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Topological gravity in genus 2 with two primary fields

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Abstract

We calculate the genus 2 correlation functions of two-dimensional topological gravity, in a background with two primary fields, using the genus 2 topological recursion relations.

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In this paper, we calculate the genus 2 correlation functions of two-dimensional topological gravity in a background with two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$; this extends the work of Eguchi, Yamada and Yang [8], who considered the case of the A_2-model.

The most interesting example of such a theory is the Gromov-Witten theory of $\mathbb{CP}^1$; in this case, there is a rigorous construction of the correlation functions (see Manin [15]). For $\mathbb{CP}^1$, our calculation may be made into a rigorous proof. One of our motivations was to confirm that the resulting potential is consistent with the Toda conjecture of Eguchi and Yang [5].

In the general case, in order to complete the proof, we must use the equation $L_1 Z = 0$, which is part of the Virasoro conjecture of Eguchi, Hori and Xiong [6]. We verify that the Virasoro conjecture then holds in genus 2 for these models.

Our results agree with those of Dubrovin and Zhang [4], who use the method of Eguchi and Xiong [9]; in particular, they use the Virasoro constraints $L_n Z = 0$, $n \leq 10$.

1 Topological recursion relations

1.1 Notation

The correlators of the theory are denoted $\langle \tau_{k_1,a_1} \dots \tau_{k_n,a_n} \rangle_g$. We denote $\tau_{0,a}$ by $\mathcal{O}_a$. The labels on the primaries are fixed in such a way that the puncture operator is $\mathcal{O}_0$. Let η_{ab} be the intersection form, η^{ab} its inverse, and let $\mathcal{O}^a = \eta^{ab}\mathcal{O}_b$. In the case of two primaries, the intersection form equals $\eta_{ab} = \delta_{a+b,1}$.

Let $\mathcal{F}_g$ be the genus g potential on the large phase space:

$$\mathcal{F}_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1 \dots k_n \\ a_1 \dots a_n}} t_{k_1}^{a_1} \dots t_{k_n}^{a_n} \langle \tau_{k_1,a_1} \dots \tau_{k_n,a_n} \rangle_g. \tag{1.1}$$

We use the summation convention with respect to the indices a_i labelling the primaries.

Denote $\partial/\partial t_k^a$ by $\partial_{k,a}$. The vector field $\partial = \partial_{0,0}$, corresponding to the puncture operator $\mathcal{O}_0$, plays a special role in the theory. The partial derivatives of the potential $\mathcal{F}_g$ are denoted

$$\langle\langle \tau_{k_1,a_1} \dots \tau_{k_n,a_n} \rangle\rangle_g = \partial_{k_1,a_1} \dots \partial_{k_n,a_n} \mathcal{F}_g.$$

1.2 The topological recursion relation in genus 0

The simplest example of a topological recursion relation is obtained by taking the relation $\psi_1 = 0$ on the zero-dimensional moduli space $\overline{\mathcal{M}}_{0,3}$. The resulting topological recursion relation is the equation

$$\langle\langle \tau_{k,a} \tau_{\ell,b} \tau_{m,c} \rangle\rangle_0 = \langle\langle \tau_{k-1,a} \mathcal{O}^d \rangle\rangle_0 \langle\langle \mathcal{O}_d \tau_{\ell,b} \tau_{m,c} \rangle\rangle_0. \tag{1.2}$$

Let Θ be the power series

$$\Theta(z)_a^b = \delta_a^b + \sum_{k=0}^{\infty} z^{k+1} \langle\langle \tau_{k,a} \mathcal{O}^b \rangle\rangle_0;$$

it is an orthogonal matrix, in the sense that $\Theta^{-1}(z) = \Theta^*(-z)$. Let $\mathcal{U}$ be the matrix with components $\mathcal{U}_a^b = \langle\langle \mathcal{O}_a \mathcal{O}^b \rangle\rangle_0$. The topological recursion relation (1.2) with $m = 0$ may be rewritten as

$$\partial_{k,a} \Theta(z) = z\, \Theta(z)\, \partial_{k,a} \mathcal{U}. \tag{1.3}$$

Let $\partial_a(z) = \sum_{k=0}^{\infty} z^k\, \partial_{k,a}$, and define vector fields $\{D_{k,a} \mid k \geq 0\}$ on the large phase space by

$$D_a(z) = \sum_{k=0}^{\infty} z^k D_{k,a} = \Theta^{-1}(z)_a^b\, \partial_b(z).$$

For example, $D_{0,a} = \partial_{0,a}$ and $D_{1,a} = \partial_{1,a} - \mathcal{U}_a^b \partial_{0,b}$.

Lemma 1.1. *We have $D_a(z)\mathcal{U} = \partial_{0,a}\mathcal{U}$ and $D_a(z)\Theta(w) = w\Theta(w)\partial_{0,a}\mathcal{U}$. In particular, $D_{k,a}\mathcal{U} = D_{k,a}\Theta = 0$ if $k > 0$.*

Proof. It follows easily from (1.2) that $D_a(z)\mathcal{U} = \partial_{0,a}\mathcal{U}$; since

$$D_a(z)\Theta(w) = w\Theta(w) D_a(z)\mathcal{U},$$

the result follows. □

Corollary 1.2. *The vector fields $D_{k,a}$ and $D_{\ell,b}$ commute if both k and ℓ are positive, while*

$$[D_{k,a}, \partial_{0,b}] = \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 \, D_{k-1,c}.$$

Proof. By Lemma 1.1,

$$\begin{aligned} D_a(w) D_b(z) &= D_a(w)\, \Theta^{-1}(z)_b^c\, \partial_c(z) \\ &= \Theta^{-1}(z)_b^c\, D_a(w)\, \partial_c(z) - z\, \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0\, D_c(z) \\ &= \Theta^{-1}(z)_b^c\, \Theta^{-1}(w)_a^d\, \partial_d(w)\, \partial_c(z) - z\, \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0\, D_c(z). \end{aligned}$$

It follows that $[D_a(w), D_b(z)] = \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 (w\, D_c(w) - z\, D_c(z))$. □

This corollary leads to an algorithm for the calculation of $D_a(z)\langle\langle \mathcal{O}_{a_1} \dots \mathcal{O}_{a_n} \rangle\rangle_g$ by induction on n in terms of $D_a(z)\mathcal{F}_g$, using the formula

$$\begin{aligned} &D_a(z)\langle\langle \mathcal{O}_{a_1} \dots \mathcal{O}_{a_n} \rangle\rangle_g \\ &= \sum_{i=1}^{n} \partial_{0,a_1} \dots [D_a(z), \partial_{0,a_i}] \dots \partial_{0,a_n} \mathcal{F}_g + \partial_{0,a_1} \dots \partial_{0,a_n} D_a(z) \mathcal{F}_g. \end{aligned} \tag{1.4}$$

1.3 The string equation in genus 0 and coordinates on the large phase space

The genus 0 string equation says that $\mathcal{L}_{-1}\mathcal{F}_0 + \frac{1}{2}\eta_{ab} t_0^a t_0^b = 0$, where $\mathcal{L}_{-1}$ is the vector field

$$\mathcal{L}_{-1} = \sum_{k=0}^{\infty} t_{k+1}^a \partial_{k,a} - \partial_{0,0}.$$

The string equation implies the following lemma.

Lemma 1.3. *The restriction of $\partial\mathcal{U}$ to the small phase space $\{t_k^a = 0 \mid k > 0\}$ equals the identity, while for $n > 1$, the restriction of $\partial^n\mathcal{U}$ to the small phase space vanishes.*

Proof. The vector fields $\partial_{0,a}$ commute with $\mathcal{L}_{-1}$; it follows that

$$\mathcal{L}_{-1}\mathcal{U}_{ab} = \mathcal{L}_{-1}\partial_{0,a}\partial_{0,b}\mathcal{F}_0 = \partial_{0,a}\partial_{0,b}\mathcal{L}_{-1}\mathcal{F}_0 = -\eta_{ab}.$$

Written out explicitly, this equation says that

$$\partial\mathcal{U}_a^b = \delta_a^b + \sum_{k=0}^{\infty} t_{k+1}^c \langle\langle \mathcal{O}_a \mathcal{O}^b \tau_{k,c} \rangle\rangle_0.$$

Applying the operator ∂^{n-1}, $n > 0$, we obtain

$$\partial^n \mathcal{U}_a^b = \sum_{k=0}^{\infty} t_{k+1}^c \partial^{n-1} \langle\langle \mathcal{O}_a \mathcal{O}^b \tau_{k,c} \rangle\rangle_0.$$

The lemma is an immediate consequence of these formulas. □

In conjunction with the genus 0 topological recursion relation, this implies the following theorem.

Theorem 1.4. *Let* $u^a = \partial\langle\langle \mathcal{O}^a \rangle\rangle_0$. *The functions* $u_n^a = \partial^n u^a$, $n \geq 0$, *form a coordinate system in a neighbourhood of the small phase space, and*

$$D_a(z) = \sum_{n=0}^{\infty} ((\partial + z\,\partial\mathcal{U})^n \partial\mathcal{U})_a^b \frac{\partial}{\partial u_n^b}. \tag{1.5}$$

Proof. Since $u^b = \mathcal{U}_0^b$, Lemma 1.1 implies that

$$D_a(z) u_n^b = \left(\Theta^{-1}(z)\,\partial^n\,\Theta(z)\,\partial\mathcal{U}\right)_a^b = \left((\Theta^{-1}(z)\cdot\partial\cdot\Theta(z))^n\,\partial\mathcal{U}\right)_a^b.$$

Since $\Theta^{-1}(z)\cdot\partial\cdot\Theta(z) = \partial + z\,\partial\mathcal{U}$ by (1.3), we conclude that $D_a(z)u_n^b = ((\partial + z\,\partial\mathcal{U})^n \partial\mathcal{U})_a^b$.

By Lemma 1.3, the restriction of $(\partial + z\,\partial\mathcal{U})^n \partial\mathcal{U}$ to the small phase space equals z^n. It follows that the restriction of $D_{k,a}u_n^b$ to the small phase space equals $\delta_{k,n}\delta_a^b$; hence the functions u_n^a form a coordinate system in a neighbourhood of the small phase space. □

In the case that there is a single primary field, we have $(\partial + z\,\partial\mathcal{U})^n \partial\mathcal{U} = z^{-1} p_{n+1}(z\partial\mathcal{U})$, where $p_{n+1}(f) = (\partial + f)^n f$ is the $(n+1)$st Faà di Bruno polynomial.

Corollary 1.5. *If* $D_{k,a} f = 0$ *for* $k > n$, *then* $\partial f / \partial u_k^a = 0$ *for* $k > n$.

Corollary 1.6. *In terms of the coordinates u_n^a, the small phase space $\{t_k^a = 0 \mid k > 0\}$ is the submanifold*

$$u_n^a = \begin{cases} \delta_0^a & n = 1, \\ 0 & n > 1. \end{cases}$$

Theorem 1.4 shows that the large phase space may be defined for any Frobenius manifold M, as the infinite jet space $J^\infty M$ (i.e. Dubrovin's "loop space"). This is seen by rewriting the matrix $\partial \mathcal{U}_b^a$ as $\partial u^c \mathcal{A}_{bc}^a$, where

$$\mathcal{A}_{bc}^a = \frac{\partial \mathcal{U}_b^a}{\partial u^c} \tag{1.6}$$

is the tensor describing the product on the tangent bundle of M.

An attractive feature of the vector fields $D_{k,a}$ is that they commute with $\mathcal{L}_{-1}$:

$$\begin{aligned} [\mathcal{L}_{-1}, D_a(z)] &= [\mathcal{L}_{-1}, \Theta^{-1}(z)_a^b \, \partial_b(z)] \\ &= [\mathcal{L}_{-1}, \Theta^{-1}(z)_a^b] \, \partial_b(z) + \Theta^{-1}(z)_a^b \, [\mathcal{L}_{-1}, \partial_b(z)] \\ &= (z \, \Theta^{-1}(z)_a^b) \, \partial_b(z) - \Theta^{-1}(z)_a^b \, (z \, \partial_b(z)) = 0. \end{aligned}$$

By the genus 0 string equation, $\mathcal{L}_{-1} u_n^a$ vanishes for $n > 0$, while $\mathcal{L}_{-1} u^a = -\delta_0^a$: it follows that in the coordinate system $\{u_n^a\}$, the vector field $\mathcal{L}_{-1}$ is given by the formula

$$\mathcal{L}_{-1} = -\frac{\partial}{\partial u^0}.$$

In the coordinate system $\{u_n^a\}$, the string equation $\mathcal{L}_{-1} \mathcal{F}_g = 0$ says that $\mathcal{F}_g$ is independent of u^0.

Lemma 1.3 shows that $\partial \mathcal{U}$ is invertible in a neighbourhood of the small phase space: denote its inverse by $\mathcal{C}$. We also see that its determinant $\Delta = \det(\partial \mathcal{U})$ equals 1 on the small phase space.

1.4 The topological recursion relation in genus 1

We now illustrate the way in which use of the vector fields $D_{k,a}$ simplifies the discussion of topological recursion relations, using as an example the topological recursion relation in genus 1:

$$\langle\langle \tau_{k,a} \rangle\rangle_1 = \langle\langle \tau_{k-1,a} \mathcal{O}^b \rangle\rangle_0 \langle\langle \mathcal{O}_b \rangle\rangle_1 + \tfrac{1}{24} \langle\langle \tau_{k-1,a} \mathcal{O}_b \mathcal{O}^b \rangle\rangle_0. \tag{1.7}$$

Multiplying by z^k and summing over k, we obtain

$$\partial_a(z)\mathcal{F}_1 = \Theta(z)_a^b \langle\langle \mathcal{O}_b \rangle\rangle_1 + \tfrac{1}{24}\, z\, \partial_a(z)\, \mathrm{Tr}(\mathcal{U}),$$

hence, by Lemma 1.1,

$$D_a(z)\mathcal{F}_1 = \langle\langle \mathcal{O}_b \rangle\rangle_1 + \tfrac{1}{24}\, z\, D_a(z)\, \mathrm{Tr}(\mathcal{U}) = \langle\langle \mathcal{O}_b \rangle\rangle_1 + \tfrac{1}{24}\, z\, \partial_{0,a}\, \mathrm{Tr}(\mathcal{U}).$$

This may be written as the sequence of differential equations

$$D_{k,a}\mathcal{F}_1 = \begin{cases} \frac{1}{24}\, \partial_{0,a}\, \mathrm{Tr}(\mathcal{U}) & k = 1, \\ 0 & k > 1. \end{cases} \tag{1.8}$$

The equations (1.8) have the particular solution $\frac{1}{24}\log(\Delta)$. Let $\psi = \mathcal{F}_1 - \frac{1}{24}\log(\Delta)$; we see that $D_{k,a}\psi = 0$ for all $k > 0$. Hence, by Corollary 1.5, ψ depends only on the coordinates u^a; by the string equation, it is independent of u^0. In this way, we recover a result of Dijkgraaf and Witten [2]: there is a function ψ of the coordinates $\{u^a\}$ such that $\mathcal{F}_1 = \frac{1}{24}\log(\Delta) + \psi$.

1.5 The dilaton equation

The dilaton equation is another important constraint on the potentials of topological gravity. Let $\mathcal{D}$ be the vector field

$$\mathcal{D} = \partial_{1,0} - \sum_{k=0}^{\infty} t_k^a \partial_{k,a}.$$

The dilaton equation says that

$$\mathcal{D}\mathcal{F}_g = \begin{cases} (2g-2)\mathcal{F}_g, & g \neq 1, \\ \chi/24, & g = 1, \end{cases}$$

where χ is the Euler characteristic of the background.

Proposition 1.7. *In the coordinate system $\{u_n^a\}$, the dilaton vector field $\mathcal{D}$ equals*

$$\mathcal{D} = \sum_{n=1}^{\infty} n\, u_n^a \frac{\partial}{\partial u_n^a}.$$

Proof. By the genus 0 dilaton equation $\mathcal{D}\mathcal{F}_0 = -2\mathcal{F}_0$, we have $\mathcal{D}u_n^a = n\, u_n^a$, and the formula for $\mathcal{D}$ follows. □

2 The A_2 and $\mathbb{CP}^1$ models in genus 2

In genus 2, there are two topological recursion relations [11]. The first is

$$\begin{aligned}
&\langle\langle \tau_{k,a} \rangle\rangle_2 \\
&= \langle\langle \tau_{k-1,a}\mathcal{O}^b \rangle\rangle_0 \langle\langle \mathcal{O}_b \rangle\rangle_2 + \langle\langle \tau_{k-2,a}\mathcal{O}^b \rangle\rangle_0 \big(\langle\langle \tau_{1,b} \rangle\rangle_2 - \langle\langle \mathcal{O}_b\mathcal{O}^c \rangle\rangle_0 \langle\langle \mathcal{O}_c \rangle\rangle_2 \big) \\
&+ \langle\langle \tau_{k-2,a}\mathcal{O}^b\mathcal{O}^c \rangle\rangle_0 \big(\tfrac{7}{10} \langle\langle \mathcal{O}_b \rangle\rangle_1 \langle\langle \mathcal{O}_c \rangle\rangle_1 + \tfrac{1}{10} \langle\langle \mathcal{O}_b\mathcal{O}_c \rangle\rangle_1 \big) \\
&+ \tfrac{13}{240} \langle\langle \tau_{k-2,a}\mathcal{O}^b\mathcal{O}^c\mathcal{O}_c \rangle\rangle_0 \langle\langle \mathcal{O}_b \rangle\rangle_1 - \tfrac{1}{240} \langle\langle \tau_{k-2,a}\mathcal{O}^b \rangle\rangle_1 \langle\langle \mathcal{O}_b\mathcal{O}^c\mathcal{O}_c \rangle\rangle_0 \\
&+ \tfrac{1}{960} \langle\langle \tau_{k-2,a}\mathcal{O}^b\mathcal{O}_b\mathcal{O}^c\mathcal{O}_c \rangle\rangle_0.
\end{aligned} \tag{2.1}$$

Using the topological recursion relations in genus 0 and 1, (2.1) may be rewritten as the sequence of differential equations

$$D_{k,a}\mathcal{F}_2 = \mathcal{R}_{k,a}, \tag{2.2}$$

where

$$\mathcal{R}_{k,a} = \begin{cases}
\langle\langle \mathcal{O}_a\mathcal{O}_b\mathcal{O}_c \rangle\rangle_0 \big(\tfrac{7}{10} \langle\langle \mathcal{O}^b \rangle\rangle_1 \langle\langle \mathcal{O}^c \rangle\rangle_1 + \tfrac{1}{10} \langle\langle \mathcal{O}^b\mathcal{O}^c \rangle\rangle_1 \big) & \\
+\tfrac{13}{240} \langle\langle \mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}^c \rangle\rangle_0 \langle\langle \mathcal{O}^b \rangle\rangle_1 & \\
-\tfrac{1}{240} \langle\langle \mathcal{O}_a\mathcal{O}^b \rangle\rangle_1 \langle\langle \mathcal{O}_b\mathcal{O}_c\mathcal{O}^c \rangle\rangle_0 & \\
+\tfrac{1}{960} \langle\langle \mathcal{O}_a\mathcal{O}_b\mathcal{O}^b\mathcal{O}_c\mathcal{O}^c \rangle\rangle_0 & k=2, \\
\langle\langle \mathcal{O}_a\mathcal{O}_b\mathcal{O}_c \rangle\rangle_0 \big(\tfrac{1}{20} \langle\langle \mathcal{O}^b \rangle\rangle_1 \langle\langle \mathcal{O}^c\mathcal{O}^d\mathcal{O}_d \rangle\rangle_0 & \\
+\tfrac{1}{480} \langle\langle \mathcal{O}^b\mathcal{O}^c\mathcal{O}^d\mathcal{O}_d \rangle\rangle_0 \big) & \\
+\tfrac{1}{1152} \langle\langle \mathcal{O}_a\mathcal{O}^b\mathcal{O}^c\mathcal{O}_c \rangle\rangle_0 \langle\langle \mathcal{O}_b\mathcal{O}^d\mathcal{O}_d \rangle\rangle_0 & k=3, \\
\tfrac{1}{1152} \langle\langle \mathcal{O}_a\mathcal{O}^b\mathcal{O}^c \rangle\rangle_0 \langle\langle \mathcal{O}_b\mathcal{O}_c\mathcal{O}^d \rangle\rangle_0 \langle\langle \mathcal{O}_d\mathcal{O}^e\mathcal{O}_e \rangle\rangle_0 & k=4, \\
0 & k>4.
\end{cases}$$

The other topological recursion relation in genus 2 is

$$\begin{aligned}
\langle\langle\tau_{k,a}\tau_{\ell,b}\rangle\rangle_2 &= \langle\langle\tau_{k,a}\mathcal{O}_c\rangle\rangle_2\langle\langle\mathcal{O}^c\tau_{\ell-1,b}\rangle\rangle_0 + \langle\langle\tau_{k-1,a}\mathcal{O}_c\rangle\rangle_0\langle\langle\mathcal{O}^c\tau_{\ell,b}\rangle\rangle_2 \\
&- \langle\langle\tau_{k-1,a}\mathcal{O}_c\rangle\rangle_0\langle\langle\tau_{\ell-1,b}\mathcal{O}_d\rangle\rangle_0\langle\langle\mathcal{O}^c\mathcal{O}^d\rangle\rangle_2 \\
&+ 3\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}^c\rangle\rangle_0\big(\langle\langle\tau_{1,c}\rangle\rangle_2 - \langle\langle\mathcal{O}_c\mathcal{O}^d\rangle\rangle_0\langle\langle\mathcal{O}_d\rangle\rangle_2\big) \\
&+ \tfrac{13}{10}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\langle\langle\mathcal{O}^c\rangle\rangle_1\langle\langle\mathcal{O}^d\rangle\rangle_1 \\
&+ \tfrac{4}{5}\big(\langle\langle\tau_{k-1,a}\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}_d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\tau_{k-1,a}\mathcal{O}_c\mathcal{O}_d\rangle\rangle_1\big)\langle\langle\tau_{\ell-1,b}\mathcal{O}^c\mathcal{O}^d\rangle\rangle_0 \\
&+ \tfrac{4}{5}\langle\langle\tau_{k-1,a}\mathcal{O}^c\mathcal{O}^d\rangle\rangle_0\big(\langle\langle\tau_{\ell-1,b}\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}_d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\tau_{\ell-1,b}\mathcal{O}_c\mathcal{O}_d\rangle\rangle_1\big) \\
&- \tfrac{4}{5}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\rangle\rangle_0\big(\langle\langle\mathcal{O}^c\mathcal{O}_d\rangle\rangle_1\langle\langle\mathcal{O}^d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_1\big) \\
&+ \tfrac{1}{48}\langle\langle\tau_{k-1,a}\mathcal{O}_c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0\langle\langle\mathcal{O}^c\tau_{\ell-1,b}\rangle\rangle_1 \\
&+ \tfrac{1}{48}\langle\langle\tau_{k-1,a}\mathcal{O}_c\rangle\rangle_1\langle\langle\tau_{\ell-1,b}\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0 \\
&+ \tfrac{23}{240}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0\langle\langle\mathcal{O}^c\rangle\rangle_1 \\
&- \tfrac{1}{80}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}^c\mathcal{O}^d\mathcal{O}_d\rangle\rangle_0 \\
&+ \tfrac{7}{30}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\langle\langle\mathcal{O}^c\mathcal{O}^d\rangle\rangle_1 + \tfrac{1}{576}\langle\langle\tau_{k-1,a}\tau_{\ell-1,b}\mathcal{O}_c\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0.
\end{aligned} \tag{2.3}$$

Taking k and ℓ equal to 1 and using the topological recursion relations in genus 0 and 1, we obtain the system of differential equations

$$D_{1,1,a,b}\mathcal{F}_2 = \mathcal{R}_{1,1,a,b}, \tag{2.4}$$

where $D_{1,1,a,b} = D_{1,a}D_{1,b} - 3\,\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}^c\rangle\rangle_0\,D_{1,c}$, and

$$\begin{aligned}
\mathcal{R}_{1,1,a,b} &= \tfrac{13}{10}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\langle\langle\mathcal{O}^c\rangle\rangle_1\langle\langle\mathcal{O}^d\rangle\rangle_1 \\
&+ \tfrac{4}{5}\big(\langle\langle\mathcal{O}_a\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}_d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\mathcal{O}_a\mathcal{O}_c\mathcal{O}_d\rangle\rangle_1\big)\langle\langle\mathcal{O}_b\mathcal{O}^c\mathcal{O}^d\rangle\rangle_0 \\
&+ \tfrac{4}{5}\langle\langle\mathcal{O}_a\mathcal{O}^c\mathcal{O}^d\rangle\rangle_0\big(\langle\langle\mathcal{O}_b\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}_d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle\rangle_1\big) \\
&- \tfrac{4}{5}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle\rangle_0\big(\langle\langle\mathcal{O}^c\mathcal{O}_d\rangle\rangle_1\langle\langle\mathcal{O}^d\rangle\rangle_1 + \tfrac{1}{24}\langle\langle\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_1\big) \\
&+ \tfrac{1}{48}\langle\langle\mathcal{O}_a\mathcal{O}_c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0\langle\langle\mathcal{O}^c\mathcal{O}_b\rangle\rangle_1 + \tfrac{1}{48}\langle\langle\mathcal{O}_a\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}_b\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0 \\
&+ \tfrac{23}{240}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0\langle\langle\mathcal{O}^c\rangle\rangle_1 - \tfrac{1}{80}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle\rangle_1\langle\langle\mathcal{O}^c\mathcal{O}^d\mathcal{O}_d\rangle\rangle_0 \\
&+ \tfrac{7}{30}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\langle\langle\mathcal{O}^c\mathcal{O}^d\rangle\rangle_1 + \tfrac{1}{576}\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}^c\mathcal{O}_d\mathcal{O}^d\rangle\rangle_0.
\end{aligned}$$

We now specialize to the case of the A_2 model. In this model, there are two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$, with intersection form $\eta_{ab} = \delta_{a+b,1}$. Denote the associated coordinates $u = \langle\langle\mathcal{O}_0\mathcal{O}_0\rangle\rangle_0 = \partial^2\mathcal{F}_0$ and $v = \langle\langle\mathcal{O}_0\mathcal{O}_1\rangle\rangle_0$. The matrix $\mathcal{U}$ is given by the formula

$$\mathcal{U} = \begin{bmatrix}\mathcal{U}_0^0 & \mathcal{U}_0^1 \\ \mathcal{U}_1^0 & \mathcal{U}_1^1\end{bmatrix} = \begin{bmatrix} v & u \\ u^2 & v\end{bmatrix},$$

and $\mathcal{F}_1 = \frac{1}{24}\log(\Delta)$. As was shown by Eguchi, Yamada and Yang [8], the genus 2 potential of the A_2-model is given by the formula

$$\begin{aligned}\mathcal{F}_2 &= \tfrac{1}{1152}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\,c^{ab}c^{cd}\\ &- \tfrac{1}{1152}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,\partial\langle\langle\mathcal{O}_c\mathcal{O}_d\mathcal{O}_e\mathcal{O}_f\rangle\rangle_0\,c^{ac}c^{bd}c^{ef}\\ &- \tfrac{1}{360}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle\rangle_0\,\partial\langle\langle\mathcal{O}_d\mathcal{O}_e\mathcal{O}_f\rangle\rangle_0\,c^{ad}c^{be}c^{cf}\\ &+ \tfrac{1}{360}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,\partial\langle\langle\mathcal{O}_c\mathcal{O}_d\mathcal{O}_e\rangle\rangle_0\,\partial\langle\langle\mathcal{O}_f\mathcal{O}_g\mathcal{O}_h\rangle\rangle_0\,c^{ac}c^{bf}c^{dg}c^{eh}.\end{aligned} \tag{2.5}$$

It may be checked that this function solves the equations (2.2) and (2.4).

For an arbitrary theory of topological gravity, let $\mathcal{F}_{2,0}$ be the function on the large phase space given by formula (2.5). For all theories of topological gravity for which we know the genus 2 potential, the function $\mathcal{F}_{2,0}$ appears to be a major contribution to this potential.

We now turn to the case of $\mathbb{CP}^1$. As in the A_2-model, there are two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$, with intersection form $\eta_{ab} = \delta_{a+b,1}$. Again, denote the associated coordinates by $u = \langle\langle\mathcal{O}_0\mathcal{O}_0\rangle\rangle_0 = \partial^2\mathcal{F}_0$ and $v = \langle\langle\mathcal{O}_0\mathcal{O}_1\rangle\rangle_0$. The matrix $\mathcal{U}$ is now given by the formula

$$\mathcal{U} = \begin{bmatrix} v & u \\ e^u & v \end{bmatrix},$$

and $\mathcal{F}_1 = \frac{1}{24}\log(\Delta) - \frac{1}{24}u$.

The correlators $\langle\tau_{1,a_1}\mathcal{O}_{a_2}\dots\mathcal{O}_{a_n}\rangle_2$ and $\langle\mathcal{O}_{a_1}\mathcal{O}_{a_2}\dots\mathcal{O}_{a_n}\rangle_2$ vanish in the $\mathbb{CP}^1$-model for dimensional reasons. It follows that the following solution to the equations (2.2) and (2.4) is the genus 2 potential:

$$\begin{aligned}\mathcal{F}_2 = \mathcal{F}_{2,0} &- \tfrac{1}{480}\,\partial^3\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,c^{ab} + \tfrac{7}{5760}\,\partial^3\langle\langle\mathcal{O}_a\rangle\rangle_0\,\partial^2\langle\langle\mathcal{O}_b\rangle\rangle_0\,c^{ab}\\ &+ \tfrac{11}{5760}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,\partial^2\langle\langle\mathcal{O}_c\mathcal{O}_d\rangle\rangle_0\,c^{ac}c^{bd}.\end{aligned} \tag{2.6}$$

The three additional terms reflect the fact that, unlike in the A_2-model, the function $\psi(u) = -\frac{1}{24}u$ is nonzero in the $\mathbb{CP}^1$-model.

The Toda conjecture of Eguchi and Yang ([5], [7], [16]) provides conjectural formulas for the functions $\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_g$, $g > 0$, of the $\mathbb{CP}^1$-model:

$$\sum_{g=0}^{\infty}\lambda^{2g}\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_g = \exp\Big(\frac{2}{\lambda^2}\big(\cosh(\lambda\partial)-1\big)\sum_{g=0}^{\infty}\lambda^{2g}\mathcal{F}_g\Big).$$

In genus 2, this yields the equation

$$\langle\langle \mathcal{O}_1\mathcal{O}_1\rangle\rangle_2 = e^u\big(\partial^2\mathcal{F}_2 + \tfrac{1}{12}\partial^4\mathcal{F}_1 + \tfrac{1}{360}\partial^6\mathcal{F}_0 + \tfrac{1}{2}(\partial^2\mathcal{F}_1 + \tfrac{1}{12}\partial^4\mathcal{F}_0)^2\big). \quad (2.7)$$

It is easily checked, using the explicit formula formula for $\mathcal{F}_2$, that this equation holds.

3 Models with two primaries

In this section, we consider topological gravity in a general background with two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$, and intersection form $\eta_{ab} = \delta_{a+b,1}$. It is not clear to what extent such a model, even if it possesses a consistent loop expansion, corresponds to a physical theory: it may be that only the A_2 and $\mathbb{CP}^1$-models are physical theories. The fact that our equations remain consistent in this setting is nevertheless very suggestive.

Denote the associated coordinates $u = \langle\langle\mathcal{O}_0\mathcal{O}_0\rangle\rangle_0$ and $v = \langle\langle\mathcal{O}_0\mathcal{O}_1\rangle\rangle_0$. The genus 0 sector is characterized by the function $\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_0$; by the string equation, this is a function of u alone, and we denote it by $\phi(u)$. The matrix $\mathcal{U}$ is given by the formula

$$\mathcal{U} = \begin{bmatrix} v & u \\ \phi(u) & v \end{bmatrix}.$$

In this section, the correlation functions $\langle\langle\tau_{k_1,a_1}\dots\tau_{k_n,a_n}\rangle\rangle_g$ are assumed to have the following form: they are holomorphic functions of $\{(v,u)\in\mathbb{C}^2 \mid u \notin (-\infty,0]\}$, Laurent polynomials in Δ, and polynomial in the remaining coordinates $\{\partial^n v, \partial^n u \mid n>0\}$.

There is a universal differential equation [10] in topological gravity relating the potentials $\mathcal{F}_0$ and $\mathcal{F}_1$. In the case of two primary fields, this equation says that

$$\tfrac{1}{24}\phi''' + \phi''\psi' - 2\,\phi'\psi'' = 0. \quad (3.1)$$

It turns out that this equation is also the necessary and sufficient condition for the system of equations (2.2) and (2.4) to have a solution. The necessity follows from the formula

$$D_{1,1,0,0}\mathcal{R}_{2,0} - D_{2,0}\mathcal{R}_{1,1,0,0} = \tfrac{2}{15}(\partial u)^3(4(\partial v)^2 + (\partial u)^2\phi')(\tfrac{1}{24}\phi''' + \phi''\psi' - 2\phi'\psi'').$$

Theorem 3.1. *Suppose that* $\frac{1}{24}\phi''' + \phi''\psi' - 2\phi'\psi'' = 0$. *Then the equations* (2.2) *and* (2.4) *have the solution* $\mathcal{F}_{2,0} + \mathcal{F}_{2,1}$, *where* $\mathcal{F}_{2,0}$ *is given by* (2.5), *and*

$$\begin{aligned}\mathcal{F}_{2,1} = \tfrac{1}{576}\Big(&(\tfrac{1}{2}\,\partial\,\partial_{0,a}\partial_{0,b}\psi + \tfrac{4}{5}\,\partial\,\partial_{0,a}\psi\,\partial_{0,b}\psi)\,\mathcal{C}^{ab} \\ &+ \partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,(\tfrac{6}{5}\,\partial_{0,c}\partial_{0,d}\psi - \tfrac{1}{10}\,\partial_{0,c}\psi\,\partial_{0,d}\psi)\,\mathcal{C}^{ac}\,\mathcal{C}^{bd} \\ &+ (\tfrac{7}{10}\,\partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle\rangle_0\,\partial_{0,d}\psi - \tfrac{3}{10}\,\partial\langle\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle\rangle_0\,\partial\,\partial_{0,d}\psi)\,\mathcal{C}^{ab}\,\mathcal{C}^{cd} \\ &+ \partial^2\langle\langle\mathcal{O}_a\mathcal{O}_b\rangle\rangle_0\,\partial\langle\langle\mathcal{O}_c\mathcal{O}_d\mathcal{O}_e\rangle\rangle_0\,\partial_{0,f}\psi\,(\tfrac{3}{10}\,\mathcal{C}^{af}\,\mathcal{C}^{bc}\,\mathcal{C}^{de} - \tfrac{23}{10}\,\mathcal{C}^{ac}\,\mathcal{C}^{bd}\,\mathcal{C}^{ef}) \\ &+ \tfrac{1}{10}\,(\partial u)^4\,\phi''\,\psi''\,\Delta^{-1}\Big).\end{aligned}$$

This solution may be characterized by the property that its restriction to the small phase space, together with the restrictions of the functions $\partial_{1,a}(\mathcal{F}_{2,0} + \mathcal{F}_{2,1})$, *vanish.*

All of the terms in the formula for $\mathcal{F}_{2,0} + \mathcal{F}_{2,1}$ except the last one $\frac{1}{5760}(\partial u)^4\,\phi''\,\psi''\,\Delta^{-1}$ are associated to Feynman graphs with propagator $\mathcal{C}$ and vertices $\partial^n\partial_{a_1}\dots\partial_{a_{k-2}}\mathcal{U}_{a_{k-1}a_k}$ and $\partial^n\partial_{a_1}\dots\partial_{a_k}\psi$. From this point of view, the last term is an instanton, which vanishes if ψ is a linear function of u, that is, for the A_2 and $\mathbb{CP}^1$-models.

One calculates that $\mathcal{F}_{2,1}$ is given by the explicit formula

$$\begin{aligned}\mathcal{F}_{2,1} = \tfrac{1}{576}\Big(&\tfrac{1}{2}\,(\partial u)^2\,\psi''' + \tfrac{9}{5}\,(\partial u)^2\,\psi''\,\psi' + \tfrac{13}{5}\,\partial^2 u\,\psi'' + \tfrac{7}{10}\,\partial^2 u\,(\psi')^2 \\ &- ((\partial v)^2 + \tfrac{7}{5}\,(\partial u)^2)\,(\partial u)^2\,\phi'\,\psi''\,\psi'\,\Delta^{-1} + \tfrac{6}{5}\,(\partial u)^4\,\phi''\,(\psi')^2\,\Delta^{-1} \\ &+ (\tfrac{2}{5}\,(\partial^2 v\,\partial v - \partial^2 u\,\partial v)\,\partial v - \tfrac{1}{10}\,(\partial u)^4\,\phi'')\,\psi''\,\Delta^{-1} \\ &+ (\tfrac{12}{5}\,\partial^3 v\,\partial v - \tfrac{12}{5}\,\partial^3 u\,\partial u\,\phi' - \tfrac{7}{5}\,\partial^2 u\,(\partial u)^2\,\phi'')\,\psi'\,\Delta^{-1} \\ &+ \tfrac{11}{5}(4\,\partial^2 v\,\partial^2 u\,\partial v\,\partial u\,\phi' - (\partial^2 v + \partial^2 u\,\phi')((\partial v)^2 + (\partial u)^2\,\phi') \\ &\quad + 2\,(\partial^2 v\,\partial u - \partial^2 u\,\partial v)\,(\partial u)^2\,\partial v\,\phi'' - \tfrac{1}{2}\,(\partial u)^6\,(\phi'')^2)\,\psi'\,\Delta^{-2}\Big).\end{aligned}$$

Now let $\mathcal{F}_2$ be a general solution of (2.2) and (2.4). Write $\mathcal{F}_2 = \mathcal{F}_{2,0} + \mathcal{F}_{2,1} + f_2$. By the equations (2.2), $D_{k,a}f_2 = 0$ for $k > 1$; thus, f_2 is a function of the coordinates $\{u, \partial v, \partial u\}$.

Theorem 3.2. *Define the functions* $h_a = h_a(u)$ *by the formula* $h_a = \left.\dfrac{\partial f_2}{\partial(\partial u^a)}\right|_{(\partial v,\partial u)=(1,0)}$. *Then*

$$f_2 = \tfrac{1}{2}\,\partial u^a\,\partial\mathcal{U}_a^b\,h_b = \tfrac{1}{2}\,\partial u^a\,\partial u^b\,\mathcal{A}_{ab}^c h_c$$
$$= \tfrac{1}{2}\big((\partial v)^2 + \phi'\,(\partial u)^2\big)\,h_0(u) + \partial v\,\partial u\,h_1(u).$$

Proof. Let $\tilde{f}_2 = \tfrac{1}{2}\,\partial u^a\,\partial\mathcal{U}_a^b\,h_b$; then $D_{1,1,a,b}\tilde{f}_2 = 0$ and $h_a = \left.\dfrac{\partial\tilde{f}_2}{\partial(\partial u^a)}\right|_{(\partial v,\partial u)=(1,0)}$.

Thus $f_2 - \tilde{f}_2$ satisfies the equations $D_{k,a}(f_2 - \tilde{f}_2) = 0$ for $k > 1$, and $D_{1,1,a,b}(f_2 - \tilde{f}_2) = 0$, as well as the dilaton equation $\mathcal{D}(f_2 - \tilde{f}_2) = 2(f_2 - \tilde{f}_2)$, and is thus determined by the restrictions of the partial derivatives $\partial_{1,a}(f_2 - \tilde{f}_2)$ to the small phase space. But these vanish; we conclude that $f_2 = \tilde{f}_2$. □

In the next section, we determine the functions h_a.

4 Virasoro constraints

We now show that the Virasoro constraints $L_0 Z = L_1 Z = 0$ of Eguchi, Hori and Xiong [6], as generalized to arbitrary Frobenius manifolds by Dubrovin and Zhang [3], may be used to complete the determination of the genus 2 potential in two-primary models of topological gravity.

The constraint $L_0 Z = 0$

According to Dubrovin and Zhang [3], an Euler vector on a Frobenius manifold determines matrices μ and $R[n]$, $n > 0$, which satisfy the commutation relations $[\mu, R[n]] = n\,R[n]$ and the symmetry conditions $\mu_{ab} + \mu_{ba} = 0$ and

$$R[n]_{ab} + (-1)^n R[n]_{ba} = 0.$$

The basis $\mathcal{O}_a$ of primary fields may be chosen in such a way that the matrix μ is diagonal

$$\mu_a^b = \delta_a^b\,\mu_a,$$

and $\mu_0 < \mu_a$ for $a \neq 0$. Setting $d_a = \mu_a - \mu_0$ and $d = -2\,\mu_0$, we have $\mu_a = d_a - d/2$.

For the Gromov-Witten invariants of a Kähler manifold X, the primaries $\mathcal{O}_a$ form a basis of the De Rham cohomology $H^*(X, \mathbb{C})$, and the number d_a is the holomorphic degree of $\mathcal{O}_a$, that is $\mathcal{O}_a \in H^{d_a,*}(X, \mathbb{C})$. (In particular, d equals the complex dimension of X.) In this case, $R[1]$ is the matrix of multiplication by $c_1(X)$, and $R[n] = 0$ for $n > 1$.

Introduce the vector field

$$\mathcal{L}_0 = \sum_{k=0}^{\infty} \left\{ (\mu_a^b + k + \tfrac{1}{2}) \tilde{t}_k^a \, \partial_{k,b} + \sum_{\ell=1}^{k} R[\ell]_a^b \, \tilde{t}_k^a \, \partial_{k-\ell,b} \right\},$$

where $\tilde{t}_k^a$ are the shifted coordinates $\tilde{t}_k^a = t_k^a - \delta_{k,1}\delta_0^a$. The Virasoro constraint $L_0 Z = 0$ in genus $g = 0$ may be expressed as the following equation:

$$\mathcal{L}_0 \mathcal{U} + \mathcal{U} + [\mu, \mathcal{U}] + R[1] = 0. \tag{4.1}$$

In genus $g > 0$, the Virasoro constraint $L_0 Z = 0$ says that

$$\mathcal{L}_0 \mathcal{F}_g + \tfrac{1}{4}\,\delta_{g,1}\,\mathrm{Tr}(\tfrac{1}{4} - \mu^2) = 0. \tag{4.2}$$

These equations are known to hold for Gromov-Witten invariants [14].

Let $\mathcal{E} = \mathcal{E}^a \partial/\partial u^a$ be the Euler vector field, where

$$\mathcal{E}^a = (1 - d_a) u^a + R[1]_0^a. \tag{4.3}$$

Then (4.1) implies that

$$\mathcal{L}_0 u^a + \mathcal{E}^a = 0. \tag{4.4}$$

In calculating the action of the vector field $\mathcal{L}_0$ in the coordinate system $\{u_n^a\}$, we use (4.4) together with the commutation relation $[\partial, \mathcal{L}_0] = \frac{1}{2}(1 - d)\partial$.

In the case of two primary fields, we have $\mu = \frac{1}{2}\left[\begin{smallmatrix} -d & 0 \\ 0 & d \end{smallmatrix}\right]$. Consider first the case in which d equals 1; then $R[1] = \left[\begin{smallmatrix} 0 & r \\ 0 & 0 \end{smallmatrix}\right]$. By (4.1), we see that $\phi = c\,e^{2u/r}$; redefining u, we may assume that $r = 2$, and we recover the $\mathbb{CP}^1$-model. Since $\mathrm{Tr}(\mu^2 - \frac{1}{4}) = 0$, we see from (4.2) that $\mathcal{L}_0 \mathcal{F}_1 = 0$; (3.1), now shows that $\psi = -\frac{1}{24} u$, consistent with the known form of $\mathcal{F}_1$ in the $\mathbb{CP}^1$-model.

The equation $\mathcal{L}_0\mathcal{F}_2 = 0$ of (4.2) constrains the functions $h_a(u)$ of Theorem 3.2; if $d = 1$, it forces them to have negative degree in e^u, and hence to vanish, as we have already observed,

If $d \neq 1$, the matrix $R[1]$ vanishes. By (4.1), we see that $\phi(u) = u^{(1+d)/(1-d)}$, up to a constant which we take to equal 1. (For example, the A_2-model, has $d = \frac{1}{3}$ and $\phi(u) = u^2$.) In genus 1, the equation (4.2) shows that $\psi(u)$ is proportional to $\log(u)$; both (3.1) and (4.2) yield the same answer for this constant,

$$\psi(u) = \frac{d(3d-1)}{24(d-1)} \log(u).$$

Note that the A_2-model, for which $d = \frac{1}{3}$, has $\psi = 0$. The equation $\mathcal{L}_0\mathcal{F}_2 = 0$ imposes the homogeneities $h_a(u) = C_a\, u^{((1+a)d-3)/(1-d)}$.

The constraint $L_1 Z = 0$

Let $\mathcal{L}_1$ be the vector field

$$\begin{aligned}\mathcal{L}_1 = &-(\mu_a - \tfrac{1}{2})(\mu_a + \tfrac{1}{2})\langle\langle \mathcal{O}^a\rangle\rangle_0 \partial_{0,a} \\ &+ \sum_{k=0}^{\infty}\Big\{ (\mu_a + k + \tfrac{1}{2})(\mu_a + k + \tfrac{3}{2})\tilde{t}^a_k \partial_{k+1,a} \\ &+ \sum_{\ell \le k+1} 2(\mu_a + k + 1) R[\ell]^b_a \tilde{t}^a_k \partial_{k+1-\ell,b} \\ &+ \sum_{\ell_1+\ell_2 \le k+1} (R[\ell_1]R[\ell_2])^b_a \tilde{t}^a_k \partial_{k+1-\ell_1-\ell_2,b} \Big\}.\end{aligned}$$

Let $\mathcal{V} = \mathcal{E}\mathcal{U}$; by (4.1), $\mathcal{V} = \mathcal{U} + [\mu, \mathcal{U}] + R[1]$. The constraint $L_1 Z = 0$ in genus 0 may written (Dubrovin and Zhang [3]; cf. Theorem 5.7 of [12])

$$\mathcal{L}_1\mathcal{U} + \mathcal{V}^2 = 0. \tag{4.5}$$

In particular, we see that

$$\mathcal{L}_1 u^a + \mathcal{E}^b\mathcal{E}^c\mathcal{A}^a_{bc} = 0. \tag{4.6}$$

In genus $g > 0$,the constraint $L_1 Z = 0$ is

$$\mathcal{L}_1\mathcal{F}_g + \tfrac{1}{2}\left(\tfrac{1}{4} - \mu^2\right)^{ab}\Big(\sum_{h=1}^{g-1}\langle\langle \mathcal{O}_a\rangle\rangle_h \,\langle\langle \mathcal{O}_b\rangle\rangle_{g-h} + \langle\langle \mathcal{O}_a\mathcal{O}_b\rangle\rangle_{g-1}\Big) = 0. \tag{4.7}$$

In the case of two primaries, this becomes

$$\mathcal{L}_1\mathcal{F}_g + \tfrac{1}{8}\,(1-d^2)\Big(\sum_{h=1}^{g-1}\langle\langle\mathcal{O}_0\rangle\rangle_h\,\langle\langle\mathcal{O}_1\rangle\rangle_{g-h} + \langle\langle\mathcal{O}_0\mathcal{O}_1\rangle\rangle_{g-1}\Big) = 0. \quad (4.8)$$

In calculating the action of the vector field $\mathcal{L}_1$ in the coordinate system $\{u^a_n\}$, we use (4.6) and the commutation relation

$$[\partial_{0,a},\mathcal{L}_1] = \big((\mu+\tfrac{1}{2})(\mu+\tfrac{3}{2})\big)^b_a D_{1,b} + \big((\mu+\tfrac{1}{2})\mathcal{V}+\mathcal{V}(\mu+\tfrac{1}{2})\big)^b_a D_{0,b}. \quad (4.9)$$

In the case of two primaries, this implies that

$$[\partial,\mathcal{L}_1] = \begin{cases} (1-d)\big(\tfrac{1}{4}\,(3-d)\,D_{1,0} + v\,D_{0,0} + u\,D_{0,1}\big), & d\neq 1,\\ 2\,D_{0,1}, & d=1. \end{cases}$$

Using these formulas, we see that the case $g=2$ of (4.8) yields the equation

$$\begin{aligned} 0 &= \mathcal{L}_1\mathcal{F}_2 + \tfrac{1}{4}(1-d^2)\big(\langle\langle\mathcal{O}_0\rangle\rangle_1\,\langle\langle\mathcal{O}_1\rangle\rangle_1 + \langle\langle\mathcal{O}_0\mathcal{O}_1\rangle\rangle_1\big)\\ &= -6\big((d+1)C_0 + \tfrac{1}{5760}\,d(3d-1)(3d-5)(d-2)\big)\,u^{-2}\,\partial v\,\partial u\\ &\quad + 3\,C_1(d-1)u^{(d-2)/(1-d)}\big((\partial v)^2 + \phi'\,(\partial u)^2\big). \end{aligned}$$

It follows that $h_1 = 0$ and

$$h_0 = -\frac{d(3d-1)(3d-5)(d-2)}{5760(d+1)}\,u^{(d-3)/(1-d)}. \quad (4.10)$$

completing the determination of $\mathcal{F}_2$.

Our formula for $\mathcal{F}_2$ agrees with that of Dubrovin and Zhang [4], who apply the method of Eguchi and Xiong [9]; in other words, they use the constraints $D_{k,a}\mathcal{F}_2 = 0$, $k>4$, and $L_nZ=0$, $n\leq 10$.

The higher Virasoro constraints

The higher Virasoro constraints are given by formulas involving a Lie algebra of vector fields $\mathcal{L}_n$, $n\geq -1$, on the large phase space, which satisfy the commutation relations

$$[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}.$$

This Lie algebra is generated by $\mathcal{L}_{-1}$ and $\mathcal{L}_n$, for any $n > 1$.

Just as for $\mathcal{L}_0$ and $\mathcal{L}_1$, we can avoid using the explicit formula for $\mathcal{L}_n$. The Virasoro constraint $L_n Z = 0$ in genus 0 may be written

$$\mathcal{L}_n \mathcal{U} + \mathcal{V}^{n+1} = 0. \tag{4.11}$$

In calculating the action of the vector field $\mathcal{L}_2$ in the coordinate system $\{u_n^a\}$, we use (4.11) and the commutation relation [13]

$$[\partial_{0,a}, \mathcal{L}_n] = \sum_{i=0}^{n} (\mathsf{B}_{n,i})_a^b D_{i,b}, \tag{4.12}$$

where the matrices $\mathsf{B}_{n,i}$ are determined by the recursion

$$\mathsf{B}_{n,i} = (\mu + i + \tfrac{1}{2})\mathsf{B}_{n-1,i-1} + \mathcal{V}\,\mathsf{B}_{n-1,i},$$

with initial condition $\mathsf{B}_{-1,i} = \delta_{i+1,0}$.

In the case of two primaries, with $n = 2$, this implies that

$$[\partial, \mathcal{L}_2] = \begin{cases} (1-d)\big(\tfrac{1}{8}(3-d)(5-d)D_{2,0} \\ +\tfrac{3}{4}(3-d)vD_{1,0} + \tfrac{1}{4}(d^2-2d+9)uD_{1,1} \\ +(\tfrac{3}{2}v^2 + \tfrac{1}{2}(1+d)(3-d)u\,\phi(u))D_{0,0} + 3uvD_{0,1}\big), & d \neq 1, \\ 6\,D_{1,1} + 4e^u D_{0,0} + 6vD_{0,1}, & d = 1. \end{cases}$$

In genus $g > 0$,the constraint $L_2 Z = 0$ is

$$\begin{aligned} \mathcal{L}_2\mathcal{F}_g &+ (\mu_a - \tfrac{3}{2})(\mu_a - \tfrac{1}{2})(\mu_a + \tfrac{1}{2})\eta^{ab} \\ &\cdot \left(\sum_{h=1}^{g-1} \langle\langle \tau_{1,a} \rangle\rangle_h \, \langle\langle \mathcal{O}_b \rangle\rangle_{g-h} + \langle\langle \tau_{1,a}\mathcal{O}_b \rangle\rangle_{g-1} \right) \\ &- \tfrac{1}{2}\,(3\mu_a^2 + 3\mu_a - \tfrac{1}{4})R[1]^{ab} \\ &\cdot \left(\sum_{h=1}^{g-1} \langle\langle \mathcal{O}_a \rangle\rangle_h \, \langle\langle \mathcal{O}_b \rangle\rangle_{g-h} + \langle\langle \mathcal{O}_a\mathcal{O}_b \rangle\rangle_{g-1} \right) = 0. \end{aligned}$$

It may be verified that $\mathcal{F}_2$ satisfies this equation.

Since the differentials operators L_n, $n > -1$, lie in the Lie algebra generated by L_{-1} and L_2, it follows that the Virasoro conjecture holds to genus 2 for two-primary models.

The Belorousski-Pandharipande equation

The Belorousski-Pandharipande equation [1] is a differential equation satisfied by the genus 2 potential, analogous to the equation (3.1) in genus 1; it may be expressed as saying that a certain cubic polynomial in the coordinates $\{u^a\}$ vanishes. It turns out that in the case of backgrounds with two primaries, the equation gives a second (and thus rigorous) derivation of the above formula for C_0, but leaves C_1 undetermined.

Taking Theorem 3.2 and the equations $\mathcal{L}_{-1}\mathcal{F}_2 = 0$ and $\mathcal{L}_0\mathcal{F}_2 = 0$ into account, the Belorousski-Pandharipande equation reduces to a single equation

$$\phi' h_0' - \tfrac{1}{2}\phi'' h_0 - \tfrac{1}{48}\psi'''' - \tfrac{3}{5}\psi'''\psi' + \tfrac{9}{10}(\psi'')^2 = 0.$$

With $\phi(u) = u^{(1+d)/(1-d)}$ and $\psi(u) = \frac{d(3d-1)}{24(d-1)}\log(u)$, the function h_0 of (4.10) satisfies this equation.

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