Adv. Theor. Math. Phys. **4** (2000) 747–789

Of McKay Correspondence, Non-linear Sigma-model and Conformal Field Theory

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Abstract

The ubiquitous A-D-E classification has induced many proposals of often mysterious correspondences both in mathematics and physics. The mathematics side includes quiver theory and the McKay Correspondence which relates finite group representation theory to Lie algebras as well as crepant resolutions

e-print archive: http://xxx.lanl.gov/hep-th/9903056
Research supported in part by the NSF and the U.S. Department of Energy under cooperative research agreement #DE-FC02-94ER40818.

of Gorenstein singularities. On the physics side, we have the graph-theoretic classification of the modular invariants of WZW models, as well as the relation between the string theory nonlinear σ-models and Landau-Ginzburg orbifolds. We here propose a unification scheme which naturally incorporates all these correspondences of the A-D-E type in two complex dimensions. An intricate web of inter-relations is constructed, providing a possible guideline to establish new directions of research or alternate pathways to the standing problems in higher dimensions.

1 Introduction

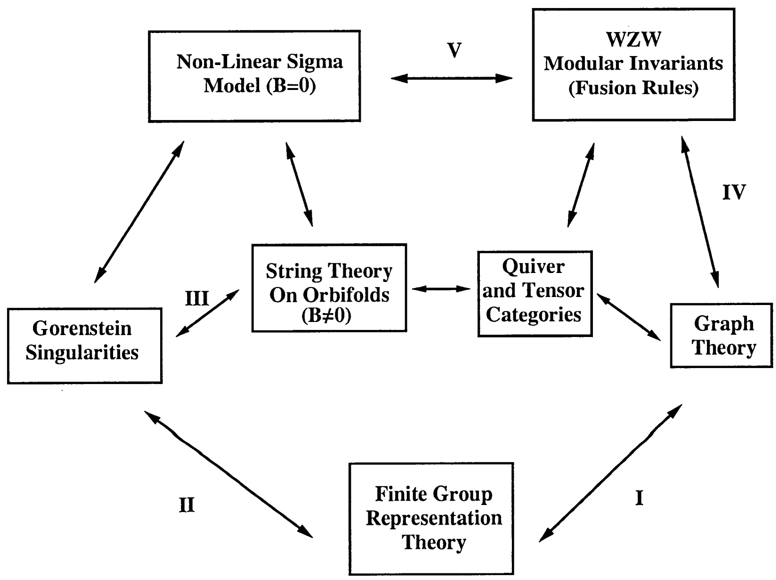

Figure 1: The Myriad of Correspondences: it is the purpose of this paper to elucidate these inter-relations in 2-dimensions, so as to motivate a similar coherent picture in higher dimensions. Most of the subsectors in this picture have been studied separately by mathematicians and physicists, but they are in fact not as disparate as they are guised.

This paper reviews the known facts about the various A-D-E classifications that arise in mathematics and string theory and organizes them into a unified picture. This picture serves as a guide for our on-going work in higher dimensions and naturally incorporates diverse concepts in mathematics.

In the course of their research on supersymmetric Yang-Mills theories resulting from the type IIB D-branes on orbifold singularities [1], as prompted by collective works in constructing (conformal) gauge theories (e.g., [4] [5] [6] and references therein), it was conjectured by Hanany and He that there may exist a McKay-type correspondence between the bifundamental matter content and the modular invariant partition functions of the Wess-Zumino-Witten (WZW) conformal field theory. Phrased in another way, the correspondence, if true, would relate the Clebsch-Gordan coefficients for tensor products of the irreducible representations of finite subgroups of $SU(n)$ with the integrable characters for the affine algebras $\widehat{SU(n)}_k$ of some integral weight k.

Such a relation has been well-studied in the case of $n = 2$ and it falls into an *A-D-E* classification scheme [7, 9, 11]. Evidences for what might occur in the case of $n = 3$ were presented in [1] by computing the Clebsch-Gordan coefficients extensively for the subgroups of $SU(3)$. Indications from the lattice integrable model perspective were given in [13].

The natural question to pose is why there should be such correspondences. Indeed, why should there be such an intricate chain of connections among string theory on orbifolds, finite representation theory, graph theory, affine characters and WZW modular invariants? In this paper, we hope to propose a unified quest to answer this question from the point of view of the conformal field theory description of Gorenstein singularities. We also observe that category theory seems to prove a common basis for all these theories.

We begin in two dimensions, where there have been numerous independent works in the past few decades in both mathematics and physics to establish various correspondences. In this case, the all-permeating theme is the *A-D-E* classification. In particular, there is the original McKay's correspondence between finite subgroups of $SU(2)$ and the *A-D-E* Dynkin diagrams [2] to which we henceforth refer as the *Algebraic McKay Correspondence.* On the geometry side, the representation rings of these groups were related to the Groethendieck (cohomology) rings of the resolved manifolds constructed from the Gorenstein singularity of the respective groups [38]; we shall refer to this as the *Geometric McKay Correspondence*. Now from physics, studies in conformal field theory (CFT) have prompted many beautiful connections among graph theory, fusion algebra, and modular invariants [7, 9, 11, 20, 21]. The classification of the modular invariant partition function associated with $\widehat{SU(2)}$ Wess-Zumino-Witten (WZW) models also mysteriously falls into an *A-D-E* type [8]. There have been some recent attempts to explain this seeming accident from the supersymmetric field theory and brane configurations perspective [15, 17]. In this paper we push from the direction of the Geometric McKay Correspondence and see how Calabi-Yau (CY) non-linear sigma models constructed on the Gorenstein singularities associated with the finite groups may be related to Kazama-Suzuki coset models [20, 21, 22, 23, 27, 28], which in turn can be related to the WZW models. This link would provide a natural setting for the emergence of the *A-D-E* classification of the modular invariants. In due course, we will review and establish a catalog of inter-relations, whereby forming a complex web and unifying many independently noted correspondences. Moreover, we find a common theme of categorical axioms that all of these theories seem to satisfy and suggest why the *A-D-E* classification and its extensions arise so naturally. This web, presented in Figure 1, is the central idea of our paper. Most of the correspondences in Figure 1 actually have been discussed in the string theory literature although not all at once in a unified manner with a mathematical tint.

Our purpose is two-fold. Firstly, we shall show that tracing through the arrows in Figure 1 may help to enlighten the links that may seem accidental. Moreover, and perhaps more importantly, we propose that this program may be extended beyond two dimensions and hence beyond *A-D-E*. Indeed, algebraic geometers have done extensive research in generalizing McKay's original correspondence to Gorenstein singularities of dimension greater than 2 ([37] to [44]); many standing conjectures exist in this respect. On the other hand, there is the conjecture mentioned above regarding the $\widehat{SU(n)}_k$ WZW and the subgroups of $SU(n)$ in general. It is our hope that Figure 1 remains valid for $n > 2$ so that these conjectures may be attacked by the new pathways we propose. We require and sincerely hope for the collaborative effort of many experts in mathematics and physics who may take interest in this attempt to unify these various connections.

The outline of the paper follows the arrows drawn in Figure 1. We begin in §2 by summarizing the ubiquitous A-D-E classifications, and §3 will be devoted to clarifying these A-D-E links, while bearing in mind how such ubiquity may permeate to higher dimensions. It will be divided in to the following subsections:

- I. The link between representation theory for finite groups and quiver graph theories (Algebraic McKay);
- II. The link between finite groups and crepant resolutions of Gorenstein singularities (Geometric McKay);
- III. The link between resolved Gorenstein singularities, Calabi-Yau manifolds and chiral rings for the associated non-linear sigma model (Stringy Gorenstein resolution);
- IV. The link between quiver graph theory for finite groups and WZW modular invariants in CFT, as discovered in the study of of string orbifold theory (Conjecture in [1]);

and finally, to complete the cycle of correspondences,

- V. The link between the singular geometry and its conformal field theory description provided by an orbifoldized coset construction which contains the WZW theory.

In §4 we discuss arrow V which fills the gap between mathematics and physics, explaining why WZW models have the magical properties that are so closely related to the discrete subgroups of the unitary groups as well as to geometry. From all these links arises §6 which consists of our conjecture that there exists a conformal field theory description of the Gorenstein singularities for higher dimensions, encoding the relevant information about the discrete groups and the cohomology ring. In §5, we hint at how these vastly different fields may have similar structures by appealing to the so-called ribbon and quiver categories.

Finally in the concluding remarks of §7, we discuss the projection for future labors.

We here transcribe our observations with the hope they would spark a renewed interest in the study of McKay correspondence under a possibly new light of CFT, and vice versa. We hope that Figure 1 will open up many interesting and exciting pathways of research and that not only some existing conjectures may be solved by new methods of attack, but also further beautiful observations could be made.

Notations and Nomenclatures

We put a $\tilde{}$ over a singular variety to denote its resolved geometry. By dimension we mean complex dimension unless stated otherwise. Also by "representation ring of Γ," we mean the ring formed by the tensor product decompositions of the irreducible representations of Γ. The capital Roman numerals, I–V, in front of the section headings correspond to the arrows in Figure 1.

2 Ubiquity of A-D-E Classifications

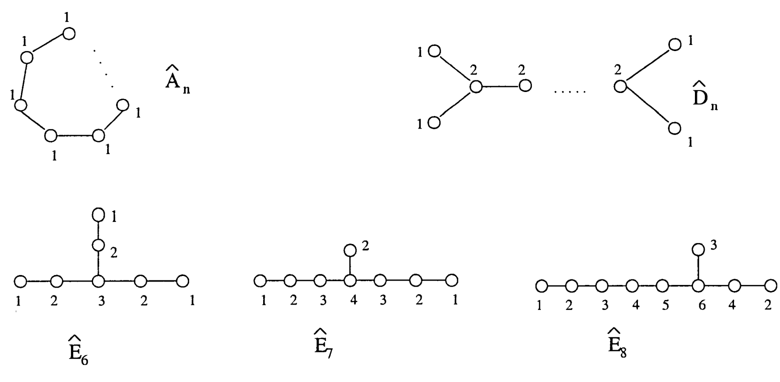

Figure 2: The Affine Dynkin Diagrams and Labels.

In this section, we summarize the appearance of the A-D-E classifications in physics and mathematics and their commonalities.

It is now well-known that the complexity of particular algebraic and

	Theory	Nodes	Matrices
(a)	Finite Subgroup Γ of $SU(2)$	Irreducible Representations	Clebsch-Gordan Coefficients
(b)	Simple Lie algebra of type ADE	Simple Roots	Extended Cartan matrix
(c)	Quiver Dynkin Diagrams	Dynkin Labels	Adjacency Matrix
(d)	Minimal Resolution $X \to \mathbb{C}^2/\Gamma$	Irreducible Components of the Exceptional Divisor (Basis of $H_2(X, \mathbb{Z})$)	Intersection Matrix
(e)	$\widehat{SU(2)}_k$ WZW Model	Modular Invariants / WZW Primary Operators	Fusion Coefficients
(f)	Landau-Ginzburg	Chiral Primary Operators	Chiral Ring Coefficients
(g)	CY Nonlinear Sigma Model	Twisted Fields	Correlation Functions

Table 1: A-D-E Correspondences in 2-dimensions. The same graphs and their affine extensions appear in different theories.

geometric structures can often be organized into classification schemes of the A-D-E type. The first hint of this structure began in the 1884 work of F. Klein in which he classified the discrete subgroups Γ of $SU(2)$ [3]. These were noted to be in 1-1 correspondence with the Platonic solids in $\mathbb{R}^3$, and with some foresight, we write them as:

- type A: the cyclic groups (the regular polygons);
- type D: the binary dihedral groups (the regular dihedrons) and
- type E: the binary tetrahedral (the tetrahedron), octahedral (the cube and the octahedron) and icosahedral (the dodecahedron and the icosahedron) groups,

where we have placed in parenthesis next to each group the geometrical shape for which it is the double cover of the symmetry group.

The ubiquity of Klein's original hint has persisted till the present day. The A-D-E scheme has manifested itself in such diverse fields as representation theory of finite groups, quiver graph theory, Lie algebra theory, algebraic geometry, conformal field theory, and string theory. It will be the intent of the next section to explain the details of the correspondences appearing in Table 1, and we will subsequently propose their extensions in the remainder of the paper.

3 The Arrows of Figure 1.

In this section, we explain the arrows appearing in Figure 1. We verify that there are compelling evidences in favor of the picture for the case of $2_{\mathbb{C}}$-dimensions, and we will propose its generalization to higher dimensions in the subsequent sections, hoping that it will lead to new insights on the McKay correspondence as well as conformal field theory.

3.1 (I) The Algebraic McKay Correspondence

In the full spirit of the omnipresent A-D-E classification, it has been noticed in 1980 by J. McKay that there exists a remarkable correspondence between the discrete subgroups of $SU(2)$ and the affine Dynkin graphs [2]. Indeed, this is why we have labeled the subgroups in the manner we have done.

Definition 3.1. For a finite group Γ, let $\{r_i\}$ be its set of irreducible representations (irreps), then we define the coefficients m_{ij}^k appearing in

$$r_k \otimes r_i = \bigoplus_j m_{ij}^k r_j \tag{3.1}$$

to be the **Clebsch-Gordan coefficients** of Γ.

For $\Gamma \subset SU(2)$ McKay chose a fixed (not necessarily irreducible) representation R in lieu of general k in 3.1 and defined matrices m_{ij}^R. He has noted that up to automorphism, there always exists a unique 2-dimensional representation, which for type A is the tensor sum of 2 dual

1-dimensional irreps and for all others the self-conjugate 2-dimensional irrep. It is this $R = 2$ which we choose and simply write the matrix as m_{ij}. The remarkable observation of McKay can be summarized in the following theorem; the original proof was on a case-to-case basis and Steinberg gave a unified proof in 1981 [2].

Theorem 3.1 (McKay-Steinberg). *For $\Gamma = A, D, E$, the matrix m_{ij} is $2I$ minus the Cartan matrix of the affine extensions of the respective simply-laced simple Dynkin diagrams $\widehat{A}, \widehat{D}$ and $\widehat{E}$, treated as undirected C_2-graphs (i.e., maximal eigenvalue of the adjacency matrix is 2).*

Moreover, the Dynkin labels of the nodes of the affine Dynkin diagrams are precisely the dimensions of the irreps. Given a discrete subgroup $\Gamma \subset SU(2)$, there thus exists a Dynkin diagram that encodes the essential information about the representation ring of Γ. Indeed the number of nodes should equal to the number of irreps and thus by a rudimentary fact in finite representation theory, subsequently equals the number of conjugacy classes of Γ. Furthermore, if we remove the node corresponding to the trivial 1-dimensional (principal) representation, we obtain the regular A-D-E Dynkin diagrams. We present these facts in the following diagram:

Clebsch-Gordan Coefficients for $\Gamma = A, D, E$	$\longleftrightarrow$	Dynkin Diagram of $\widehat{A}, \widehat{D}, \widehat{E}$	$\longleftrightarrow$	Cartan matrix and dual Coxeter labels of $\widehat{A}, \widehat{D}, \widehat{E}$

This is Arrow I of Figure 1.

Proofs and extension of McKay's results from geometric perspectives of this originally combinatorial/graph-theoretic theorem soon followed; they caused fervent activities in both algebraic geometry and string theory (see e.g., [30, 37, 38]). Let us first turn to the former.

3.2 (II) The Geometric McKay Correspondence

In this section, we are interested in crepant resolutions of Gorenstein quotient singularities.

Definition 3.2. The singularities of $\mathbb{C}^n/\Gamma$ for $\Gamma \subset GL(n,\mathbb{C})$ are called **Gorenstein** if there exists a nowhere-vanishing holomorphic n-form[1] on regular points.

Restricting Γ to $SU(n)$ would guarantee that the quotient singularities are Gorenstein.

Definition 3.3. We say that a smooth variety $\widetilde{M}$ is a **crepant** resolution of a singular variety M if there exists a birational morphism $\pi : \widetilde{M} \to M$ such that the canonical sheaves $\mathcal{K}_M$ and $\mathcal{K}_{\widetilde{M}}$ are the same, or more precisely, if $\pi^*(\mathcal{K}_M) = \mathcal{K}_{\widetilde{M}}$.

For $n \leq 3$, Gorenstein singularities always admit crepant resolutions [38]. On the other hand, in dimensions greater than 3, there are known examples of terminal Gorenstein singularities which do not admit crepant resolutions. It is believed, however, that when the order of Γ is sufficiently larger than n, there exist crepant resolutions for most of the groups.

The traditional A-D-E classification is relevant in studying the discrete subgroups of $SU(2)$ and resolutions of Gorenstein singularities in two complex-dimensions. Since we can choose an invariant Hermitian metric on $\mathbb{C}^2$, finite subgroups of $GL(2,\mathbb{C})$ and $SL(2,\mathbb{C})$ are conjugate to finite subgroups of $U(2)$ and $SU(2)$, respectively. Here, motivated by the string compactification on manifolds of trivial canonical bundle, we consider the linear actions of non-trivial discrete subgroups Γ of $SU(2)$ on $\mathbb{C}^2$. Such quotient spaces $M = \mathbb{C}^2/\Gamma$, called *orbifolds*, have fixed points which are isolated Gorenstein singularities of the A-D-E type studied by Felix Klein.

As discussed in the previous sub-section, McKay[2] has observed a 1-1 correspondence between the non-identity conjugacy classes of discrete subgroups of $SU(2)$ and the Dynkin diagrams of A-D-E simply-

[1] Gorenstein singularities thus provide local models of singularities on Calabi-Yau manifolds.

laced Lie algebras, and this relation in turn provides an indirect correspondence between the orbifold singularities of M and the Dynkin diagrams. In fact, there exists a direct geometric correspondence between the crepant resolutions of M and the Dynkin diagrams. Classical theorems in algebraic geometry tell us that there exists a unique crepant resolution $(\widetilde{M}, \pi)$ of the Gorenstein singularity of M for all $\Gamma \subset SU(2)$. Furthermore, the exceptional divisor $E = \pi^{-1}(0)$ is a compact, connected union of irreducible $1_{\mathbb{C}}$-dimensional curves of genus zero[2] such that their intersection matrix is represented by the simply-laced Dynkin diagram associated to Γ. More precisely, each node of the diagram corresponds to an irreducible $\mathbb{P}^1$, and the intersection matrix is negative of the Cartan matrix of the Dynkin diagram such that two $\mathbb{P}^1$'s intersect transversely at one point if and only if the two nodes are connected by a line in the diagram. In particular, we see that the curves have self-intersection numbers -2 which exhibits the singular nature of the orbifold upon blowing them down. Simple consideration shows that these curves form a basis of the homology group $H_2(\widetilde{M}, \mathbb{Z})$ which is seen to coincide with the root lattice of the associated Dynkin diagram by the above identification. Now, combined with the algebraic McKay correspondence, this crepant resolution picture yields a 1-1 correspondence between the basis of $H_2(\widetilde{M}, \mathbb{Z})$ and the non-identity conjugacy classes of Γ. We recapitulate the above discussion in the following diagram:

$H_2(\widetilde{M}, \mathbb{Z})$ of the blow-up	$\longleftrightarrow$	Dynkin Diagram of Γ	$\longleftrightarrow$	Non-identity Conjugacy Classes of Γ

This is Arrow II in Figure 1. Note incidentally that one can think of irreducible representations as being dual to conjugacy classes and hence as basis of $H^2(\widetilde{M}, \mathbb{Z})$. This poses a subtle question of which correspondence is more natural, but we will ignore such issues in our discussions.

It turns out that M is not only analytic but also algebraic; that is, M is isomorphic to $f^{-1}(0)$, where $f : \mathbb{C}^3 \to \mathbb{C}$ is one of the polynomials in Table 2 depending on Γ. The orbifolds defined by the zero-loci of the polynomials are commonly referred to as the singular ALE spaces.

[2] We will refer to them as $\mathbb{P}^1$ blow-ups.

$f(x,y,z)$	Subgroup Γ	Order of Γ
$x^2+y^2+z^{k+1}$	A_k Cyclic	$k+1$
$x^2+y^2z+z^{k-1}$	D_k Binary Dihedral	$4(k-2)$
$x^2+y^4+z^3$	E_6 Binary Tetrahedral	24
$x^2+y^3z+z^3$	E_7 Binary Octahedral	48
$x^2+y^5+z^3$	E_8 Binary Icosahedral	120

Table 2: Algebraic Surfaces with Quotient Singularities.

3.3 (II, III) McKay Correspondence and SCFT

One of the first relevance of A-D-E series in conformal field theory appeared in attempts to classify $N=2$ superconformal field theories (SCFT) with central charge $c<3$ [20]. Furthermore, the exact forms of the A-D-E polynomials in Table 2 appeared in a similar attempt to classify certain classes of $N=2$ SCFT in terms of Landau-Ginzburg (LG) models. The LG super-potentials were precisely classified by the polynomials, and the chiral ring and quantum numbers were computed with applications of singularity theory [23]. The LG theories which realize coset models would appear again in this paper to link the WZW to geometry.

In this subsection, we review how string theory, when the B-field is non-vanishing, resolves the orbifold singularity and how it encodes the information about the cohomology of the resolved manifold. Subsequently, we will consider the singular limit of the conformal field theory on orbifolds by turning off the B-field, and we will argue that, in this singular limit, the $\widehat{SU(2)}_k$ WZW fusion ring inherits the information about the cohomology ring from the smooth theory.

3.3.1 Orbifold Resolutions and Cohomology Classes

Our discussion here will be general and not restricted to $n = 2$. Many remarkable features of string theory stem from the fact that we can "pull-back" much of the physics on the target space to the world-sheet, and as a result, the resulting world-sheet conformal field theory somehow encodes the geometry of the target space. One example is that CFT is often[3] insensitive to Gorenstein singularities and quantum effects revolve the singularity so that the CFT is smooth. More precisely, Aspinwall [33] has shown that non-vanishing of the *NS-NS B*-field makes the CFT smooth. In fact, string theory predicts the Euler characteristic of the *resolved* orbifold [30]; the local form of the statement is

Conjecture 3.1 (Stringy Euler characteristic). *Let $M = \mathbb{C}^n/\Gamma$ for $\Gamma \subset SU(n)$ a finite subgroup. Then, there exists a crepant resolution $\pi : \widetilde{M} \to M$ such that*

$$\chi(\widetilde{M}) = |\{\textit{Conjugacy Classes of}\,\Gamma\}| \,. \tag{3.2}$$

Furthermore, the Hodge numbers of resolved orbifolds were also predicted by Vafa for CY manifolds realized as hypersurfaces in weighted projective spaces and by Zaslow for Kähler manifolds [29]. In dimension three, it has been proved [38, 40] that every Gorenstein singularity admits a crepant resolution[4] and that every crepant resolution satisfies the Conjecture 3.1 and the Vafa-Zaslow Hodge number formulae. For higher dimensions, there are compelling evidences that the formulae are satisfied by all crepant resolutions, when they exist.

As the Euler Characteristic in mathematics is naturally defined by the Hodge numbers of cohomology classes, motivated by the works of

[3]Not all CFT on singular geometry are smooth. For example, there are examples of singular CFT's defined on singular backgrounds, such as in the case of gauge symmetry enhancement of the Type IIA string theory compactified on singular $K3$ where the B-field vanishes [34]. Later, we will discuss a tensored coset model [36] describing this singular non-linear sigma model and relate it to the algebraic McKay Correspondence.

[4]In fact, a given Gorenstein singularity generally admits many crepant resolutions [48]. String theory so far has yielded two distinguished desingularizations: the traditional CFT resolution without discrete torsion and deformation with discrete torsion [32]. In this paper, we are concerned only with Kähler resolutions without discrete torsion.

string theorists and the fact that $\widetilde{M}$ has no odd-dimensional cohomology[5], mathematicians have generalized the classical McKay Correspondence [38, 40, 41] to geometry.

The geometric McKay Correspondence in 2-dimensions actually identifies the cohomology ring of $\widetilde{M}$ and the representation ring of Γ not only as vector spaces but as rings. Given a finite subgroup $\Gamma \subset SU(2)$, the intersection matrix of the irreducible components of the exceptional divisor of the resolved manifold is given by the negative of the Cartan matrix of the associated Dynkin diagram which is specified by the algebraic McKay Correspondence. Hence, there exists an equivalence between the tensor product decompositions of conjugacy classes and intersection pairings of homology classes. Indeed in [42], Ito and Nakajima prove that for all $\Gamma \subset SU(2)$ and for abelian $\Gamma \subset SU(3)$, the Groethendieck (cohomology) ring of $\widetilde{M}$ is isomorphic as a $\mathbb{Z}$-module to the representation ring of Γ and that the intersection pairing on its dual, the Groethendieck group of coherent sheaves on $\pi^{-1}(0)$, can be expressed as the Clebsch-Gordan coefficients. Furthermore, string theory analysis also predicts a similar relation between the two ring structures [31].

The geometric McKay Correspondence can thus be stated as

Conjecture 3.2 (Geometric McKay Correspondence). *Let* Γ, M, *and* $\widetilde{M}$ *be as in Conjecture 3.1. Then, there exist bijections*

$$\boxed{\textit{Basis of } H^*(\widetilde{M},\mathbb{Z})} \quad \longleftrightarrow \quad \boxed{\{\textit{Irreducible Representations of } \Gamma\}}$$
$$\boxed{\textit{Basis of } H_*(\widetilde{M},\mathbb{Z})} \quad \longleftrightarrow \quad \boxed{\{\textit{Conjugacy Classes of } \Gamma\}},$$

and there is an identification between the two ring structures.

3.3.2 Question of Ito and Reid and Chiral Ring

In [40], Ito and Reid raised the question whether the cohomology ring[6] $H^*(\widetilde{M})$ is generated by $H^2(\widetilde{M})$. In this subsection, we rephrase the

[5]See [40] for a discussion on this point.

[6]Henceforth, $\dim M = n$ is not restricted to 2.

question in terms of $N = 2$ SCFT on $M = \mathbb{C}^n/\Gamma$, $\Gamma \subset SU(n)$. String theory provides a way[7] of computing the cohomology of the resolved manifold $\widetilde{M}$. Let us briefly review the method for the present case [30]:

The cohomology of $\widetilde{M}$ consists of those elements of $H^*(\mathbb{C}^n)$ that survive the projection under Γ and new classes arising from the blow-ups. In this case, $H^0(\mathbb{C}^n)$ is a set of all constant functions on $\mathbb{C}^n$ and survives[8] the projection, while all other cohomology classes vanish. Hence, all other non-trivial elements of $H^*(\widetilde{M})$ arise from the blow-up process; in string theory language, they correspond to the twisted chiral primary operators, which are not necessarily all marginal. In the $N = 2$ SCFT of non-linear sigma-model on a compact CY manifold, the $U(1)$ spectral flow identifies the chiral ring of the SCFT with the cohomology ring of the manifold, modulo quantum corrections. For non-compact cases, by considering a topological non-linear σ-model, the A-model chiral ring matches the cohomology ring and the blow-ups still correspond to the twisted sectors.

An $N = 2$ non-linear sigma model on a CY n-fold X has two topological twists called the A and B-models, of which the "BRST" non-trivial observables [52] encode the information about the Kähler and complex structures of X, respectively. The correlation functions of the A-model receive instanton corrections whereas the classical computations of the B-model give exact quantum answers. The most efficient way of computing the A-model correlation functions is to map the theory to a B-model on another manifold Y which is a mirror[9] of X [49]. Then, the classical computation of the B-model on Y yields the full quantum answer for the A-model on X.

In this paper, we are interested in Kähler resolutions of the Gorenstein singularities and, hence, in the A-model whose chiral ring is a quantum deformation of the classical cohomology ring. Since all non-

[7] It is believed that string theory somehow picks out a distinguished resolution of the orbifold, and the following discussion pertains to such a resolution when it exists.

[8] This cohomology class should correspond to the trivial representation in the McKay correspondence.

[9] Mirror symmetry has been intensely studied by both mathematicians and physicists for the past decade, leading to many powerful tools in enumerative geometry. A detailed discussion of mirror symmetry is beyond the scope of this paper, and we refer the reader to [49] for introductions to the subject and for references.

trivial elements of the cohomology ring, except for H^0, arise from the twisted sector or blow-up contributions, we have the following reformulation of the Geometric McKay Correspondence which is well-established in string theory:

Proposition 3.1 (String Theory McKay Correspondence). *Let Γ be a discrete subgroup of $SU(n)$ such that the Gorenstein singularities of $M = \mathbb{C}^n/\Gamma$ has a crepant resolution $\pi : \widetilde{M} \to M$. Then, there exists a following bijection between the cohomology and A-model data:*

$$\boxed{\textit{Basis of } \bigoplus_{i>0} H^i(\widetilde{M})} \longleftrightarrow \boxed{\{\textit{Twisted Chiral Primary Operators}\}}\,, \tag{3.3}$$

or equivalently, by the Geometric McKay Correspondence,

$$\boxed{\{\textit{Conjugacy classes of } \Gamma\}} \longleftrightarrow \boxed{\{\textit{Twisted Elements of the Chiral Ring}\}}\,. \tag{3.4}$$

Thus, since all $H^i, i > 0$ arise from the twisted chiral primary but not necessarily marginal fields and since the marginal operators correspond to H^2, we can now reformulate the question of whether H^2 generates H^* as follows:

> *Do the marginal twisted chiral primary fields generate the entire twisted chiral ring?*

This kind of string theory resolution of orbifold singularities is Arrow III in Figure 1. In §4, we will see how a conformal field theory description of the singular limit of these string theories naturally allows us to link geometry to representation theory. In this way, we hint why McKay correspondence and the discoveries of [8] are not mere happy flukes of nature, as it will become clearer as we proceed.

3.4 (I, IV) McKay Correspondence and WZW

When we calculate the partition function for the WZW model with its energy momentum tensor associated to an algebra $\widehat{g}_k$ of level k, it will

Dynkin Diagram of Modular Invariants	Level of WZW
A_n	$n-1$
D_n	$2n-4$
E_6	10
E_7	16
E_8	28

Table 3: The ADE-Dynkin diagram representations of the modular invariants of the $\widehat{SU(2)}$ WZW.

be of the form[10]:

$$Z(\tau) = \sum_{\widehat{\lambda},\widehat{\xi}\in P_+^{(k)}} \chi_{\widehat{\lambda}}(\tau)\mathcal{M}_{\widehat{\lambda},\widehat{\xi}}\overline{\chi_{\widehat{\xi}}}(\overline{\tau})$$

where $P_+^{(k)}$ is the set of dominant weights and $\chi_{\widehat{\lambda}}$ is the affine character of $\widehat{g}_k$. The matrix $\mathcal{M}$ gives the multiplicity of the highest weight modules in the decomposition of the Hilbert space and is usually referred to as the *mass matrix.* Therefore the problem of classifying the modular invariant partition functions of WZW models is essentially that of the integrable characters χ of affine Lie algebras.

In the case of $\widehat{g}_k = \widehat{SU(2)}_k$, all the modular invariant partition functions are classified, and they fall into an A-D-E scheme ([7] to [11]). In particular, they are of the form of sums over modulus-squared of combinations of the weight k Weyl-Kac character χ_{λ}^k for $\widehat{SU(2)}$ (which is in turn expressible in term of Jacobi theta functions), where the level k is correlated with the rank of the ADE Dynkin diagrams as shown in Table 3 and λ are the eigenvalues for the adjacency matrices of the A-D-E Dynkin diagrams.

Not only are the modular invariants classified by these graphs, but some of the fusion ring algebra can be reconstructed from the graphs.

Though still largely a mystery, the reason for this classification can

[10]we henceforth use the notation in [7]

be somewhat traced to the so-called *fusion rules*. In a rational conformal field theory, the fusion coefficient $N^{\phi_k^*}_{\phi_i\phi_j}$ is defined by

$$\phi_i \times \phi_j = \sum_{\phi_k^*} \mathcal{N}^{\phi_k^*}_{\phi_i\phi_j} \phi_k^* \tag{3.5}$$

where $\phi_{i,j,k}$ are chiral[11] primary fields. This fusion rule provides such vital information as the number of independent coupling between the fields and the multiplicity of the conjugate field ϕ_k^* appearing in the operator product expansion (OPE) of ϕ_i and ϕ_j. In the case of the WZW model with the energy-momentum tensor taking values in the algebra $\widehat{g}_k$ of level k, we can recall that the primary fields have integrable representations $\widehat{\lambda}$ in the dominant weights of $\widehat{g}_k$, and subsequently, (3.5) reduces to

$$\widehat{\lambda} \otimes \widehat{\mu} = \bigoplus_{\widehat{\nu} \in P_+^k} \mathcal{N}^{\widehat{\nu}}_{\widehat{\lambda}\widehat{\mu}}\, \widehat{\nu}.$$

Indeed now we see the resemblance of (3.5) coming from conformal field theory to (3.1) coming from finite representation theory, hinting that there should be some underlying relation. We can of course invert (3.1) using the properties of finite characters, just as we can extract $\mathcal{N}$ by using the Weyl-Kac character formula (or by the Verlinde equations).

Conformal field theorists, inspired by the A-D-E classification of the minimal models, have devised similar methods to treat the fusion coefficients. It turns out that in the simplest cases the fusion rules can be generated entirely from one special case of $\widehat{\lambda} = f$, the so-called fundamental representation. This is of course in analogy to the unique (fundamental) 2-dimensional representation R in McKay's paper. In this case, all the information about the fusion rule is encoded in a matrix $[N]_{ij} = \mathcal{N}^{j}_{fi}$, to be treated as the adjacency matrix of a finite graph. Conversely we can define a commutative algebra to any finite graph, whose adjacency matrix is defined to reproduce the fusion rules for the so-called *graph algebra*. It turns out that in the cases of A_n, D_{2n}, E_6 and E_8 Dynkin diagrams, the resulting graph algebra has an subalgebra

[11] Chirality here means left- or right-handedness not chirality in the sense of $N = 2$ superfields.

which reproduces the (extended) fusion algebra of the respective *A*-*D*-*E* $\widehat{SU(2)}$ WZW models.

From another point of view, we can study the WZW model by quotienting it by discrete subgroups of $SU(2)$; this is analogous to the twisted sectors in string theory where for the partition function we sum over all states invariant under the action of the discrete subgroup. Of course in this case we also have an *A*-*D*-*E*-type classification for the finite groups due to the McKay Correspondence, therefore speculations have risen as to why both the discrete subgroups and the partition functions are classified by the same graphs [7, 13], which also reproduce the associated ring structures. The reader may have noticed that this connection is somewhat weaker than the others hitherto considered, in the sense that the adjacency matrices do not correspond 1-1 to the fusion rules. This subtlety will be addressed in §4 and §5.

Indeed, the graph algebra construction has been extended to $\widehat{SU(3)}$ and a similar classification of the modular invariants have in fact been done and are shown to correspond to the so-called *generalized Dynkin Diagrams* [7, 10, 13]. On the other hand, the Clebsch-Gordan coefficients of the McKay type for the discrete subgroups of $SU(3)$ have been recently computed in the context of studying D3-branes on orbifold singularities [1]. It was noted that the adjacency graphs drawn in the two different cases are in some form of correspondence and was conjectured that this relationship might extend to $\widehat{SU(n)}_k$ model for n other than 2 and 3 as well. It is hoped that this problem may be attacked by going through the other arrows.

We have now elucidated arrows I and IV in Figure 1.

4 The Arrow V: σ-model/LG/WZW Duality

We here summarize the link V in Figure 1 for ALE spaces, as has been established in [36].

It is well-known that application of catastrophe theory leads to the *A*-*D*-*E* classification of Landau-Ginzburg models [23]. It has been subsequently shown that the renormalization group fixed points of these theories actually provide the Lagrangian formulations of $N = 2$ discrete minimal models [27]. What is even more surprising and beautiful is Gepner's another proposal [21] that certain classes of $N = 2$ non-linear sigma-models on CY 3-folds are equivalent to tensor products of $N = 2$ minimal models with the correct central charges and $U(1)$ projections. Witten has successfully verified the claim in [28] using a gauged linear-sigma model which interpolates between Calabi-Yau compactifications and Landau-Ginzburg orbifolds.

In a similar spirit, Ooguri and Vafa have considered LG orbifolds[12] of the tensor product of $SL(2,\mathbb{R})/U(1)$ and $SU(2)/U(1)$ Kazama-Suzuki models[13] [61] and have shown that the resulting theory describes the singular conformal field theory of the non-linear sigma-model with the B-field turned off. In particular, they have shown that the singularity on A_{n-1} ALE space is described by the

$$\frac{\frac{SL(2)_{n+2}}{U(1)} \times \frac{SU(2)_{n-2}}{U(1)}}{\mathbb{Z}_n} \tag{4.1}$$

orbifold model which contains the $\widehat{SU(2)}_{n-2}$ WZW theory at level $k = n-2$. The coset descriptions of the non-linear σ-models on D and E-type ALE spaces also contain the corresponding WZW theories whose modular invariants are characterized by the D and E-type resolution graphs of the ALE spaces. The full orbifoldized Kazama-Suzuki model has fermions as well as an extra Feigin-Fuchs scalar, but we will be interested only in the WZW sector of the theory, for this particular

[12]The universality classes of the LG models are completely specified by their superpotentials W, and such a simple characterization leads to very powerful methods of detailed computations [22, 25]. Generalizations of these models have many important applications in string theory, and the OPE coefficients of topological LG theories with judiciously chosen non-conformal deformations yield the fusion algebra of rational conformal field theories. In [59], Gepner has shown that the topological LG models with deformed Grassmannian superpotentials yield the fusion algebra of the $\widehat{SU(n)}_k$ WZW, illustrating that much information about non-supersymmetric RCFT can be extracted from their $N = 2$ supersymmetric counterparts. Gepner's superpotential could be viewed as a particular non-conformal deformation of the superpotential appearing in Ooguri and Vafa's model.

[13]The $SL(2,\mathbb{R})/U(1)$ coset model describes the two-dimensional black hole geometry [62], while the $SU(2)/U(1)$ Kazama-Suzuki model is just the $N = 2$ minimal model.

ALE Type	Level of WZW
A_n	$n-1$
D_n	$2n-4$
E_6	10
E_7	16
E_8	28

Table 4: The WZW subsector of the Ooguri-Vafa conformal field theory description of the singular non-linear sigma-model on ALE.

sector contains the relevant information about the discrete group Γ and the cohomology of $\widetilde{\mathbb{C}^2/\Gamma}$. We summarize the results in Table 4.

We now assert that many amazing A-D-E -related properties of the $\widehat{SU(2)}$ WZW conformal field theory and the McKay correspondence can be interpreted as consequences of the fact that the conformal field theory description of the singularities of ALE spaces contains the $\widehat{SU(2)}$ WZW. That is, we argue that the WZW theory inherits most of the geometric information about the ALE spaces.

4.1 Fusion Algebra, Cohomology and Representation Rings

Comparing the Table 4 with the Table 3, we immediately see that the graphical representations of the homology intersections of $H_2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ and the modular invariants of the associated $\widehat{SU(2)}$ WZW subsector are identical.

Let us recall how $\widehat{SU(2)}_k$ WZW model has been historically related to the finite subgroups of $SU(2)$. Meanwhile we shall recapitulate some of the key points in §3.4. The finite subgroups Γ of $SU(2)$ have two infinite and one finite series. The Algebraic McKay Correspondence showed that the representation ring of each finite group admits a graphical representation such that the two infinite series have the precise A and D Dynkin diagrams while the finite series has the $E_{6,7,8}$ Dynkin diagrams. Then, it was noticed that the same Dynkin diagrams classify the modular invariants of the $\widehat{SU(2)}_k$ WZW model, and this observation was interesting but there was no *a priori* connection to the representation theory of finite subgroups. It was later discovered that the Dynkin diagrams also encode the $\widehat{SU(2)}_k$ WZW fusion rules or their extended versions[14]. Independently of the WZW models, the Dynkin diagrams are also well-known to represent the homological intersection numbers on $\widetilde{\mathbb{C}^2/\Gamma}$, which are encoded the chiral ring structure of the sigma-model when $B \neq 0$. What Ooguri and Vafa have shown us is that when the B-field is set to zero, the information about the chiral ring and the discrete subgroup Γ do not get destroyed but get transmitted to the orbifoldized Kazama-Suzuki model which contains the $\widehat{SU(2)}_k$ WZW.

Let us demonstrate the fusion/cohomology correspondence for the A-series. Let C_i be the basis of $H^2(\widetilde{\mathbb{C}^2/\mathbb{Z}_n}, \mathbb{Z})$ and Q_{ij} their intersection matrix inside the A_{n-1} ALE space. The $\widehat{SU(2)}_k$ WZW at level $k = n-2$ has $k+1$ primary fields $\phi_a, a = 0, 1, \ldots n-2$. Then, the fusion of the fundamental field ϕ_1 with other primary fields

$$\phi_1 \times \phi_a = \mathcal{N}_{1a}{}^b \, \phi_b \tag{4.2}$$

is precisely given by the intersection matrix, i.e. $\mathcal{N}_{1a}{}^b = Q_{ab}$. Now, let N_1 be the matrix whose components are the fusion coefficients $(N_1)_{ab} = \mathcal{N}_{1a}{}^b$, and define $k-1$ matrices $N_i, i = 2, \ldots, k$ recursively by the following equations

$$\begin{aligned}
N_1 N_1 &= N_0 + N_2 \\
N_1 N_2 &= N_1 + N_3 \\
N_1 N_3 &= N_2 + N_4 \; \cdots \\
N_1 N_{k-1} &= N_{k-2} + N_k \\
N_1 N_k &= N_{k-1}
\end{aligned} \tag{4.3}$$

where $N_0 = \mathrm{Id}_{(k+1)\times(k+1)}$. That is, multiplication by N_1 with N_j just lists the neighboring nodes in the A_{k+1} Dynkin diagram with a sequential labeling. Identifying the primary fields ϕ_i with the matrices N_i, it

[14]See [7] for a more complete discussion of this point.

is easy to see that the algebra of N_i generated by the defining equations (4.3) precisely reproduces the fusion algebra of the ϕ_i for the $\widehat{SU(2)}_k$ WZW at level $k = n - 2$. This algebra is the aforementioned graph algebra in conformal field theory. The graph algebra has been known for many years, but what we are proposing in this paper is that the graph algebra is a consequence of the fact that the WZW contains the information about the cohomology of the corresponding ALE space.

Furthermore, recall from §3.1 that the intersection matrix is identical to the Clebsch-Gordan coefficients m_{ij}, ignoring the affine node. This fact is in accordance with the proof of Ito and Nakajima [42] that the cohomology ring of $\widetilde{\mathbb{C}^2/\Gamma}$ is isomorphic to the representation ring $\mathcal{R}(\Gamma)$ of Γ. At first sight, it appears that we have managed to reproduce only a subset of Clebsch-Gordan coefficients of $\mathcal{R}(\Gamma)$ from the cohomology or equivalently the fusion ring. For the A-series, however, we can easily find all the Clebsch-Gordan coefficients of the irreps of $\mathbb{Z}_n$ from the fusion algebra by simply relabeling the irreps and choosing a different self-dual 2-dimensional representation. This is because the algebraic McKay correspondence produces an A_{n-1} Dynkin diagram for any self-dual 2-dimensional representation R and choosing a different R amounts to relabeling the nodes with different irreps. The graph algebras of the $\widehat{SU(2)}_k$ WZW theory for the D and E-series actually lead not to the fusion algebra of the original theory but to that of the extended theories, and these cases require further investigations.

String theory is thus telling us that the cohomology ring of $\widetilde{\mathbb{C}^2/\Gamma}$, fusion ring of $\widehat{SU(2)}$ WZW and the representation ring of Γ are all equivalent. We summarize the noted correspondences and our observations in Figure 3.

4.2 Quiver Varieties and WZW

In this subsection, we suggest how affine Lie algebras may be arising so naturally in the study of two-dimensional quotient spaces.

Based on the previous studies of Yang-Mills instantons on ALE spaces as in [45], Nakajima has introduced in [46] the notion of a quiver

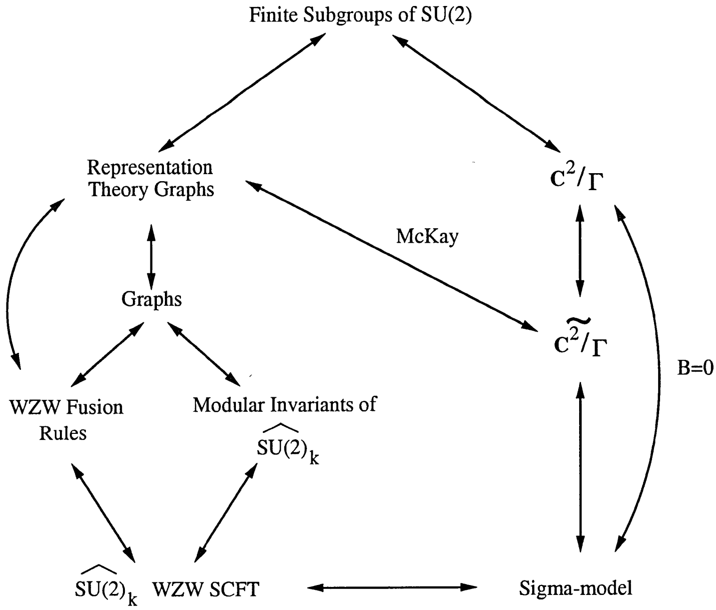

Figure 3: Web of Correspondences: Each finite group $\Gamma \subset SU(2)$ gives rise to an isolated Gorenstein singularity as well as to its representation ring $\mathcal{R}$. The cohomology ring of the resolved manifold is isomorphic to $\mathcal{R}$. The $\widehat{SU(2)}_k$ WZW theory at level $k = \#$ Conjugacy classes of $\Gamma - 2$ has a graphical representation of its modular invariants and its fusion ring. The resulting graph is precisely the non-affine version of McKay's graph for Γ. The WZW model arises as a subsector of the conformal field theory description of the quotient singularity when the B-field has been set to zero. We further note that the three rings in the picture are equivalent.

variety which is roughly a hyper-Kähler moduli space of representations of a quiver associated to a finite graph (We shall turn to quivers in the next section). There, he presents a beautiful geometric construction of representations of affine Lie algebras. In particular, he shows that when the graph is of the A-D-E type, the middle cohomology of the quiver variety is isomorphic to the weight space of integrable highest-weight representations. A famous example of a quiver variety with this kind of affine Lie algebra symmetry is the moduli space of instantons over ALE spaces.

In a separate paper [42], Nakajima also shows that the quotient space $\mathbb{C}^2/\Gamma$ admits a Hilbert scheme resolution X which itself can be identified with a quiver variety associated with the affine Dynkin diagram of Γ. The analysis of [46] thus seems to suggest that the second cohomology of the resolved space X is isomorphic to the weight space of some affine Lie algebra. We interpret Nakajima's work as telling physicists that the $\widehat{SU(2)}_k$ WZW has every right to be present and carries the geometric information about the second cohomology. Let us demonstrate our thoughts when $\Gamma = \mathbb{Z}_n$. In this case, we have $\dim H^2 = n-1$, consisting of $n-1$ $\mathbb{P}^1$ blow-ups in a linear chain. We interpret the H^2 basis as furnishing a representation of the $\widehat{SU(2)}_k$ WZW at level $k = n-2$, as the basis matches the primary fields of the WZW. This interpretation agrees with the analysis of Ooguri and Vafa, but we are not certain how to reproduce the result directly from Nakajima's work.

4.3 T-duality and Branes

In [14, 15, 16, 17], the $\widehat{SU(2)}_k$ WZW theory arose in a different but equivalent context of brane dynamics. As shown in [36], the type IIA (IIB) string theory on an A_{n-1} ALE space is T-dual to the type IIB (IIA) theory in the background of n $NS5$-branes. The world-sheet description of the near-horizon geometry of the colliding $NS5$-branes is in terms of the $\widehat{SU(2)}_k$ WZW, a Feigin-Fuchs boson, and their superpartners. More precisely, the near-horizon geometry of n $NS5$-branes is given by the WZW at level $n-2$, which is consistent with the analysis of Ooguri and Vafa.

It was conjectured in [15], and further generalized in [16], that the string theory on the near horizon geometry of the $NS5$-branes is dual to the decoupled theory on the world-volume of the $NS5$-branes. In this paper, our main concern has been the singularity structure of the ALE spaces, and we have thus restricted ourselves only to the transverse directions of the $NS5$-branes in the T-dual picture.

5 Ribbons and Quivers at the Crux of Correspondences

There is a common theme in all the fields relevant to our observations so far. In general we construct a theory and attempt to encode its rules into some matrix, whether it be fusion matrices, Clebsch-Gordan coefficients, or intersection numbers. Then we associate this matrix with some graph by treating the former as the adjacency matrix of the latter and study the properties of the original theory by analyzing the graphs[15].

Therefore there appears to be two steps in our program: firstly, we need to study the commonalities in the minimal set of axioms in these different fields, and secondly, we need to encode information afforded by these axioms by certain graphical representations. It turns out that there has been some work done in both of these steps, the first exemplified by the so-called ribbon categories and the second, quiver categories.

5.1 Ribbon Categories as Modular Tensor Categories

Prominent work in the first step has been done by A. Kirillov [53] and we shall adhere to his notations. We are interested in monoidal additive categories, in particular, we need the following:

Definition 5.1. A **ribbon category** is an additive category $\mathcal{C}$ with the following additional structures:

- BRAIDING: A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ along with functorial associativity and commutativity isomorphisms for objects V and W:

$$a_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),$$
$$\check{R}_{V,W} : V \otimes W \to W \otimes V;$$

[15]There is interesting work done to formalize to sub-factors and to investigate the graphs generated [58].

- MONOIDALITY: A unit object $\mathbf{1} \in \mathrm{Obj}\, \mathcal{C}$ along with isomorphisms $\mathbf{1} \otimes V \to V, V \otimes \mathbf{1} \to V$;
- RIGIDITY of duals: for every object V we have a (left) dual V^* and homomorphisms

$$\begin{aligned} e_V &: V^* \otimes V \to \mathbf{1}, \\ i_V &: \mathbf{1} \to V \otimes V^*; \end{aligned}$$

- BALANCING: functorial isomorphisms $\theta_V : V \to V$, satisfying the compatibility condition

$$\theta_{V \otimes W} = \check{R}_{W,V} \check{R}_{V,W} (\theta_V \otimes \theta_W).$$

Of course we see that all the relevant rings in Figure 1 fall under such a category. Namely, we see that the representation rings of finite groups, chiral rings of non-linear σ-models, Groethendieck rings of exceptional divisors or fusion rings of WZW, together with their associated tensor products, are all different realizations of a ribbon category [16]. This fact is perhaps obvious from the point of view of orbifold string theory, in which the fusion ring naturally satisfies the representation algebra of the finite group and the WZW arises as a singular limit of the vanishing B-field. The ingredients of each of these rings, respectively the irreps, chiral operators and cohomology elements, thus manifest as the objects in $\mathcal{C}$. Moreover, the arrows of Figure 1, loosely speaking, become functors among these various representations of $\mathcal{C}$ whereby making our central diagram a (meta)graph associated to $\mathcal{C}$. What this means is that as far as the ribbon category is concerned, all of these theories discussed so far are axiomatically identical. Hence indeed any underlying correspondences will be natural.

What if we impose further constraints on $\mathcal{C}$?

Definition 5.2. We define $\mathcal{C}$ to be **semisimple** if

- It is defined over some field $\mathbb{K}$ and all the spaces of homomorphisms are finite-dimensional vector spaces over $\mathbb{K}$;

[16] Of course they may possess additional structures, e.g., these rings are all finite. We shall later see how finiteness becomes an important constraint when going to step two.

- Isomorphism classes of simple objects X_i in $\mathcal{C}$ are indexed by elements i of some set I. This implies involution $^* : I \to I$ such that $X_i^* \simeq X_{i^*}$ (in particular, $0^* = 0$);
- "Schur's Lemma": $\hom(X_i, X_j) = \mathbb{K}\delta_{ij}$;
- Complete Finite Reducibility: $\forall\ V \in \text{Obj}\ \mathcal{C}$, $V = \bigoplus\limits_{i \in I} N_i X_i$, such that the sum is finite, i.e., almost all $N_i \in \mathbb{Z}_+$ are zero.

Clearly we see that in fact our objects, whether they be WZW fields or finite group irreps, actually live in a semisimple ribbon category. It turns out that semisimplicity is enough to allow us to define composition coefficients of the "Clebsch-Gordan" type:

$$X_i \oplus X_j = \bigoplus N_{ij}^k X_k,$$

which are central to our discussion.

Let us introduce one more concept, namely the matrix s_{ij} mapping $X_i \to X_j$ represented graphically by the simple ribbon tangle, i.e., a link of 2 closed directed cycles of maps from X_i and X_j respectively into themselves. The remarkable fact is that imposing that

- s_{ij} be invertible and that
- $\mathcal{C}$ have only a finite number of simple objects (i.e., the set I introduced above is finite)

naturally gives rise to modular properties. We define such semisimple ribbon category equipped with these two more axioms as a **Modular Tensor Category**. If we define the matrix $t_{ij} = \delta_{ij}\theta_i$ with θ_i being the functorial isomorphism introduced in the balancing axiom for $\mathcal{C}$, the a key result is the following [53]:

Theorem 5.1. *In the modular tensor category $\mathcal{C}$, the matrices s and t generate precisely the modular group $SL(2, \mathbb{Z})$.*

Kirillov remarks in [53] that it might seem mysterious that modular properties automatically arise in the study of tensor categories and argues in two ways why this may be so. Firstly, a projective action of

$SL(2,\mathbb{Z})$ may be defined for certain objects in $\mathcal{C}$. This is essentially the construction of Moore and Seiberg [54] when they have found new modular invariants for WZW, showing how WZW primary operators are objects in $\mathcal{C}$. Secondly, he points out that geometrically one can associate a topological quantum field theory (TQFT) to each tensor category, whereby the mapping class group of the Riemann surface associated to the TQFT gives rise to the modular group. If the theories in Figure 1 are indeed providing different but equivalent realizations of $\mathcal{C}$, we may be able to trace the origin of the $SL(2,\mathbb{Z})$ action on the category to the WZW modular invariant partition functions. That is, it seems that in two dimensions the A-D-E scheme, which also arises in other representations of $\mathcal{C}$, naturally classifies some kind of modular invariants. In a generic realization of the modular tensor category, it may be difficult to identify such modular invariants, but they are easily identified as the invariant partition functions in the WZW theories.

5.2 Quiver Categories

We now move onto the second step. Axiomatic studies of the encoding procedure (at least a version thereof) have been done even before McKay's result. In fact, in 1972, Gabriel has noticed that categorical studies of quivers lead to A-D-E-type classifications [55].

Definition 5.3. We define the **quiver category** $\mathcal{L}(\Gamma,\Lambda)$, for a finite connected graph Γ with orientation Λ, vertices Γ_0 and edges Γ_1 as follows: The objects in this category are any collection (V,f) of spaces $V_\alpha, \alpha \in \Gamma_0$ and mappings $f_l, l \in \Gamma_1$. The morphisms are $\phi : (V,f) \to (V',f')$ a collection of linear mappings $\phi_\alpha : V_\alpha \to V'_\alpha$ compatible with f by $\phi_{e(l)} f_l = f'_l \phi_{b(l)}$ where $b(l)$ and $e(l)$ are the beginning and the ending nodes of the directed edge l.

Finally we define decomposability in the usual sense that

Definition 5.4. The object (V,f) is **indecomposable** iff there do not exist objects $(V_1,f_1),(V_2,f_2) \in \mathcal{L}(\Gamma,\Lambda)$ such that $V = V_1 \oplus V_2$ and $f = f_1 \oplus f_2$.

Under these premises we have the remarkable result:

Theorem 5.2 (Gabriel-Tits). *The graph Γ in $\mathcal{L}(\Gamma, \Lambda)$ coincides with one of the graphs $A_n, D_n, E_{6,7,8}$, if and only if there are only finitely many non-isomorphic indecomposable objects in the quiver category.*

By this result, we can argue that the theories, which we have seen to be different representations of the ribbon category $\mathcal{C}$ and which all have *A-D-E* classifications in two dimensions, each must in fact be realizable as a finite quiver category $\mathcal{L}$ in dimension two. Conversely, the finite quiver category has representations as these theories in 2-dimensions. To formalize, we state

Proposition 5.1. *In two dimensions, finite group representation ring, WZW fusion ring, Gorenstein cohomology ring, and non-linear σ-model chiral ring, as representations of a ribbon category $\mathcal{C}$, can be mapped to a finite quiver category $\mathcal{C}$. In particular the "Clebsch-Gordan" coefficients $\mathcal{N}_{ij}^{k}$ of $\mathcal{C}$ realize as adjacency matrices of graphs in $\mathcal{L}$* [17].

Now $\mathcal{L}$ has recently been given a concrete realization by the work of Douglas and Moore [5], in the context of investigating string theory on orbifolds. The objects in the quiver category have found representations in the resulting $\mathcal{N} = 2$ Super Yang-Mills theory. The modules V (nodes) manifest themselves as gauge groups arising from the vector multiplet and the mappings f (edges which in this case are really bidirectional arrows), as bifundamental matter. This is the arrow from graph theory to string orbifold theory in the center of Figure 1. Therefore it is not surprising that an *A-D-E* type of result in encoding the physical content of the theory has been obtained. Furthermore, attempts at brane configurations to construct these theories are well under way (e.g. [6]).

Now, what makes *A-D-E* and two dimensions special? A proof of the theorem due to Tits [55] rests on the fact that the problem can essentially be reduced to a Diophantine inequality in the number of nodes and edges of Γ, of the general type:

$$\sum_i \frac{1}{p_i} \geq c$$

[17]Here the graphs are *A-D-E* Dynkin diagrams. For higher dimension we propose that there still is a mapping, though perhaps not to a *finite* quiver category.

where c is some constant and $\{p_i\}$ is a set of integers characterizing the problem at hand. This inequality has a long history in mathematics [57]. In our context, we recall that the uniqueness of the five perfect solids in $\mathbb{R}^3$ (and hence the discrete subgroups of $SU(2)$) relies essentially on the equation $1/p + 1/q \geq 1/2$ having only 5 pairs of integer solutions. Moreover we recall that Dynkin's classification theorem of the simple Lie algebras depended on integer solutions of $1/p + 1/q + 1/r \geq 1$.

Since Gabriel's theorem is so restrictive, extensions thereto have been done to relax certain assumptions (e.g., see [56]). This will hopefully give us give more graphs and in particular those appearing in finite group, WZW, orbifold theories or non-linear σ-models at higher dimensions. A vital step in the proof is that a certain quadratic form over the $\mathbb{Q}$-module of indices on the nodes (effectively the Dynkin labels) must be positive-definite. It was noted that if this condition is relaxed to positive semi-definity, then Γ would include the affine cases $\widehat{A}, \widehat{D}, \widehat{E}$ as well. Indeed we hope that further relaxations of the condition would admit more graphs, in particular those drawn for the $SU(3)$ subgroups. This inclusion on the one hand would relate quiver graphs to Gorenstein singularities in dimension three due to the link to string orbifolds[18] and on the other hand to the WZW graph algebras by the conjecture of Hanany and He [1]. Works in this direction are under way. It has been recently suggested that since the discrete subgroups of SU(4,5,6,7) have also been classified [11], graphs for these could be constructed and possibly be matched to the modular invariants corresponding to $\widehat{SU(n)}$ for $n = 4, \ldots, 7$ respectively. Moreover, proposals for unified schemes for the modular invariants by considering orbifolds by abelian Γ in SU$(2, 3, \ldots, 6)$ have been made in [12].

Let us summarize what we have found. We see that the representation ring of finite groups with its associated $(\otimes, \oplus)$, the chiral ring of nonlinear σ-model with its $(\otimes, \oplus)$, the fusion ring of the WZW model with its $(\times, \oplus)$ and the Groethendieck ring of resolved Gorenstein singularities with it $(\otimes, \oplus)$ manifest themselves as different realizations of a semisimple ribbon category $\mathcal{C}$. Furthermore, the requirement of finiteness and an invertible s-matrix makes $\mathcal{C}$ into a modular tensor category. The A-D-E schemes in two dimensions, if they arise in one representation of $\mathcal{C}$, might naturally appear in another. Furthermore,

[18]In this case we get $\mathcal{N} = 1$ Super-Yang-Mills theory in 4 dimension.

the quiver category $\mathcal{L}$ has a physical realization as bifundamentals and gauge groups of SUSY Yang-Mills theories. The mapping of the Clebsch-Gordan coefficients in $\mathcal{C}$ to the quivers in $\mathcal{L}$ is therefore a natural origin for the graphical representations of the diverse theories that are objects in $\mathcal{C}$.

6 Conjectures

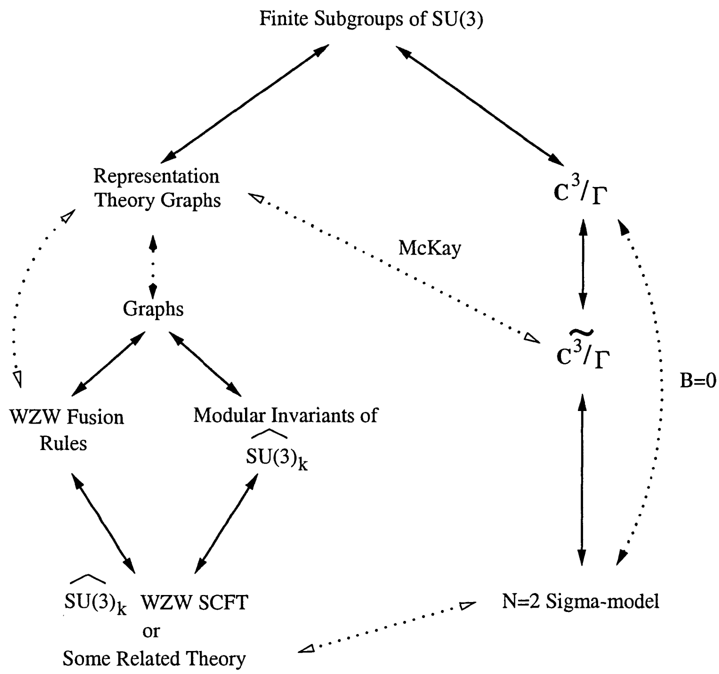

Figure 4: Web of Conjectures: Recently, the graphs from the representation theory side were constructed and were noted to resemble those on WZW $\widehat{SU(3)}_k$ side [1]. The solid lines have been sufficiently well-established while the dotted lines are either conjectural or ill-defined.

We have seen that there exists a remarkably coherent picture of inter-relations in two dimensions among many different branches of mathematics and physics. The organizing principle appears to be the mathematical theory of quivers and ribbon category, while the crucial bridge between mathematics and physics is the conformal field theory description of the Gorenstein singularities provided by the orbifoldized coset construction.

Surprisingly, similar features have been noted in three dimensions. The Clebsch-Gordan coefficients for the tensor product of irreducible representations for all discrete subgroups of $SU(3)$ were computed in [1, 50], and a possible correspondence was noted, and conjectured for $n \geq 3$, between the resulting Dynkin-like diagrams and the graphic representations of the fusion rules and modular invariants of $\widehat{SU(3)}_k$ WZW models. Furthermore, as discussed previously, the Geometric McKay Correspondence between the representation ring of the abelian discrete subgroups $\Gamma \subset SU(3)$ and the cohomology ring of $\widetilde{\mathbb{C}^3/\Gamma}$ has been proved in [42]. Hence, the situation in 3-dimensions as seen in Figure 4 closely resembles that in 2-dimensions.

Now, one naturally inquires:

> *Are there graphical representations of the fusion rules and modular invariants of the $\widehat{SU(n)}_k$ WZW model or some related theory that contain the Clebsch-Gordan coefficients for the representations of $\Gamma \subset SU(n)$? And, in turn, are the Clebsch-Gordan coefficients related to the (co)-homological intersections on the resolved geometry $\widetilde{\mathbb{C}^n/\Gamma}$ that are contained in the chiral ring of the $N = 2$ σ-model on $\mathbb{C}^n/\Gamma$ with a non-vanishing B-field?* **Most importantly, what do these correspondences tell us about the two conformal field theories and their singular limits?**

As physicists, we believe that the McKay correspondence and the classification of certain modular invariants in terms of finite subgroups are consequences of orbifolding and of some underlying quantum equivalence of the associated conformal field theories.

We thus believe that a picture similar to that seen in this paper for 2-dimensions persists in higher dimensions and conjecture that there exists a conformal field theory description of the Gorenstein singularities in higher dimensions. If such a theory can be found, then it would explain the observation made in [1] of the resemblance of the graphical representations of the representation ring of the finite subgroups of $SU(3)$ and the modular invariants of the $\widehat{SU(3)}_k$ WZW. We have checked that the correspondence, if any, between the finite subgroups of $SU(3)$ and the $\widehat{SU(3)}_k$ WZW theory is not one-to-one. For example, the number of primary fields generically does not match the number of conjugacy classes of the discrete subgroups. It has been observed in [1], however, that some of the representation graphs appear to be subgraphs of the graphs encoding the modular invariants. We hope that the present paper serves as a motivation for finding the correct conformal field theory description in three dimensions which would tell us how to "project" the modular invariant graphs to retrieve the representation graphs of the finite graphs.

Based on the above discussions, we summarize our speculations, relating geometry, generalizations of the A-D-E classifications, representation theory, and string theory in Figure 4.

6.1 Relevance of Toric Geometry

It is interesting to note that the toric resolution of certain Gorenstein singularities also naturally admits graphical representations of fans. In fact, the exceptional divisors in the Geometric McKay Correspondence for $\Gamma = \mathbb{Z}_n \subset SU(2)$ in 2-dimensions can be easily seen as the vertices of new cones in the toric resolution, and these vertices precisely form the A_{n-1} Dynkin digram. Thus, at least for the abelian case in 2-dimensions, the McKay correspondence and the classification of $\widehat{SU(2)}$ modular invariants seem to be most naturally connected to geometry as toric diagrams of the resolved manifolds $\widetilde{C^2/\Gamma}$.

Surprisingly—perhaps not so much so in retrospect—we have noticed a similar pattern in 3-dimensions. That is, the toric resolution diagrams of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ singularities reproduce the graphs that classify the $\mathcal{A}$-type modular invariants of the $\widehat{SU(3)}_k$ WZW models. For which k? It has been previously observed in [63] that there seems to be a correspondence, up to some truncation, between the subgroups $\mathbb{Z}_n \times \mathbb{Z}_n \subset SU(3)$ and the $\mathcal{A}$-type $\widehat{SU(3)}_{n-1}$ modular invariants, which do appear as subgraphs of the $\widetilde{\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}}$ toric diagrams. On the other hand, a precise formulation of the correspondence with geometry and the conformal field theory description of Gorenstein singularities still remains as an unsolved problem and will be presented elsewhere [64].

7 Conclusion

Inspired by the ubiquity of *A-D-E* classification and prompted by an observation of a mysterious relation between finite groups and WZW models, we have proposed a possible unifying scheme. Complex and intricate webs of connections have been presented, the particulars of which have either been hinted at by collective works in the past few decades in mathematics and physics or are conjectured to exist by arguments in this paper. These webs include the McKay correspondences of various types as special cases and relate such seemingly disparate subjects as finite group representation theory, graph theory, string orbifold theory and sigma models, as well as conformal field theory descriptions of Gorenstein singularities. We note that the integrability of the theories that we are considering may play a role in understanding the deeper connections.

This paper catalogs many observations which have been put forth in the mathematics and physics literature and presents them from a unified perspective. Many existing results and conjectures have been phrased under a new light. We can summarize the contents of this paper as follows:

1. In two dimensions, all of the correspondences mysteriously fall into an *A-D-E* type. We have provided, via Figure 1, a possible setting how these mysteries might arise naturally. Moreover, we have pointed out how axiomatic works done by category theorists may demystify some of these links. Namely, we have noted that the relevant rings of the theories can be mapped to the quiver category.

2. We have also discussed the possible role played by the modular tensor category in our picture, in which the modular invariants arise very naturally. Together with the study of the quiver category and quiver variety, the ribbon category seems to provide the reasons for the emergence of affine Lie algebra symmetry and the *A-D-E* classification of the modular invariants.

3. We propose the validity of our program to higher dimensions, where the picture is far less clear since there are no A-D-E schemes, though some hints of generalized graphs have appeared.

4. There are three standing conjectures:

   - We propose that there exists a conformal field theory description of the Gorenstein singularities in dimensions greater than two.
   - As noted in [1], we conjecture that the modular invariants and the fusion rings of the $\widehat{SU(n)}, n > 2$ WZW, or their generalizations, may be related to the discrete subgroups of the $SU(n)$.
   - Then, there is the mathematicians' conjecture that there exits a McKay correspondence between the cohomology ring $H^*(\widetilde{C^n/\Gamma}, \mathbb{Z})$ and the representation ring of Γ, for finite subgroup $\Gamma \subset SU(n)$.

   We have combined these conjectures into a web so that proving one of them would help proving the others.

We hope that Figure 1 essentially commutes and that the standing conjectures represented by certain arrows therewithin may be solved by investigating the other arrows. In this way, physics may provide us with a possible method of attack and explanation for McKay's correspondence and many other related issues, and likewise mathematical structures may help to clarify and rigorize some observations made from string theory.

It is the purpose of this writing to inform the physics and mathematics community of a possibly new direction of research which could harmonize ostensibly different and diverse branches of mathematics and physics into a unified picture.

Acknowledgments

We would like to express our sincere gratitude to A. Hanany and Y.S. Song for reviewing our paper, B. Feng for valuable discussions, and

G. Tian for his patience and explaining the mathematical subtleties of ALE spaces. We are especially grateful to C. Vafa for helpful comments and for correcting our mistake on the preliminary version of the paper. We also thank H. Nakajima for clarifying his work.

Y.H.H. would like to express his thanks to I. Savonije, M. Spradlin and the Schmidts and above all *Ad Catharinae Sanctae Alexandria et Ad Maiorem Dei Gloriam.* J.S.S. would like to thank A.M.W. for inspiration. And indeed we are eternally indebted to our parents for their ever-loving support.

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