Theta}_{\dot{\lambda}, h}[s_1, s_2](\tau + l_{\dot{L}}, z) = e^{2\pi i l_{\dot{L}}(\frac{(\dot{\lambda}, \dot{\lambda})}{2h} - \frac{h}{2}(s_1, s_1))} \dot{\Theta}_{\dot{\lambda}, h}[s_1, l_{\dot{L}}s_1 + s_2](\tau, z). \quad (\text{A.10})$$

The relation (A.8) is indeed satisfied, since $2\dot{\beta} \in L \equiv (L^*)^*$, while (A.9) reduces to $2(s_1, \dot{\beta}) \in \mathbb{Z}$ because the lattice $\sqrt{2}\dot{L}$ is even, which in turn is valid whenever (A.7) holds, again as a consequence of $2\dot{\beta} \in L$.

■ In the previous reasoning we have used explicitly the requirement that $\sqrt{2}L$ is an even lattice. As already noted above, this is not fulfilled in the case where $\mathfrak{g} = A_{2n}$ and $\omega_o = \omega_c$. Now the relevant lattice L is in this special case spanned by the elements $\dot{\alpha}^{(i)} = \alpha^{(i)} + \alpha^{(2n+1-i)}$, $i = 1, 2, \dots, n$, of the A_{2n} root lattice. Inspection shows that the condition (A.8) is then still satisfied, for the same reason as before, but condition (A.9) no longer reduces to (A.7). Rather, one must distinguish between the cases where $(\dot{\beta}, \dot{\beta})$ is integral and those where it lies in $\mathbb{Z} + 1/2$. In the former case the coefficient of $\dot{\alpha}^{(n)}$ in an expansion of $\dot{\beta}$ with respect to the $\dot{\alpha}^{(i)}$ is even, while in the latter case it is odd. It follows that the requirement on s_1 is that

$$s_1 = s_o + \frac{1}{2} \sum_{i=1}^n m_i (\Lambda_{(i)} + \Lambda_{(2n+1-i)}) \quad (\text{A.11})$$

with $m_i \in \mathbb{Z}$ for $i = 1, 2, \dots, n$ and

$$s_o = \frac{1}{4} (\Lambda_{(n)} + \Lambda_{(n+1)}), \quad (\text{A.12})$$

which is the A_{2n} -weight already encountered in (3.61).

■ To study also the S-transformation $S: \tau \mapsto -1/\tau$, $z \mapsto z/\tau$, we consider both types of functions together. Poisson resummation shows that

twisted Theta functions of one type transform into linear combinations of twisted Theta functions of the other type, according to

$$\begin{aligned}
 \Theta_{\lambda,h}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |L^*/L|^{-1/2} h^{-r/2} (-i\tau)^{r/2} e^{i\pi h(z,z)/\tau} \\
 &\quad e^{2\pi i h(s_1, s_2)} \sum_{\dot{\mu} \in L^*/h\dot{L}} e^{-2\pi i(\dot{\mu}, \lambda)/h} \dot{\Theta}_{\dot{\mu}, h}[s_2, -s_1](\tau, z) \\
 \dot{\Theta}_{\dot{\lambda}, h}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |\dot{L}^*/\dot{L}|^{-1/2} h^{-r/2} (-i\tau)^{r/2} e^{i\pi h(z,z)/\tau} \\
 &\quad e^{2\pi i h(s_1, s_2)} \sum_{\mu \in \dot{L}^*/hL} e^{-2\pi i(\mu, \dot{\lambda})/h} \Theta_{\mu, h}[s_2, -s_1](\tau, z).
 \end{aligned} \tag{A.13}$$

The behavior of the shift parameters is again in accordance with the action of the mapping class group element on the fundamental cycles. Notice that the factor $\exp(-2\pi i(\dot{\mu}, \lambda)/h)$ is single-valued when $\dot{\mu}$ is defined modulo $h\dot{L}$, because $\lambda \in \dot{L}^*$, and also when λ is defined modulo hL , because $\dot{\mu} \in L^*$; then the isomorphisms $\dot{L}^*/L \cong (L^*/\dot{L})^* \cong L^*/\dot{L}$ (compare (3.32)) imply that there are as many Θ -functions as $\dot{\Theta}$ -functions.

■ The matrices⁸

$$\begin{aligned}
 \mathcal{S}_{\lambda, \dot{\mu}}^{(0)} &= |L^*/L|^{-1/2} h^{-r/2} e^{2\pi i h(s_1, s_2)} e^{-2\pi i(\dot{\mu}, \lambda)/h}, \\
 \mathcal{S}_{\dot{\lambda}, \mu}^{(1)} &= |\dot{L}^*/\dot{L}|^{-1/2} h^{-r/2} e^{2\pi i h(s_1, s_2)} e^{-2\pi i(\mu, \dot{\lambda})/h}
 \end{aligned} \tag{A.14}$$

that arise in the S-transformation (A.13) are not unitary; rather,

$$\begin{aligned}
 \sum_{\dot{\mu} \in L^*/h\dot{L}} \mathcal{S}_{\lambda, \dot{\mu}}^{(0)} (\mathcal{S}_{\dot{\lambda}', \dot{\mu}}^{(0)})^* &= |\dot{L}/L|^{-1} \delta_{\lambda, \dot{\lambda}'}, \quad \sum_{\lambda \in \dot{L}^*/hL} \mathcal{S}_{\lambda, \dot{\mu}}^{(0)} (\mathcal{S}_{\dot{\mu}, \dot{\lambda}'}^{(0)})^* = |L^*/\dot{L}^*|^{-1} \delta_{\dot{\mu}, \dot{\mu}'}, \\
 \sum_{\dot{\mu} \in L^*/h\dot{L}} \mathcal{S}_{\dot{\mu}, \lambda}^{(1)} (\mathcal{S}_{\dot{\mu}, \dot{\lambda}'}^{(1)})^* &= |L^*/\dot{L}^*| \delta_{\lambda, \dot{\lambda}'}, \quad \sum_{\lambda \in \dot{L}^*/hL} \mathcal{S}_{\dot{\mu}, \lambda}^{(1)} (\mathcal{S}_{\dot{\mu}', \lambda}^{(1)})^* = |\dot{L}/L| \delta_{\dot{\mu}, \dot{\mu}'}.
 \end{aligned} \tag{A.15}$$

Because of $|L^*/\dot{L}^*| = |\dot{L}/L|$ (compare formula (3.33)), both for $\mathcal{S}^{(0)}$ and $\mathcal{S}^{(1)}$ the two relations come with the same factor as they should.

⁸ Just like for the ordinary characters, the further factor $e^{i\pi h(z,z)/\tau}$ gets absorbed by the inhomogeneous transformation $u \mapsto u - (z, z)/2\tau$ of a third variable u on which these functions should depend. For brevity that variable is suppressed throughout this paper.

A.2 Twisted Ξ -functions

Using the \mathcal{W} -character ε we now antisymmetrize the Θ - and $\dot{\Theta}$ -functions according to

$$\begin{aligned}\Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_2](\tau, z) &:= \sum_{w \in \mathcal{W}} \varepsilon(w) \Theta_{w(\lambda),h}[s_1, s_2](\tau, z), \\ \dot{\Xi}_{\dot{\lambda},h}^{\mathcal{W}}[s_1, s_2](\tau, z) &:= \sum_{w \in \mathcal{W}} \varepsilon(w) \dot{\Theta}_{w(\dot{\lambda}),h}[s_1, s_2](\tau, z).\end{aligned}\tag{A.16}$$

These functions inherit most of their properties from those of the twisted Theta functions. Let us list some of them.

- They are invariant under shifts by the lattice hL and $h\dot{L}$, respectively:

$$\begin{aligned}\Xi_{\lambda+h\beta}^{\mathcal{W}}[s_1, s_2] &= \Xi_{\lambda}^{\mathcal{W}}[s_1, s_2] \quad \text{for all } \beta \in L, \\ \dot{\Xi}_{\dot{\lambda}+h\dot{\beta}}^{\mathcal{W}}[s_1, s_2] &= \dot{\Xi}_{\dot{\lambda}}^{\mathcal{W}}[s_1, s_2] \quad \text{for all } \dot{\beta} \in \dot{L}.\end{aligned}\tag{A.17}$$

- They depend only on the combination $\lambda + hs_1$ and $\dot{\lambda} + h\dot{s}_1$, respectively. Thus as a parameter of $\Xi^{\mathcal{W}}$, s_1 can be regarded as being defined modulo L , and as a parameter of $\dot{\Xi}^{\mathcal{W}}$, modulo \dot{L} ; still, in both cases we will keep both arguments.

- Because of (A.4) we have

$$\begin{aligned}\Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_2 + \mu](\tau, z) &= e^{2\pi i h(\mu, s_1)} \sum_{w \in \mathcal{W}} \varepsilon(w) e^{2\pi i(\mu, w(\lambda))} \Theta_{w(\lambda),h}[s_1, s_2](\tau, z), \\ \dot{\Xi}_{\dot{\lambda},h}^{\mathcal{W}}[s_1, s_2 + \dot{\mu}](\tau, z) &= e^{2\pi i h(\dot{\mu}, s_1)} \sum_{w \in \mathcal{W}} \varepsilon(w) e^{2\pi i(\dot{\mu}, w(\dot{\lambda}))} \dot{\Theta}_{w(\dot{\lambda}),h}[s_1, s_2](\tau, z)\end{aligned}\tag{A.18}$$

for all $\mu \in \dot{L}^*$ and $\dot{\mu} \in L^*$.

- At $z=0$ those $\Xi^{\mathcal{W}}$ -functions for which s_2 lies in L have integral coefficients in the expansion in powers of $q = \exp(2\pi i \tau)$.
- As the lattices L and \dot{L} are mapped to themselves under \mathcal{W} , the $\Xi^{\mathcal{W}}$ - and $\dot{\Xi}^{\mathcal{W}}$ -functions are \mathcal{W} -odd (that is, ε -twisted):

$$\begin{aligned}\Xi_{w(\lambda)}^{\mathcal{W}}[s_1, s_2] &= \varepsilon(w) \Xi_{\lambda}^{\mathcal{W}}[s_1, s_2] \quad \text{for all } w \in \mathcal{W}, \\ \dot{\Xi}_{w(\dot{\lambda})}^{\mathcal{W}}[s_1, s_2] &= \varepsilon(w) \dot{\Xi}_{\dot{\lambda}}^{\mathcal{W}}[s_1, s_2] \quad \text{for all } w \in \mathcal{W}.\end{aligned}\tag{A.19}$$

■ Owing to (A.6), (A.10) and (A.18), respectively, under the above-mentioned restrictions on the shift vector s_1 we have

$$\begin{aligned}\Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_2](\tau+1, z) &= e^{2\pi i(\frac{(\lambda,\lambda)}{2h}-\frac{h}{2}(s_1,s_1))} \Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_1+s_2](\tau, z), \\ \dot{\Xi}_{\lambda,h}^{\mathcal{W}}[s_1, s_2](\tau+l_{\dot{L}}, z) &= e^{2\pi i l_{\dot{L}}(\frac{(\lambda,\lambda)}{2h}-\frac{h}{2}(s_1,s_1))} \dot{\Xi}_{\lambda,h}^{\mathcal{W}}[s_1, l_{\dot{L}}s_1+s_2](\tau, z)\end{aligned}\tag{A.20}$$

for the behavior under the T-operation.

A.3 The S-transformation

The transformation of $\Xi^{\mathcal{W}}$ and $\dot{\Xi}^{\mathcal{W}}$ under the S-operation could still be discussed for the general situation studied so far, provided that appropriate properties of the group \mathcal{W} are imposed. For brevity we now restrict, however, our attention to the specific case that is of interest in the main text. Thus in particular $\mathcal{W} = W_{\omega_0}$ is the Weyl group of the orbit Lie algebra of \mathfrak{g} , while $\varepsilon = \varepsilon_{\omega_0}$ is the sign function of the Coxeter group W_{ω_0} . Combining the result (A.13) with the fact that the group W_{ω_0} acts on all four lattices by isometries and with the representation property of ε , it follows that in this case we have

$$\begin{aligned}\Xi_{\lambda,h}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) \\ = |L_{\omega_0}^{\vee*}/L_{\omega_0}^{\vee}|^{-1/2} h^{-r/2} (-i\tau)^{r/2} e^{i\pi h(z,z)/\tau} e^{2\pi i h(s_1,s_2)} \\ \sum_{\dot{\mu} \in \dot{P}_h^{\omega_0++}} \sum_{w \in W_{\omega_0}} \varepsilon_{\omega_0}(w) e^{-2\pi i(\dot{\mu}, w(\lambda))/h} \dot{\Xi}_{\dot{\mu},h}^{\omega_0}[s_2, -s_1](\tau, z)\end{aligned}\tag{A.21}$$

and

$$\begin{aligned}\dot{\Xi}_{\lambda,h}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) \\ = |\dot{L}_{\omega_0}^{\vee*}/\dot{L}_{\omega_0}^{\vee}|^{-1/2} h^{-r/2} (-i\tau)^{r/2} e^{i\pi h(z,z)/\tau} e^{2\pi i h(s_1,s_2)} \\ \sum_{\mu \in P_h^{\omega_0++}} \sum_{w \in W_{\omega_0}} \varepsilon_{\omega_0}(w) e^{-2\pi i(\mu, w(\lambda))/h} \Xi_{\mu,h}^{\omega_0}[s_2, -s_1](\tau, z),\end{aligned}\tag{A.22}$$

respectively. Moreover, we have $P_{\mathfrak{g}^\vee}^{\omega_0++} = \{\rho\}$ and $\dot{P}_{\mathfrak{g}^\vee}^{\omega_0++} = \{\dot{\rho}\}$, so that according to (A.21) and (A.22) we find

$$\begin{aligned} \Xi_{\rho, \mathfrak{g}^\vee}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |L_{\omega_0}^{\vee*}/L_{\omega_0}^\vee|^{-1/2} (g^\vee)^{-r_{\omega_0}/2} (-i\tau)^{r_{\omega_0}/2} e^{i\pi g^\vee(z, z)/\tau} \\ &\quad e^{2\pi i g^\vee(s_1, s_2)} \kappa_{\omega_0} \dot{\Xi}_{\dot{\rho}, \mathfrak{g}^\vee}^{\omega_0}[s_2, -s_1](\tau, z), \\ \dot{\Xi}_{\dot{\rho}, \mathfrak{g}^\vee}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |\dot{L}_{\omega_0}^{\vee*}/\dot{L}_{\omega_0}^\vee|^{-1/2} (g^\vee)^{-r_{\omega_0}/2} (-i\tau)^{r_{\omega_0}/2} e^{i\pi g^\vee(z, z)/\tau} \\ &\quad e^{2\pi i g^\vee(s_1, s_2)} \kappa_{\omega_0} \Xi_{\rho, \mathfrak{g}^\vee}^{\omega_0}[s_2, -s_1](\tau, z), \end{aligned} \quad (\text{A.23})$$

where

$$\kappa_{\omega_0} := \sum_{w \in W_{\omega_0}} \varepsilon_{\omega_0}(w) e^{-2\pi i(\dot{\rho}, w(\rho))/g^\vee}. \quad (\text{A.24})$$

The absolute value of the number κ_{ω_0} follows from the general orthogonality relations (A.15):

$$|\kappa_{\omega_0}| = |L_{\omega_0}^{\vee*}/g^\vee \dot{L}_{\omega_0}^\vee|^{-1/2} \equiv |\dot{L}_{\omega_0}^{\vee*}/g^\vee L_{\omega_0}^\vee|^{-1/2}. \quad (\text{A.25})$$

To determine its phase, we employ the denominator identity of the finite-dimensional orbit Lie algebra to find

$$\kappa_{\omega_0} = \prod_{\dot{\alpha} \in \Delta_{\omega_0}} \left(-2i \sin\left(\frac{\pi(\dot{\rho}, \dot{\alpha})}{g^\vee}\right)\right) \quad (\text{A.26})$$

with Δ_{ω_0} the set of symmetric \mathfrak{g} -roots, which implies that $\kappa_{\omega_0} \cdot i^{|\Delta_{\omega_0}|}$ is a positive real number. Collecting these results, we can conclude that

$$\begin{aligned} \Xi_{\rho, \mathfrak{g}^\vee}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |\dot{L}_{\omega_0}^\vee/L_{\omega_0}^\vee|^{-1/2} (-i)^{d_{\omega_0}/2} \tau^{r_{\omega_0}/2} e^{i\pi g^\vee(z, z)/\tau} \\ &\quad e^{2\pi i g^\vee(s_1, s_2)} \dot{\Xi}_{\dot{\rho}, \mathfrak{g}^\vee}^{\omega_0}[s_2, -s_1](\tau, z), \\ \dot{\Xi}_{\dot{\rho}, \mathfrak{g}^\vee}^{\omega_0}[s_1, s_2]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= |\dot{L}_{\omega_0}^\vee/L_{\omega_0}^\vee|^{1/2} (-i)^{d_{\omega_0}/2} \tau^{r_{\omega_0}/2} e^{i\pi g^\vee(z, z)/\tau} \\ &\quad e^{2\pi i g^\vee(s_1, s_2)} \Xi_{\rho, \mathfrak{g}^\vee}^{\omega_0}[s_2, -s_1](\tau, z), \end{aligned} \quad (\text{A.27})$$

where d_{ω_0} is the dimension of the finite-dimensional orbit Lie algebra.

Finally we consider the relation (A.18) for the special case where the vectors $\mu \in \dot{L}^*$ and $\dot{\mu} \in L^*$ lie in addition in the coweight lattice of \mathfrak{g} . In

this case we have $(\mu, w(\lambda)) = (\mu, \lambda) \bmod \mathbb{Z}$ and $(\dot{\mu}, w(\lambda)) = (\dot{\mu}, \lambda) \bmod \mathbb{Z}$ for all $w \in W$, and hence

$$\begin{aligned}\Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_2 + \mu](\tau, z) &= e^{2\pi i(h(\mu, s_1) + (\mu, \lambda))} \Xi_{\lambda,h}^{\mathcal{W}}[s_1, s_2](\tau, z), \\ \dot{\Xi}_{\lambda,h}^{\mathcal{W}}[s_1, s_2 + \dot{\mu}](\tau, z) &= e^{2\pi i(h(\dot{\mu}, s_1) + (\dot{\mu}, \dot{\lambda}))} \dot{\Xi}_{\lambda,h}^{\mathcal{W}}[s_1, s_2](\tau, z).\end{aligned}\tag{A.28}$$

Acknowledgement

C.S. would like to thank UC Berkeley for hospitality while part of this work was done.

References

- [1] F.A. Bais and P. Bouwknegt, *A classification of subgroup truncations of the bosonic string*, Nucl. Phys., **B 279** (1987), 561.
- [2] P. Bantay, *The Frobenius–Schur indicator in conformal field theory*, Phys. Lett., **B 394** (1997), 87.
- [3] A. Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, in: *Hirzebruch 65 Conference on Algebraic Geometry* [Israel Math. Conf. Proc. 9], M. Teicher, ed. (Bar-Ilan University, Ramat Gan 1996), p. 75.
- [4] L.A. Borisov, M.B. Halpern, and C. Schweigert, *Systematic approach to cyclic orbifolds*, Int. J. Mod. Phys., **A 13** (1998), 125.
- [5] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys., **B 324** (1989), 581.
- [6] C. Dong, H. Li, and G. Mason, *Twisted representations of vertex operator algebras*, Math. Annal., **310** (1998), 571.
- [7] J.K. Freericks and M.B. Halpern, *Conformal deformation by the currents of affine g* , Ann. Phys., **188** (1988), 258. [*ibid.*, **190** (1989), 212, *Erratum*]
- [8] J. Fuchs, U. Ray, and C. Schweigert, *Some automorphisms of Generalized Kac–Moody algebras*, J. Algebra, **191** (1997), 518.
- [9] J. Fuchs and C. Schweigert, *Orbifold analysis of broken bulk symmetries*, Phys. Lett., **B 447** (1999), 266.

- [10] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries I. General theory*, Nucl. Phys., **B 558** (1999), 419; hep-th/9902132.
- [11] J. Fuchs and C. Schweigert, *Branes: from free fields to general backgrounds*, Nucl. Phys., **B 530** (1998), 99.
- [12] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, Phys. Lett., **B 414** (1997), 251.
- [13] J. Fuchs and C. Schweigert, *The action of outer automorphisms on bundles of chiral blocks*, preprint hep-th/9805026. Commun. Math. Phys., **206** (1999), 691.
- [14] J. Fuchs, A.N. Schellekens, and C. Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Commun. Math. Phys., **180** (1996), 39.
- [15] J. Fuchs and D. Gepner, *On the connection between WZW and free field theories*, Nucl. Phys., **B 294** (1987), 30.
- [16] N. Gorman, L. O’Raifeartaigh, and W. McGlinn, *Cartan preserving automorphisms of untwisted and twisted Kac-Moody algebras*, J. Math. Phys., **30** (1989), 1921.
- [17] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* [Pure and Applied Mathematics, **80**] (Academic Press, New York, 1978).
- [18] V.G. Kac, *Infinite-dimensional Lie Algebras*, third edition (Cambridge University Press, Cambridge, 1990).
- [19] V.G. Kac and I.T. Todorov, *Affine orbifolds and rational conformal field theory extensions of $W_{1+\infty}$* , Commun. Math. Phys., **190** (1997), 57.
- [20] W. Lerche, C. Vafa, and N.P. Warner, *Chiral rings in $N=2$ superconformal theories*, Nucl. Phys., **B 324** (1989), 427.
- [21] B. Mühlherr, *Coxeter groups in Coxeter groups*, in: *Finite Geometry and Combinatorics* [London Math. Soc. Lecture Note Series # 191], F. De Clerck et al., eds. (Cambridge University Press, Cambridge, 1993), p. 277.
- [22] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *The open descendants of non-diagonal $SU(2)$ WZW models*, Phys. Lett., **B 356** (1995), 230.
- [23] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *Planar duality in $SU(2)$ WZW models*, Phys. Lett., **B 354** (1995), 279.

- [24] A.N. Schellekens and N.P. Warner, *Conformal subalgebras of Kac-Moody algebras*, Phys. Rev., **D 34** (1986), 3092.
- [25] A.N. Schellekens and S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, Nucl. Phys., **B 327** (1989), 673.
- [26] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants, and fixed points*, Int. J. Mod. Phys., **A 5** (1990), 2903.
- [27] A. Tsuchiya, K. Ueno, and H. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Studies in Pure Math., **19** (1989), 459.