Adv. Theor. Math. Phys. 4 (2000) 791–822

Three-Dimensional Gorenstein Singularities and $\widehat{SU(3)}$ Modular Invariants

Jun S. Song

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
jssong@ctp.mit.edu

Abstract

Using $N = 2$ Landau-Ginzburg theories, we examine the recent conjectures relating the $\widehat{SU(3)}$ WZW modular invariants, finite subgroups of $SU(3)$ and Gorenstein singularities. All isolated three-dimensional Gorenstein singularities do not appear to be related to any known Landau-Ginzburg theories, but we present some curious observations which suggest that the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model may be related to a deformed geometry of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$. The toric resolution

e-print archive: http://xxx.lanl.gov/hep-th/9908008
Research supported in part by the NSF Graduate Fellowship and the U.S. Department of Energy under cooperative research agreement #DE-FC02-94ER40818.

diagrams of those particular singularities are also seen to be classifying the diagonal modular invariants of the $\widehat{SU(3)}_n$ as well as the $\widehat{SU(2)}_{n+1}$ WZW models.

1 Introduction

Study of integrable lattice models has previously led physicists to speculate a possible connection between the finite subgroups of $SU(3)$ and the modular invariants of the $\widehat{SU(3)}$ WZW models. In particular, it has been observed in [3] that the representation theory graphs of the $\mathbb{Z}_n \times \mathbb{Z}_n$ finite subgroups are closely related to the diagonal $\mathcal{A}$-modular invariants of the $\widehat{SU(3)}$ WZW models at level $(n-1)$. Such a relation, if it exists, would not be an absolute surprise to those who are conversant with the ubiquitous A-D-E classifications of the finite subgroups of $SU(2)$, modular invariants of the $\widehat{SU(2)}$ WZW, two-dimensional Gorenstein singularities, and $N=2$ minimal models [1, 9, 14, 17]. The current situation for $SU(3)$, however, is not nearly as good as that for $SU(2)$. That is, unlike the case of $SU(2)$, where the A-D-E Dynkin diagrams precisely classify both the modular invariants and the finite subgroups, the graphs characterizing the $\widehat{SU(3)}$ modular invariants are not quite the same as those encoding the irreducible representations of the finite subgroups of $SU(3)$; the former graphs appear to be subgraphs of the latter with many lines and nodes deleted. In [3], it has thus been suggested that there should be a way of truncating the representation theory graphs of the $SU(3)$ finite subgroups in order to reproduce the $\widehat{SU(3)}$ modular invariant graphs, but no particular algorithm has yet been put forth satisfactorily.

In this paper, we provide evidences for a slightly different correspondence using geometry as our main tool. The essential motivation for our study stems from the fact that a lot of information about a given finite subgroup $\Gamma \subset SU(N)$ are encoded in the geometry of $\mathbb{C}^N/\Gamma$ and its resolution[1] $\pi : M \to \mathbb{C}^N/\Gamma$. In particular, string theory predicts

[1]Throughout the paper, we use the standard mathematical notions of resolution, deformation, and their combination as means of desingularizations.

the Hodge numbers and the Euler characteristics of the resolution[2] of such Gorenstein orbifolds in terms of group theoretic data. It has subsequently led mathematicians to conjecture that there exists a so-called McKay's correspondence between the irreducible representation ring of a finite subgroup $\Gamma \subset SU(N)$ and the cohomology ring of the resolved manifold M, with certain maps between two ring structures. Thus, Gorenstein singularities are intrinsically connected to finite subgroups of $SU(N)$. One of the merits of string theory is that (super)-algebra and geometry often play complementary roles both in its quantization and compactifications. More precisely, certain non-linear sigma-models on Calabi-Yau (CY) manifolds admit purely algebraic descriptions in terms of $N = 2$ superconformal field theories. Therefore, since Gorenstein orbifolds are intrinsically related to finite groups, we could perhaps understand the vague connection between the modular invariants of $\widehat{SU(N)}$ WZW and the finite subgroups of $SU(N)$ if we could find a superconformal field theory (SCFT) which both encodes the WZW theory and describes certain Gorenstein singularities. This line of thinking furnishes the main theme in the work of Ooguri and Vafa [17] and in the present paper: Geometry and its SCFT description can provide a possible explanation for the mysterious connections among the coset models of the integrable lattices[3], the WZW modular invariants and the finite subgroups.

We will restrict ourselves to the case in which the relevant group is $SU(3)$ and the SCFT is a Landau-Ginzburg formulation of the $N = 2$ Kazama-Suzuki models. We will address two complementary questions: "Given a three-dimensional Gorenstein singularity, is there a LG theory which describes the non-linear sigma-model on this geometry and also encodes the $\widehat{SU(3)}$ modular invariants in some way?" and "Given a LG theory, is there a corresponding geometry?" At first sight, it appears that the singularities must be isolated in order to be related to the well-known Landau-Ginzburg formalism, and we are consequently

[2]Assuming that the crepant resolutions actually exist, that is.

[3]We will not explicitly address the integrable lattice models in this paper, but it should be possible to relate our discussion to those cases which are, in many aspects, non-supersymmetric cousins of the Kazama-Suzuki theories. We note that both the Kazama-Suzuki models and the continuum limit of the integrable lattice models based on $\widehat{SU(3)}$ diagonal modular invariants can serve as minimal matters for the (super) W_3-algebra. Furthermore, both of them are related to Toda and affine Toda theories under relevant perturbations.

led to search for all isolated Gorenstein singularities. We find that the requirement of isolated singularity imposes a strong constraint on the possible types of three-dimensional quotient singularities. It turns out that isolated Gorenstein singularities in three-dimensions cannot be realized as a hypersurface in $\mathbb{C}^4$ but only as complete intersections in higher dimensions, implying that they do not correspond to simple LG theories of the known type. In the opposite direction of pursuit, we face a similar problem; since LG superpotentials have isolated singularities, there is no direct way of relating the LG theory to a finite subgroup and its associated orbifold. We thus take an indirect approach to the problem of relating, if it is really possible, the LG theory to finite subgroups of $SU(3)$. What we are able to do is to look for the defining signatures of the LG theory in the resolved geometry of various Gorenstein orbifolds; in particular, we look for the chiral ring structure in the cohomology of the resolved manifold. As a result, we observe some peculiar matchings between the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki models at level n and the resolutions of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ Gorenstein orbifolds which actually have non-isolated singularities. Note that the level is different from the conjecture in [3]. We also observe that the toric resolution of the $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ Gorenstein orbifold yields a natural connection of the $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ finite subgroups to the modular invariants. For example, the compact exceptional divisors, which are degree 6 del Pezzo surfaces[4], are in one-to-one correspondence with the chiral primary operators of the Kazama-Suzuki model, and their intersections reproduce the Verlinde algebra and the associated diagonal modular invariant graphs of $\widehat{SU(3)}_n$ WZW at level n. There are also non-compact exceptional divisors which are ruled surfaces and which seem to classify the diagonal $\widehat{SU(2)}_{n+1}$ modular invariants. We give an explanation of these observations by noting that $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ has three $\mathbb{Z}_{n+3} \subset SU(2) \subset SU(3)$ subgroups whose quotient singularities are resolved by the ruled surfaces. Thus, our approach, if correct, seems to give us a canonical way of identifying the relevant elements of the $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ subgroups with the $\widehat{SU(3)}$ modular invariants. Namely, those non-trivial elements not contained in a subgroup of $SU(2) \subset SU(3)$ are in one-to-one correspondence with the del Pezzo surfaces and two-cycles in the resolved manifold, and they are the ones that classify the $\widehat{SU(3)}_n$ modular invariants. It thus seems that there is a mysterious connection between

[4]By a degree 6 del Pezzo surface, we mean $\mathbb{P}^2$ blown up at three points.

$N = 2$ SCFT and generally non-isolated three-dimensional Gorenstein singularities yet to be made more precise.

This paper is organized as follows: We begin by discussing what kind of subgroups $\Gamma \subset SU(3)$ leads to an isolated singularity $\mathbb{C}^3/\Gamma$ and the embedding of the resulting geometry as affine varieties in $\mathbb{C}^n$. We argue that none of these cases leads to known LG theories. §3 initiates the opposite view point and discusses the relevant details of the $\widehat{SU(3)}$ WZW and the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model. The following section §4 is a search for a corresponding geometry of the Kazama-Suzuki model. We analyze the blow-up geometry in both toric and algebraic geometry set-ups and show that the diagonal modular invariant graphs appear in the resolution diagrams. The paper concludes by addressing some open questions and puzzles. In Appendix, we prove a few facts regarding the three-dimensional isolated Gorenstein singularities.

Notations

We will adhere to the following conventions throughout this paper: We define the action of $\omega := \frac{1}{n}(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_n$ on $\mathbb{C}^3$ by $(z_1, z_2, z_3) \mapsto (q^{\alpha_1} z_1, q^{\alpha_2} z_2, q^{\alpha_3} z_3)$ where $q = \exp(2\pi i/n)$. By $\langle \frac{1}{n}(\alpha_1, \alpha_2, \alpha_3) \rangle$, we mean a cyclic group of order n generated by the element $\frac{1}{n}(\alpha_1, \alpha_2, \alpha_3)$. The subscripts in $SU(3)_n/SU(2) \times U(1)$ and $\widehat{SU(N)}_n$ represent the levels of the Kac-Moody algebras. Finally, by "dimension", we always mean complex dimension.

2 Sigma-Models and Landau-Ginzburg Theories

2.1 Isolated Singularities and LG Superpotentials

It is by now a familiar concept[5] that certain Calabi-Yau (CY) sigma-models admit equivalent descriptions in terms of exactly solvable $N = 2$ superconformal field theories (SCFT). Traditionally, such an equivalence has usually involved compact projective CY manifolds and $N = 2$ A-D-E minimal models. In [17], the consideration has been extended to ALE spaces which are non-compact two-dimensional CY manifolds. The natural question to ask then is whether one can extend the situation to three-dimensional non-compact CY manifolds obtained from desingularizations of Gorenstein singularities[6].

Let us first briefly review the essential ideas of [17] and see how to generalize them. The main ingredient is the fact that all ALE spaces can be represented as hypersurfaces in $\mathbb{C}^3$, and one can use the defining equations of those hypersurfaces as superpotentials in Landau-Ginzburg (LG) theories. For example, the singular limit of the A_n ALE space, which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_{n+1}$, can be represented as $x^2 + y^2 + z^{n+1} = 0$. The corresponding superpotential is then

$$W = \mu w^{-n-1} + x^2 + y^2 + z^{n+1} \tag{2.1}$$

where μ is a moduli parameter and the power of w has been chosen[7] to yield the correct total central charge $\hat{c} = 2$ of the sigma-model on the ALE space. Treating w as a parameter, we see that (2.1) describes a deformation of the A_n singularity. The claim is that in the degenerating limit, $\mu \to 0$, the above SCFT captures the physics of string theory on a singular ALE space.

[5]One of the first examples that have been studied describes the quintic in $\mathbb{P}^4$ as a tensor product of A_4 minimal models and plays an important role in mirror symmetry.

[6]For an elementary introduction to these terminologies, we refer the reader to [9].

[7]Recall that the central charge of a Landau-Ginzburg theory with a superpotential $W(x_1, \ldots, x_k)$ is given by $c = 3\sum_{i=1}^{k}(1 - 2q_i)$, where q_i is the $U(1)$ charge of the field x_i. As usual, we define $3\hat{c} = c$.

In a similar spirit, we now search for a possible Gorenstein orbifold in 3-dimensions whose defining equation can be used as a part of the LG superpotential. More precisely, we look for a desingularization of some quotient space of the form $\mathbb{C}^3/\Gamma$, where Γ is a discrete subgroup of $SU(3)$, whose certain degenerating limit admits a description in terms of a tensor product of $N = 2$ Kazama-Suzuki models [12]. One of the criteria for the superpotentials appearing in Landau-Ginzburg theories is that they must have only isolated singularities[8]. Thus, if we insist upon a naive generalization of the correspondence between the superpotential and the hypersurface equation, we want to consider only those orbifolds with isolated Gorenstein singularities. Incidentally in three-dimensions, it turns out that the requirement of isolated singularity imposes a very restrictive constraint on the possible types of orbifolds. Indeed in [23], Yau and Yu prove the following theorem:

Theorem 2.1. *The three-dimensional Gorenstein singularities of $\mathbb{C}^3/\Gamma$ are isolated if and only if Γ is abelian and every non-trivial element $g \in \Gamma$ does not have 1 as an eigenvalue.*

From this theorem, we can determine more precisely which three-dimensional orbifolds of the form $\mathbb{C}^3/\Gamma, \Gamma \subset SU(3)$, have only isolated singularities. Noting that any abelian finite group Γ can be written as a product of cyclic groups, i.e.

$$\Gamma = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_n} \ ,$$

we summarize the results, which are proven in the Appendix, as follows:

Corollary 2.1. *The Gorenstein orbifold $\mathbb{C}^3/\Gamma$ has only isolated singularities if and only if $\Gamma = \mathbb{Z}_k = \langle \frac{1}{k}(\alpha_1, \alpha_2, \alpha_3) \rangle$ such that $GCD(k, \alpha_i) = 1, \forall i$. This in particular implies that k has to be odd.*

We would now like to study a possible connection of these Gorenstein orbifolds with isolated singularities to $N = 2$ Landau-Ginzburg theories.

---

[8]Having non-isolated singularities can lead to problems such as an infinite dimensional chiral ring.

2.2 $\mathbb{C}^3/\Gamma$ as Hypersurfaces and Complete Intersections

In order to make a direct connection to Landau-Ginzburg models, as previously discussed for two-dimensions, we need to determine whether the Gorenstein orbifolds can be represented by hypersurfaces in $\mathbb{C}^4$ so that their equations can play the role of LG superpotentials. We will now show that an isolated 3-dimensional Gorenstein singularity actually cannot be embedded as a hypersurface in $\mathbb{C}^4$ but rather only as a complete intersection affine variety in higher dimensions.

Let us briefly describe how to embed a Gorenstein orbifold as an affine algebraic variety in some $\mathbb{C}^n$. We will closely follow [23], restricting our attention to three-dimensions. Let $S = \mathbb{C}[z_1, z_2, z_3]$ be a polynomial ring and S^Γ its subring of polynomials which are invariant under the action of $\Gamma : (z_1, z_2, z_3) \mapsto \Gamma(z_1, z_2, z_3)$. One first needs to find the minimal set $\{f_1, f_2, \ldots, f_n\}$ of generators of S^Γ as a $\mathbb{C}$-algebra. Let $\mathbb{C}[y_1, \ldots, y_n]$ be a polynomial ring associated to the generators, then there exists a ring homomorphism

$$v : \mathbb{C}[y_1, \ldots, y_n] \longrightarrow S \tag{2.2}$$

defined by the substitution map

$$v(F(y_1, \ldots, y_n)) = F(f_1, \ldots, f_n) \tag{2.3}$$

where $F(y_1, \ldots, y_n) \in \mathbb{C}[y_1, \ldots, y_n]$. Then, we have $\mathrm{Im}(v) = S^\Gamma \cong \mathbb{C}[y_1, \ldots, y_n]/K$, where $K := \ker(v)$ is an ideal with a minimal set of generators $\mathcal{R}_i$ called *relations*. Now, the relations define an affine algebraic subvariety $V_\Gamma \subset \mathbb{C}^n$:

$$V_\Gamma = \{(y_1, \ldots, y_n) \in \mathbb{C}^n |\ \mathcal{R}_i(y_1, \ldots, y_n) = 0, \forall i\} . \tag{2.4}$$

It has been shown in [2] that there exists a biholomorphism $\phi : \mathbb{C}^3/\Gamma \to V_\Gamma$, yielding the desired embedding of Gorenstein orbifolds as complete intersections in $\mathbb{C}^n$.

Now, using the above prescription, we have explicitly checked for many of the isolated Gorenstein singularities given in Corollary 2.1 that they cannot be embedded simply as a hypersurface in $\mathbb{C}^4$. For example, we find that $\mathbb{C}^3/\mathbb{Z}_3$ embeds in $\mathbb{C}^{10}$. The computations are very laborious and discouraging. In fact, it turns out that isolated singularities of hypersurfaces in four or higher dimensions can never be quotient singularities[9] [24].

Thus, we encounter a difficult problem that the isolated Gorenstein singularities cannot be simply represented as hypersurfaces in $\mathbb{C}^4$, and thus, there is no obvious Landau-Ginzburg description of the singularities. At best, we can realize them as complete intersections in higher dimensional embedding spaces $\mathbb{C}^n, n > 4$. It is still possible to study complete intersections in the LG approach, but it usually requires many extra variables and a non-trivial change of variables in the path integral. A more serious problem arises from the fact that many of the subgroups of $SU(3)$ seem to be "missing" in this analysis in the sense that they have non-isolated singularities and therefore cannot be related to the $\widehat{SU(3)}$ WZW modular invariants in this way. In particular, we have seen that $\mathbb{Z}_k$, for k even, does not give rise to isolated singularities and hence cannot be related to a LG superpotential in a direct way.

3 Modular Invariants and Kazama-Suzuki Models

It is well-known that the modular invariants of the $\widehat{SU(2)}$ WZW theories fall under an A-D-E classification [1], which also governs the classification of the finite subgroups of $SU(2)$ [14]. The connection between the two *a priori* unrelated classifications has been explained in [17] by using[10] $\frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)}$ Kazama-Suzuki models to describe[11] a certain de-

[9]This may be related to Schlessinger's Rigidity Theorem which states that quotient singularities of codimension three or greater does not have non-trivial deformations.

[10]up to $U(1)$ projections.

[11]Roughly speaking, the $\frac{SL(2)}{U(1)}$ factor describes the Feigin-Fuchs boson emanating from the singularity while the $\frac{SU(2)}{U(1)}$ factor describes the transverse directions surrounding the singularity. Furthermore, the $\frac{SU(2)}{U(1)}$ Kazama-Suzuki model is closely related to the $\widehat{SU(2)}$ WZW theory, and indeed, they arise from almost identical parafermionic representations. It is thus not very surprising that the partition functions of the $\frac{SU(2)}{U(1)}$ Kazama-Suzuki theory naturally contains the $\widehat{SU(2)}$ WZW modular invariants, up to different contributions from the $U(1)$ theta functions.

generating limit of the orbifolds $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2)$. More precisely, the sigma-model/Kazama-Suzuki correspondence states that the subgroup Γ is specified by the same *A-D-E* Dynkin diagram which classifies the $\widehat{SU(2)}$ modular invariant appearing in the partition function of the Kazama-Suzuki model. For example, the A_{n-1} diagonal modular invariant arises in the Kazama-Suzuki model at level $(n-2)$ for the $\frac{SU(2)}{U(1)}$ sector, and the same Dynkin diagram classifies the $\mathbb{Z}_n$ subgroup. It has been argued in [17] that the tensored Kazama-Suzuki model at this level captures the physics of the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ when the B-field has been turned off. We will consider here the simplest generalization[12] of [17], namely $\frac{SL(2)}{U(1)} \times \frac{SU(3)}{SU(2)\times U(1)}$, and see whether there exists a corresponding geometry of some quotient singularities. In this section, we briefly review some useful facts about the Kazama-Suzuki models, and we will devote the next section to finding the candidate geometry. As previously mentioned, the hypersurface with an isolated singularity defined by the superpotential cannot be a quotient singularity in three-dimensions. Thus, if the Kazama-Suzuki model is related to quotient singularities at all, then matching the equations of the affine subvarieties representing Gorenstein orbifolds with superpotentials, as done in [17] for the ALE spaces, does not work here, and we need to consider more indirect paths.

3.1 $\widehat{SU(3)}$ WZW and Kazama-Suzuki Models

The simplest $N = 2$ coset models are based on hermitian symmetric spaces, more specifically complex Grassman manifolds $SU(n+m)/SU(n) \times SU(m) \times U(1)$. Algebraically, these types of Kazama-Suzuki (KS) models are based on the GKO coset construction of

$$G(k,m,n) = \frac{SU(k+m)_n \times SO(2km)_1}{SU(k)_{m+n} \times SU(m)_{k+n} \times U(1)_{km(k+m)(k+m+n)}} \tag{3.1}$$

where n is the level of $SU(k+m)$ and so on [12]. These models are manifestly symmetric in k and m, and they actually turn out to be also symmetric in any permutation of $k, m,$ and n, generalizing the level-rank duality of WZW models. The $\widehat{SU(3)}_n$ WZW theory is related[13]

[12]Generalizations to higher dimensions will appear in [19].

[13]In fact, in many ways, G(k,1,n) Kazama-Suzuki models are $N = 2$ generalizations of the $\widehat{SU(k+1)}_n$ WZW theories.

to the $G(2,1,n) = SU(3)_n \times SO(4)_1/SU(2)_{n+1} \times U(1)_{6(n+3)}$ model [5], which is the one that we will consider in this paper and denote henceforth by $SU(3)_n/SU(2) \times U(1)$. Before analyzing its Landau-Ginzburg formulation, we will briefly discuss the states and partition functions of the model in a more general context.

Let Λ be a highest weight of $SU(3)$ at level n, a that of $SO(4)$ at level 1, and λ that of $SU(2) \times U(1)$ at level $n+1$. Then, a general field $\Phi_\lambda^{\Lambda,a}$ of the coset theory is defined by the decomposition

$$G^\Lambda V^a = \sum_\lambda \Phi_\lambda^{\Lambda,a} H^\lambda , \tag{3.2}$$

where G, V and H are fields in the indicated representations of $SU(3)$, $SO(4)$ and $SU(2)\times U(1)$, respectively. The affine characters decompose in a similar way, and the character of the coset theory is the branching function $\chi_\lambda^{\Lambda,a}$ in

$$\chi^\Lambda \chi^a = \sum_\lambda \chi_\lambda^{\Lambda,a} \chi^\lambda . \tag{3.3}$$

The modular invariant partition function of the coset theory is then obtained by taking products of left- and right-handed sectors

$$Z = \frac{1}{K} \sum_{\substack{\Lambda,\bar{\Lambda},a,\lambda,\bar{\lambda} \\ C(\Lambda,\lambda),\, C(\bar{\Lambda},\bar{\lambda})}} \chi_\lambda^{\Lambda,a} \mathcal{N}_{\Lambda,\bar{\Lambda}} \mathcal{M}_{\lambda,\bar{\lambda}} \chi_{\bar{\lambda}}^{\bar{\Lambda},a}, \tag{3.4}$$

where $\mathcal{N}$ and $\mathcal{M}$ are matrices defining the modular invariants of $\widehat{SU(3)}_n$ and $\widehat{SU(2)}_{n+1}$ WZW, the summation is restricted to satisfy a certain condition $C(\Lambda, \lambda)$, and K is the order of the proper external automorphism group[14] which identifies fields in the coset [6].

The situation simplifies if we consider only the chiral[15] scalars. Then, the restriction $C(\Lambda, \lambda)$ just requires that $\Lambda = \lambda$ such that picking the $SU(3)$ integrable highest weight, after the field identifications, uniquely fixes the weights of $SU(2) \times U(1)$ in the decomposition (3.2).

[14]The factor of $1/K$ is included to take care of the so-called field identification problem. For the $SU(m)_n/SU(m-1)_{n+1} \times U(1)$ theory, $K = m(m-1)$.

[15]As usual, a field is chiral if its $N = 2$ $U(1)$ charge is twice its conformal dimension.

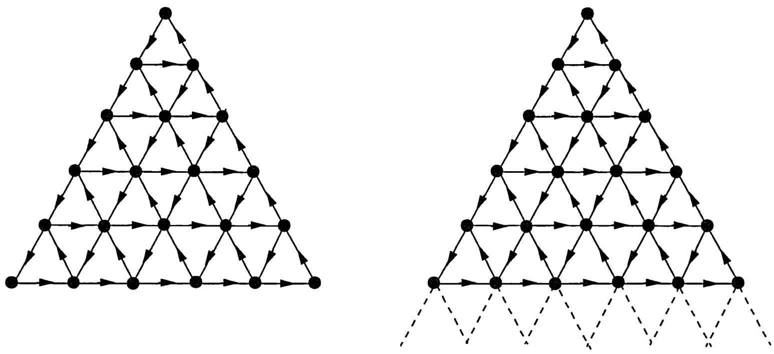

Figure 1: The diagonal $\widehat{SU(3)}_5$ WZW $\mathcal{A}$-modular invariant graph at level 5. For general level n, the diagram extends until there are $n+1$ nodes on each side of the boundary. In these diagrams, the nodes represent the primary operators.

At level n, there are $(n+1)(n+2)/2$ such chiral scalars corresponding to the integrable highest weights, or primary fields, of the $\widehat{SU(3)}$ WZW, and they are precisely the scalar components of the $N=2$ LG superfields realizing the $SU(3)_n/SU(2)\times U(1)$ KS theory. We should also note that the LG formulation corresponds to only the diagonal modular invariants, i.e. to diagonal matrices $\mathcal{N}, \mathcal{M}$. For non-diagonal modular invariants, there are generally no corresponding LG descriptions. Hence, the LG theory to be discussed below captures the diagonal modular invariants of the $\widehat{SU(3)}$ WZW theory.

The WZW models comprise a very vast and rich subject, and for the relevant facts regarding the modular invariants and the Verlinde algebras of the $\widehat{SU(N)}$ WZW theories, we refer the reader to [4, 9]. As an illustration, Figure 1 shows an $\widehat{SU(3)}$ $\mathcal{A}$-modular invariant at level 5. These diagrams will appear again in the toric resolution of $\mathbb{C}^3/\mathbb{Z}_8\times\mathbb{Z}_8$.

3.2 $SU(3)/SU(2)\times U(1)$ Kazama-Suzuki Model

Not all Kazama-Suzuki theories have Landau-Ginzburg realizations, but as mentioned in the previous subsection, the particular model of our interest does have one [6, 13]. Landau-Ginzburg theories capture the chiral aspects of $N=2$ SCFT, and in the case of $SU(3)_n/SU(2)\times U(1)$, it is the chiral part that is closely related to the $\widehat{SU(3)}_n$ WZW theory[16]. Furthermore, LG theories are known to describe a different phase of CY non-linear sigma-models [21]. Thus, the $N=2$ LG formulation of the $SU(3)_n/SU(2)\times U(1)$ KS model is the most natural setting for studying the connection among finite subgroups, WZW theories, and geometry.

The superpotentials W_n for the $SU(3)/SU(2)\times U(1)$ Kazama-Suzuki coset models at level n are given by

$$W_n(x,y) = q_1^{n+3} + q_2^{n+3} \,, \tag{3.5}$$

where it is understood that the superpotential is actually a function of the symmetric polynomials $x = q_1 + q_2$ and $y = q_1 q_2$. In terms of the x, y variables, the expressions for a few low n are

$$\begin{aligned}
W_1 &= x^4 - 4x^2y + 2y^2 \\
W_2 &= x^5 - 5x^3y + 5xy^2 \\
W_3 &= x^6 - 6x^4y + 9x^2y^2 - 2y^3 \\
W_4 &= x^7 - 7x^5y + 14x^3y^2 - 7xy^3 \\
W_5 &= x^8 - 8x^6y + 20x^4y^2 - 16x^2y^3 + 2y^4 \\
W_6 &= x^9 - 9x^7y + 27x^5y^2 - 30x^3y^3 + 9xy^4 \\
W_7 &= x^{10} - 10x^8y + 35x^6y^2 - 50x^4y^3 + 25x^2y^4 - 2y^5 \,,
\end{aligned} \tag{3.6}$$

where we have rescaled by an over-all normalization. We see that W_n is quasi-homogeneous if we assign x and y of weights 1 and 2, respectively. The number of chiral primary fields at level n is $\frac{(n+2)(n+1)}{2}$ which, as explained before, matches the number of primary fields of the $\widehat{SU(3)}_n$ WZW theory.

In the rest of the paper, we will be interested in a possible relation between the above $N=2$ LG theory and Gorenstein singularities of the type $\mathbb{C}^3/\mathbb{Z}_{n+3}\times\mathbb{Z}_{n+3}$. To produce the correct total central charge $\hat{c}=3$, we need to tensor the $SU(3)/SU(2)\times U(1)$ KS to an extra KS model. Since we are interested in non-compact orbifolds, with the hindsight from the two-dimensional black holes [22], we consider the $SL(2,\mathbb{R})_k/U(1)$ model whose central charge at level k is $1+2/(k-2)$.

[16]It has been shown in [5] that generalized Chebyshev integrable deformations of the $SU(N)_n/SU(N-1)\times U(1)$ LG theories yield the correct Verlinde algebra of the $\widehat{SU(N)}_n$ WZW.

Since the central charge of $SU(3)/SU(2)\times U(1)$ at level n is $2n/(n+3)$,

$$\hat{c}_{\text{Total}} = \hat{c}_{\frac{SU(3)_n}{SU(2)\times U(1)}} + \hat{c}_{\frac{SL(2)_k}{U(1)}} = 3 \implies k = \frac{n+9}{3} \ . \tag{3.7}$$

The superpotential for the LG realization of the $SL(2,\mathbb{R})_k/U(1)$ model is $W_k = t^{2-k}$. Hence, the total superpotential for the tensor product theory $\frac{SL(2,\mathbb{R})_{(n+9)/3}}{U(1)} \times \frac{SU(3)_n}{SU(2)\times U(1)}$ is

$$W_{\text{T}}(t,x,y,z,w) = \mu t^{-\frac{n+3}{3}} + q_1^{n+3} + q_2^{n+3} + z^2 + w^2 \ , \tag{3.8}$$

where again the expression should be understood as a function of x and y, and μ is a moduli parameter which we set equal to zero in the singular limit. Note also that we have added two more variables whose quadratic terms do not affect the chiral ring. Now, as in [17], we will go to the patch where $t \neq 0$ and think of $W_{\text{T}}(1,x,y,z,w) = 0$ as defining a hypersurface in $\mathbb{C}^4$, which we hope to relate to certain Gorenstein orbifolds.

4 $\mathbb{C}P^2$ Kazama-Suzuki and $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$

In this section, we present some evidences that the Landau-Ginzburg formulation of the $SU(3)_n/SU(2) \times U(1)$ KS model may be related to a desingularization of the $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ orbifolds. Because of Theorem 2.1, given a KS LG superpotential, or a hypersurface with an isolated singularity for that matter, there is no canonical way of relating it to a finite subgroup of $SU(3)$ and its associated Gorenstein orbifold.

It is important to note at this point that the natural physical degrees of freedom in a LG theory are the complex deformations of the superpotential by chiral ring elements. The KS LG theories thus, if they are related to Gorenstein orbifolds in some way, describe a deformation, not resolution, of the singularities. Deformation and resolution of singular orbifolds in three-dimensions are two very different processes, generally leading to topologically distinct manifolds. Thus, a resolution of an orbifold does not necessarily carry information about the structure of a deformed manifold; but amazingly, it sometimes does. An example is the phenomenon occurring in [20] in the context of discrete torsion.

In that case, $T^2 \times T^2 \times T^2/\mathbb{Z}_2 \times \mathbb{Z}_2$ can be either completely resolved without discrete torsion or deformed in terms of the invariant variables in the presence of discrete torsion, resulting in 64 remaining conifold singularities. The two desingularizations are argued to be mirror pairs. Motivated by this interesting case and the ease with which resolutions can be studied, we propose a simple but naive step towards finding possible subgroups of $SU(3)$ that can be related to the KS LG theories: We search for resolutions of $\mathbb{C}^3/\Gamma, \Gamma \subset SU(3)$, whose classical cohomology encodes the chiral ring structure of the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model. An *ad hoc* proposal such as ours is worth considering if and perhaps only if it meets a success; but surprisingly, it has. In this section, we will see that the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ seems to be closely related to our coset theory.

It turns out that considering the subgroups of the form $\mathbb{Z}_n \times \mathbb{Z}_n$ cures some of the difficulties previously encountered, while, at the same time, introducing new obstacles. The group

$$\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3} = \langle \frac{1}{n+3}(\alpha, -\alpha, 0) \ , \ \frac{1}{n+3}(0, \alpha, -\alpha) \rangle \ , \qquad (4.1)$$

where $\exp(2\alpha\pi i/(n+3))$ is a primitive $(n+3)$-th root of unity, is actually a maximal finite subgroup of $SU(3)$ consisting of all cyclic elements of order $(n+3)$. Thus, we seem to have incorporated many of the "missing" subgroups of $SU(3)$ in this approach. On the other hand, the orbifold $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ has singularities along subvarieties, and thus, their defining hypersurface equations cannot be used as LG superpotentials at first sight[17]. We nevertheless argue that the resolved geometry of that type appears to classify the $\widehat{SU(3)}$ as well as $\widehat{SU(2)}$ WZW modular invariants.

We first study the geometry of the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ orbifold singularities both from the toric geometry and algebraic geometry points of view. We then comment on the desingularization by complex structure deformations and speculate that the LG theories describe the geometry of certain deformations of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$.

[17]Such superpotentials lead to degenerate conformal field theories, and the chiral ring is infinite dimensional.

4.1 Cohomology of the Resolution and McKay Correspondence

There are several ways to study the resolution of three-dimensional Gorenstein singularities and compute the Hodge numbers of the resolution. It is known that all crepant resolutions are related to each other by flops. Let $\pi : M \to \mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ be a resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$. Then, using the *age grading* of the $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ subgroup introduced in [10] in the context of McKay correspondence[18], we find:

$$\begin{aligned} \#\{\text{elements of age } 0\} &= h^0(M,\mathbb{Q}) = 1 \\ \#\{\text{elements of age } 1\} &= h^2(M,\mathbb{Q}) = \frac{(n+7)(n+2)}{2} \\ \#\{\text{elements of age } 2\} &= h^4(M,\mathbb{Q}) = \frac{(n+1)(n+2)}{2}, \end{aligned} \tag{4.2}$$

where h^i are the Hodge numbers of the resolved manifold. These numbers can also be obtained from toric geometry, as will be discussed in the next subsection.

The sum of the Hodge numbers, which is just the Euler characteristics of M in this case, is equal to $(n+3)^2$ which is precisely the order of the group $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$. The number $(n+3)^2$ is also the number of irreducible representations of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, and thus, we see that the predictions of the McKay correspondence–or string theory, whichever the reader prefers–are well satisfied.

4.2 Toric Resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$

The Hodge numbers (4.2) can also be computed directly from geometry by using toric blow-ups and Poincaré duality. For background materials on toric varieties, see [16]. The cone for the unresolved $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ is defined over a convex polygon given in Figure 2 which is a hyperplane cross-section of the first "quadrant" of the standard integral lattice $\mathbb{Z}^3$ by a plane passing through $(n+3,0,0), (0,n+3,0)$, and $(0,0,n+3)$.

To resolve the singularities, we need to add in all lattice points lying on the polygon and triangulate the cone to produce triangles with unit

[18]For a review on McKay correspondence, see [9, 10].

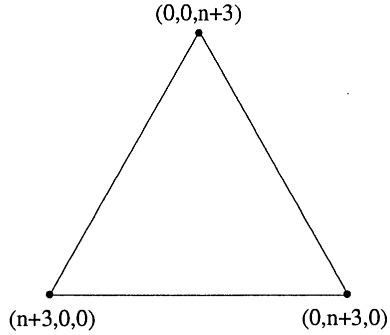

Figure 2: The polygon for unresolved $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$.

area in appropriate units. As an example, Figure 3 gives the polygon for a particular complete resolution of $\mathbb{C}^3/\mathbb{Z}_8 \times \mathbb{Z}_8$, other crepant resolutions being related to this one by a sequence of flops.

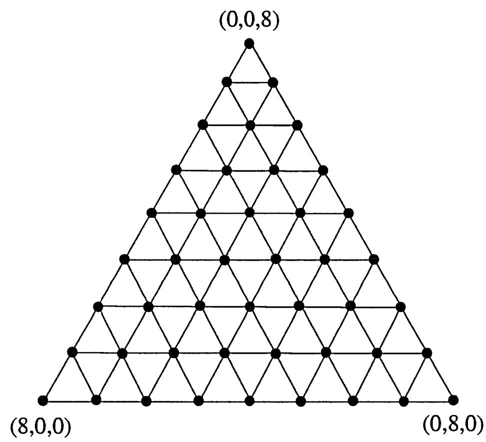

Figure 3: The polygon for completely resolved $\mathbb{C}^3/\mathbb{Z}_8 \times \mathbb{Z}_8$. The toric fan consists of cones defined over this polygon.

Here, the new points on the outer boundary of the polygon represent non-compact exceptional divisors which are ruled surfaces lying along the three coordinate axes of $\mathbb{C}^3/\mathbb{Z}_8 \times \mathbb{Z}_8$. In the general case of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, we will have $3(n+2)$ ruled surfaces as non-compact exceptional divisors. We can understand their origin as follows: The group $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ has two generators $\omega = \frac{1}{n+3}(\alpha, -\alpha, 0)$ and $\eta = \frac{1}{n+3}(0, \alpha, -\alpha)$. Then, there are three $\mathbb{Z}_{n+3} \subset SU(2)$ subgroups of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, namely $\langle\eta\rangle$, $\langle\omega\eta\rangle$, and $\langle\omega\rangle$, which fix z_1, z_2, and z_3 coordinates of $\mathbb{C}^3$, respectively, and produce the familiar A_{n+2} ALE singularities along those axes. The A_{n+2} singularities in ALE spaces are resolved by a chain of $(n+2)$ $\mathbb{P}^1$ blow-ups which, upon fibration over the coordinate axes, become the $3(n+2)$ ruled surfaces that we see in the toric resolution. Now, since A_{n+2} ALE singularities classify the diagonal $\widehat{SU(2)}_{n+1}$ WZW modular invariants, what we have just found is that the ruled surfaces in the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ classify the same invariants.

The remaining $(n+1)(n+2)$ non-trivial elements of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ not contained in those three $\mathbb{Z}_{n+3} \subset SU(2)$ subgroups are all seen not to possess an eigenvalue 1, and thus, they lead to an isolated quotient singularity at the origin which is resolved by introducing $(n+1)(n+2)/2$ degree 6 del Pezzo surfaces. These compact exceptional divisors are the 21 interior points in Figure 3, which is the correct number of del Pezzo surfaces for $n = 5$. Immediate from the toric digram is another observation that the $\widehat{SU(3)}_n$ modular invariants are classified by the $(n+1)(n+2)/2$ compact del Pezzo exceptional divisors. The fourth Hodge number h^4 can be computed by applying the Poincaré duality to the compact[19] second cohomology $H^2_c(M, \mathbb{Q})$, which is dual to the del Pezzo surfaces, and thus we have $(n+1)(n+2)/2$ two-cycles which appear in the intersections of the exceptional divisors. We now see that of $(n+1)(n+2)$ non-trivial elements of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ not contained in the $\mathbb{Z}_{n+3} \subset SU(2)$ subgroups, half of them corresponds to the del Pezzo surfaces and the other half to the non-trivial two-cycles. Furthermore, the aforementioned McKay correspondence is clearly satisfied, and there is indeed a one-to-one correspondence between the irreducible representations of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ and the cohomology elements of the resolved manifold [10].

4.3 Intersection Homologies and Blow-ups

We now study the classical intersection theory on the resolved manifold and see that it reproduces a perturbed chiral ring structure of the coset theory. We do not know whether the full quantum intersection theory would correspond to the unperturbed chiral ring, which happens to be the cohomology ring of the Grassmannian $U(n+2)/U(2) \times U(n)$

[19]Because we are dealing with non-compact spaces, we need to take some caution when applying mathematical facts that are familiar from studying compact spaces.

satisfying the Schubert calculus.

In this section, we argue that the classical intersection homology of the resolved $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ captures the diagonal Verlinde algebras of the $\widehat{SU(3)}_n$ and $\widehat{SU(2)}_{n+1}$ WZW theories. There are two ways of illustrating this. The first way is to use the Stanley-Reisner relations in toric geometry to find the cohomology ring structure. In fact, without going into details, a rough picture of the intersection homology comes from the following facts:

1. Two divisors, which are represented by points on the polygon, intersect along a two-cycle if and only if their points are connected by a line in the polygon.

2. Three divisors intersect at a point if and only if their corresponding points form three vertices of a triangle in the triangulation of the polygon.

Hence, just by looking at the toric resolution diagrams, the simple-minded rule that two primary fields fuse if and only if their corresponding exceptional divisors intersect gives a relation[20] between the fusion matrices of the $\widehat{SU(3)}$ and $\widehat{SU(2)}$ WZW theories and the intersection homology of the resolved manifold. That is, the intersections of the ruled surfaces give the Verlinde algebra of the $\widehat{SU(2)}_{n+1}$ WZW while the intersections of the del Pezzo surfaces yield that of the $\widehat{SU(3)}_n$ WZW.

The second way to see the details of the resolved geometry of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ is to blow up explicitly along subvarieties which are loci of the singularities. We will just sketch the main ideas by blowing up along one of the coordinate axes. As will be subsequently discussed, the equation $xyz = w^{n+3}$ describes the orbifold $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ as an affine subvariety $M \subset \mathbb{C}^4$. It has three lines of singularities along $xy = 0, xz = 0$, and $yz = 0$. Without a loss of generality, let us take

[20]This rule is a generalization of that in two-dimensions where the intersection matrices of exceptional $\mathbb{P}^1$ divisors reproduce the fusion matrices of $\widehat{SU(2)}$ WZW. But, here the analogy is not completely satisfactory, because the divisors intersect along any one of the $(n+1)(n+2)/2$ two-cycles in the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$.

the first locus $xy = 0$ and blow up along the z-axis. Define

$$\Delta = \{(x,y,z,w) \times (s,t,u) \in \mathbb{C}^4 \times \mathbb{P}^2 \mid xt = ys,\ xu = ws,\ yu = wt\}\ . \tag{4.3}$$

We have effectively replaced $\mathbb{C}^4$ by $\mathbb{C}^1 \times \mathcal{O}_{\mathbb{P}^2}(-1)$, or equivalently, we have replaced the origin of $\mathbb{C}^4$ with an exceptional divisor $E = \mathbb{P}^2$ so that $\pi : \Delta \backslash E \to \mathbb{C}^4 \backslash \{0\}$ is an isomorphism. Away from $x = y = w = 0$, $\pi^* M$ is just its 1-1 pre-image. To understand the blow-up geometry at $x = y = w = 0$ and $z \neq 0$, we consider a line passing through a point (x,y,z,w) on the hypersurface and $(0,0,z,0)$ at constant z. By construction (4.3), each such a line defines a point on the $\mathbb{P}^2$. Now, we take the limit where the point on the hypersurface approaches $(0,0,z,0)$ in all possible directions and determine the corresponding limiting points on the $\mathbb{P}^2$. Intuitively, each tangent line at the singular point $(0,0,z,0)$ along the constant z subvariety becomes a point on the exceptional divisor $\mathbb{P}^2$, and thus in the blow-up picture the hypersurface will end on a one-dimensional subvariety on $\mathbb{P}^2$ as follows: Pick a point $[s,t,u] \in \mathbb{P}^2$ with a fixed length $|s|^2 + |t|^2 + |u|^2 = 1$, and consider the point $(\epsilon s, \epsilon t, z, \epsilon u) \in \mathbb{C}^4$, where $|\epsilon| << 1$. Now, demanding that the point $(\epsilon s, \epsilon t, z, \epsilon u)$ lies on the hypersurface $xyz = w^{n+3}$, we have

$$stz\epsilon^2 \ = \ \epsilon^{n+3} u^{n+3} \approx 0 \quad \Longrightarrow \quad s = 0 \ \text{ or } \ t = 0. \tag{4.4}$$

Hence, we see that the hypersurface ends on two intersecting $\mathbb{P}^1$'s in $\mathbb{P}^2$ defined by the homogeneous coordinates $s = 0$ and $t = 0$. Since the singularities actually occur along the z-axis, each $\mathbb{P}^1$ is fibrated over the axis and thus defines a ruled surface.

We also want to know how the hypersurface has transformed near the exceptional $\mathbb{P}^2$, and for that purpose, we need to view Δ as $\mathbb{C}^1 \times \mathcal{O}_{\mathbb{P}^2}(-1)$, where $\mathcal{O}_{\mathbb{P}^2}(-1)$ is the universal bundle of $\mathbb{P}^2$. On the patch where $u \neq 0$, the good coordinates of the $\mathbb{P}^2$ are

$$\alpha = \frac{s}{u} \ \text{ and } \ \beta = \frac{t}{u}. \tag{4.5}$$

In these variables, we have $x = w\alpha$ and $y = w\beta$, and the hypersurface becomes

$$xyz = w^{n+3} \quad \Longrightarrow \quad w^2(\alpha\beta z - w^{n+1}) = 0\ . \tag{4.6}$$

In the new coordinates, $w = 0$ corresponds to $x = y = w = 0$ and thus to the entire $\mathbb{P}^2$. So, the intersection of the hypersurface with $\mathbb{P}^2$ is given by

$$\alpha\beta z - w^{n+1} = 0, \tag{4.7}$$

which is just the original equation with its degree diminished by 2, and the resulting singularity structures (4.7) are the same as before. That is, it again has singularities along the z-axis at $\alpha\beta = w = 0$, which is precisely at $s = t = 0$ where two $\mathbb{P}^1$'s meet on $\mathbb{P}^2$. This kind of singularities is exactly the same as that appearing in the resolution of A_{n+2} ALE spaces, except here we introduce a pair of ruled surfaces rather than $\mathbb{P}^1$'s with each blow up. Repeating this procedure until we have resolved all the singularities along the z-axis will thus produce $(n+2)$ ruled surfaces intersecting in a chain, which clearly corresponds to the points on the outer edge of the toric picture. We have now understood the 3 chains of $(n+2)$ ruled surfaces whose intersections clearly resemble the Verlinde algebra of the $\widehat{SU(2)}_{n+1}$ WZW theory. In retrospect, we should have expected this phenomenon because, as previously discussed, these non-isolated singularities result from the $\mathbb{Z}_{n+3} \subset SU(2) \subset SU(3)$ and we know from [17] that A_{n+2} ALE spaces classify the $\widehat{SU(2)}_{n+1}$ diagonal modular invariants.

Analysis of the extra singularity at the origin can be performed in a similar way, in principle, but the computation is highly non-trivial and we omit its presentation in this paper. For that purpose, it is much easier to resort to the toric resolutions. We have previously seen that the extra quotient singularities at the origin arise from those elements of $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ which are not contained in a $SU(2)$ subgroup of $SU(3)$. Half of those non-trivial elements corresponds to the $(n+1)(n+2)/2$ del Pezzo surfaces, and the other half should correspond to $(n+1)(n+2)/2$ two-cycles, which we have not been able to analyze in any detail[21]. Thus, if we were to relate the finite subgroups $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ to the diagonal $\widehat{SU(3)}_n$ WZW modular invariants, we need to look at the del Pezzo divisors over the origin. Indeed, the intersections of those del Pezzo surfaces seem to reproduce the Verlinde algebra of $\widehat{SU(3)}_n$ WZW at level n and classify the corresponding diagonal modular invariant. A more precise formulation of the correspondence would have to realize the fusion coefficients as some functions[22] of the intersection homology, which would require an understanding of the two-cycles.

---

[21] To do so, one really needs to study the Stanley-Reisner ideals to find the cohomology generators.

[22] Our speculation that the primaries fuse if and only if the del Pezzo surfaces intersect along some two cycle seems too naive at the moment.

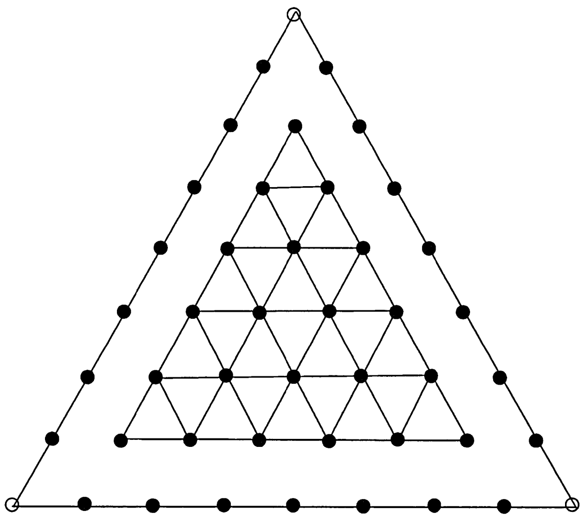

Figure 4: The fan for the resolved $\mathbb{C}^3/\mathbb{Z}_8 \times \mathbb{Z}_8$ encodes the $\widehat{SU(3)}_5$ and $\widehat{SU(2)}_6$ modular invariants.

We now need to know how the $\widehat{SU(2)}_{n+1}$ and $\widehat{SU(3)}_n$ WZW theories are related to the $SU(3)_n/SU(2)\times U(1)$ Kazama-Suzuki model. Besides the fact that these theories arise in the GKO coset construction of the KS theory, recall the following facts about the WZW models and the LG formulation of the KS theory [5]: The $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model has $(n+1)(n+2)/2$ chiral primary fields and under the perturbation of the LG superpotential by generalized Chebyshev polynomials, the chiral ring structure reproduces the $\widehat{SU(3)}_n$ WZW fusion coefficients. Hence, the chiral primaries of the KS theory are in one-to-one correspondence with the primary fields of the $\widehat{SU(3)}_n$ WZW theory, and the chiral algebra is the homogeneous part of the Verlinde algebra. This correspondence between the deformed chiral ring and the WZW Verlinde algebra actually generalizes to all $\mathbb{C}P^N$ KS and $\widehat{SU(N+1)}$ WZW. Furthermore, the Landau-Ginzburg formulation of the Kazama-Suzuki theory at level n is obtained from the diagonal modular invariants of the $\widehat{SU(3)}_n$ WZW at level n, which is classified by diagrams such as Figure 1, and those of the $\widehat{SU(2)}_{n+1}$ WZW at level $(n+1)$, which is classified by the A_{n+2} Dynkin diagram. Incidentally, we have just seen that the toric resolution diagram for $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ contains both the $\widehat{SU(3)}_n$ WZW modular invariant graph as a subgraph describing the del Pezzo surfaces and the $\widehat{SU(2)}_{n+1}$ WZW modular invariant graphs on the outer edges. For $n = 5$, we display the observation in Figure 4. Whether this strange, but general, phenomenon is a complete fluke or is actually in line with the attempt to classify the $\widehat{SU(3)}_n$ WZW modular invariants using finite subgroups remains to be seen. In order to prove that there actually exist underlying relations among the WZW modular invariants, the subgroups $\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ and the orbifolds $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, we would need a SCFT description of the Gorenstein orbifolds. Turning things around, we speculate that, given the close connections between the KS and WZW models, the above observations seem to suggest that the SCFT is likely to contain the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model as one of its factors. The LG SCFT however does not describe the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, but rather some deformation or a combination of both.

We devote the remainder of the paper to examining the above observations and their consequences.

4.4 Hypersurfaces and LG Theories

Our ultimate interest in studying any kind of correspondences among seemingly unrelated objects lies in understanding the *a priori* reason for such occurrences. The status of the $SU(2)$ cases is much more well-founded than the fairly untouched $SU(3)$ counterparts. Thus, without any pretense of rigor or fallacious confidence, we want to devote the rest of this paper in making several comments which may shed some light for future efforts.

For our study, the work of Joyce plays an important role [11]. Given a Gorenstein orbifold $\mathbb{C}^3/\Gamma$, we can often find a family of desingularizations $\pi : M \to \mathbb{C}^3/\Gamma$ carrying geometric structures that are compatible with Calabi-Yau conditions and which approach the orbifold geometry at a degenerating limit. Using deformations of codimension two singularities, Joyce has shown that there are in fact many topologically distinct families of Calabi-Yau desingularizations, with different[23] Hodge numbers and Euler characteristics. Indeed, mathematically, desingularizing orbifold singularities is an extremely complicated and laborious

[23]It is believed that all crepant resolutions in any dimension give rise to the string theory orbifold Euler characteristics and Hodge numbers, when things are properly defined.

process. Even for the simplest case of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, Joyce has found nine distinct Calabi-Yau desingularizations. Similarly, in the case of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, there are thousands of different Calabi-Yau desingularizations, two of which are selected out by string theory with and without a discrete torsion [20]. Why do other desingularizations not have physical realizations? Is it possible that there are physical desingularizations of which we are presently unaware?

So far, we have seen that the resolution of the $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ Gorenstein orbifold seems to reproduce the Verlinde algebras and the modular invariants of the $\widehat{SU(3)}_n$ and $\widehat{SU(2)}_{n+1}$ WZW theories, which arise in the construction of the $SU(3)_n/SU(2) \times U(1)$ KS model. We interpret this phenomenon as implying that the KS LG theory could be describing some kind of a deformation of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ in such a way that upon blowing down the the exceptional divisors in the resolution and then deforming the singularities, the information about the intersection homology on the resolved manifold gets transmitted to the chiral ring of the KS theory. It is not clear to us how to interpret the chiral ring geometrically in terms of the deformed manifold, although as will be subsequently discussed, studying the intersections of vanishing cycles on the Milnor lattice of the KS LG affine variety gives a suggestion. Interestingly, we have found a coordinate transformation which transforms the KS superpotential into the hypersurface equation for $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, but the transformation is not everywhere well-defined.

The first step in trying to desingularize the Gorenstein singularity by deformation is to embed the orbifold in $\mathbb{C}^4$ as a hypersurface, and then, deform the algebraic equation while maintaining the Calabi-Yau properties. Using the procedure of [23], we can represent the orbifold $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ as a hypersurface in $\mathbb{C}^4$ as follows: The independent, invariant monomials that can be constructed out of $(z_1, z_2, z_3) \in \mathbb{C}^3$ are

$$\begin{aligned} x &= z_1^{n+3} \\ y &= z_2^{n+3} \\ z &= z_3^{n+3} \\ w &= z_1 z_2 z_3 \ , \end{aligned} \tag{4.8}$$

and it can be checked that they generate the invariant subring $S^{\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}}$ of the polynomial ring $\mathbb{C}[z_1, z_2, z_3]$. The relation among the invariants is

$$xyz = w^{n+3} \ , \tag{4.9}$$

which embeds $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ in $\mathbb{C}^4$.

Now, note the curious fact that there is a singular coordinate transformation that maps the Kazama-Suzuki superpotential to an expression similar to the form (4.9): Consider the redefinitions

$$x = \beta^{-\frac{1}{n+3}}(1+\alpha) \quad , \quad y = \alpha\beta^{-\frac{2}{n+3}} \tag{4.10}$$

in the superpotentials (3.6) or (3.5). Then, the superpotentials can be written as

$$W_n = \frac{1}{\beta}\left(1 + \alpha^{n+3} + \beta z w\right) \ , \tag{4.11}$$

where the term inside the parenthesis is a deformation of the expression (4.9). It seems to suggest that the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model is indeed describing a complex deformation of the Gorenstein orbifold $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$, but we do not understand the validity of this argument since the coordinate transformation has a non-constant Jacobian

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{\beta^{-\frac{3}{n+3}-1}}{n+3}(1-\alpha) \tag{4.12}$$

which is not everywhere well-defined.

The motivation for the coordinate transformation (4.10) comes from the fact that the $SU(3)_n/SU(2) \times U(1)$ Kazama-Suzuki model possesses a discrete $\mathbb{Z}_{n+3}$ symmetry and thus, as it is familiar from standard Gepner constructions, that we really need to consider a $\mathbb{Z}_{n+3}$ orbifold of the KS LG theory. It is easy to see that the coordinates α and β are $\mathbb{Z}_{n+3}$ invariant coordinates.

Besides the deformed chiral ring structure of the KS theories, there are other evidences that the KS model is an $N = 2$ analogue of the $\widehat{SU(3)}_n$ WZW theory. For example, regarding the $SU(3)_n/SU(2) \times U(1)$ KS LG superpotential as defining an affine variety actually leads to very interesting results. In [8], it has been shown that the intersection form of vanishing cycles in the Milnor lattice of the affine variety defined by the superpotential reproduces the $\widehat{SU(3)}_n$ Verlinde algebra. Thus, we see that the classical intersection theory of the del Pezzo surfaces in the resolved Gorenstein orbifold is encoded in the intersection forms of vanishing cycles in the affine variety of the KS LG superpotential.

5 Conclusion: Much Ado about Nothing?

The idea of classifying the $\widehat{SU(3)}$ WZW modular invariants in terms of finite subgroups of $SU(3)$ has not met much success to date. In this paper, we have reduced the problem into geometry and $N = 2$ superconformal field theory–that is, into finding pairs of Gorenstein singularities and associated LG superconformal field theories which have *a priori* connections to the finite subgroups and the modular invariants. Unlike the situation for $SU(2)$, such an attempt to find SCFT descriptions of Gorenstein singularities is hindered by the complexity of singularity structures and their desingularizations. Furthermore, we have seen that there is no direct correspondence between LG superpotentials and isolated quotient singularities; even though LG superpotentials have isolated singularities, isolated quotient singularities cannot be represented as hypersurfaces and thus cannot be LG superpotentials. We have been thus led to more indirect methods of analyzing the possible connections between LG theories and Gorenstein orbifolds. We have chosen the $SU(3)_n/SU(2) \times U(1)$ KS model as our particular LG theory and searched for evidences for a correspondence with a Gorenstein orbifold $\mathbb{C}^3/\Gamma$ by studying the intersection homology of its resolution. Among many subgroups $\Gamma \subset SU(3)$ that we have considered, we have found some surprising matches between the resolution of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ and the $SU(3)_n/SU(2) \times U(1)$ KS LG theory at level n.

Using different techniques of desingularization generally leads to topologically inequivalent Calabi-Yau manifolds with different Hodge numbers. Furthermore, we know that the KS LG theories describe deformations and not resolutions of Gorenstein orbifolds. At first sight, one might expect some kind of mirror symmetry between the resolved $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ and the LG orbifold. The complete resolution of the singularity produces a rigid Calabi-Yau with no complex deformations. On the other hand, one can embed the orbifold in $\mathbb{C}^4$ by the equation $xyz = w^{n+3}$ and try to deform the algebraic equation to remove the singularities, in which case the Hodge numbers are completely different from the resolution picture. In fact, with judicious choices of deformations, it is possible to produce a CY manifold with no second cohomology. After all, certain rigid manifolds are known to have as their mirrors Landau-Ginzburg theories, whose physical modes have a natural interpretation as complex deformations of the superpotential but which have no explicit, geometric Kähler modes. Thus, it may appear to be not all inconceivable that there exist Landau-Ginzburg mirrors of the complete or partial resolutions of the Gorenstein orbifolds. A careful investigation, however, shows that there are problems with this picture. In particular, the Kähler classes of the resolved manifolds correspond to CFT moduli fields, but the chiral ring in general consists of relevant, marginal, and irrelevant operators[24]. They are thus not mirror pairs.

What is more likely to be true is that the KS theory actually describes some deformation of $\mathbb{C}^3/\mathbb{Z}_{n+3} \times \mathbb{Z}_{n+3}$ which is quite different from the picture that traditional string theory techniques yield. Thus, besides the two known string theory Calabi-Yau desingularizations, with and without discrete torsion, among thousands of other ones allowed by mathematics, perhaps certain KS theories can provide us with exactly solvable descriptions of new desingularizations of Gorenstein orbifolds.

We have dismissed many issues in this paper, mainly because not many things are known about the interpretation of KS LG theories as describing non-compact Calabi-Yau manifolds. It would be interesting to see whether the full quantum cohomology of the resolved Gorenstein orbifold can be related to the chiral ring of some KS LG model, but this subject is clearly beyond the scope of this paper. The reader may have noticed that, in our presentation, we have largely ignored the non-compact factor $SL(2,\mathbb{R})/U(1)$ and the $U(1)$ projection onto integral charges. It is because the main focus of the paper lies in classifying the WZW modular invariants using geometry and because the relevant information is contained in the $SU(3)_n/SU(2) \times U(1)$ sector of the tensored theory[25]. Despite our effort, there is yet no explanation of why we should *a priori* expect the finite subgroups to classify certain modular invariants of conformal systems beyond the $SU(2)$ case.

Among the questions that we have ignored are: What would the integrable deformations of the KS theory correspond to in terms of

[24]In LG theories, there exists a unique state whose $U(1)$ charge is equal to $\hat{c}$. Since $\hat{c} = 2n/(n+3)$ for the $SU(3)_n/SU(2) \times U(1)$ KS model, there will always be irrelevant chiral ring elements for $n > 3$.

[25]Furthermore, imitating the ideas of [17], we want to send the coefficient μ in (3.8) to zero and argue that the tensored theory describes the degenerating limit of an affine variety defined by the superpotential.

geometry? Can we include D-branes in our study? What KS fields correspond to the non-compact ruled surfaces in the resolution; are they related to the $SL(2,\mathbb{R})/U(1)$ sector?

Finally, note that the Gorenstein orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ have also appeared in the context of AdS/CFT correspondence [15]. The "non-spherical near horizon" geometry of a $D3$-brane located at the origin of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$, for $n = 2, 3$, has been shown to be given by $U(1)$ bundles over certain del Pezzo surfaces. It is also known that the $SU(2)/U(1)$ Kazama-Suzuki model is closely related to the near horizon geometry of NS5-branes. It would be worth studying whether a similar picture exists for other Grassman Kazama-Suzuki models. Furthermore, regarding the modern string compactifications on AdS backgrounds, we are reminded of the situation in 1980's where the validity of Calabi-Yau compactifications was justified to some extent by replacing geometric compactifications with algebraic counterparts using exactly solvable $N = 2$ minimal models [5]. Similarly, it would be very interesting to see whether there exist exactly solvable superconformal systems[26] describing the AdS compactifications. We hope that our present study may well be pertinent to such directions of pursuit.

Acknowledgments

We would like to thank C. Vafa for many valuable discussions and suggestions. We also gratefully acknowledge Y.-H.E. He, D. Morrison, G. Tian, R. Vakil, and Stephen S.-T. Yau for discussions and Y.S. Song for comments on the preliminary version of this paper.

A Appendix

In this appendix, we determine which abelian quotient singularities are isolated. From Theorem 2.1, we can prove the following simple results:

Corollary A.1. *The orbifold $\mathbb{C}^3/\mathbb{Z}_k, \mathbb{Z}_k \subset SU(3)$, has only an isolated singularity if and only if $\mathbb{Z}_k = \langle \frac{1}{k}(\alpha_1, \alpha_2, \alpha_3)\rangle$ has GCD$(k, \alpha_i) = 1, \forall i$. In particular, $\mathbb{C}^3/\mathbb{Z}_k$ always has non-isolated singularities for k even.*

[26]We suspect that RR-backgrounds may cause difficulties.

Proof. Let $\mathbb{Z}_k = \langle\omega\rangle$, where $\omega = \frac{1}{k}(\alpha_1, \alpha_2, \alpha_3)$. It is easy to see that any non-trivial element $\omega^n \in \mathbb{Z}_k$ does not have an eigenvalue 1 if and only if $\mathrm{GCD}(k, \alpha_i) = 1, \forall i$, and the first claim thus follows from Theorem 2.1. Now, since $\mathbb{Z}_k \subset SU(3)$, the α_i's satisfy the condition $\alpha_1 + \alpha_2 + \alpha_3 = 0 \pmod{k}$, which implies that for k even, at least one α_i, say α_1, also has to be even. Let $m = \mathrm{GCD}(\alpha_1, k) \geq 2$. Then, the action of the non-trivial element $\omega^{k/m}$ on $\mathbb{C}^3$ fixes the first coordinate and thus produces a non-isolated singularity along this axis. □

Furthermore, a product of two cyclic groups satisfying the above conditions yields an isolated singularity if and only if their orders are coprime:

Corollary A.2. *Let* $\mathbb{Z}_k = \langle\frac{1}{k}(\alpha_1, \alpha_2, \alpha_3)\rangle \subset SU(3)$ *and* $\mathbb{Z}_{k'} = \langle\frac{1}{k'}(\alpha'_1, \alpha'_2, \alpha'_3)\rangle \subset SU(3)$, *where* $0 < \alpha_i < k$ *and* $0 < \alpha'_i < k'$. *Assume that* k, k' *are odd and that* $GCD(k, \alpha_i) = GCD(k', \alpha'_i) = 1$, $\forall i$. *Then,* $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_{k'}$ *has only isolated singularities if and only if* k *and* k' *are coprime.*

Proof. Without a loss of generality, assume that $k' < k$. In the diagonal basis, we can represent any non-trivial elements $g \in \mathbb{Z}_k$ and $g' \in \mathbb{Z}_{k'}$ as $g = \frac{1}{k}(a_1, a_2, a_3)$ and $g' = \frac{1}{k'}(b_1, b_2, b_3)$, for some integers $0 < a_i < k$ and $0 < b_i < k'$. We see that gg' has an eigenvalue 1 if and only if

$$\frac{a_i}{k} + \frac{b_i}{k'} = 1 \tag{A.1}$$

or equivalently,

$$a_i = \frac{k(k' - b_i)}{k'} \tag{A.2}$$

for some i. Now, assume that $\mathrm{GCD}(k, k') = 1$. Then, (A.2) tells us that in order for a_i to be an integer, k' has to divide $k' - b_i$, which is impossible.

Conversely, suppose that $\mathrm{GCD}(k, k') = c > 1$, such that $k = cm$ and $k' = cn$ for some positive coprime integers m, n. But, because we have assumed that $\mathrm{GCD}(k, \alpha_i) = \mathrm{GCD}(k', \alpha'_i) = 1\ \forall i$, there will be elements $g = \frac{1}{k}(a_1, a_2, a_3)$ and $g' = \frac{1}{k'}(b_1, b_2, b_3)$ for any $d, 1 \leq d < c$, such that $a_1 = dm$ and $b_1 = n(c - d)$. Then, we have

$$\frac{a_1}{k} + \frac{b_1}{k'} = \frac{a_1 n + b_1 m}{cmn} = 1$$

and thus, gg' has an eigenvalue 1. □

Corollary A.2 just means that $\mathbb{Z}_k \times \mathbb{Z}_{k'} \cong \mathbb{Z}_{kk'}$, where $\mathbb{Z}_{kk'}$ must satisfy the conditions of Corollary A.1. Corollary 2.1 now follows.

References

[1] A. Cappelli, C. Itzykson, and J.-B. Zuber, Commun. Math. Phys., **113** (1987), 1.

[2] H. Cartan, *Quotient d'un espace analytique par an groupe d'automorphismes*, in 'Algebraic Geometry and Topology,' Princeton Mathematical Series, **12** (1957), 90-102.

[3] P. Di Francesco, *Integrable Lattice Models, Graphs and Modular Invariant Conformal Field Theories*, Int. J. Mod. Phys., **A7** (1992), 407.

[4] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer-Verlag, NY, 1997.
S. Ketov, *Conformal Field Theory*, World Scientific, Singapore (1994).

[5] D. Gepner, *Fusion Rings and Geometry*, Comm. Math. Phys., **141** (1991), 381-411.

[6] D. Gepner, *Scalar Field Theory and String Compactification*, Nucl. Phys., **B322** (1989), 65.
——, *Field Identification in Coset Conformal Field Theories*, Phys. Lett., **B222** (1989,) 207.

[7] D. Gepner, *Exactly Solvable String Compactifications on Manifolds of $SU(N)$ Holonomy*. Phys. Lett. **B199** (1987) 380.
——, *Space-Time Supersymmetry in Compactified String Theory and Superconformal Models*, Nucl. Phys., **B296** (1988), 757.

[8] S.M. Gusein-Zade and A. Varchenko, *Verlinde Algebras and the Intersection Form on Vanishing Cycles*, hep-th/9610058.

[9] Y.-H. He and J.S. Song, *Of McKay Correspondence, Nonlinear Sigma-model and Conformal Field Theory*, hep-th/9903056.

[10] Y. Ito and M. Reid, *The McKay Correspondence for Finite Subgroups of $SL(3,\mathbb{C})$*, alg-geom/9411010.
M. Reid, *McKay Correspondence*, alg-geom/9702016.

[11] D. Joyce, *On the Topology of Desingularizations of Calabi-Yau Orbifolds*, alg-geom/9806146.

[12] Y. Kazama and H. Suzuki, *New $N = 2$ Superconformal Field Theories and Superstring Compactification*, Nucl. Phys., **B321** (1989), 232.

[13] W. Lerche, C. Vafa, and N.P. Warner, *Chiral Rings in $N = 2$ Superconformal Theories*, Nucl. Phys., **B324** (1989), 427.

[14] J. McKay, *Graphs, singularities, and finite groups*, Proc. Symp. Pure. Math., **37** (1980), 183.
R., Steinberg, *subgroups of SU(2) and Dynkin Graphs*, UCLA preprints, 1981.

[15] D.R. Morrison and M.R. Plesser, *Nonspherical Horizons I*, hep-th/9810201.

[16] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, New York (1985).
W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton (1993).

[17] H. Ooguri and C. Vafa, *Two-dimensional Black Hole and Singularities of CY Manifolds*, Nucl. Phys., **B463** (1996), 55.

[18] M. Schlessinger, *Rigidity of Quotient Singularities*, Inv. Math., **14** (1971), 17.

[19] C. Vafa, in preparation.

[20] C. Vafa and E. Witten, *On Orbifolds with Discrete Torsion*, J. Geom. Phys., **15** (1995), 189, hep-th/9409188.

[21] E. Witten, *Phases of $N = 2$ Theories in Two-Dimensions*, Nucl. Phys., **B403** (1993), 159.

[22] E. Witten, *On String Theory and Black Holes*, Phys. Rev., **D44** (1991), 159.

[23] Stephen S.-T. Yau and Y. Yu, *Gorenstein Quotient Singularities in Dimension Three*, Memoirs of the AMS, **505**, Providence, 1993.

[24] Stephen S.-T. Yau, private communication.