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Observables of Non-Commutative Gauge Theories

David J. Gross [a], Akikazu Hashimoto [a], and N. Itzhaki [b]

[a]Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106
gross@itp.ucsb.edu
aki@itp.ucsb.edu

[b]Department of Physics
University of California
Santa Barbara, CA 93106
sunny@physics.ucsb.edu

Abstract

We construct gauge invariant operators in non-commutative gauge theories which in the IR reduce to the usual operators of ordinary field theories (e.g. $\mathrm{Tr}\, F^2$). We show that in the deep UV the two-point functions of these operators admit a universal

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exponential behavior which fits neatly with the dual supergravity results. We also consider the ratio between n-point functions and two-point functions to find exponential suppression in the UV which we compare to the high energy fixed angle scattering of string theory.

1 Introduction

In the study of gauge field theories, one is often interested in computing the correlation functions of gauge invariant local operators such as $\mathrm{Tr}\, F^2(x)$. These correlation functions contain a great deal of information about the dynamics of the theory. In particular, they are localized probes of the gauge theory dynamics. Localized probes are especially interesting in the context of gauge theories on non-commutative geometries. Non-commutativity introduces a new physical scale to the problem, which gives rise to new physics which one would like to study.

With respect to gauge invariant local operators non-commutative gauge theory is quite different from its commutative counterparts. In a non-commutative gauge theory, operators such as $\mathrm{Tr}\, F^2(x)$, are not gauge invariant, but rather they transform non-trivially. In order to render such an operator gauge invariant out of these operators, one must integrate over all space. In other words, the same operator in momentum space, $\mathrm{Tr}\, F^2(k)$, is gauge invariant only if $k = 0$. But then the operator is not useful as a localized probe.

At first sight, it appears that gauge invariant operators which carry non-zero momentum simply do not exist in non-commutative gauge theories and thus the set of observables of NC gauge theories is much smaller than in ordinary gauge theories. There are different ways to see that this possibility is not satisfactory. Perhaps the most compelling is due to the existence of a supergravity dual to NCSYM [1, 2]. The supergravity dual automatically captures the gauge invariant dynamics of NCSYM. The excitations of supergravity modes are not restricted to the zero momentum sector, implying that there is a momentum dependent gauge invariant observables in the theory. Moreover, the fact that the supergravity solution in the near horizon limit asymptotes to $AdS_5 \times S_5$ implies that the operators corresponding to these supergravity modes should approach ordinary commutative SYM operators in the small momentum limit. This raises the question: what are the non-commutative generalizations of gauge invariant operator such as $\mathrm{Tr}\, F^2(k)$?

Ishibashi, Iso, Kawai, and Kitazawa have recently constructed a set of gauge invariant operators in non-commutative gauge theories carrying non-vanishing momentum [3]. (See also [4, 5, 6].) These authors showed that an *open* Wilson line with momentum p_μ is gauge invariant if the distance between the end-points of the line is

$$l^\nu = p_\mu \theta^{\mu\nu}. \tag{1.1}$$

Roughly speaking, the gauge dependence of the open Wilson line is canceled by the gauge dependence due to the momentum.

Just like the Wilson loops in ordinary gauge theories, these operators constitute an over-complete set of gauge invariant operators of non-commutative gauge theories [6]. However, it is not clear at first sight how these operators correspond to the excitations of the dual supergravity modes, and how these operators reduce to the set of ordinary gauge invariant local operators in the small momentum limit. The goal of this paper is to address these issues. It turns out that a simple generalization of the IIKK construction will give rise to the desirable set of operators. These are the non-commutative generalizations of the gauge invariant local operators! We will investigate the correlation functions of these operators in perturbation theory and compare the result with supergravity.

The organization of this paper is as follows. We will begin in section 2 by reviewing the construction of IIKK Wilson lines. We will then describe how IIKK construction can be modified to give rise to the set of operators that are "closest" to the ordinary gauge invariant local operators. In section 3, we investigate the two-point function of these operators in perturbation theory. We find a universal exponential behavior in the UV. In section 4, we find similar behavior using the supergravity dual description. In section 5, we calculate the ratio between the n-point function and the two point function to find an exponential suppression which we compare in section 6 to the well known behavior of the high energy fixed angle scattering of string theory.

The relation between open Wilson lines and supergravity was also studied recently, using a different approach, in [7].

2 Gauge invariant operators in NCYM

In this section we construct the non-commutative generalization of gauge invariant operators. The construction is based on the open Wilson lines of IIKK which we review below. Let us first set the notation that we use throughout the paper. For simplicity, and to avoid problems with Wick rotation, we take θ^{23} to be the only non-vanishing component of the non-commutativity parameter. We will take the gauge group to be $U(N)$. The action for this theory is

$$S = \frac{1}{4}\int d^4x \mathrm{Tr}\,(F_{\mu\nu}(x) * F_{\mu\nu}(x)), \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig(A_\mu * A_\nu - A_\nu * A_\mu), \tag{2.2}$$

and $*$ is the familiar star product

$$f(x) * g(x) \equiv e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{x=y}. \tag{2.3}$$

The action is invariant under the non-commutative gauge transformation,

$$A_\mu(x) \to U(x) * A_\mu(x) * U(x)^\dagger - \frac{i}{g}U(x) * \partial_\mu U(x)^\dagger, \tag{2.4}$$

where $U(x)$ is the non-commutative gauge parameter, with $U(x) * U(x)^\dagger = 1$. Under this transformation law, $\mathrm{Tr}\, F^2(x)$ is not gauge invariant but transforms according to

$$\mathrm{Tr}\, F^2(x) \to \mathrm{Tr}\, U(x) * F^2(x) * U^\dagger(x). \tag{2.5}$$

However, integrating over all space will give rise to a gauge invariant operator, since integrals of $*$-products can be cyclically permuted.

Similarly a Wilson line can be generalized to non-commutative gauge theories

$$W(x, C) = P_* \exp\left(ig\int_0^1 d\sigma \frac{d\zeta^\mu}{d\sigma} A_\mu(x + \zeta(\sigma))\right), \tag{2.6}$$

where C is the curve which parameterized by $\zeta^{\mu}(\sigma)$ with $0 \leq \sigma \leq 1$, $\zeta(0) = 0$, and $\zeta(1) = l$. P_* denotes path ordering with respect to the star product

$$\begin{aligned} &W(x, C) \\ &= \sum_{n=0}^{\infty} (ig)^n \int_0^1 d\sigma_1 \int_{\sigma_1}^1 d\sigma_2 ... \int_{\sigma_{n-1}}^1 d\sigma_n \, \zeta'_{\mu_1}(\sigma_1) ... \zeta'_{\mu_n}(\sigma_n) A_{\mu_1}(x + \zeta(\sigma_1)) \\ &\quad * \cdots * A_{\mu_n}(x + \zeta(\sigma_n)). \end{aligned} \tag{2.7}$$

Under the gauge transformation, the Wilson lines transform according to

$$W(x, C) \to U(x) * W(x, C) * U(x + l)^{\dagger}. \tag{2.8}$$

Just as in the ordinary gauge theories, an open Wilson line by itself is not gauge invariant. However, unlike in ordinary gauge theories, closing the line does not make the Wilson line gauge invariant. On the other hand, in non-commutative gauge theories, one can construct a gauge invariant operator out of the open Wilson lines in the following way. Consider the operator

$$W(k, C) = \int d^4x \, \mathrm{Tr} \, W(x, C) * e^{ikx}, \tag{2.9}$$

which is simply the Fourier transform of the open Wilson line. The integration is over the base point while keeping the path fixed. Under gauge transformations, this operator maps to

$$W(k, C) \to \int d^4x \, \mathrm{Tr} \, U(x) * W(x, C) * U^{\dagger}(x + l) * e^{ikx}. \tag{2.10}$$

The reason why it is useful to Fourier transform to momentum space is that in non-commutative geometry, e^{ikx} is a translation operator. That is,

$$e^{ikx} * f(x) = f(x + k\theta) * e^{ikx}. \tag{2.11}$$

Eq. (2.10) can, therefore, be written as

$$W(k, C) \to \int d^4x \, \mathrm{Tr} \, U(x) * W(x, C) * e^{ikx} * U^{\dagger}(x + l - k\theta), \tag{2.12}$$

and hence $W(k, C)$ is gauge invariant if C satisfies the condition

$$l^{\nu} = k_{\mu} \theta^{\mu\nu}. \tag{2.13}$$

Notice that this fixes only the distance between the end points of the Wilson line but does not put any additional constraint on the shape of the line. Notice further that in the commutative limit ($\theta \to 0$ keeping k fixed), we find that gauge invariance requires loops to be closed, as expected.

The set of open Wilson lines satisfying the condition (2.13) constitutes an over complete set of gauge invariant operators, just like the closed loops in ordinary gauge theories [6]. We will now describe how one constructs a convenient set of gauge invariant operators which is a natural generalization of the standard local gauge theory operators in the commutative limit.

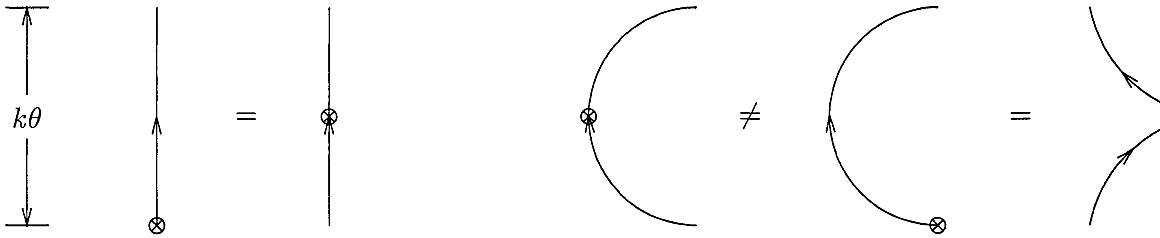
Consider an operator which consists of the usual local operator attached at one end of the Wilson line with non-vanishing momentum. An example of such an operator is

$$\operatorname{Tr} \tilde{F}^2(k) = \int d^4x \, \operatorname{Tr} F^2(x) * W(x, C) * e^{ikx}. \tag{2.14}$$

As long as C satisfies the condition (2.13), such an operator will be gauge invariant. In fact, any local operator in the adjoint representation can be attached at the end of the Wilson line satisfying (2.13) to give rise to a gauge invariant operator. See figure 1 for an illustration. Clearly, the set of all operators consisting of all local operators attached to Wilson lines of all shapes is an over-complete set. Since information about the shape of the lines can be absorbed into the local operators, one can drastically reduce the redundancies in the set of operators by requiring the Wilson line to take on a definite shape, but allowing arbitrary local operators to be attached at its endpoint. The simplest choice is to take these lines to be a straight line satisfying the condition (2.13). Just like the closed straight Wilson loop on a torus, such an operator acts as an order parameter for the large gauge transformations.

It may seem somewhat ad-hoc to attach the operator at one end of the Wilson line. Why not attach it in the middle, or smear it evenly throughout the line? One could have taken any of these prescriptions in defining the set of operators. It turns out, in the case of a straight Wilson line, that all of these prescriptions give rise to the same operator[1]. This is yet another reason why the straight Wilson lines are the

[1]We thank J. Maldacena for pointing this out to us.



(a) (b)

Figure 1: Moving the location of the local operator on the Wilson line does not change the definition of the gauge invariant operator only if the line is straight (a). For a generic line one gets a different gauge invariant operator (b).

most natural lines to consider. This follows from the fact that

$$\int d^4x\ \mathcal{O}(x) * W(x,C) * e^{ikx}$$
$$= \int d^4x\ W^\dagger(x,-C_2) * \mathcal{O}(x) * W(x,C_1) * e^{ikx}\ , \quad (2.15)$$

where

$$C = C_1 + C_2\ . \quad (2.16)$$

Only for the straight line will the shape of the line remain invariant with respect to this transformation. See figure 1 for an illustration.

To summarize: For any local operator of ordinary gauge theories $\mathcal{O}(x)$ in the adjoint of the gauge group we have constructed a non-commutative generalization

$$\tilde{\mathcal{O}}(k) = \mathrm{Tr} \int d^4x\ \mathcal{O}(x) * P_* \exp\left(ig \int_C d\zeta^\mu A_\mu(x+\zeta)\right) * e^{ikx}, \quad (2.17)$$

where C is a straight path

$$\zeta^\nu(\sigma) = k_\mu \theta^{\mu\nu}\sigma, \qquad 0 \le \sigma < 1. \quad (2.18)$$

The tilde is used to emphasize the fact that we have attached a Wilson line to the operator. This should be thought of as the generalization of Fourier transforms of gauge invariant operators to the non-commutative case.

Several comments are in order regarding the set of operators (2.17).

• Due to the antisymmetry of $\theta^{\mu\nu}$, k_μ and l^ν are orthogonal to one another. In other words, the Wilson line is extended in the direction transverse to the momentum.

• For small values of k or θ, the length of the Wilson line goes to zero and (2.17) is reduced to operators in the ordinary field theory

$$\tilde{\mathcal{O}}(k) \to \mathcal{O}(k) = \int d^4x\, \mathcal{O}(x) e^{ikx}. \tag{2.19}$$

• The fact that $\tilde{\mathcal{O}}(k)$ contains Wilson lines whose length depends on k means that one should not think of these operators as the "same" operator with different momentum as we usually do in commutative theories. One should think of $\tilde{\mathcal{O}}(k)$ as genuinely different operators at different momentum. One should therefore not expect to obtain a local operator by Fourier transforming to position space. This is closely related to the fact that these operators require momentum dependent regularization in perturbation theory and in the supergravity dual [2]. They are simply different operators.

• At large values of k, the length of the Wilson line becomes large as dictated by the non-commutativity relation. In this limit, the operator is dominated by the Wilson line regardless of what operator is attached at the end. We therefore expect the correlation function of these operators to exhibit a universal large k behavior.

In section 3 we will explain the computation of correlation functions for these operators in perturbation theory and elaborate on their universal features at large momentum.

2.1 Operator formulation

For completeness we describe in this sub-section an alternative derivation of such gauge invariant observables using the operator formulation of non-commutative gauge theory. We introduce

$$c = \frac{1}{\sqrt{2\theta}}\left(x^2 - ix^3\right), \quad c^\dagger = \frac{1}{\sqrt{2\theta}}\left(x^2 + ix^3\right), \tag{2.20}$$

which obey

$$[c, c^\dagger] = 1. \tag{2.21}$$

Since $c, c^\dagger$ satisfy the commutation relations of the annihilation and creation operators we can identify functions $f(x^2, x^3)$ with operator functions of $\hat{c}$, $\hat{c}^\dagger$ acting in the standard Fock space of the creation and annihilation operators by Weyl ordering, as defined by

$$\begin{aligned} f(x) &= f\left(z = x^2 - ix^3, \bar{z} = x^2 + ix^3\right) \\ \mapsto \hat{f}(\hat{c}, \hat{c}^\dagger) &= \int \frac{d^2x d^2p}{(2\pi)^2} f(x) \exp\left(i\bar{p}(\sqrt{2\theta}\,\hat{c} - z) + ip(\sqrt{2\theta}\,\hat{c}^\dagger - \bar{z})\right). \end{aligned} \tag{2.22}$$

It is easy to see that if $f \mapsto \hat{f}$, and $g \mapsto \hat{g}$, then $f * g \mapsto \hat{f}\hat{g}$, and

$$\int dx^2 dx^3 f(x) = 2\theta \,\mathrm{tr}[\hat{f}(\hat{c}, \hat{c}^\dagger)], \tag{2.23}$$

where tr is a trace over the Fock space. Translations in the Hilbert space are generated by $\hat{\partial}_i$, where

$$2\theta\,\hat{\partial}_2 = \sqrt{2\theta}\,(\hat{c} - \hat{c}^\dagger) = -2i\hat{x}^3, \quad 2\theta\,\hat{\partial}_3 = i\sqrt{2\theta}\,(\hat{c} + \hat{c}^\dagger) = 2i\hat{x}^2. \tag{2.24}$$

Thus, if $f(x) \mapsto \hat{f}$, then $f(x+l) \mapsto \exp(l\cdot\hat{\partial})\hat{f}\exp(-l\cdot\hat{\partial})$. The covariant derivative of a $U(N)$ gauge field is then represented as the operator,

$$\hat{D}_2 = -i\hat{\partial}_2 + g\hat{A}_2, \qquad \hat{D}_3 = -i\hat{\partial}_3 + g\hat{A}_3, \tag{2.25}$$

where $\hat{A}_\mu$ are $N \times N$ Hermitian matrix operators in the Fock space that represent the components of the gauge field. Under a gauge transformation the covariant derivative transforms like a field in the adjoint representation:

$$\hat{D}_\mu \to \hat{U}\hat{D}_\mu\hat{U}^\dagger \quad \hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1\ . \tag{2.26}$$

In this formalism it is easy to see why observables localized in position space are not gauge invariant. This is because translations of operators in the non-commutative directions are equivalent, up to constant shifts of the gauge field, to gauge transformations. Translations are generated by the operators $\hat{\partial}$, defined in eq. (2.24). Thus the translation, by an amount $l = (l^2, l^3)$, of a field, $\hat{\Phi}$, in the adjoint representation and of $\hat{A}_\mu$, are given by

$$\hat{\Phi} \to \exp(l \cdot \hat{\partial})\hat{\Phi} \exp(-l \cdot \hat{\partial}), \quad \hat{A}_\mu \to \exp(l \cdot \hat{\partial})\hat{A}_\mu \exp(-l \cdot \hat{\partial}). \tag{2.27}$$

This is a gauge transformation of the Higgs field, $\hat{\Phi} \to \hat{U}(l)\hat{\Phi}\hat{U}^\dagger(l)$, where

$$\hat{U}(l) = \exp(l \cdot \hat{\partial}) = \exp(il^i\theta_{ij}^{-1}\hat{x}^j). \tag{2.28}$$

Acting on the gauge field this transformation yields

$$\begin{aligned}\hat{A} \to \hat{U}(l)\hat{A}\hat{U}^\dagger(l) &= \left[\hat{U}(l)(\hat{A} - \hat{c}^\dagger)\hat{U}^\dagger(l) + \hat{c}^\dagger\right] + \hat{U}(l)[\hat{c}^\dagger, \hat{U}^\dagger(l)] \\ &\equiv \delta_1\hat{A} + \delta_2\hat{A}\ .\end{aligned} \tag{2.29}$$

The first term, $\delta_1\hat{A}$, is a gauge transformation and the second term, $\delta_2\hat{A}$, is a constant shift of the gauge field,

$$\delta_2\hat{A} = \hat{U}(l)[\hat{c}^\dagger, \hat{U}^\dagger(l)] = -(l^2 + il^3)\ . \tag{2.30}$$

Both of these, gauge transformations and constant shifts of the gauge field, are symmetries of the action.

What is unusual about non-commutative gauge theories is that *translations in the non-commutative directions are equivalent to a combination of a gauge transformation and a constant shift of the gauge field.* This explains why in NC gauge theories there do not exist local gauge invariant observables in position space, since by a gauge transformation we can effect a spatial translation! This is analogous to the situation in general relativity, where translations are also equivalent to gauge transformations (general coordinate transformations) and one cannot construct local gauge invariant observables. The fact that spatial translations are equivalent to gauge transformations (up to global symmetry transformations) is one of the most interesting features of NC gauge theories. These theories are thus toy models of general relativity—the only other theory that shares this property. However, unlike the case of general relativity, it is easy enough to derive a large set on non-local, gauge invariant, observables.

Suppressing the dependence on the commuting coordinates, a complete set of gauge invariant loops is given by

$$\operatorname{Tr}\operatorname{tr}\left[\prod_i \exp\left(i\hat{D}\cdot l_i\right)\right]. \tag{2.31}$$

The double trace, (Tr) over the $N \times N$ matrices and (tr) over the Fock space states, is necessary to ensure gauge invariance. A Wilson line segment, from x to $x + l$, is represented by the operator

$$\begin{aligned} W(x, x+l) &= P_* \exp\left(ig\int_x^{x+l} d\zeta^\mu A_\mu(\zeta)\right) \mapsto \\ \hat{W}(\hat{x}, \hat{x}+l) &\equiv \exp(\hat{x}\cdot\hat{\partial})\exp\left(i\hat{D}\cdot l\right)\exp(-(\hat{x}+l)\cdot\hat{\partial}). \end{aligned} \tag{2.32}$$

Therefore, using eq. (2.24) we get,

$$\begin{aligned} &\operatorname{Tr}\operatorname{tr}\left[\prod_i \exp\left(i\hat{D}\cdot l_i\right)\right] \\ &= \operatorname{Tr}\operatorname{tr}\left[\hat{W}(0, l_1)\hat{W}(l_1, l_1+l_2)\ldots\hat{W}(L-l_n, L)\exp iL\cdot\hat{\partial}\right] \\ &\mapsto \int dx^2 dx^3 \operatorname{Tr} P_* \exp\left(ig\int_{C(l_i)} d\zeta^\mu A_\mu(\zeta)\right)\exp\left(iL^i\theta^{-1}_{ij}x^j\right), \end{aligned} \tag{2.33}$$

where $L = \sum_i l_i$ and $C(l_i)$ is the contour from x to $x + L$ composed out of the n line-segments l_i. These are precisely the operators considered previously.

3 Correlation function of gauge invariant observables

Let us take the coupling constant to be small so that we can use perturbation theory. In the next section we compare our results with the supergravity results which are valid at large coupling to find a nice agreement. We consider $\mathcal{N} = 4$ $U(N)$ theory which contains adjoint scalars and fermions in addition to the gauge bosons. We will take the 't Hooft limit so that only the planer graphs contribute even though this is not crucial for the main results of this section.

Effects due to non-commutativity are weak in the infra red and grows as one increases the energy. In the extreme ultra violet, the correlation functions exhibit a universal exponential dependence on the energies which does not depend on the choice of $\mathcal{O}(x)$. Let us illustrate this with a concrete example. Perhaps the simplest operator to consider is to take $\mathcal{O}(x) = \operatorname{Tr}\phi(x)$. The relevant corresponding non-commutative operator is

$$\operatorname{Tr}\tilde{\phi}(k) = \operatorname{Tr}\int d^4x\,\phi(x) * P_* \exp\left(ig\int_0^1 d\sigma\zeta'^{\mu}A_\mu(x+\zeta(\sigma))\right) * e^{ikx}, \tag{3.1}$$

with

$$\zeta^\nu(\sigma) = k_\mu\theta^{\mu\nu}\sigma. \tag{3.2}$$

In perturbation theory, one can expand (3.1) order by order in g,

$$\begin{aligned}&\operatorname{Tr}\tilde{\phi}(k) \\ &= \operatorname{Tr}\left(\phi(k) + ig\int d^4x\int d\zeta_1\,\phi(x) * A(x+\zeta_1) * e^{ikx}\right. \\ &\quad\left. +(ig)^2\int d^4x\int d\zeta_1 d\zeta_2\,\phi(x) * A(x+\zeta_1) * A(x+\zeta_2)) * e^{ikx} + \ldots\right).\end{aligned} \tag{3.3}$$

Under complex conjugation, the path ordering is reversed

$$\begin{aligned}&\operatorname{Tr}\tilde{\phi}^\dagger(k) \\ &= \operatorname{Tr}\left(\phi(k) - ig\int d^4x\int d\zeta_1\,A(x+\zeta_1) * \phi(x) * e^{ikx}\right. \\ &\quad\left. +(ig)^2\int d^4x\int d\zeta_1 d\zeta_2\,A(x+\zeta_2)) * A(x+\zeta_1) * \phi(x) * e^{ikx} + \ldots\right).\end{aligned} \tag{3.4}$$

To facilitate the perturbative calculation using standard Feynman rules, it will be convenient to express the fields in momentum space

$$\phi(x) = \int \frac{d^4p_0}{(2\pi)^4}\phi(k)e^{-ip_0x}, \qquad A_\mu(x+\zeta_i) = \int\frac{d^4p_i}{(2\pi)^4}A_\mu(p_i)e^{-ip_i(x+\zeta_i)}. \tag{3.5}$$

The $*$-product will only act on the exponential factor and generate the usual non-commutative phase factors. Integrating over x will constrain

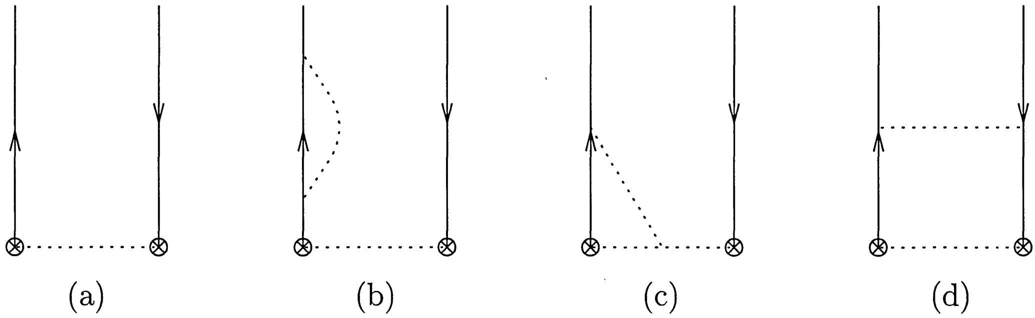


Figure 2: Leading contributions to the two-point functions. (a) is the g^0 order which agrees with the ordinary gauge theory results. (b), (c) and (d) are the non-commutative corrections at order g^2.

$\sum p_i = k$, and gives

$$\begin{aligned}\mathrm{Tr}\,\tilde{\phi}(k) = \mathrm{Tr}\,\Bigg(\phi(k) &+ ig\int\frac{d^4p_1}{(2\pi)^4}\int d\zeta_1\,\phi(k-p_1)A(p_1)e^{ik\theta p_1/2}\\ &+ (ig)^2\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int d\zeta_1 d\zeta_2\,\phi(k-p_1-p_2)\\ &\cdot A(p_1)A(p_2)e^{i(k\theta p_1+k\theta p_2+p_1\theta p_2)/2} + \ldots\Bigg).\end{aligned} \quad (3.6)$$

An interesting dynamical quantity to study is the two point function

$$\langle \mathrm{Tr}\,\tilde{\phi}(k_1)\mathrm{Tr}\,\tilde{\phi}^\dagger(k_2)\rangle. \quad (3.7)$$

The leading order contribution at order g^0 comes from the diagram (a) in figure 2. This is simply the commutative result

$$\langle \mathrm{Tr}\,\phi(k_1)\mathrm{Tr}\,\phi(-k_2)\rangle = \frac{N}{k_1^2}(2\pi)^4\delta^4(k_1-k_2). \quad (3.8)$$

We will drop the momentum conserving δ-function from most of the discussion below.

At order g^2, one gets a contribution from diagrams (b), (c), and (d) in figure 2. Let us evaluate these diagrams explicitly. The contribution to diagram (b) comes from crossing the first and the third term of (3.6). The gauge bosons in the third term of (3.6) is contracted to each other, so that we can replace

$$A(p_1)A(p_2) \to (2\pi)^4\delta^4(p_1+p_2)\frac{1}{p_1^2}. \quad (3.9)$$

This will cause the non-commutative phase factors to cancel, and the remaining p integrals can be done explicitly to give

$$\frac{(ig)^2}{k^2}\int d\zeta_1 d\zeta_2 \frac{1}{4\pi^2(\zeta_1-\zeta_2)^2}. \tag{3.10}$$

The integral diverges when ζ_1 and ζ_2 approach one another. If we regulate this integral at ultra-violet scale Λ, we find

$$(\mathrm{b}) = \frac{g^2N^2}{4\pi^2k^2}|k\theta|\Lambda, \tag{3.11}$$

as the contribution from the diagram (b).

Similar techniques can be used to compute the contribution from the diagram (c). The phase factor in (3.6) cancels the phase factor from the three point vertices, and we find

$$(\mathrm{c}) = \frac{(ig)^2N^2}{k^2}\int \frac{d^4p_1}{(2\pi)^4}\frac{1}{(k-p_1)^2}\frac{1}{p_1^2}\int_0^1 d\sigma_1\, i(k\theta p_1)e^{-i(k\theta p_1)\sigma_1}. \tag{3.12}$$

After doing the p_1 and σ_1 integrals, we find that (c) scales according to

$$(\mathrm{c}) \sim \begin{cases} \dfrac{g^2N^2}{k^2}\log(|k\theta||k|) & , \quad \mathrm{IR} \\[2ex] \dfrac{g^2N^2}{k^2} & , \quad \mathrm{UV} \end{cases} \tag{3.13}$$

where the IR and UV refers to $|k||\theta k| \ll 1$ and $|k||\theta k| \gg 1$, respectively. Finally, from diagram (d), we find the contribution

$$(\mathrm{d}) = -(ig)^2N^2|k\theta|^2\int \frac{d^4p_1}{(2\pi)^4}\frac{1}{(k-p_1)^2}\frac{1}{p_1^2}\int_0^1 d\sigma_1\int_0^1 d\sigma_2\, e^{-i(k\theta p_1)(\sigma_1-\sigma_2)}. \tag{3.14}$$

Doing the p_1 and $\sigma_{1,2}$ integral shows that (d) scales as follows

$$(\mathrm{d}) \sim \begin{cases} \dfrac{g^2N^2}{k^2}(|k||k\theta|)^2\log(|k||k\theta|) & , \quad \mathrm{IR} \\[2ex] \dfrac{g^2N^2}{k^2}|k||k\theta| & , \quad \mathrm{UV}. \end{cases} \tag{3.15}$$

We see that in the IR the corrections are small, as expected. The more interesting question is how these correlation functions behave at energies much larger than the non-commutativity scale. From the behavior of (3.13) and (3.15) at large momentum, we see that the leading contribution is due to diagram (d). The perturbative correction due to (3.15) is controlled by the dimensionless parameter

$$g^2N|k||\theta k|. \tag{3.16}$$

Even for small coupling constant, at sufficiently large k, this parameter will be greater than one. Therefore, contributions from higher order diagrams must be taken into account. Fortunately, it is possible to identify the diagrams which dominate at each order in the perturbative expansion and perform the resummation to reliably compute the behavior for large $|k||\theta k|$.

To gain some intuition on why the resummation is possible it is useful to recall that something very similar happens in the calculation of quark anti-quark force using Wilson loops in ordinary commutative gauge theories. There, one computes the expectation value of a rectangular loop of size $T \times L$ where T is in the time direction and L is the distance between quarks and $T \gg L$. If the gauge theory is in the Coulomb phase, the Wilson loop expectation value takes the form

$$\begin{aligned} W(T,L) &= \exp(g^2NT/L) \\ &= 1 + \frac{g^2NT}{L} + \frac{1}{2!}\left(\frac{g^2NT}{L}\right)^2 + \frac{1}{3!}\left(\frac{g^2NT}{L}\right)^3 + \dots . \end{aligned} \tag{3.17}$$

Again it seems that even for small 't Hooft coupling, the expansion parameter becomes too big for large enough T. But in fact, the exponential form of (3.17) is obtained using perturbation theory by summing over ladder diagrams which dominate when $T \gg L$.

The problem of computing the two point function of the open Wilson lines at large momentum is very similar to the problem of computing the expectation value of rectangular Wilson loops. The length of the line $|k\theta|$ plays the role of T and the momentum $|k|$ plays the role of $1/L$. It is clear that taking $T \gg L$ corresponds to $|k||\theta k| \gg 1$. In the calculation of $\langle \tilde{\phi}(k_1)\tilde{\phi}^\dagger(k_2)\rangle$ to order g^2, we saw that the leading large momentum contribution came from the ladder diagram, and we expect this pattern to persist to higher orders. This suggests that we should

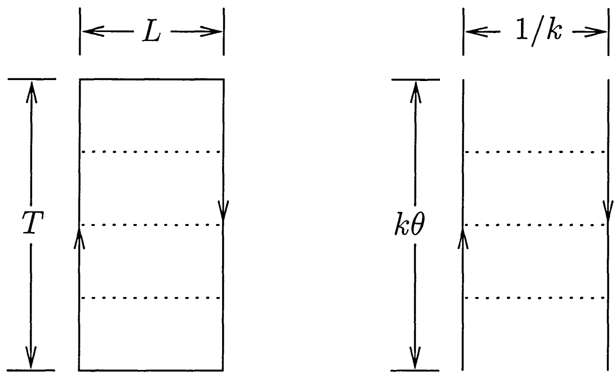


Figure 3: Ladder diagram contributions to the rectangular Wilson loop in ordinary field theories and to the open Wilson lines in non-commutative gauge theories.

resum the ladder diagrams in order to evaluate the large momentum behavior of these expectation values, just as one does in the rectangular Wilson loops.

In fact, one can compute explicitly the leading large k contribution from the planer ladder diagrams to the two point correlation function of the non-commutative gauge invariant operators. The general features will not depend very much on the choice of operator $\mathcal{O}(x)$ one attaches to the Wilson line. For the sake of simplicity, let us take $\mathcal{O}(x)$ to be an identity operator so that our non-commutative gauge invariant operator is a pure Wilson line. We then find that $\langle W(k)W^\dagger(k)\rangle$ receives contribution from the n-th ladder diagram of the form

$$g^{2n}N^n \int d^3r \frac{2}{(n+1)!}\left(\frac{1}{4\pi|r|}\right)^n |k\theta|^{n+1}e^{ikr}. \tag{3.18}$$

The steps leading to this expression are summarized in the appendix A. The integrand of (3.18) will resum to an exponential. The integral over r is essentially a Fourier transform. However, there are some subtleties associated with this Fourier transform which we will now explain[2]. For $n \leq 1$, the integral over r converges fine, but for $n \geq 2$, the integral receives strong divergent contributions from the region near $r = 0$. One can regulate this integral by introducing a small distance cut-off $r > 1/\Lambda$.

[2]These subtleties are not special to non-commutative theories and also appear if one tries to Fourier transform eq. (3.17) in ordinary gauge theories

In momentum space, we are only interested in terms which are non-analytic in k^2. Terms analytic in k^2 corresponds to contact terms in the position space and do not contribute when the operators are separated, say, in one of the commutative directions. In fact, we could have chosen to Fourier transform only in the x_2 and x_3 directions and continue to work with position space in the x_0 and x_1 directions. It turns out that all terms which diverge as we remove the cut-off Λ are analytic in k^2. We are therefore justified in dropping these terms. The non-analytic contribution to the Fourier transform of x^{-n} is simply

$$\begin{array}{ll} n \geq 2 \quad \text{even} & -\dfrac{2\pi^2 i}{(-k^2)^{3/2}} \dfrac{(-k^2)^{n/2}}{(n-2)!}, \\ n \geq 3 \quad \text{odd} & -\dfrac{2\pi}{(-k^2)^{-3/2}} \log(k^2/\Lambda^2) \dfrac{(-k^2)^{n/2}}{(n-2)!}. \end{array} \tag{3.19}$$

Applying (3.19) and (3.20) to (3.18), we find that the non-analytic contribution to the ladder diagrams are given by

$$\begin{array}{ll} n \geq 2 \quad \text{even} & -\dfrac{4\pi^2 i|k\theta|}{(-k^2)^{3/2}} \dfrac{z^n}{(n+1)!(n-2)!}, \\ n \geq 3 \quad \text{odd} & -\dfrac{4\pi|k\theta|}{(-k^2)^{3/2}} \log(k^2/\Lambda^2) \dfrac{z^n}{(n+1)!(n-2)!}, \end{array} \tag{3.20}$$

where

$$z = \left(\frac{g^2 N |k\theta| \sqrt{-k^2}}{4\pi} \right). \tag{3.21}$$

This can be resummed to

$$\begin{array}{ll} n \geq 2 \text{ even} & -\dfrac{2\pi^2 i|k\theta|}{(-k^2)^{3/2}} \left(\sqrt{z} I_3(2\sqrt{z}) + \sqrt{z} J_3(2\sqrt{z}) \right), \\ n \geq 3 \text{ odd} & -\dfrac{2\pi|k\theta|}{(-k^2)^{3/2}} \log(k^2/\Lambda^2) \left(\sqrt{z} I_3(2\sqrt{z}) - \sqrt{z} J_3(2\sqrt{z}) \right). \end{array} \tag{3.22}$$

Particularly simple quantity one can consider is the imaginary part of this expression when $k^2 < 0$. They are

$$\begin{array}{ll} n \geq 2 \text{ even} & -\dfrac{2\pi^2|k\theta|}{(-k^2)^{3/2}} \left(\sqrt{z} I_3(2\sqrt{z}) + \sqrt{z} J_3(2\sqrt{z}) \right), \\ n \geq 3 \text{ odd} & -\dfrac{2\pi^2|k\theta|}{(-k^2)^{3/2}} \left(\sqrt{z} I_3(2\sqrt{z}) - \sqrt{z} J_3(2\sqrt{z}) \right). \end{array} \tag{3.23}$$

Combining the odd and the even parts give rise to a very simple expression

$$\mathrm{Im}\langle W(k)W^{\dagger}(k)\rangle = -\frac{4\pi^2|k\theta|}{(-k^2)^{3/2}}\sqrt{z}I_3(2\sqrt{z}). \tag{3.24}$$

Using the fact that $I_\nu(x) \approx e^x/\sqrt{2\pi x}$ we find that this expression has an asymptotic behavior

$$\mathrm{Im}\langle W(k)W^{\dagger}(k)\rangle \approx \frac{2\sqrt{\pi}^{3/2}|k\theta|}{(-k^2)^{3/2}}z^{1/4}e^{2\sqrt{z}}. \tag{3.25}$$

What we found is that the two point function of gauge invariant operators in non-commutative gauge theories *grows* exponentially. So far we have ignored the contribution from the UV divergent diagram (a) in figure 2. In the case of the square Wilson loop, these diagrams resum like $e^{-g^2N\Lambda T/4\pi^2}$. The same is happening here and hence the two point function of the non-commutative Wilson lines behaves like

$$\langle W(k)W^{\dagger}(k)\rangle \sim \exp\left(-\frac{g^2N|k\theta|\Lambda}{4\pi^2} + \sqrt{\frac{g^2N|k\theta||k|}{4\pi}}\right), \tag{3.26}$$

at large momenta. The main contribution to this behavior comes from the long Wilson line which is part of all the gauge invariant operators carrying large momenta. Therefore, this is a universal property of gauge invariant operators in non-commutative gauge theories. Attaching an operator to the Wilson line will only modify the power of momentum in front of the universal multiplicative exponential factor.

There are several corrections to this leading result which one should take into account. One is the finite k correction which is suppressed by $1/|k||\theta k|$. Such a correction is the analog of the L/T corrections to the rectangular Wilson loops in ordinary gauge theories and corrects the ladder approximation. The other corrections are due to finite 't Hooft coupling. These corrections come about from summing up ladder diagrams which are sub-leading in the coupling constant. The effect of these corrections is to replace the 't Hooft coupling in the exponent by some function of the coupling constant while leaving the dependence on $|k\theta|$ and $|k|$ (or T and L in the ordinary field theory case) intact. At large coupling it was shown in [8, 9], in the ordinary field theory case, that this function is $\sqrt{g^2N}$. In the next section we shall find a similar behavior in our case.

4 Supergravity dual of NCSYM

So far we have concentrated on the properties of the non-commutative gauge invariant operators from the point of view of field theory. We found a remarkable universal behavior for the two point functions at large momentum due to the presence of the large Wilson line in these operators. We will show in this section that much of these features can also be seen from the dual supergravity description of NCSYM. The supergravity dual of NCSYM was found in [1, 2] and takes the form

$$ds^2 = \alpha' \left\{ \frac{U^2}{\sqrt{\lambda}}(-dt^2 + dx_1^2) + \frac{\sqrt{\lambda}U^2}{\lambda + U^4\Delta^4}(dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2}dU^2 + \sqrt{\lambda}d\Omega_5^2 \right\}, \tag{4.1}$$

$$e^\phi = \frac{\lambda}{4\pi N}\sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}.$$

We have only written the background metric in string frame and the dilaton here. $\lambda = 2\pi g^2 N$ is the 't Hooft coupling constant. We have taken the non-commutativity parameter to be non-vanishing only in the $\theta^{23} = 2\pi\Delta^2$ component.

In the AdS/CFT correspondence, one compares the fluctuations of the supergravity background on the supergravity side to the correlation function of gauge invariant operators on the field theory side. In order to make such a comparison precise, one must understand the fluctuations on the supergravity side along the lines of [10]. Unfortunately, the presence of a non-trivial dilaton and other supergravity field background gives rise to more mixing of the small fluctuations than in the $AdS_5 \times S_5$ case. This mixing has to be diagonalized. The complete treatment of this issue appears to be a rather non-trivial task. Note however that in the small U limit, the supergravity background approaches $AdS_5 \times S_5$, and all the effect of the mixing will be suppressed by $\lambda^{-1}\Delta^4 U^4 \sim \lambda\Delta^4 E^4$ where we have used the relation $E = U/\sqrt{\lambda}$. The mixing of the supergravity fluctuations and the mixing of the gauge invariant operators due to the Wilson line attached to them are related. Though hard to handle, this mixing is an important ingredient in the duality between NCSYM and string theory on the relevant background (4.1).

4.1 Non-decoupling of the $U(1)$

One effect of the mixing is that the $U(1)$ sector of $U(N) = U(1) \times SU(N)/Z_N$ no longer decouples from the $SU(N)/Z_N$ when the effect of non-commutativity is taken into account. Let us explain how to see this explicitly in the language of the supergravity dual. First recall that $AdS_5 \times S_5$ is dual to $SU(N)/Z_N$ rather than $U(N)$ gauge theory. There are many different ways to argue this [11, 12, 13, 14]. Perhaps the most intuitive argument is the fact that gravity couples to everything, so the $U(1)$ cannot be part of the dynamics of the supergravity dual [11]. A concrete way to see this is to consider $U(N)$ SYM on $R^2 \times T^2$ with non-zero 't Hooft flux along the T^2. All of the energies associated with the 't Hooft flux is contained in the $U(1)$ part of the gauge group [15]. From the point of view of supergravity, turning on a 't Hooft flux corresponds to turning on a constant B_{NS} background [14]. (See also [16] for a discussion of 't Hooft fluxes in supergravity in relation to Morita equivalence.) The fact that a constant B_{NS} can be turned on without any cost in energy implies that the supergravity dual knows only about the $SU(N)/Z_N$.

In the case of the supergravity dual of the NCSYM, on the other hand, one can not turn on B_{NS} without costing energy. This is due to the Chern-Simons term in type IIB supergravity

$$\int F_5 \wedge F_3 \wedge B_{NS}, \tag{4.2}$$

where F_5 and F_3 are the Ramond-Ramond five and three form field strengths, respectively. In pure $AdS_5 \times S_5$ this term does not prevent us from turning on a constant B field since $F_3 = 0$.[3] In the non-commutative case, F_3 is non-zero, so a constant B field will couple to the background and cannot be turned on without costing energy. In pure $AdS_5 \times S_5$, constant B_{NS} is part of the singleton multiplet. So it decouples from the rest of the fields in the small U region where the background (4.1) is effectively $AdS_5 \times S_5$, but they mix at large U. This is the sense in which the singleton fields and other supergravity modes are mixed, which can be viewed as the mixing of the $U(1)$ from the field theory side.

[3]On the other hand this term is very important for the understanding of many related issues in AdS_5 including the existence of a baryon vertex operator [12] and the quantization of B_{NS} and B_{RR} [13, 14].

4.2 Two point function and the absorption cross section

Clearly, disentangling the mixing of supergravity modes is a highly non-trivial task. However, since we are primarily interested in understanding the properties of the correlation functions which are universal, let us consider a generic fluctuation on the supergravity side. In a given supergravity background, there are generally scalars which couple minimally to the background. The field equation for such a field is given by

$$\frac{1}{\sqrt{g^E}}\partial_\mu \sqrt{g^E} g_E^{\mu\nu} \partial_\nu \Phi(x, U, \Omega) = 0 \tag{4.3}$$

where g^E is the Einstein metric $g^E_{\mu\nu} = e^{-\phi/2} g^S_{\mu\nu}$. Working with a fixed momentum mode

$$\Phi(x, U, \Omega) = e^{ikx}\Phi(U), \tag{4.4}$$

and writing in the form of a Schroedinger equation by redefinition of the fields $\Phi(U) = U^{-5/2}\Psi(U)$, one finds

$$-\frac{\partial^2}{\partial U^2}\Psi(U) + V(U)\Psi(U) = 0, \qquad V(U) = \frac{\lambda k^2}{U^4} + \frac{15}{4U^2} + \left|\frac{k\theta}{2\pi}\right|^2. \tag{4.5}$$

If $k^2 < 0$ the region near $U = 0$ is classically allowed. Therefore, one can tunnel through the barrier from $U = \infty$. The tunneling amplitude will then correspond to the imaginary part of the two point function of the operator associated with the minimal scalar and will be of the order of

$$\mathrm{Im}\langle \mathcal{O}(k)\mathcal{O}^\dagger(k)\rangle \sim \exp(-\int dU \sqrt{V(U)}). \tag{4.6}$$

If we turn off the non-commutativity effect by setting $\theta = 0$ then $V \sim 1/U^2$ and we find that $\int \sqrt{V}$ has logarithmic divergence at large U. Therefore, the imaginary part is a polynomial of the ratio between the momentum and the cut-off. This, of course, is just the WKB approximation to the standard two point function computation in $AdS_5 \times S_5$ in momentum space [17].

The case with non-vanishing $|k\theta|$ is the interesting case from the point of view of the non-commutative gauge invariant operators. As long as $k^2 < 0$, there will be a classically allowed region near $U = 0$, and one can compare the tunneling amplitude to the imaginary part

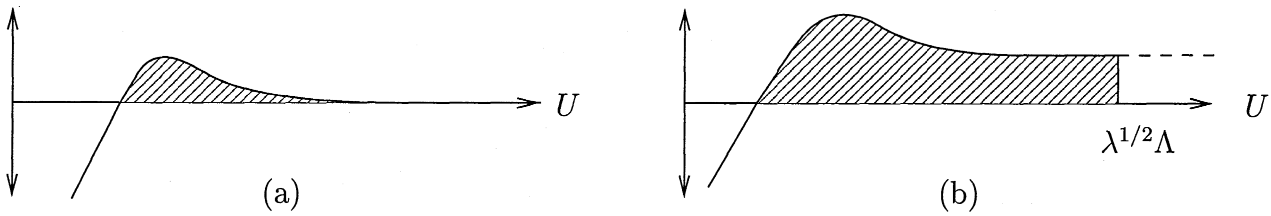


Figure 4: WKB approximation to the absorption cross section. (a) In AdS the potential falls at infinity like $1/U^2$ and hence the absorption cross section is suppressed like a power of the cutoff. (b) In the NC version of AdS the potential goes to a constant at infinity and so the absorption cross section is suppressed like an exponential of the cutoff.

of the two point function as we did above for AdS. There is one main difference. The potential does not approach zero at infinity. Therefore, when we terminate the space at some large $U = \sqrt{\lambda}\Lambda$ the tunneling amplitude will be exponentially suppressed. Keeping the leading terms we get

$$\exp\left(-\frac{1}{2\pi}\sqrt{\lambda}|k\theta|\Lambda + \frac{2\pi}{\Gamma(\frac{1}{4})^2}\sqrt{\sqrt{\lambda}|k\theta||k|}\right). \tag{4.7}$$

It is very interesting that this quantity behaves identically to the field theory result (3.26) up to replacing λ at weak coupling by $\sim \sqrt{\lambda}$ at strong coupling. It is especially interesting that the field theory divergences due to diagram (a) in figure 3 have such a simple role on the supergravity side. As was mentioned in the previous section, replacing the 't Hooft constant with its square root is a familiar effect of strong coupling which we have encountered many times before. The fact that the supergravity agrees with the field theory result, up to the 't Hooft coupling, is a strong indication that we have identified the correct supergravity counterpart of the correlation function of Wilson lines on the field theory side.

The universality can be seen by turning on excitations on S^5 which correspond to changing the R-charge of the operator. A short calculation shows that this will only change the power of momentum in front of the universal exponent.

5 Higher point functions

So far, we have considered only the two point function. Although we have found a remarkable agreement between the field theory and supergravity calculations, there is some cause for caution in interpreting such a result. In ordinary field theories, the relative normalization of operators such as $\mathrm{Tr}\, F^2(k)$ for different values of k is fixed completely by the fact that it is the Fourier transform of some definite operator in position space. This, however, is not the case in non-commutative gauge theories. As we saw in section 2, gauge invariant operators in NCSYM contain a piece corresponding to a Wilson line of definite length which depends on the momentum k. In principle, nothing prevents one from choosing to normalize these operators in an arbitrarily k dependent manner. Unlike the commutative theories, operators like $\mathrm{Tr}\, \tilde{F}^2(k)$ for different values of k are really different operators. By changing the normalization of the operators, however, one can conclude anything about the k-dependence of two point function of any operators

$$\langle \tilde{\mathcal{O}}(k)\tilde{\mathcal{O}}^\dagger(k)\rangle \to f(k)^2 \langle \tilde{\mathcal{O}}(k)\tilde{\mathcal{O}}^\dagger(k)\rangle. \tag{5.1}$$

To avoid the ambiguities associated with operator normalizations, one should consider instead the ratio of n-point function to product of two-point functions of the form

$$\frac{\langle \tilde{\mathcal{O}}_1(k_1)\tilde{\mathcal{O}}_2(k_2)\ldots\tilde{\mathcal{O}}_n(k_n)\rangle}{\sqrt{\langle \tilde{\mathcal{O}}_1(k_1)\tilde{\mathcal{O}}_1^\dagger(k_1)\rangle\langle \tilde{\mathcal{O}}_2(k_2)\tilde{\mathcal{O}}_2^\dagger(k_2)\rangle\ldots\langle \tilde{\mathcal{O}}_n(k_n)\tilde{\mathcal{O}}_n^\dagger(k_n)\rangle}}. \tag{5.2}$$

It is clear that (5.2) is invariant with respect to the change in the normalization of the operators. This is the appropriate quantity to use in comparing non-commutative and ordinary gauge theories.

In ordinary field theories, (5.2) will scale like a power for large k. What happens to (5.2) for non-commutative theories? We saw in equation (3.26) that the two point function of operators whose normalization are fixed by (2.17) decays exponentially with respect to the cut-off Λ and grows exponentially with respect to $|k||\theta k|$. How do the n-point functions of the operators with the same normalization scale? The n-point functions contain an identical dependence on the cut-off Λ due to the UV divergence of gluon propagator starting and ending on the same Wilson line. Therefore, the dependence on Λ will cancel out in (5.2).

What about the exponential dependence on $|k||\theta k|$? This was due to the resummation of ladder diagrams in the case of the two-point function. Can something similar happen to the n-point function? The answer is no. To see this, note that the origin of the exponential dependence in the two point function can be traced to the fact that the Wilson lines are parallel. Since the distance between two parallel lines is a constant, the contribution from the gluon exchange propagator gets enhanced by the size of the Wilson line, which gets larger as we increase the momentum. In the n-point function, however, the lines are not generically parallel, and the contribution from the propagators are suppressed when the distances between the points on the Wilson line gets large, even when the Wilson line itself is very long.

Some sample calculation of the three point function of the Wilson lines are presented in Appendix B. Unlike the case of the two point function, perturbative expansion appears to be controlled in the UV by the dimensionless parameter $g^2N/k^2\theta$ which is certainly a small parameter. Therefore, there is no need to take the contribution of the higher order diagrams into consideration. In other words, the n-point function will not grow exponentially, contrary to the behavior of the two-point functions.

Consequently, the ratio between the n-point function and the two point function is suppressed in the UV like,

$$\frac{\langle \tilde{O}_1(k_1)\tilde{O}_2(k_2)\ldots\tilde{O}_n(k_n)\rangle}{\sqrt{\langle \tilde{O}_1(k_1)\tilde{O}_1^\dagger(k_1)\rangle\langle \tilde{O}_2(k_2)\tilde{O}_2^\dagger(k_2)\rangle\ldots\langle \tilde{O}_n(k_n)\tilde{O}_n^\dagger(k_n)\rangle}} \sim \exp\left(-\sum_{i=1}^{n}\sqrt{\frac{g^2N|k_i\theta||k_i|}{16\pi}}\right). \tag{5.3}$$

What we have found here is an intriguing exponential suppression of the ratio (5.2) of n-point function to the product of the two point functions. There are absolutely no ambiguities in the evaluation of this quantity, and the behavior of this quantity in non-commutative theory exhibits a clear departure from the behavior of the commutative theory.

6 Comparison to string theory

In the previous section, we described a universal feature of the gauge invariant observables of non-commutative gauge theories: the ratio of n-point function to the product of two-point functions are suppressed exponentially at large momentum. This is strongly reminiscent of the behavior of form factors in high energy fixed angle scattering of string theory [18], and seems like a tantalizing hint for a connection between string theory and field theories on non-commutative spaces.

There have been other hints of relation between string theory and non-commutative field theories. Both are theories with intrinsic non-locality scale set by a dimensionful parameter: α' in the case of string theory and θ in the case of non-commutative gauge theories. In fact, these two scales are intimately related since under the open string/closed string map of [19], objects whose size are of the order of α' in closed string metric has the size of the order of θ in the open string metric. Consequently, the string scale in the bulk corresponds to the non-commutativity scale at the boundary.

In the analysis of the n-point function of gauge invariant observables in non-commutative gauge theories, one finds further tantalizing similarities to string theory:

• Both string theory and non-commutative gauge theory exhibits universality in the exponential suppression at large momentum. On the string theory side, the exponential suppression of the scattering amplitude is independent of the choice of the vertex operator. On the non-commutative gauge theory side, the exponential suppression of (5.3) is independent of the choice of operators $\mathcal{O}_i$.

• Exponential suppression at large k can be thought of as an on-shell phenomenon for both string theory and non-commutative gauge theory. On the string theory side, this is obvious since one is referring to the behavior of the S-matrix. On the gauge theory side, this may seem strange since (5.3) is an off-shell correlation function. However, in light of AdS/CFT correspondence which maps off-shell quantities on the boundary to on-shell quantities on the bulk, one can think of (5.3) as an on-shell observable of the bulk theory.

• In both cases fixing the angle while taking the UV limit is essential to get the exponential suppression. In string theory this is well known. On the field theory side, this corresponds to fixing the relative orientations of k_i as we take k_i to be large, which fixes the orientation of the Wilson lines to be non-parallel.

• Both string theory and non-commutative gauge theory requires resummation of diagrams in order for the universal large k behavior to manifest itself. On the non-commutative gauge theory side, the resummation of ladder diagrams was necessary in order to obtain the suppression by an exponential factor $\exp(-|k|)$. On the string theory side, at each order in the world sheet genus expansion, the amplitudes are suppressed by $\exp(-k^2)$. The Borel resummation of these genus expansion amplitudes [20], valid for $\alpha' k^2 \gg \log(1/g)$, gives rise to a suppression by $\exp(-|k|)$ in agreement with the field theory result.

These similarities are indeed suggestive of a deep connection between the exponential suppression found in (5.3) and the exponential suppression of high-energy fixed angle scattering in string theory. However, there is one crucial difference which immediately rules out the correspondence, at least in its most naive form. The point is that in string theory the suppression is due to large relative momentum of the different particles involved in the scattering. This corresponds to large momentum on the *internal* legs. Here on the other hand the suppression is due to large momentum on the *external* legs. In this sense, exponential suppression of (5.3) is quite different from the exponential suppression of high energy fixed angle scattering of strings. In the language of supergravity dual, the exponential suppression of (5.3) is *not* due to Gross-Mende like behavior in the bulk. Instead, it is due to the propagation from the boundary to the bulk in the geometry given by (4.1). Clearly, it would be very interesting to better understand the physical meaning of the "suppression by external legs."

7 Discussion

The aim of the present paper is to initiate a systematic treatment of gauge invariant operators in non-commutative gauge theories. We saw that there is a simple and natural generalization of gauge invariant operators that carry momentum. The loss of gauge invariance due to the non-zero momentum was compensated by attaching a Wilson line of definite size. Simple straight Wilson line does the job. This is a very natural generalization since the operators reduce to ordinary gauge invariant local operators in the IR. Though formally this is a very simple modification, we found that it has drastic consequences in the UV. For example the Wilson line gives rise to a universal enhancement factor to the two-point function which depends exponentially on $|k|$, $|k\theta|$ and the cutoff Λ. Exactly the same dependence on these parameters was found from the dual supergravity description.

The n-point functions (with $n > 2$) do not receive such an exponential enhancement and hence the ratio between the n-point function and the two-point function is exponentially suppressed in the UV. This looks very similar at first sight to the exponential dependence of high energy fixed angle string scattering amplitudes. Closer look revealed, on the contrary, that this is a completely different effect. The suppression is due to the external leg factors rather than the internal leg factors.

Clearly, many questions remain to be answered. For example, to make further progress it seems important to understand better the mixing issue both from the field theory side (due to the Wilson lines) and from the supergravity side (due to the non-trivial background fields). Understanding the mixing issue is essential to extract the precise mapping of the degrees of freedom. It might also be important for the comparison to string theory. A somewhat related question is the interpretation of eq. (5.3) from the point of view of the supergravity dual. Since (5.3) does not depend on the normalization, it should also have an unambiguous formulation in the language of supergravity. This can be useful for understanding better the way holography works in that background which is somewhat similar to flat space-time. Refs. [21, 22] seems to provide a good starting point to study that question.

A different kind of question is the relation between the open Wilson lines considered here and the supergravity description of the Wilson lines (in the sense of string world sheet minimal surfaces [8, 9, 23]). In [24] it was shown that to reach the boundary the fundamental strings must carry momentum. Even though these Wilson lines involve the scalar fields (unlike the ones considered in this article), they transform in a similar manner with respect to the gauge group, and it is natural to expect a close relation between the two.

It should also be very interesting to try to generalize the gauge invariant operators of non-commutative gauge theories to string field theory. Perhaps the generalization is simpler in the large B field limit where the string field theory algebra factors out [25, 26, 27]. A different approach to learn about gauge invariant operators in string theories might be to consider the S-dual description of the operators considered here which should be the natural operators in NCOS theories [28, 29, 30].

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Appendix A: Ladder diagrams for the Wilson line two point function

In this appendix, we will explain how one systematically computes the ladder diagrams contributing to the $\langle W(k)W^\dagger(k)\rangle$ correlation function, where

$$W(k) = \int d^4x\, P_* e^{ig\int_0^L d\zeta A(x+\zeta)} * e^{ikx} \ . \tag{A.1}$$

We begin by expanding these operators

$$\begin{aligned} W(k) = \sum_n (ig)^n \int d^4x \int_{\zeta_n>\zeta_{n-1}>\ldots\zeta_1} d\zeta_i\, A(x+\zeta_1) \\ * A(x+\zeta_2) * \ldots * A(x+\zeta_n) * e^{ikx} \end{aligned} \tag{A.2}$$

$$\begin{aligned} W^\dagger(k) = \sum_n (-ig)^n \int d^4x \int_{\zeta'_n>\zeta'_{n-1}>\ldots\zeta'_1} d\zeta'_i\, A(x+\zeta'_n) \\ * A(x+\zeta'_{n-1}) * \ldots * A(x+\zeta'_1) * e^{-ikx} \ . \end{aligned} \tag{A.3}$$

This can be rewritten in momentum space as

$$W(k) = \sum_n (ig)^n \int d^4x \int \frac{d^4p_i}{(2\pi)^4} \int d\zeta_i A(p_1) e^{-ip_1(x+\zeta_1)} \\ * A(p_2) e^{-ip_2(x+\zeta_2)} * \ldots * A(p_n) e^{-ip_n(x+\zeta_n)} * e^{ikx} \tag{A.4}$$

$$W^\dagger(k) = \sum_n (-ig)^n \int d^4x \int \frac{d^4p_i}{(2\pi)^4} \int d\zeta_i' A(-p_n) e^{ip_n(x+\zeta_n')} \\ * A(-p_{n-1}) e^{ip_{n-1}(x+\zeta_{n-1}')} * \ldots * A(-p_1) e^{ip_1(x+\zeta_1')} * e^{-ikx} . \tag{A.5}$$

Taking the $*$ product and integrating over x, we find

$$W(k) = \sum_n (ig)^n \int \frac{d^4p_i}{(2\pi)^4} \int d\zeta_i (2\pi)^4 \delta\left(\sum p_i - k\right) A(p_1) e^{-ip_1\zeta_1} \\ \cdot A(p_2) e^{-ip_2\zeta_2} \ldots A(p_n) e^{-ip_n\zeta_n} e^{i\sum_{i<j} p_i\theta p_j/2} \tag{A.6}$$

$$W^\dagger(k) = \sum_n (-ig)^n \int \frac{d^4p_i}{(2\pi)^4} \int d\zeta_i' (2\pi)^4 \delta\left(\sum p_i - k\right) A(-p_n) e^{ip_n\zeta_n'} \\ \cdot A(-p_{n-1}) e^{ip_{n-1}\zeta_{n-1}'} \ldots A(-p_1) e^{ip_1\zeta_1'} e^{-i\sum_{i<j} p_i\theta p_j/2} . \tag{A.7}$$

Contracting the A's that form the ladder diagram, we find

$$\langle W(k) W^\dagger(k) \rangle = g^{2n} N^n \int d\zeta_i \int \frac{d^4p_i}{(2\pi)^4} (2\pi)^4 \delta\left(\sum p_i - k\right) \\ \cdot \frac{1}{p_1^2} \frac{1}{p_2^2} \cdots \frac{1}{p_n^2} e^{-i\sum_i p_i(\zeta_i - \zeta_i')} . \tag{A.8}$$

Note that the factors of $\exp(ip_i\theta p_j)$ canceled out as to be expected for a planer diagram.

We can now re-express the correlation function in position space

$$\langle W(k) W^\dagger(k) \rangle = g^{2n} N^n \int d^4x d\zeta_i \prod_i \left(\frac{1}{4\pi^2(r^2 + (x_3 + \zeta_i - \zeta_i')^2)} \right) e^{ikr} \tag{A.9}$$

where

$$r^2 = x_0^2 + x_1^2 + x_2^2 . \tag{A.10}$$

Now let us do the ζ_i, ζ_i', and the x_3 integrals. We will first do the ζ' integral by simply noting the fact that they receive most of their contribution from the region

$$x_3 + \zeta_i - \zeta_i' = 0 \tag{A.11}$$

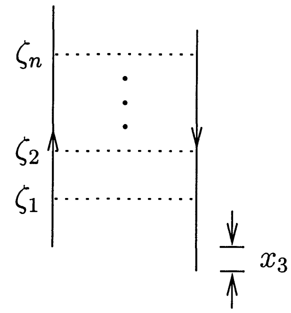

Figure 5: The ladder diagrams dominate at large momentum due to the integration over ζ_i which grows linearly with the size of the line.

or

$$\zeta_i' = x_3 + \zeta_i \ . \tag{A.12}$$

Since $0 < \zeta_i' < L$, we find that x_3 must satisfy the condition

$$-\zeta_1 < x_3 < L - \zeta_n \tag{A.13}$$

in order for all the ζ_i' integrals to contribute. (See figure 5.) Each ζ_i' integral will contribute a factor

$$\int d\zeta_i' \frac{1}{4\pi^2(r^2 + \zeta_i'^2)} = \frac{1}{4\pi|r|} \tag{A.14}$$

where we have used the fact that, for $L \gg k^{-1}$, the integral over ζ can be approximated by an integral over the entire real axis. The rest of the ζ_i and x_3 integrals are relatively straight forward

$$\int_0^L d\zeta_n \int_0^{\zeta_n} d\zeta_{n-1} \ldots \int_0^{\zeta_2} d\zeta_1 \int_{-\zeta_1}^{L-\zeta_n} dx_3 = \frac{2L^{n+1}}{(n+1)!} \ . \tag{A.15}$$

This then shows that the correlation function can be expressed in the form

$$\langle W(k) W^\dagger(k) \rangle = g^{2n} N^n \int d^3 r \left(\frac{1}{4\pi|r|} \right)^n \frac{2L^{n+1}}{(n+1)!} e^{ikr} \ . \tag{A.16}$$

Appendix B: Some three point functions

In this appendix, we will describe the computation of some diagrams that contribute to the three point function of the Wilson line operators in non-commutative gauge theory. We will consider the diagrams illustrated in figure 6.

We will begin by considering the diagram (a) in figure 6. The terms in the perturbative expansion of the Wilson line operators which contribute to this diagram are

$$
\begin{aligned}
W_1 &= \int d^4x\, da\, \zeta_1' \cdot A(x+\zeta_1(a)) * e^{ik_1x} \\
&= \int da\, \zeta_1' \cdot A(k) e^{-ik_1\zeta_1(a)} = \zeta_1' \cdot A(k_1) && \text{(B.1)} \\
W_2 &= \int d^4x\, db\, db'\; \zeta_2' \cdot A(x+\zeta_2(b)) * \zeta_2' \cdot A(x+\zeta_2(b')) * e^{ik_2x} \\
&= \int db\, db' \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \zeta_2' \cdot A(p)\, \zeta_2' \\
&\quad \cdot A(q) e^{ip\theta q/2 - ip\zeta_2(b) - iq\zeta_2(b')} (2\pi)^4 \delta^4(p+q-k_2) && \text{(B.2)} \\
W_3 &= \int dc\; \zeta_3' \cdot A(k_3) e^{-ik_3\zeta_3(c)} = \zeta_3' \cdot A(k_3) && \text{(B.3)}
\end{aligned}
$$

so that

$$
\langle W_1 W_2 W_3 \rangle = \int db\, db'\, \frac{\zeta_1' \cdot \zeta_2' \zeta_3' \cdot \zeta_2'}{k_1^2 k_3^2} e^{ik_1\theta k_3/2 + ik_1\zeta_2(b) + ik_3 z_2(b')} \tag{B.4}
$$

where a, b, b' and c take values between 0 and 1, and

$$
\zeta_1(a) = k_1\theta a \qquad \zeta_2(b) = k_2\theta b \qquad \zeta_3(c) = k_3\theta c\ . \tag{B.5}
$$

Therefore, we have

$$
\langle W_1 W_2 W_3 \rangle = \int db\, db'\, \frac{(k_1\theta^2 k_2)(k_3\theta^2 k_2)}{k_1^2 k_3^2} e^{ik_1\theta k_3/2 + ik_1\theta k_2 b + ik_3\theta k_2 b'}\ . \tag{B.6}
$$

Using

$$
k_1\theta k_2 = -k_1\theta k_3, \qquad k_3\theta k_2 = -k_2\theta k_3 = k_1\theta k_3 \tag{B.7}
$$

this becomes

$$
\langle W_1 W_2 W_3 \rangle = \int db\, db'\, \frac{(k_1\theta^2 k_2)(k_3\theta^2 k_2)}{k_1^2 k_3^2} e^{ik_1\theta k_3/2 - ik_1\theta k_3(b-b')}\ . \tag{B.8}
$$

Doing the integral over b and b', we find

$$
\langle W_1 W_2 W_3 \rangle = \frac{(k_1\theta^2 k_2)(k_3\theta^2 k_2)}{k_1^2 k_3^2} e^{ik_1\theta k_3/2} \left(\frac{i}{k_1\theta k_3} + \frac{1 - e^{ik_1\theta k_3}}{(k_1\theta k_3)^2} \right)\ . \tag{B.9}
$$

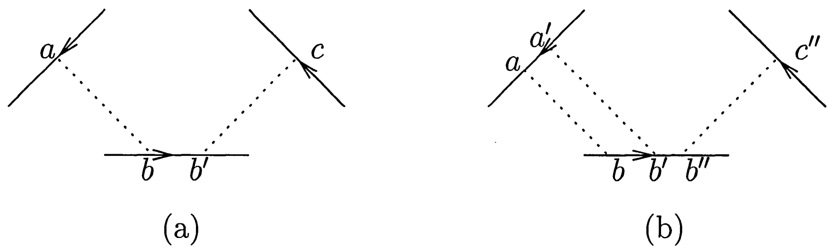

Figure 6: A typical three point diagrams. Since the lines are not parallel (b) will not dominate over (a) even at large momenta.

For large $k^2\theta$, the first term in the parenthesis dominates, and its overall scaling with respect to k and θ is

$$\langle W_1 W_2 W_3\rangle \approx g^4 N^2 \frac{\theta^3}{k^2} \; . \tag{B.10}$$

Let us now consider the contribution from diagram (b) in figure 6. The operators are

$$\begin{aligned}
W_1 &= \int d^4x \int da \int^a da' \zeta_1' \cdot A(x+\zeta_1(a')) * \zeta_2' \cdot A(x+\zeta(a)) * e^{ik_1 x} \\
&= \int da \int da' \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \zeta_1' \cdot A(q) \zeta_1' \\
&\quad \cdot A(p) e^{iq\theta p/2 - iq\zeta_1(a') - ip\zeta_1(a)} (2\pi)^4 \delta^4(p+q-k_1)
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
W_2 &= \int d^4x \int db'' \int^{b''} db' \int^{b'} db \\
&\quad \cdot \zeta_2' \cdot A(x+\zeta_2(b)) * \zeta_2' \cdot A(x+\zeta_2(b')) * \zeta_2' \cdot A(x+\zeta_2(b'')) * e^{ik_2 x} \\
&= \int db'' \int db' \int db \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} \\
&\quad \cdot (2\pi)^4 \delta^4(k_2+p+q+r) \\
&\quad \cdot \zeta_2' \cdot A(-p)\, \zeta_2' \cdot A(-q)\, \zeta_2' \cdot A(-r) \\
&\quad \cdot e^{ip\theta q/2 + ip\theta r/2 + iq\theta r/2 + ip\zeta_2(b) + iq\zeta_2(b') + ir\zeta_2(b'')}
\end{aligned} \tag{B.12}$$

$$W_3 = \zeta_3' \cdot A(k_3) \; . \tag{B.13}$$

Then $\langle W_1 W_2 W_3\rangle$ goes as

$$\int da da' db'' db' db \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} \frac{1}{(k_1-q)^2}$$

$$\cdot\, e^{-iqa'-i(k_1-q)a+i(k-q)b+iqb'-ik_3b''}e^{ik_1\theta k_3/2}\frac{(\zeta_1'\cdot\zeta_2')^2\zeta_2'\cdot\zeta_3'}{k_3^2}\,. \quad \text{(B.14)}$$

This can be written in the form

$$\langle W_1W_2W_3\rangle = \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{(k_1-q)^2}F(q)e^{ik_1\theta k_3/2}\frac{(\zeta_1'\cdot\zeta_2')^2\zeta_2'\cdot\zeta_3'}{k_3^2}\,, \quad \text{(B.15)}$$

where

$$F(q) = \int dada'db''db'db\, e^{-iq\theta k_1a'-i(k_1-q)\theta k_1a+i(k_1-q)\theta k_2b+iq\theta k_2b'-ik_3\theta k_2b''}\,. \quad \text{(B.16)}$$

Since $F(q)$ is again damping function with a width of the order $q \approx 1/k\theta$, to leading order we can approximate

$$\langle W_1W_2W_3\rangle = \int \frac{1}{k_1^2}\left(\int d^2qF(q)\right)e^{ik_1\theta k_3/2}\frac{(\zeta_1'\cdot\zeta_2')^2\zeta_2'\cdot\zeta_3'}{k_3^2}\,. \quad \text{(B.17)}$$

The d^2q integral will give rise to a δ-function. One finds that

$$\int d^2qF(q) = \int dada'db''db'db\; \delta^2(\theta k_1(a'-a)+\theta k_2(b-b'))e^{ik_1\theta k_2b-ik_3\theta k_2b''}\,. \quad \text{(B.18)}$$

This will go like

$$\int d^2qF(q) \approx \frac{1}{(\theta k)^2}\frac{1}{(k\theta k)^2} \approx \frac{1}{k^6\theta^4}, \quad \text{(B.19)}$$

where the first factor comes from integrating over the δ function and the second factor comes from integrating over the phase factors. Plugging this back into (B.17) leads to the conclusion that

$$\langle W_1W_2W_3\rangle \approx g^6N^3\frac{\theta^2}{k^4}\,. \quad \text{(B.20)}$$

Comparing (B.10) and (B.20), we find that the dimensionless parameter controlling the perturbative expansion is $g^2N/k^2\theta$.

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