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Supersymmetric Index In Four-Dimensional Gauge Theories

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Abstract

This paper is devoted to a systematic discussion of the supersymmetric index $\mathrm{Tr}\,(-1)^F$ for the minimal supersymmetric Yang-Mills theory – with any simple gauge group G – primarily in four spacetime dimensions. The index has refinements that probe confinement and oblique confinement and the possible spontaneous breaking of chiral symmetry and of global symmetries, such as charge conjugation, that are derived from outer automorphisms of the gauge group. Predictions for the index and its refinements are obtained on the basis of standard hypotheses about the infrared behavior of gauge theories. The predictions are confirmed via microscopic calculations which involve a Born-Oppenheimer computation of the spectrum as well as mathematical formulas involving triples of commuting elements of G and the Chern-Simons invariants of flat bundles on the three-torus.

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1 Introduction

A constraint on the dynamics of supersymmetric field theories is provided by the supersymmetric index $\mathrm{Tr}\,(-1)^F$ [1]. It is defined as follows. One formulates an n-dimensional supersymmetric theory of interest on a manifold $\mathbf{T}^{n-1}\times\mathbf{R}$, where $\mathbf{R}$ parametrizes the "time" direction and $\mathbf{T}^{n-1}$ is an $(n-1)$-torus (endowed with a flat metric and with a spin structure that is invariant under supersymmetry, that is with periodic boundary conditions for fermions in all directions). If the original theory is such that the classical energy grows as one goes to infinity in field space in any direction,[1] then the spectrum of the theory in a finite volume is discrete. Under such conditions, let n_B and n_F be the number of supersymmetric states of zero energy that are bosonic or fermionic, respectively. The supersymmetric index is defined to be $n_B - n_F$ and is usually written as $\mathrm{Tr}\,(-1)^F$, where $(-1)^F$ is the operator that equals $+1$ or -1 for bosonic or fermionic states and the trace is taken in the space of zero energy states. Alternatively, as states of nonzero energy are paired (with equally many bosons and fermions), one can define the index as

$$\mathrm{Tr}\,(-1)^F e^{-\beta H}, \tag{1.1}$$

where H is the Hamiltonian, β is any positive real number, and the trace is now taken in the full quantum Hilbert space. This definition of the index shows that it can be computed as a path integral on an n-torus $\mathbf{T}^n$, with a positive signature metric of circumference β in the time direction (and periodic boundary conditions on fermions in the time direction – to reproduce the $(-1)^F$ in the trace – as well as in the space directions).

The index is invariant under any smooth deformations of a supersymmetric theory that leave fixed the behavior of the potential at infinity (so that supersymmetric vacua cannot move to or from infinity). The reason is that since the states of nonzero energy are paired, any change in n_B – resulting from a state moving to or from zero energy – is accompanied by an equal change in n_F. Because of this, it is usually possible to effectively compute the index by perturbing to some sufficiently simple situation while preserving supersymmetry. A nonzero index implies that the ground state energy is exactly zero for any volume of the spatial torus $\mathbf{T}^{n-1}$, and hence that the ground state energy per unit volume vanishes in the infinite volume limit. It follows that if the infinite volume theory has a stable ground state (which will be so if the potential grows at infinity) then that state is supersymmetric.

[1] This condition notably excludes theories that have a noncompact flat direction in the classical potential.

This way of thinking illuminates supersymmetric dynamics in many situations. But application to four-dimensional gauge theories was hampered for some years by the fact that, though one could readily obtain attractive results for $SU(n)$ and $Sp(n)$ gauge groups, the results for other groups appeared to clash with expectations based on chiral symmetry breaking. Ultimately (see the appendix to [2]), it became clear that the discrepancy arose from overlooking the fact that certain moduli spaces of triples of commuting elements of a Lie group are not connected.

The purpose of the present paper is to systematically develop the theory of the supersymmetric index for the minimal supersymmetric gauge theories in 2+1 and 3+1 dimensions. Actually, for the (2+1)-dimensional theories, we only consider the special case that the Chern-Simons coupling is $k = \pm h/2$, where confinement and a mass gap are expected. The index in the $(2+1)$-dimensional theories with more general Chern-Simons coupling has been analyzed elsewhere [3].

In either $2+1$ or $3+1$ dimensions, we consider supersymmetric theories with the smallest possible number of supercharges (two or four in $2+1$ or $3+1$ dimensions) and with the minimal field content: only the gauge fields and their supersymmetric partners. We will compute the index for bundles of any given topological type on $\mathbf{T}^2$ or $\mathbf{T}^3$. Because Yang-Mills theory of a product group $G' \times G''$ is locally the product of the G' theory and the G'' theory, the index for a semisimple G can be inferred from the index for simple G (provided that in the simple case one has results for all possible G-bundles). So we will assume that G is simple.

The index has a number of important refinements (some of which were treated in [1]) that we will explain. It is possible, by letting G be non-simply-connected, to include a discrete electric or magnetic flux and thereby probe the hypothesis of confinement. In four dimensions, it is possible to refine the index to take into account a discrete chiral symmetry group and to give evidence that the chiral symmetry is spontaneously broken. It is also possible, by allowing G to be disconnected, to give evidence that discrete symmetries associated with outer automorphisms of G (such as charge conjugation for $G = SU(n)$) are unbroken.

In sections 2 and 3, we explain what predictions about the supersymmetric index and its refinements follow from standard claims about gauge dynamics. In section 4, we compute the index and its refinements microscopically, following the strategy of [1] but (in four dimensions) including the contributions of all components of the moduli space of flat connections. Full agreement is obtained in all cases; in $3+1$ dimensions, obtaining such agreement depends on a result counting the moduli spaces of commuting triples that was proposed in [2] and has subsequently been justified. For connected and simply-connected G, the moduli spaces of commuting triples have been analyzed in [4–7]. A generalization to non-simply-connected G, necessary for including the discrete fluxes, and an analysis of the Chern-Simons invariants of flat bundles on $\mathbf{T}^3$, necessary for testing the claims about chiral symmetry breaking, have been made in [7]. In section 5, we give some details about moduli spaces of commuting triples.

2 Expectations In $2+1$ Dimensions

2.1 Preliminaries

The minimal $(2+1)$-dimensional supersymmetric theory has a field content consisting of the gauge field A of some compact connected simple Lie group G, along with a Majorana fermion λ in the adjoint representation of the gauge group. The usual kinetic energy for these fields is

$$L = \frac{1}{g^2}\int d^3x \,\mathrm{Tr}\left(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\overline{\lambda}i\Gamma\cdot D\lambda\right). \tag{2.1}$$

However, there is a crucial subtlety in $2+1$ dimensions: it is possible to add an additional Chern-Simons coupling while preserving supersymmetry. The additional interaction is

$$-\frac{ik}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \overline{\lambda}\lambda\right). \tag{2.2}$$

Here k must, for topological reasons, obey a quantization condition. For G simply-connected, k must be congruent to $h/2$ mod 1, where h is the dual Coxeter number of G; if G is not simply-connected, k must be congruent to $h/2$ mod s where s is an integer that depends on G. The $h/2$ term in the quantization law comes from an anomaly involving the fermions [3, 8].

The supersymmetric index for general k has been studied in [3]. Our intent here is to describe what properties of the index, and related invariants, can be deduced if we assume that in bulk the theory has a unique vacuum with a mass gap and confinement. As has been explained in [3], it is reasonable to believe that these properties hold precisely if $k = \pm h/2$. Hence in the present paper, when we consider the $(2+1)$-dimensional theory, we will always specialize to these values of k.

First we consider the theory with a gauge group G that is simply-connected. Since we assume a mass gap, there is no Goldstone fermion, so that the unique vacuum has unbroken supersymmetry. One expects that an isolated vacuum with mass gap will contribute $+1$ to the supersymmetric index, so one expects $\mathrm{Tr}\,(-1)^F = 1$. [2] The logic in this statement is that one can compute $\mathrm{Tr}\,(-1)^F$ by formulating the theory on a two-torus of very large radius R with $1/R$ much smaller than the mass gap; then on the length scale R, the system is locked in its ground state, which contributes 1 to the index. The mass gap is important here; an isolated vacuum without a mass gap makes a contribution to the index that is not necessarily equal to 1 (or even ± 1), as was seen in some examples in [1].

Now we consider the case that G is not simply-connected. Its fundamental group $\pi_1(G)$ is necessarily a finite abelian group (cyclic except in the special case $G = SO(4n)/\mathbf{Z}_2$, for which $\pi_1(G) = \mathbf{Z}_2 \times \mathbf{Z}_2$). From a Hamiltonian point of view, in quantizing on a two-torus $\mathbf{T}^2$, there are basically two changes that occur when G is not simply-connected. First of all, a G-bundle over $\mathbf{T}^2$ can be topologically non-trivial if G is not simply-connected. In the present paper, it will be important that if X is a two-manifold or a three-manifold, the possible G-bundles are classified by a "discrete magnetic flux," a characteristic class m of the G-bundle which takes values in

$$M = H^2(X, \pi_1(G)). \tag{2.3}$$

(On a manifold of dimension higher than three, a G-bundle has the characteristic class m plus additional invariants such as instanton number.) All values of m can occur. To gain as much information as possible, we do not want to sum over m; we want to compute the index as a function of m.

The second basic consequence of G not being simply-connected is that the group of gauge transformations, that is the group of maps of $\mathbf{T}^2$ to G (or more generally the group of bundle automorphisms if the G-bundle is non-trivial) has different components. Restricted to a non-contractible loop in $\mathbf{T}^2$, a gauge transformation determines an element of $\pi_1(G)$ which may or may not be trivial. By restricting it to cycles generating $\pi_1(\mathbf{T}^2)$, a gauge transformation g determines a homomorphism γ_g from $\pi_1(\mathbf{T}^2)$ to $\pi_1(G)$; g is continuously connected to the identity if and only if $\gamma_g = 0$.

Let $\mathcal{W}_0$ be the group of gauge transformations which are continuously connected to the identity, and $\mathcal{W}$ the group of all gauge transformations. In quantizing a gauge theory, we must impose Gauss's law, which quantum

[2] There is actually a small subtlety here. In $2+1$ dimensions, after compactifying on a torus, there is no natural definition of the sign of the operator $(-1)^F$, and hence the contribution of a massive vacuum to the index might be $+1$ or -1. It was shown in [3] that with a natural convention, if the index is $+1$ for $k = h/2$, then it is $(-1)^r$ at $k = -h/2$, where r is the rank of G. For our purposes, we work at, say, $k = h/2$ and define the sign of $(-1)^F$ so that the index is $+1$.

mechanically asserts that physical states must be invariant under $\mathcal{W}_0$. We need not impose invariance under $\mathcal{W}$, and it will soon be apparent that we can obtain more information if we do not impose such invariance.

The quotient

$$\Gamma = \mathcal{W}/\mathcal{W}_0 \tag{2.4}$$

can be identified as follows:

$$\Gamma = \mathrm{Hom}(\pi_1(\mathbf{T}^2), \pi_1(G)) = H^1(\mathbf{T}^2, \pi_1(G)). \tag{2.5}$$

In particular, Γ is a finite abelian group. We define the physical Hilbert space $\mathcal{H}_m$ by quantizing the space of connections on a bundle having characteristic class m and requiring invariance under $\mathcal{W}_0$ only. Then Γ acts on $\mathcal{H}_m$. We can decompose $\mathcal{H}_m$ in characters of Γ, in fact

$$\mathcal{H}_m = \oplus_e \mathcal{H}_{e,m}, \tag{2.6}$$

where $\mathcal{H}_{e,m}$ is the subspace of $\mathcal{H}_m$ transforming in the character e of Γ. The character e is usually called a "discrete electric flux." It takes values in

$$E = \mathrm{Hom}(\Gamma, U(1)) = \mathrm{Hom}(H^1(\mathbf{T}^2, \pi_1(G)), U(1)). \tag{2.7}$$

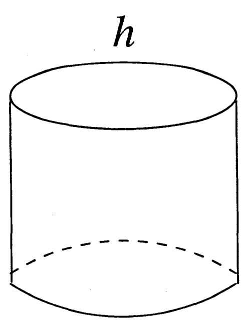

Fig. 1. A nontrivial bundle on $\mathbf{T}^2$ can be built from a trivial bundle on a cylinder by gluing together the two ends, using a gluing function h. The magnetic flux of the bundle is determined by the homotopy class of h in $\pi_1(G)$.

Constructions Of Non-Trivial Bundles

For future reference, it will help to know that the two facts just mentioned are closely related. One way to make a non-trivial bundle on $\mathbf{T}^2$ is to begin (figure 1) with a bundle on a cylinder $\mathbf{S}^1 \times I$ (here $I = [0,1]$ is a unit interval) and glue the two ends together via a gauge transformation. The bundle on $\mathbf{S}^1 \times I$ is inevitably trivial ($\mathbf{S}^1 \times I$ is contractible to $\mathbf{S}^1$, and since G is connected, a G-bundle on $\mathbf{S}^1$ is trivial). But in gluing the two ends together, one can identify the fiber over $\mathbf{S}^1 \times \{0\}$ with the fiber over $\mathbf{S}^1 \times \{1\}$ via an arbitrary gauge transformation, that is an arbitrary map of $\mathbf{S}^1$ to G. Up to homotopy, the map of $\mathbf{S}^1$ to G is determined by an element $f \in \pi_1(G)$, and for every such f, we get a non-trivial G-bundle on $\mathbf{T}^2$. Its characteristic class is an element of $H^2(\mathbf{T}^2, \pi_1(G))$ that we will call $m(f)$. Note in fact that, as there is essentially only one two-cycle on $\mathbf{T}^2$, namely $\mathbf{T}^2$ itself, $H^2(\mathbf{T}^2, \pi_1(G))$ is naturally isomorphic to $\pi_1(G)$; the map from f to $m(f)$ is an isomorphism. f, if lifted to the universal cover $\widehat{G}$ of G, can be regarded as a path from the identity in $\widehat{G}$ to an element $\widehat{f}$ of the center of $\widehat{G}$.

A second and related construction is as follows. Delete a point P from $\mathbf{T}^2$. As $\pi_0(G) = 0$, a G-bundle W is trivial on the one-skeleton of any manifold and hence on $\mathbf{T}^2 - P$. Likewise it is trivial in a small neighborhood U of P. Let V be the intersection $V = (\mathbf{T}^2 - P) \cap U$. Comparing the two trivializations on V gives a "transition function" $f : V \to G$. As V is homotopic to a circle, this gives an element of $\pi_1(G)$, and thus, again, an element $\widehat{f}$ of the center of $\widehat{G}$.

As a variant of this, W, being trivial on either $\mathbf{T}^2 - P$ or U, can be lifted to a $\widehat{G}$-bundle $\widehat{W}$ on either $\mathbf{T}^2 - P$ or U. However, the two lifts cannot be glued together to make a $\widehat{G}$-bundle over $\mathbf{T}^2$; the lifted transition function is not single-valued, but is multiplied by an element $\widehat{f}$ of the center of $\widehat{G}$ in going around the circle.

In this paper, a third interpretation of G-bundles on a torus will also be helpful. Every G-bundle on $\mathbf{T}^2$ admits a flat connection. [3] A flat connection has holonomies around the two directions in $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$; the holonomies are two elements of G, say a and b, which commute. Suppose, however, that one lifts a and b to elements, say A and B, in the universal cover $\widehat{G}$ of G. There is no natural way to do this; pick any choice. Then

$$\widehat{f} = ABA^{-1}B^{-1} \tag{2.8}$$

is an element of $\widehat{G}$ that projects to the identity in G. Thus, $\widehat{f}$ is in particular an element of the center of $\widehat{G}$. ($\widehat{f}$ is independent of the choice of lifting of a

[3]Pick a complex structure on $\mathbf{T}^2$. Any connection on a G-bundle $W \to \mathbf{T}^2$ defines a holomorphic structure on the complexification of W; this holomorphic structure is semi-stable if the connection is chosen generically. Such a semi-stable holomorphic bundle over a Riemann surface admits a flat unitary connection by a theorem of Narasimhan and Seshadri; this can be interpreted as a flat connection on the original G-bundle.

and b, since different liftings differ by multiplication of A and B by central elements of $\widehat{G}$ which cancel out in the definition of $\widehat{f}$.)

We thus have three different ways to associate a flat G-bundle over $\mathbf{T}^2$ with an element $f \in \pi_1(G)$ or $\widehat{f}$ in the center of $\widehat{G}$:

(1) Pull the bundle back to a flat bundle on a cylinder, as in figure 1, and identify the gluing data with f.

(2) Trivialize the bundle away from and near a point $P \in \mathbf{T}^2$, and find the f that appears in comparing the two trivializations.

(3) Find a flat connection and measure $\widehat{f}$ as in (2.8).

It can be shown that the three approaches are equivalent.

Following [7], we will call group elements $A, B \in \widehat{G}$ that commute if projected to a quotient G of $\widehat{G}$ (and hence $ABA^{-1}B^{-1}$ is in the center of $\widehat{G}$) "almost commuting."

2.2 The Index

Supersymmetry acts in each $\mathcal{H}_{e,m}$, so we can define the index $\mathrm{Tr}\,(-1)^F$ for each value of e and m; we call this index function $I(e, m)$. Our goal, in the $(2+1)$-dimensional case, is to predict $I(e, m)$ based on the usual hypotheses about the infrared dynamics of this theory. In section 4.1, we will compare the predictions to a microscopic computation. Of course, the predictions will only hold for those values of k (namely $k = \pm h/2$) at which the theory has confinement, a mass gap, and a unique vacuum. The reason for analyzing this case in so much detail is largely that it is good background for $3+1$ dimensions.

I will first explain why one intuitively expects $I(e, 0) = 0$ for $e \neq 0$ on the basis of confinement. We let $|\Omega\rangle$ be the supersymmetric ground state in $\mathcal{H}_0$. In lattice gauge theory (which is difficult to implement technically for the supersymmetric theory, but should give us a guide about the meaning of confinement), one associates $|\Omega\rangle$ with a state that in the strong coupling limit is independent of the connection A. Such a state is certainly Γ-invariant. Thus, $|\Omega\rangle$ should in particular be a Γ-invariant state. Hence, when we decompose $\mathcal{H}_0 = \oplus_e \mathcal{H}_{e,0}$, the supersymmetric state appears for $e = 0$, so that $I(0, 0) = 1$, $I(e, 0) = 0$ for $e \neq 0$.

To see what is going on more explicitly, we will describe a typical state in $\mathcal{H}_{e,0}$ for $e \neq 0$. Let C be a noncontractible loop in $\mathbf{T}^2$ that wraps once around one of the two circles in $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. We let V be an irreducible representation of the simply-connected universal cover $\widehat{G}$ of G that is not a representation of G. Consider the Wilson loop operator around C in the V representation:

$$W_V(C) = \mathrm{Tr}_V P \exp \oint_C A. \tag{2.9}$$

It is invariant under $\mathcal{W}_0$, but not under $\mathcal{W}$; it transforms in some nontrivial character e of Γ. (e is determined by the action of the center of $\widehat{G}$ on V.) Consider the state

$$W_V(C)|\Omega\rangle. \tag{2.10}$$

It lies in $\mathcal{H}_{e,0}$. Now, intuitively confinement means[4] that the energy of the state (2.10), or any state carrying the same electric flux, should grow linearly with the radius R of $\mathbf{T}^2$. In particular, in the space $\mathcal{H}_{e,0}$, there should be no zero energy supersymmetric state if R is large enough, so $I(e,0)$ should vanish if the theory is confining.

Now let us reformulate the vanishing of $I(e,0)$ in terms of path integrals (essentially following an argument due to 't Hooft [10].) Let $|\Gamma|$ denote the number of elements of Γ. The projection operator onto states that transform in the character e is

$$P_e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{-1}(\gamma)\,\gamma. \tag{2.11}$$

We can therefore write

$$\begin{aligned} I(e,0) &= \mathrm{Tr}_{\mathcal{H}_{e,0}}(-1)^F e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}_0} P_e (-1)^F e^{-\beta H} \\ &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{-1}(\gamma)\mathrm{Tr}_{\mathcal{H}_0}\gamma(-1)^F e^{-\beta H}. \end{aligned} \tag{2.12}$$

Here β is an arbitrary positive real number.

Now, let us consider a computation of

$$\mathrm{Tr}_{\mathcal{H}_0}\gamma(-1)^F e^{-\beta H} \tag{2.13}$$

via path integrals. For $\gamma = 1$, this was already explained in the introduction: the trace in question is represented by a path integral on a three-torus $\mathbf{T}^3 =$

[4]This is an oversimplification, for the classification of massive phases of gauge theories is more complicated than traditionally supposed, as shown in [3, 9], but the statement holds in the simplest universality class of confining theories. We will test the hypothesis that the theories under discussion are in this universality class.

$\mathbf{T}^2 \times \mathbf{S}^1$, where $\mathbf{S}^1$ is a circle of circumference β and the boundary conditions for the fermions are periodic in all directions. To insert γ in the trace, we simply proceed as in figure 2. We begin with $\mathbf{T}^2 \times I$, where I is the closed interval $[0, \beta]$ of length β. We glue together $\mathbf{T}^2 \times \{0\}$ with $\mathbf{T}^2 \times \{\beta\}$ to make a trace. But in the gluing, we identify the two ends via the gauge transformation γ (that is, via any gauge transformation that is in the homotopy class of γ). Such gluing is a standard way, introduced in connection with figure 1, to construct a non-trivial G-bundle over $\mathbf{T}^3$.

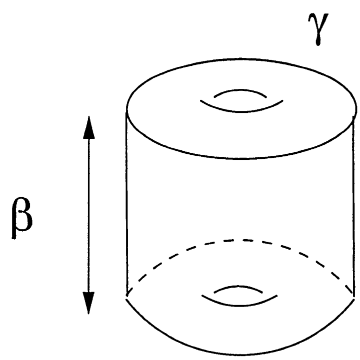

Fig. 2. A path integral computation of $\mathrm{Tr}_{\mathcal{H}_0}\gamma(-1)^F e^{-\beta H}$ is made by performing a path integral on $\mathbf{T}^2 \times \mathbf{S}^1$. The G-bundle on $\mathbf{T}^2 \times \mathbf{S}^1$ is built from a G-bundle on $\mathbf{T}^2 \times I$, with I a unit interval of length β, by using the gauge transformation γ as a gluing function.

Let $m_0(\gamma)$ be the characteristic class of the bundle obtained in this way. To describe $m_0(\gamma)$ more fully, let x^1 and x^2 be angular coordinates on the "space" manifold $\mathbf{T}^2$, and x^3 an angular coordinate on $\mathbf{S}^1$. Then $m_0(\gamma) \in H^2(\mathbf{T}^2 \times \mathbf{S}^1, \pi_1(G))$ has vanishing 1-2 component, while the 1-3 and 2-3 components are determined by γ. For any $\widehat{m} \in H^2(\mathbf{T}^2 \times \mathbf{S}^1, \pi_1(G))$ we call the 1-2 component "magnetic" and the 1-3 and 2-3 components "electric." So $m_0(\gamma)$ is purely electric. We will write $\widehat{m}$ for a general element of $H^2(\mathbf{T}^2 \times \mathbf{S}^1, \pi_1(G))$, and m for its magnetic part, that is, the restriction of $\widehat{m}$ to $\mathbf{T}^2$.

For any G-bundle on $\mathbf{T}^3$ with characteristic class $\widehat{m}$, let us write $Z(\widehat{m})$ for the partition function on $\mathbf{T}^3$, with periodic boundary conditions for fermions in all directions. Then from the discussion of (2.13),

$$\mathrm{Tr}_{\mathcal{H}_0}\gamma(-1)^F e^{-\beta H} = Z(m_0(\gamma)) \tag{2.14}$$

and hence from (2.12) we get

$$I(e,0) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{-1}(\gamma)Z(m_0(\gamma)). \tag{2.15}$$

Now, $m_0(\gamma)$ is not a completely general element of $H^2(\mathbf{T}^2\times\mathbf{S}^1,\pi_1(G))$, because its magnetic part, that is the restriction to $\mathbf{T}^2$, vanishes. This is because in our above construction, we started with a trivial bundle on the original $\mathbf{T}^2$. But $m_0(\gamma)$ is subject to no other restriction. A general $\widehat{m}\in H^2(\mathbf{T}^2\times\mathbf{S}^1,\pi_1(G))$ has components $\widehat{m}_{12}$, $\widehat{m}_{13}$, and $\widehat{m}_{23}$. Vanishing of the magnetic part means that $\widehat{m}_{12}=0$.

Next, as discussed above, we interpret confinement to mean that $I(e,0)=0$ for all non-zero e. Then (2.15) implies that $Z(m_0(\gamma))$ is independent of γ. Since $Z(m_0(0))=Z(0)$ is the original index, which is expected to equal 1 since the theory in infinite volume has a unique vacuum with mass gap, we get

$$Z(\widehat{m})=1 \text{ if } \widehat{m}_{12}=0. \tag{2.16}$$

2.3 Action Of $SL(3,\mathbf{Z})$

So far, on $\mathbf{T}^3=\mathbf{T}^2\times\mathbf{S}^1$, we have considered a flat metric for which the two factors are orthogonal. More generally, we could consider any flat metric on $\mathbf{T}^3$; because of the interpretation of $Z(\widehat{m})$ as an index, it is independent of which flat metric we pick. To justify the last statement in detail, note that the path integral on $\mathbf{T}^3=\mathbf{T}^2\times\mathbf{S}^1$ with trivial spin structure and specified $\widehat{m}$ can be interpreted as

$$\mathrm{Tr}_{\mathcal{H}_m}\gamma(-1)^F e^{-\beta H+i\vec{a}\cdot\vec{P}}. \tag{2.17}$$

Here notation is as follows: m is the 1-2 component of $\widehat{m}$, which we regard as the characteristic class of a bundle on $\mathbf{T}^2$; $\mathcal{H}_m$ is the Hilbert space for quantization with this bundle; γ is such that $m_0(\gamma)$ as defined earlier has the same 13 and 23 components as $\widehat{m}$; $\vec{P}$ are the momentum operators; and β and $\vec{a}$ are constants that contain the information needed to specify the $\mathbf{T}^3$ metric once the $\mathbf{T}^2$ metric is given. ($\vec{a}=0$ if and only if the two factors in $\mathbf{T}^2\times\mathbf{S}^1$ are orthogonal.) As H and $\vec{P}$ commute with supersymmetry and vanish for supersymmetric states, this trace can as usual be computed just in the space of supersymmetric states and is independent of $\vec{a}$ and β; it similarly is independent of the metric on $\mathbf{T}^2$ because supersymmetric states can only appear or disappear in bose-fermi pairs when that metric is varied. This justifies the claim that $Z(\widehat{m})$ is independent of the flat metric on $\mathbf{T}^3$.[5]

$\mathbf{T}^3$ has a group of orientation-preserving diffeomorphisms isomorphic to $SL(3,\mathbf{Z})$, acting on the angles x^1, x^2, x^3 in the natural way. An $SL(3,\mathbf{Z})$ transformation certainly leaves invariant the path integral if we allow for the action of $SL(3,\mathbf{Z})$ on both the metric of $\mathbf{T}^3$ and $\widehat{m}$. As $Z(\widehat{m})$ is independent of metric, it is invariant under the action of $SL(3,\mathbf{Z})$ just on $\widehat{m}$.

Since we are in three dimensions, a two-form is dual to a vector, and we can identify $\widehat{m}$ with a three-plet $\vec{w} = (w_1, w_2, w_3) = (\widehat{m}_{23}, \widehat{m}_{31}, \widehat{m}_{12})$ of elements of $\pi_1(G)$. $SL(3,\mathbf{Z})$ acts on $\vec{w}$ in the natural three-dimensional representation, tensored with $\pi_1(G)$. If $\pi_1(G)$ is a cyclic group (isomorphic to $\mathbf{Z}_n$ for some n), then $\vec{w}$ is just a single vector with coefficients in $\mathbf{Z}_n$. Let μ be the greatest common divisor of the w_i and n (an integer prime to n is invertible mod n, so only divisors of the w_i that divide n are meaningful). One can lift $\vec{w}/\mu$ to a vector $\vec{w}'$ with integer coefficients that is "primitive" (the components are relatively prime). One can find a basis of the lattice in which $\vec{w}'$ is the first vector; this basis is related by an $SL(3,\mathbf{Z})$ transformation to the original basis. This $SL(3,\mathbf{Z})$ transformation, which we have constructed assuming $\pi_1(G)$ is cyclic, puts $\vec{w}$ in the form $\vec{w} = (\mu, 0, 0)$; in particular, this shows that when $\pi_1(G)$ is cyclic, the greatest common divisor μ is the only $SL(3,\mathbf{Z})$ invariant of $\widehat{m}$. $\vec{w} = (\mu, 0, 0)$ means that $\widehat{m}_{12} = \widehat{m}_{31} = 0$.

Actually, $\pi_1(G)$ is cyclic for all simple G unless $G = SO(4n)/\mathbf{Z}_2$, in which case $\pi_1(G) = \mathbf{Z}_2 \times \mathbf{Z}_2$. For this group, $\vec{w}$ consists of a pair of $\mathbf{Z}_2$-valued vectors. (Since 2 is prime, we can assume in this case that $\mu = 1$.) By an $SL(3,\mathbf{Z})$ transformation we can put the first vector in the form $(w, 0, 0)$, and then, using the $SL(2,\mathbf{Z})$ that leaves this vector invariant, we can put the second, if not zero, in the form $(w', w'', 0)$. In particular, we can set $\widehat{m}_{12} = 0$. In short, in all cases we can set $\widehat{m}_{12} = 0$ by an $SL(3,\mathbf{Z})$ transformation.

We can thus extend (2.16):

$$Z(\widehat{m}) = 1 \text{ for all } \widehat{m} \text{ and } G. \tag{2.18}$$

General Prediction For The Index

Now we can state the general prediction for the index in three spacetime dimensions.

[5] This claim could alternatively be proved by noting that the derivative of $Z(\widehat{m})$ with respect to the metric is given by a one-point function of the stress tensor T_{mn}. But this one-point function vanishes because T is of the form $\{Q, \dots\}$; in fact $T_{mn} = \gamma_m^{\alpha\beta}\{Q_\alpha, S_{n\beta}\} + m \leftrightarrow n$, with $S_{n\beta}$ the supercurrent and Q_α the supercharges.

First consider $I(0,m)$. This is the same as the path integral $Z(\widehat{m})$ with $\widehat{m}_{12} = m$, $\widehat{m}_{13} = \widehat{m}_{23} = 0$. Hence, by virtue of (2.18) we get

$$I(0,m) = 1 \text{ for all } m. \tag{2.19}$$

To compute $I(e,m)$ in general, we introduce some notation. Given $m \in H^2(\mathbf{T}^2, \pi_1(G))$, and $\gamma \in \mathrm{Hom}(\pi_1(\mathbf{T}^2), \pi_1(G))$, we write $\widehat{m}(m,\gamma)$ for the element of $H^2(\mathbf{T}^2 \times \mathbf{S}^1, \pi_1(G))$ such that $\widehat{m}_{12} = m$ (that is, the restriction of $\widehat{m}$ to $\mathbf{T}^2$ equals m) while the 1-3 and 2-3 components of $\widehat{m}(m,\gamma)$ equal those of $m_0(\gamma)$ as defined above. Then arguing as we did in connection with figure 2 but starting with a bundle whose characteristic class on $\mathbf{T}^2$ is m, we get

$$I(e,m) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{-1}(\gamma) Z(\widehat{m}(m,\gamma)). \tag{2.20}$$

Using (2.18), we therefore get

$$I(e,m) = 0 \text{ for } e \neq 0 \tag{2.21}$$

for all m and G.

(2.19) and (2.21) are the basic predictions concerning the index in three dimensions. In section 4.1, we will confront them with microscopic computations. For now, we move on to a discussion of the analogous predictions in four dimensions.

3 The Four-Dimensional Case

3.1 Simply-Connected Gauge Group

We will now consider the minimal (3 + 1)-dimensional super Yang-Mills theory in a similar way. Again we start with the case that the gauge group G is simply-connected. The Lagrangian is

$$L = \frac{1}{g^2} \int d^3x \, \mathrm{Tr} \left(\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \overline{\lambda} i \Gamma \cdot D \lambda \right) + \frac{\theta}{8\pi^2} \int \mathrm{Tr} F \wedge F. \tag{3.1}$$

The last term is a "topological" term that has no close analog in 2 + 1 dimensions, while on the other hand the (2 + 1)-dimensional Chern-Simons term has no analog in 3 + 1 dimensions. Also λ was real in 2 + 1 dimensions, but in 3 + 1 dimensions, λ is a positive chirality fermion in the adjoint representation of G, and $\overline{\lambda}$ is its complex conjugate.

That last remark leads at once to an important difference between the $2+1$ and $3+1$-dimensional problems. In $3+1$ dimensions, we have at the classical level the $U(1)$ chiral symmetry

$$\lambda \to e^{-i\alpha}\lambda, \quad \overline{\lambda} \to e^{i\alpha}\overline{\lambda}, \tag{3.2}$$

with α an angular parameter. We describe this transformation law by saying that λ and $\overline{\lambda}$ have respectively charge -1 and charge 1 under the symmetry. Quantum mechanically, $U(1)$ is broken by an anomaly to a subgroup $\mathbf{Z}_{2h}$, where h is the dual Coxeter number of G. This is computed by showing that in an instanton field, the index of the Dirac operator for the λ field is

$$\mathrm{ind}(\lambda) = 2h. \tag{3.3}$$

The surviving $\mathbf{Z}_{2h}$ is an exact symmetry group of the quantum theory. It contains a subgroup $\mathbf{Z}_2$, which is generated by a group element that multiplies both λ and $\overline{\lambda}$ by -1. This element is equivalent to a 2π rotation in spacetime and thus cannot be spontaneously broken. By analogy with chiral symmetry breaking in the strong interactions, it is believed that if the theory is formulated on flat $\mathbf{R}^4$, the $\mathbf{Z}_{2h}$ symmetry is spontaneously broken to $\mathbf{Z}_2$; this is its maximal subgroup that would allow a bare mass for λ. If so, there are at least h vacua. It is believed that there are precisely h vacua and that these vacua all have a mass gap and confinement.

If then the theory is formulated on $\mathbf{T}^3 \times \mathbf{R}$, where $\mathbf{T}^3$ is a very large three-torus and $\mathbf{R}$ parametrizes "time," then these vacua (as they differ by a symmetry rotation) all make the same contribution to the index, and this contribution is ± 1 because of the mass gap. With one natural normalization of the sign of $(-1)^F$, which we will describe in section 4, the sign is $(-1)^r$, where r is the rank of G, and the index is $\mathrm{Tr}\,(-1)^F = (-1)^r h$. (For many purposes, one might redefine the sign of $(-1)^F$ to eliminate the factor $(-1)^r$.) However, in $3+1$ dimensions, we can define an invariant more refined than the index.

Let $\mathcal{H}$ be the Hilbert space of the theory on $\mathbf{T}^3$. Of course, the supersymmetry algebra requires that states of zero energy also have zero three-momentum. We can decompose $\mathcal{H}$ in the form

$$\mathcal{H} = \oplus_{k=0}^{2h-1}\mathcal{H}_k, \tag{3.4}$$

where $\mathcal{H}_k$ consists of states that transform under the chiral symmetry with charge k, that is by

$$\psi \to e^{ik\alpha}\psi. \tag{3.5}$$

Since an anomaly breaks the chiral $U(1)$ to $\mathbf{Z}_{2h}$, k is only defined modulo $2h$, a fact that is incorporated in the notation in (3.4). Since elementary fermi fields all have charge ± 1 while elementary bosons are invariant under the chiral symmetry, states in $\mathcal{H}_k$ are all bosonic or all fermionic for even or odd k:

$$(-1)^F = (-1)^k. \tag{3.6}$$

Actually, we should be more careful here. All we really know about $(-1)^F$ and $(-1)^k$ is that they transform all local fields in the same way. This would allow a sign factor in the relation between these two operators, but with the conventions we will use for the microscopic computation in section 4, the sign is $+1$.

Let Q_α and $\overline{Q}_{\dot\alpha}$ be the supercharges of positive and negative chirality (here α and $\dot\alpha$ are spinor indices of the two chiralities). Then Q_α and $\overline{Q}_{\dot\alpha}$ have respectively charge 1 and charge -1 under $\mathbf{Z}_{2h}$. Let Q be any generic linear combination of the Q_α, and $\overline{Q}$ its hermitian conjugate. Then the supersymmetry algebra implies that

$$Q^2 = 0, \tag{3.7}$$

and because Q has charge 1, Q maps $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$. In this situation, we can define the *cohomology groups* of Q:

$$H^k(Q) = \ker(Q : \mathcal{H}_k \to \mathcal{H}_{k+1})/\mathrm{im}(Q : \mathcal{H}_{k-1} \to \mathcal{H}_k). \tag{3.8}$$

Let $h^k(Q)$ be the dimension of $H^k(Q)$.

The supersymmetry algebra implies that (after multiplying Q by a suitable scalar)

$$\{\overline{Q}, Q\} = 2(H + \vec{a}\cdot\vec{P}) \tag{3.9}$$

with H the Hamiltonian, $\vec{P}$ the momentum, and $\vec{a}$ a unit three-vector that depends on the choice of Q. Given (3.9), a standard "Hodge theory" argument says that the $H^k(Q)$ consist of states in $\mathcal{H}_k$ annihilated by $H + \vec{a}\cdot\vec{P}$. If there is a mass gap, this means that the $H^k(Q)$ consist of the states annihilated by both H and $\vec{P}$ (massive particles have $H > |\vec{P}|$).[6] This shows that $H^k(Q)$ is independent of Q. It also implies, given (3.6), that

$$\mathrm{Tr}\,(-1)^F = \sum_k (-1)^k h^k(Q). \tag{3.10}$$

[6]Even if there is no mass gap, states annihilated by $H + \vec{a}\cdot\vec{P}$ are annihilated separately by H and $\vec{P}$ for generic $\vec{a}$ – and hence for generic Q – as H and $\vec{P}$ both have discrete spectra.

Let us compute the $h^k(Q)$ for the case of a large volume on $\mathbf{T}^3$. Let v be the generator of $\mathbf{Z}_{2h}$ obtained by setting $\alpha = \pi/h$ in (3.2). Let $|\Omega\rangle$ be any one supersymmetric state of the system that obeys cluster decomposition in the infinite volume limit, and let $|\Omega_s\rangle = v^s|\Omega\rangle$ for any integer s. Note that $|\Omega_{s+h}\rangle = (-1)^r|\Omega_s\rangle$, where the sign arises because $v^h = (-1)^F$ and all of these states have $(-1)^F = (-1)^r$. The $|\Omega_s\rangle$ are thus cyclically permuted by the broken symmetry, and spontaneous symmetry breaking implies that they are linearly independent. Let

$$|\Theta_k\rangle = \sum_{s=0}^{h-1} e^{-\pi iks/h}|\Omega_s\rangle \tag{3.11}$$

for any integer k such that $k+r$ is even. $|\Theta_k\rangle$ thus has charge k. The $|\Theta_k\rangle$ span the space of zero energy states, so

$$h^k(Q) = \begin{cases} 1, & \text{for even } k+r \\ 0, & \text{for odd } k+r. \end{cases} \tag{3.12}$$

This is clearly compatible with the previous claim that $\mathrm{Tr}\,(-1)^F = (-1)^r h$.

In this theory, the chiral supercharges Q_α vary holomorphically as a function of

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{3.13}$$

(The anti-chiral supercharges vary anti-holomorphically.) θ and g are the theta angle and coupling constant of the theory defined with a specific renormalization scheme; one could use the renormalization group to replace g with a mass parameter used in the renormalization. Because of the holomorphy, one expects values of τ at which there appear extra zero energy states, not present generically, to be isolated points in τ space.

The result obtained in (3.12) follows from the existence of a mass gap and holds for fixed coupling and any sufficiently large volume, or equivalently for fixed volume and sufficiently large coupling. Since it holds not just at an isolated set of points, it must be the generic result. This important statement can also be justified in either of the following two ways:

(1) Any "jumping" phenomenon consists of the appearance of extra zero energy states in bose-fermi pairs.[7] Continuity of the spectrum as a function

[7]In fact, jumping involves appearance of extra states with adjacent values of k. Any states that come to or from zero energy can be naturally grouped in pairs of states related by the action of Q. Since Q has charge 1, such a pair consists of a boson of charge k and a fermion of charge $k+1$ or $k-1$, for some k.

of coupling implies that zero energy states present generically cannot disappear at a special value of the coupling. The generic value of the $h^k(Q)$ is thus the minimum value that they attain for any value of the coupling. Since according to (3.12), the $h^k(Q)$ vanish for odd k, this must be the correct generic value for odd k, and hence also for even k.

(2) By arguments in [1] that involve showing that Q changes by conjugation under a change in volume or coupling, it can be shown in this particular problem that there is actually no jumping for any τ.

Since (3.12) is the generic value, it can be compared to a weak coupling computation peformed for sufficiently small g and any θ. Such a comparison will be the goal of section 4.

Before going on to the case that G is not simply connected, we recall the Hamiltonian interpretation of the theta angle in gauge theories. The group of gauge transformations on $\mathbf{T}^3$ is not connected. If $\pi_1(G) = 0$, the components are labeled by a "winding number" (or degree) associated with

$$\pi_3(G) = \mathbf{Z}. \tag{3.14}$$

Let T be a gauge transformation of "winding number one." Just as in the discussion of discrete electric flux in section 2, in quantizing the theory we must consider states that are invariant under the group $\mathcal{W}_0$ of gauge transformations that are continuously connected to the identity; but we need not require invariance under T. Rather, we can select an arbitrary angle θ and require that our quantum states $|\psi\rangle$ obey

$$T|\psi\rangle = e^{i\theta}|\psi\rangle. \tag{3.15}$$

3.2 Non-Simply Connected Case

If G is not simply-connected (but, for simplicity, still connected), then, as in $2+1$ dimensions, quantum states can be labeled by discrete magnetic and electric charges. As in (2.3) and (2.9), the discrete magnetic charge is a characteristic class of the G-bundle that takes values in

$$M = H^2(\mathbf{T}^3, \pi_1(G)), \tag{3.16}$$

and the discrete electric charge is a quantum number that takes values in

$$E = \mathrm{Hom}(H^1(\mathbf{T}^3, \pi_1(G)), U(1)). \tag{3.17}$$

Actually, there is a subtlety already at this point, because on $\mathbf{T}^3$, unlike $\mathbf{T}^2$, an element of $H^1(\mathbf{T}^3, \pi_1(G))$ does not uniquely determine the homotopy class of a gauge transformation. There is also the winding number discussed in the last paragraph. For the moment, we will avoid this issue by assuming that $\theta = 0$, so that the winding number does not affect how a gauge transformation acts on physical states. We restore the theta dependence in section 3.4 when we analyze oblique confinement.

For some purposes, there is no essential loss to suppose that G, if not simply connected, is the adjoint form of the group, which is the form of the group with trivial center and the largest possible $\pi_1(G)$. For this choice of G, we meet the widest possible set of values of m and e. Other choices of G correspond to considering only a subset of conceivable values of m and e.

Identification Of E And M

Before discussing the predictions for the supersymmetric index and the $h^k(Q)$'s, we will make a small digression (which the reader may choose to omit). $(3+1)$-dimensional gauge theories can sometimes have nonperturbative dualities that exchange electric and magnetic fields. We will not actually in this paper study any example in which this happens. But for any such example to exist, it must be that in three dimensions the spaces M and E are the same, so that it is possible to have a nonperturbative duality exchanging m and e.[8]

First of all, by Poincaré duality, (3.17) can be rewritten in the form

$$E = H^2(\mathbf{T}^3, \widetilde{\pi_1(G)}), \tag{3.18}$$

where

$$\widetilde{\pi_1(G)} = \mathrm{Hom}(\pi_1(G), U(1)). \tag{3.19}$$

Now, if Γ is any finite abelian group, then $\widetilde{\Gamma} = \mathrm{Hom}(\Gamma, U(1))$ is isomorphic to Γ. So the groups E and M are isomorphic as abstract abelian groups. However, there is for general Γ no *natural* isomorphism between Γ and $\widetilde{\Gamma}$ (generally no such isomorphism is invariant under all automorphisms of Γ). For nonperturbative dualities exchanging e and m to be possible, it should be that there is a natural isomorphism between E and M, which will be so if there is a natural isomorphism between $\pi_1(G)$ and $\widetilde{\pi_1(G)}$. This can be constructed as follows:[9]

[8]For instance, in [11], it was shown that Montonen-Olive duality for $N = 4$ super Yang-Mills theory exchanges m and e.

[9]The following was pointed out by J. Morgan.

(1) If G is not simply-laced, then $\pi_1(G)$ is trivial or $\mathbf{Z}_2$, and either way there is only one possible isomorphism between $\pi_1(G)$ and $\widetilde{\pi_1(G)}$. Being unique, this isomorphism is natural.

(2) In discussing the simply-laced case, we first assume that G is the adjoint group, with trivial center. Then if Λ is the root lattice of G and $\Lambda^\vee$ (its dual) is the weight lattice, we have $\pi_1(G) = \Lambda^\vee/\Lambda$. Let $(\ , \ )$ be the inner product on $\Lambda^\vee$ (normalized as usual so that roots of G have length squared two). It has the property that $(x, y) \in \mathbf{Z}$ for $x \in \Lambda^\vee$, $y \in \Lambda$, but not, in general, if $x, y \in \Lambda^\vee$. Hence, if reduced mod 1, (x, y) is a well-defined pairing on $\Lambda^\vee/\Lambda = \pi_1(G)$. This pairing takes values in $\mathbf{Q}/\mathbf{Z}$, which we can regard as a subgroup of $\mathbf{R}/\mathbf{Z} = U(1)$. This pairing lets us define a map from $\pi_1(G)$ to $\widetilde{\pi_1(G)}$ by sending x to the element $\phi_x \in \widetilde{\pi_1(G)}$ that is defined by $\phi_x(y) = (x, y)$ (or $\exp(2\pi i(x, y))$ in a multiplicative notation). This map is an isomorphism by general facts about lattices.

This is easily extended to cases that G is a non-trivial cover of the adjoint group. If the adjoint group is $SO(4n)/\mathbf{Z}_2$, then G has fundamental group that is trivial or $\mathbf{Z}_2$, and either way as in (1) above there is only one isomorphism of $\pi_1(G)$ with its dual. In other cases, the adjoint group has a cyclic fundamental group $\mathbf{Z}_s$, for some s, and the fundamental group of G consists of elements of $\mathbf{Z}_s$ that are divisible by some divisor t of s. Then by setting $\phi_x(y) = \exp(2\pi i(x, y)/t)$ we get the desired natural isomorphism of $\pi_1(G)$ with its dual.

3.3 Predictions For The Index

We wish to describe the predictions for $\mathrm{Tr}\,(-1)^F$ and the more refined invariants $H^k(Q)$ that follow from standard claims about the infrared behavior of this theory. As before, we write $I(e, m)$ for the index as a function of e and m. We temporarily postpone discussion of the $H^k(Q)$ because of a subtlety involving oblique confinement.

The first basic point is that if confinement holds, then there are no zero energy states on a sufficiently big $\mathbf{T}^3$ for $e \neq 0$ and $m = 0$. Hence

$$I(e, 0) = 0, \quad \text{for } e \neq 0. \tag{3.20}$$

By considering the case of simply-connected G, we have already deduced that $I(0, 0) = (-1)^r h$. So (3.20) completes the story for the index if $m = 0$.

To understand what happens for $m \neq 0$, we begin as in section 2.2. We consider a path integral on $\mathbf{T}^4$ with supersymmetric boundary conditions for fermions and with a G-bundle of characteristic class $\widehat{m}$. We let $Z(\widehat{m})$ be the path integral for this G-bundle. We have

$$Z(0) = (-1)^r h, \tag{3.21}$$

since it equals $\mathrm{Tr}\,(-1)^F$ for the simply-connected cover of G. We want to determine $Z(\widehat{m})$ for other $\widehat{m}$'s. If we regard the first three directions in $\mathbf{T}^4$ as the "space" directions, and the fourth direction as "time," then it is natural to consider the components $\widehat{m}_{ij}$ of $\widehat{m}$ with $i, j = 1, \ldots, 3$ as "magnetic" and the $\widehat{m}_{i4}$ components as "electric." We consider $\widehat{m}$ to be "purely magnetic" if the electric components vanish, and "purely electric" if the magnetic components vanish.

Now, as in $2+1$ dimensions, we have the formula (2.15),

$$I(e,0) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{-1}(\gamma) Z(\widehat{m}_0(\gamma)). \tag{3.22}$$

$\widehat{m}_0(\gamma)$ is defined as in $2+1$ dimensions to have magnetic components zero and electric components determined by γ. Since $I(e,0) = 0$ for $e \neq 0$, this formula implies that $Z(\widehat{m}_0(\gamma))$ is independent of γ. Since $\widehat{m}_0(\gamma)$ is an arbitrary purely electric element of $H^2(\mathbf{T}^4, \pi_1(G))$, we find, using (3.21) to fix the constant, that

$$Z(\widehat{m}) = (-1)^r h \text{ if } \widehat{m} \text{ is purely electric.} \tag{3.23}$$

Now suppose that $\widehat{m}$ is purely magnetic. The non-zero components of $\widehat{m}$ are the 1-2, 1-3, and 2-3 components, but just as in section 2.3, we can make an $SL(3,\mathbf{Z})$ transformation to set $\widehat{m}_{12} = 0$. Then after an $SL(4,\mathbf{Z})$ transformation that exchanges the 3 and 4 directions, $\widehat{m}$ becomes purely electric and we can use (3.23). So we learn

$$Z(\widehat{m}) = (-1)^r h \text{ if } \widehat{m} \text{ is purely magnetic.} \tag{3.24}$$

The path integral $Z(\widehat{m})$ for a purely magnetic $\widehat{m}$ has a natural Hamiltonian interpretation. It is the index for states with magnetic flux $\widehat{m}$. This can be further decomposed according to the value of the electric flux:

$$Z(\widehat{m}) = \sum_e I(e,m). \tag{3.25}$$

Hence

$$\sum_e I(e,m) = (-1)^r h \tag{3.26}$$

for all m. Just as in (2.20), the general index can be written

$$I(e,m) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{-1}(\gamma)Z(\widehat{m}(m,\gamma)). \qquad (3.27)$$

(The definition of $\widehat{m}(m,\gamma)$ is as before. Writing $\mathbf{T}^4 = \mathbf{T}^3 \times \mathbf{S}^1$ with the last factor the "time" direction, the magnetic part of $\widehat{m}(m,\gamma)$ equals $m \in H^2(\mathbf{T}^3, \pi_1(G))$, and the electric part is determined by γ.)

One might think that confinement would mean that $I(e,m) = 0$ for all nonzero e. In fact, in $2+1$ dimensions, we proved this by virtue of the assumption that $I(e,0) = 0$ for nonzero e plus $SL(3,\mathbf{Z})$ symmetry. However, in $3+1$ dimensions, confinement does not imply that $I(e,m)$ always vanishes when e is nonzero.

The physical reason for this has to do with 't Hooft's classification [10] of massive phases of gauge theories. [10] According to this classification, such phases are described by condensation of a linear combination of electric and magnetic charge. If the condensed charge is electric, one has a Higgs phase; if the condensed charge is magnetic, one has a conventional confining phase. If the condensed charge is a mixture of electric and magnetic charge, one has a phase with "oblique confinement." [11]

In assuming that $I(e,0) = 0$ for all non-zero e, we have incorporated the idea that arbitrary external electric charge is confined. Vanishing of $I(e,0)$ is expected whether the confinement is ordinary or oblique confinement. Once one turns on $m \neq 0$, however, it is different; for $m \neq 0$, oblique confinement might mean $I(e,m) \neq 0$ for certain nonzero e.

To determine $I(e,m)$, we need to know, in the specific $N = 1$ supersymmetric gauge theories that we are studying, which of the abstractly posssible massive phases are actually realized. For this, we proceed as follows. First of all, we recall from the derivation of (3.11) that the vacua of this theory are all of the form $v^s|\Omega\rangle$, where v is a generator of the broken chiral symmetry $\mathbf{Z}_{2h}$. Moreover, the vacua are permuted in an adiabatic increase of θ by 2π. In other words, if one adiabatically increases θ by 2π, the state $v^s|\Omega\rangle$ is transformed to $v^{s-1}|\Omega\rangle$; the vacua are transformed by v^{-1}.[12]

[10] As we have noted in a footnote above, this is not a complete classification of conceivable massive phases of gauge theories, but it describes the massive phases that are believed to be relevant to the models under discussion here.

[11] A slight elaboration of the classification, explained in section 4 of [12], includes "mixed phases," in which the gauge group is Higgsed to a subgroup that is then confined or obliquely confined. We will not recall the details here as they are not relevant to the models studied in the present paper.

[12] This standard result is obtained as follows. The one instanton amplitude is propor-

Now we recall 't Hooft's intuitive explanation of oblique confinement: under a 2π increase in θ, a magnetic monopole may acquire electric charge (as one sees explicitly in Higgs phases [13]). In the present context, this means that under an adiabatic increase of θ by 2π, e may not be invariant but will, in general, change by an amount depending on m. We will call this change in e under a 2π increase in θ the "spectral flow" and denote it by $\Delta(m)$. As we compute later, $\Delta(m)$ is trivial for some groups, and nontrivial for others. But at any rate, we propose that the appropriate physical prediction in this theory is

$$I(e,m) = 0 \text{ unless } e \text{ is a multiple of } \Delta(m). \tag{3.28}$$

Thus, the phases are the ordinary confining phase and others obtained from it by spectral flow. Moreover, if $\Delta(m)$ is of order c in the finite group E, then we must have

$$I(e,m) = (-1)^r \frac{h}{c} \text{ if } e \text{ is a multiple of } \Delta(m). \tag{3.29}$$

In fact, since the different allowed e's are obtained from each other continuously by adiabatic increase in θ, they must all have the same index.

This obviously leaves us with the problem of computing $\Delta(m)$ for various G's and m's. But first, we pause to extend our discussion from the index to the more refined invariants, the Q cohomology groups.

The Cohomology Groups

We formulate the theory, in a Hamiltonian framework, on $\mathbf{T}^3$ with a bundle of some fixed m. We want to take $\mathbf{T}^3$ large and compare to the bulk behavior on $\mathbf{R}^3$. For this purpose, we should not project onto a particular value of e, because the projection onto definite e is a global operation on $\mathbf{T}^3$. Rather, we will sum over all e, and construct the Q cohomology in $\mathcal{H}_m = \oplus_e \mathcal{H}_{e,m}$.

We can reason just as we did at $m = 0$. Each vacuum of the infinite volume theory will, because of the mass gap, contribute at most one supersymmetric state on $\mathbf{T}^3$. As there are h vacua, the total number of supersymmetric states is at most h. On the other hand, the index formula (3.26) says that the number of supersymmetric states is at least h. Hence the number is precisely h, and they are in one-to-one correspondence with the vacua of the infinite volume theory.

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tional to $e^{i\theta}\lambda^h$, as there are h zero modes of λ in an instanton field. Hence $\theta \to \theta + c$ is equivalent to $\lambda \to e^{ic/h}\lambda$. So an adiabatic increase in θ by 2π gives back the same theory with $\lambda \to e^{2\pi i/h}\lambda$, which is the action of v^{-1}.

Those are, of course, permuted by the broken chiral symmetry. Hence the reasoning that we used to get (3.12) gives for all m

$$h^k(Q;m) = \begin{cases} 1, & \text{for even } k+r \\ 0, & \text{for odd } k+r. \end{cases} \tag{3.30}$$

Here, of course, the $h^k(Q;m)$ are dimensions of the cohomology groups for the action of Q on the Hilbert space $\mathcal{H}_m$ with specified m.

Thus the picture at nonzero m is similar to what it is at zero m. There are h supersymmetric states, each inherited from the bulk, and arising at arbitrary even values of k. If one wishes to further define Q cohomology groups in a sector in which both e and m are specified, one runs into the following subtlety. For groups G such that $\Delta(m) = 0$, one can define the Q cohomology groups in each $\mathcal{H}_{e,m}$, and these groups vanish for $e \neq 0$. It is different when there is oblique confinement. The index k labeling the cohomology groups is essentially the eigenvalue of the generator v of the chiral symmetry group $\mathbf{Z}_{2h}$. To be more precise, a state $\psi \in H^k(Q)$ transforms under v as $\psi \to e^{\pi i k/h}\psi$. But v is equivalent to $\theta \to \theta - 2\pi$, and this induces $e \to e - \Delta(m)$. So one cannot simultaneously measure both k and e. One way to proceed is to regard e as an element of $E/\Delta(m)$ (in other words, we consider e and e' equivalent if $e - e'$ is a multiple of $\Delta(m)$), and then we have the prediction

$$h^k(Q;0,m) = \begin{cases} 1, & \text{for even } k+r \\ 0, & \text{for odd } k+r, \end{cases} \tag{3.31}$$

as well as $h^k(Q;e,m) = 0$ if e is a nonzero element of $E/\Delta(m)E$ (in other words, if e is not divisible by $\Delta(m)$).

Now let us briefly discuss the physics of (3.30). As 't Hooft showed, in a massive Higgs vacuum, a nonzero magnetic flux on $\mathbf{T}^3$ produces a "defect" of strictly positive energy. So a massive Higgs vacuum would not contribute to $\mathrm{Tr}\,(-1)^F$ or the $H^k(Q)$. But (3.30) and our previous formulas for the index (both in $2+1$ and $3+1$ dimensions) show that for confining vacua, the magnetic flux produces no such effect; it "spreads out" and produces no contribution to the energy.

Action Of $SL(4,\mathbf{Z})$

In $2+1$ dimensions, we used $SL(3,\mathbf{Z})$, plus the assumption that $I(e,0) = 0$ for all nonzero e, to prove that $I(e,m) = 0$ for all nonzero e. Here we would like to explain why one cannot prove the analogous statement in $3+1$ dimensions by use of $SL(4,\mathbf{Z})$.

In $2+1$ dimensions, we proved in section 2.3 that by an $SL(3,\mathbf{Z})$ transformation, an arbitrary $m \in H^2(\mathbf{T}^3, \pi_1(G))$ can be made purely electric (one can set $m_{12} = 0$). The key step was to argue that if $\pi_1(G)$ is a cyclic group $\mathbf{Z}_s$ for some s, then the only invariant of m is the greatest common divisor, μ, of the m_{ij} and s.

The difference in $3+1$ dimensions is that, even for $\pi_1(G) = \mathbf{Z}_s$, it is not true that μ is the only invariant. An element $\widehat{m} \in H^2(\mathbf{T}^4, \mathbf{Z}_s)$ has another invariant, the "Pfaffian"

$$\mathrm{Pf}(\widehat{m}) = \widehat{m}_{12}\widehat{m}_{34} + \widehat{m}_{13}\widehat{m}_{42} + \widehat{m}_{14}\widehat{m}_{23}. \tag{3.32}$$

It is well-defined modulo μ and related to the cup product as follows. In general, for $\widehat{m}_1, \widehat{m}_2 \in H^2(\mathbf{T}^4, \mathbf{Z}_s)$, we define

$$(\widehat{m}_1, \widehat{m}_2) = \int_{\mathbf{T}^4} \widehat{m}_1 \cup \widehat{m}_2. \tag{3.33}$$

Then because $\mathbf{T}^4$ is spin, $(\widehat{m}, \widehat{m})$ is divisible by 2 in a natural fashion, and

$$\mathrm{Pf}(\widehat{m}) = \frac{(\widehat{m}, \widehat{m})}{2}. \tag{3.34}$$

It can be shown that μ and $\mathrm{Pf}(\widehat{m})$ classify $\widehat{m}$ up to the action of $SL(4,\mathbf{Z})$.[13]

In view of (3.27), to get $I(e,m) = 0$ for all m, one needs $Z(\widehat{m}(m,\gamma))$ to be independent of γ. But $Z(\widehat{m}(m,0)) = (-1)^r h$ by virtue of (3.24), so $Z(\widehat{m}(m,\gamma))$ is independent of γ if and only if $Z(\widehat{m})$ is entirely independent of $\widehat{m}$.

For G and m such that the spectral flow $\Delta(m)$ is nonzero, the instanton number is nonintegral, as we will see, for certain $\widehat{m}$, and in particular cannot vanish. Hence, the classical action is strictly positive for such $\widehat{m}$, and the partition function vanishes for $g \to 0$ as $\exp(-\mathrm{const}/g^2)$. Consequently, $Z(\widehat{m}) = 0$ for such $\widehat{m}$; but $Z(0) = (-1)^r h$. So the fractionality of the instanton number is an obstruction to having $Z(\widehat{m})$ independent of $\widehat{m}$ and thus to having $I(e,m)$ vanish for all nonzero e. As we have already explained, and as we will see in more detail in section 3.4, the correct statement is that $I(e,m) = 0$ unless e is a multiple of $\Delta(m)$. From our computations below, this assertion is equivalent to the statement that $Z(\widehat{m})$ depends only on $\mathrm{Pf}(\widehat{m})$ and not on μ. It would be interesting to try to verify this by direct

[13]The idea of the proof is to first reduce to $\mu = 1$ by replacing $\widehat{m}$ by $\widehat{m}/\mu$ and s by s/μ. Then one shows that one can by an $SL(4,\mathbf{Z})$ transformation set $\widehat{m}_{12} = 1$, after which by an $SL(4,\mathbf{Z})$ transformation one can set $\widehat{m}_{ij} = 0$ for $i = 1,2$ and $j = 3,4$ and (therefore) $\widehat{m}_{34} = \mathrm{Pf}(\widehat{m})$.

study of path integrals, but we will not do that in the present paper. Section 4 of the paper is devoted to microscopic calculations verifying the predictions that we have presented up to this point, but these calculations will be done from a Hamiltonian point of view.

3.4 Evaluation Of The Spectral Flow

The purpose of the present section is to evaluate the spectral flow. We consider a G-bundle over $\mathbf{T}^4$, with a given $\widehat{m} \in H^2(\mathbf{T}^4, \pi_1(G))$, and compute the deviation of the instanton number from being an integer. After doing this computation for $SU(n)/\mathbf{Z}_n$, we explain why it determines the spectral flow. Then – in a slightly lengthy case by case analysis – we make the computation for all simple Lie groups.

$SU(n)$

We begin with $SU(n)$. For an $SU(n)$ bundle V on $\mathbf{T}^4$, the instanton number or second Chern class is an integer k. Let $\mathrm{ad}(V)$ be the bundle derived from V in the adjoint representation. The ratio of the quadratic Casimir of the adjoint and fundamental representations of $SU(n)$ is $2n$, so the instanton number of $\mathrm{ad}(V)$ is $k' = 2nk$.

Suppose now that there is a nonzero magnetic flux $\widehat{m}$. Then the gauge group is really $SU(n)/\mathbf{Z}_n$, and the $SU(n)$ bundle V does not exist. The adjoint bundle $\mathrm{ad}(V)$ still exists, and its instanton number is an integer, say k'. The curvature integral $(1/8\pi^2)\int \mathrm{Tr} F \wedge F$, which measures the instanton number for an honest $SU(n)$ bundle, assigns the value $k = k'/2n$ to V. So the instanton number, if measured in the usual units, is a rational number rather than an integer, once there is a magnetic flux.

The value of the instanton number modulo 1 depends only on $\widehat{m}$. The reason for this is that any two $SU(n)/\mathbf{Z}_n$ bundles on $\mathbf{T}^4$ of the same $\widehat{m}$ can be obtained from one another by gluing in an ordinary instanton, of integer instanton number.[14] Thus to determine the value of the instanton number, modulo 1, for given $\widehat{m}$, we need only compute for a particular example.

Such an example can be constructed as follows. Let $\mathcal{L}$ be a line bundle over $\mathbf{T}^4$ whose first Chern class $c_1(\mathcal{L}) \in H^2(\mathbf{T}^4, \mathbf{Z})$ reduces mod n to $-\widehat{m}$.

[14]This statement can be proved by obstruction theory, building up the bundle on the r-skeleton of $\mathbf{T}^4$ for increasing r, and using the fact that the bundle is determined up to the two-skeleton by $\widehat{m}$, that $\pi_2(SU(n)) = 0$ so nothing new happens on the three-skeleton, and that $\pi_3(SU(n)) = \mathbf{Z}$, which labels the extensions from the three-skeleton over all of $\mathbf{T}^4$, is associated with the instanton number.

Now consider the "$SU(n)$ bundle" $V = \mathcal{L}^{-1/n} \otimes (\mathcal{L} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O})$, with $n-1$ trivial line bundles $\mathcal{O}$. Because of the fractional exponents, V does not really exist as an $SU(n)$ bundle. However, the associated adjoint bundle $\mathrm{ad}(V)$ does exist as an $SU(n)/\mathbf{Z}_n$ bundle.[15] Its magnetic flux equals $\widehat{m}$. (The basic reason for this is that the nonzero value of $-c_1(\mathcal{L})/n$ mod 1 or equivalently of $\widehat{m} = -c_1(\mathcal{L})$ mod n is the obstruction to defining $\mathcal{L}^{-1/n}$ as a line bundle; this is the same as the obstruction to defining V as a vector bundle since $\mathcal{L}^{-1/n}$ is the only problematic factor in the definition of V.)

Note that

$$\int_{\mathbf{T}^4} c_1(\mathcal{L}) \cup c_1(\mathcal{L}) = 2\widehat{m}_{12}\widehat{m}_{34} = 2\,\mathrm{Pf}(\widehat{m}). \tag{3.35}$$

The instanton number of V (which as noted above is $1/2n$ times the instanton number of $\mathrm{ad}(V)$) can be correctly computed, despite the fractional exponents in the definition of V, using the fact that for any $SU(n)$ bundle $W = \oplus_{i=1}^{n}\mathcal{L}_i$, the instanton number is

$$\int_{\mathbf{T}^4} c_2(W) = \sum_{i<j} \int_{\mathbf{T}^4} c_1(\mathcal{L}_i) \cup c_1(\mathcal{L}_j). \tag{3.36}$$

Using (3.36) and (3.35), a small computation shows that the instanton number of V is

$$k = -\mathrm{Pf}(\widehat{m})\left(1 - \frac{1}{n}\right). \tag{3.37}$$

We will denote the deviation of the instanton number from being an integer as $\Delta'(\widehat{m})$. So

$$\Delta'(\widehat{m}) = \frac{\mathrm{Pf}(\widehat{m})}{n} \text{ modulo } 1 \tag{3.38}$$

for $SU(n)$.

Our computations below for other groups will always involve reducing to an $SU(n)$ subgroup for some n. It will help to introduce the following notation. For any $n \geq 2$ and $\widehat{m} \in H^2(\mathbf{T}^4, \mathbf{Z}_n)$, we let $V_n(\widehat{m})$ be an "$SU(n)$ bundle" whose existence is obstructed by the magnetic flux $\widehat{m}$. This means, to be more precise, that $\mathrm{ad}(V_n(\widehat{m}))$ is an $SU(n)/\mathbf{Z}_n$ bundle of magnetic flux $\widehat{m}$.

Relation To The Spectral Flow

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[15]Concretely, $\mathrm{ad}(V)$ is the direct sum of $n-1$ copies of $\mathcal{L}$, $n-1$ copies of $\mathcal{L}^{-1}$, and $(n-1)^2$ copies of $\mathcal{O}$.

Before computing the deviation $\Delta'(\widehat{m})$ from integral instanton number for other groups, we will explain why this deviation determines the spectral flow $\Delta(m)$.

Consider first a G-bundle X over $\mathbf{T}^3$ with some magnetic flux $m \in H^2(\mathbf{T}^3, \pi_1(G))$ and some connection A. The connection has a Chern-Simons invariant

$$CS(A) = \frac{1}{8\pi^2} \int_{\mathbf{T}^3} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{3.39}$$

$CS(A)$ is invariant under gauge transformations that are continuously connected to the identity but not under all gauge transformations. It has been normalized so that it is shifted by 1 under the gauge transformation T associated with the generator of $\pi_3(\widehat{G}) = \mathbf{Z}$.[16]

$CS(A)$ is not a topological invariant. But it has the following very important property. If A is flat, that is if $F = dA + A \wedge A$ is zero, then $CS(A)$ is invariant under continuous deformations of A that preserves the flatness. For under any first order deformation $A \to A + \delta A$, the variation of $CS(A)$ is a multiple of $\int \mathrm{Tr}\, \delta A \wedge F$, which vanishes for $F = 0$.

Let $\widetilde{\Gamma}$ be the group of components of the group of gauge transformations of X. $\pi_3(G)$ gives some information about $\widetilde{\Gamma}$, but if the gauge group is not simply-connected, $\pi_1(G)$ also enters in describing $\widetilde{\Gamma}$. As we will now explain, the two mix in an interesting way. Let γ be a gauge transformation whose restriction to loops in $\mathbf{T}^3$ determines an element, which we will call $\overline{\gamma}$, of $\Gamma = \mathrm{Hom}(\pi_1(\mathbf{T}^3), \pi_1(G))$. To determine how $CS(A)$ transforms under γ, we proceed as follows. Rather as in the example sketched in figure 1, we can use the gauge transformation γ as gluing data in the $\mathbf{S}^1$ or "time" direction to extend the G-bundle $X \to \mathbf{T}^3$ to a G-bundle $X' \to \mathbf{T}^4 = \mathbf{T}^3 \times \mathbf{S}^1$. This bundle has a characteristic class in $H^2(\mathbf{T}^4, \pi_1(G))$ that we write as $\widehat{m}(m, \gamma)$; its magnetic components coincide with m, and the electric components are determined by γ. Moreover, the change $\Delta_\gamma(CS(A))$ in $CS(A)$ under γ is simply the instanton number of the bundle X'. This is $\Delta'(\widehat{m})$ modulo 1:[17]

$$\Delta_\gamma(CS(A)) = \Delta'(\widehat{m}) \text{ modulo } 1. \tag{3.40}$$

[16] This definition of $CS(A)$ assumes that the bundle is trivial and so A can be regarded globally as a Lie algebra valued one-form. More generally, we let X be a four-manifold of boundary $\mathbf{T}^3$ over which the bundle and connection extend, and we define $CS(A) = (1/8\pi^2) \int_X \mathrm{Tr} F \wedge F$.

[17] The reason that we only get a nice formula for $\Delta_\gamma(CS(A))$ mod 1 is that to determine Δ_γ precisely, we would need a precise description of the gauge transformation γ, and not just its image $\overline{\gamma}$ in $\Gamma = \mathrm{Hom}(\pi_1(\mathbf{T}^2), \pi_1(G))$. Without changing $\overline{\gamma}$, we could substitute $\gamma \to \gamma T^k$ for any integer k; this would add k to $\Delta_\gamma(CS(A))$.

Now suppose that γ is such that $\overline{\gamma}$ is of order s. Thus γ^s determines a trivial homomorphism of $\pi_1(\mathbf{T}^3)$ to $\pi_1(G)$. This does not mean that γ^s is homotopic to the identity gauge transformation; it only means that up to homotopy

$$\gamma^s = T^r \tag{3.41}$$

for some r. We can determine r modulo s by noting that γ^s shifts $CS(A)$ by $s\Delta_\gamma(CS(A)) = s\Delta'(\widehat{m})$ mod s. Since T shifts $CS(A)$ by 1, we have

$$r = s\Delta'(\widehat{m}) \text{ modulo } s. \tag{3.42}$$

The undetermined multiple of s in (3.42) is inessential, as r can be shifted by a multiple of s by $\gamma \to \gamma T^k$, as in the footnote.

If r is not always zero modulo s, we have the following mathematical situation. The group $\widetilde{\Gamma}$ of all homotopy classes of gauge transformations of the bundle $X \to \mathbf{T}^3$ maps to Γ by mapping a gauge transformation γ to its image $\overline{\gamma} \in \Gamma$. It contains a subgroup $\mathbf{Z} = \pi_3(G)$ of gauge transformations that map trivially to Γ. However, if $\Delta'(\widehat{m})$ is not zero for some γ, then $\widetilde{\Gamma}$ is not a product but a nontrivial extension:

$$0 \to \mathbf{Z} \to \widetilde{\Gamma} \to \Gamma \to 0. \tag{3.43}$$

Indeed, since T is a generator of $\mathbf{Z}$, the relation $\gamma^s = T^r$ has no solution in $\mathbf{Z} \times \Gamma$ if r is not a multiple of s, so having a $\gamma \in \widetilde{\Gamma}$ that obeys this relation means that $\widetilde{\Gamma}$ is a nontrivial extension rather than a product.

Now, we are almost ready to compute the spectral flow. In this situation, we must be more careful about defining the electric flux e. So far (keeping to $\theta = 0$) we have regarded the electric flux as a character of the group Γ. If we want to generalize this to arbitrary θ, we should instead regard e as a character of the full group $\widetilde{\Gamma}$. However, to define the theory at given θ, we want to allow only those characters e which on the subgroup $\mathbf{Z}$ act as follows: T is mapped to $e^{i\theta}$.

To compute the spectral flow, we fix a character e of $\widetilde{\Gamma}$ for fixed θ, and follow it continuously under $\theta \to \theta + 2\pi$. Let us take an arbitrary $\gamma \in \widetilde{\Gamma}$ and follow $e(\gamma)$ while θ is increased. We have from the above $\gamma^s = T^r = T^{s\Delta'}$, so $e(\gamma)^s = e(\gamma^s) = e(T^{s\Delta'}) = e(T)^{s\Delta'} = e^{i\theta s\Delta'}$. Taking the s^{th} root of this, we learn that

$$e(\gamma) = Ce^{i\theta\Delta'}, \tag{3.44}$$

where C is a root of unity that is independent of θ. Hence under $\theta \to \theta + 2\pi$, we have

$$e(\gamma) \to e(\gamma)e^{2\pi i\Delta'}. \tag{3.45}$$

Switching to an additive notation, this means that under $\theta \to \theta + 2\pi$, e is shifted by

$$e \to e + \Delta(m) \tag{3.46}$$

where $\Delta(m)$ is defined as follows: $\Delta(m)$ maps $\gamma \in \widetilde{\Gamma}$ to $\Delta'(\widehat{m}(m,\gamma))$ or more briefly to $\Delta'(m,\gamma)$.

More informally, we might say that $\Delta = \Delta'$; the deviation Δ' from integral instanton number determines the spectral flow Δ. In the remainder of this section, we compute Δ' and hence Δ for the remaining simple Lie groups, starting with the symplectic group.

$Sp(n)$

The center of $Sp(n)$ is $\mathbf{Z}_2$, generated by the element that acts as -1 on the fundamental $2n$-dimensional representation of $Sp(n)$. The adjoint group is therefore $G = Sp(n)/\mathbf{Z}_2$, with $\pi_1(G) = \mathbf{Z}_2$.

A simple example of an $Sp(n)/\mathbf{Z}_2$ bundle with given $\widehat{m}$ can be constructed as follows. For $n = 1$, $Sp(1) = SU(2)$, and we take the $SU(2)/\mathbf{Z}_2$ bundle $\mathrm{ad}(V_2(\widehat{m}))$ that was defined in the discussion of $SU(n)$. Now, for general n, $Sp(n)$ contains the subgroup $SU(2)^n$ generated by the "diagonal quaternions." Consider the "$Sp(n)$ bundle" $W = \oplus_{i=1}^n V_2^{(i)}(\widehat{m})$, with $V_2^{(i)}(\widehat{m})$ a copy of $V_2(\widehat{m})$ for the i^{th} factor in $SU(2)^n \subset Sp(n)$. Once again, it is only the associated adjoint bundle $\mathrm{ad}(W)$ that really exists globally.[18] The characteristic class that restricts the lifting of $\mathrm{ad}(W)$ to W is $\widehat{m}$, since this is the characteristic class that obstructs lifting the $\mathrm{ad}(V_2^{(i)}(\widehat{m}))$ to $SU(2)$ bundles $V_2^{(i)}(\widehat{m})$; defining W is of course equivalent to defining the summands $V_2^{(i)}(\widehat{m})$.

Using (3.38), the instanton number of W receives a contribution $\mathrm{Pf}(\widehat{m})/2$ for each $SU(2)$ factor, so for $Sp(n)$ we get

$$\Delta'(\widehat{m}) = \frac{n}{2}\mathrm{Pf}(\widehat{m}) \text{ modulo } 1. \tag{3.47}$$

So oblique confinement occurs in $Sp(n)$ for n odd but not for n even.

E_6

[18] In fact, $\mathrm{ad}(W) = \oplus_i \mathrm{ad}(V_2^{(i)}(\widehat{m})) \oplus_{i<j} V_2^{(i)}(\widehat{m}) \otimes V_2^{(j)}(\widehat{m})$, which is isomorphic to the sum of n copies of $\mathrm{ad}(V_2(\widehat{m}))$ and $n(n-1)/2$ copies of $\mathcal{O}$. This is worked out by looking at the decomposition of the adjoint representation of $Sp(n)$ under $SU(2)^n$.

We consider next the exceptional group E_6. The center is $\mathbf{Z}_3$, and the adjoint group is $G = E_6/\mathbf{Z}_3$.

E_6 contains a subgroup locally isomorphic to $SU(3) \times SU(3) \times SU(3)$. An $E_6/\mathbf{Z}_3$ bundle X with given $\widehat{m}$ can be constructed from the bundle that is respectively $V_3(0)$, $V_3(-\widehat{m})$, and $V_3(\widehat{m})$ for the three $SU(3)$'s. First of all, to see that this does give an $E_6/\mathbf{Z}_3$ bundle, we look at the decomposition of the adjoint representation of E_6 under $SU(3) \times SU(3) \times SU(3)$:

$$\mathbf{78} = (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}) \oplus (\mathbf{3},\mathbf{3},\mathbf{3}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}}). \tag{3.48}$$

With the given $SU(3) \times SU(3) \times SU(3)$ bundle, we see that our adjoint bundle is

$$\begin{aligned}&\mathrm{ad}(V_3(0)) \oplus \mathrm{ad}(V_3(-\widehat{m})) \oplus \mathrm{ad}(V_3(\widehat{m}))\\ &\oplus V_3(0) \otimes V_3(-\widehat{m}) \otimes V_3(\widehat{m}) \oplus \overline{V_3(0)} \oplus \overline{V_3(-\widehat{m})} \oplus \overline{V_3(\widehat{m})}.\end{aligned} \tag{3.49}$$

(For a bundle W, $\overline{W}$ is the complex conjugate bundle; $\overline{W}$ has opposite magnetic flux to W.) This is a well-defined $E_6/\mathbf{Z}_3$ bundle, because the total magnetic flux vanishes for each summand. To show that the magnetic flux of X is $\widehat{m}$, we note that the $\mathbf{27}$ of E_6 decomposes under $SU(3)^3$ as

$$(\mathbf{3},\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}},\mathbf{1},\mathbf{3}). \tag{3.50}$$

Hence, a hypothetical lift of X to a bundle whose fiber transforms in the $\mathbf{27}$ of E_6 would be

$$X' = V_3(0) \otimes \overline{V_3(-\widehat{m})} \oplus V_3(-\widehat{m}) \otimes \overline{V_3(\widehat{m})} \oplus V_3(\widehat{m}) \otimes \overline{V_3(0)}. \tag{3.51}$$

Since $\overline{V_3(-\widehat{m})} = V_3(\widehat{m})$, and $3\widehat{m} = 0$ (as the fundamental group is $\mathbf{Z}_3$), the "total magnetic flux" of each summand is $\widehat{m}$. Hence, using any of the constructions of nontrivial bundles given at the end of section 2.1, the obstruction to lifting X to an E_6 bundle X' is the same as the obstruction to lifting $\mathrm{ad}(V_3(\widehat{m}))$ to an $SU(3)$ bundle $V_3(\widehat{m})$. The characteristic class of X is therefore $\widehat{m}$. When we consider other groups below, we will not describe the sort of arguments presented in the present paragraph in such detail.

The instanton number of the $E_6/\mathbf{Z}_3$ bundle X, according to (3.38), receives contributions 0, $\mathrm{Pf}(-\widehat{m})/3 = \mathrm{Pf}(\widehat{m})/3$, and $\mathrm{Pf}(\widehat{m})/3$ from the three $SU(3)$'s, so

$$\Delta'(\widehat{m}) = \frac{2}{3}\mathrm{Pf}(\widehat{m}) \text{ modulo } 1 \tag{3.52}$$

for $E_6/\mathbf{Z}_3$. E_6 will therefore exhibit oblique confinement.

E_7

Next we consider the other exceptional group with a nontrivial center, which is E_7, with $\pi_1(E_7) = \mathbf{Z}_2$. The adjoint group is hence $G = E_7/\mathbf{Z}_2$.

E_7 contains a subgroup that is locally $SU(4) \times SU(4) \times SU(2)$. The **56** transforms as

$$(\mathbf{4}, \overline{\mathbf{4}}, \mathbf{1}) \oplus (\overline{\mathbf{4}}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{6}, \mathbf{2}). \tag{3.53}$$

A convenient $E_7/\mathbf{Z}_2$ bundle of given $\widehat{m}$ can be constructed by taking the "$SU(4) \times SU(4) \times SU(2)$ bundle" $V_4(2\widehat{m}) \otimes V_4(0) \otimes V_2(\widehat{m})$. (Note that as $\widehat{m}$ is defined modulo 2, $2\widehat{m}$ is defined modulo 4, so $V_4(2\widehat{m})$ is defined.) That this $E_7/\mathbf{Z}_2$ bundle is well-defined and has magnetic flux $\widehat{m}$ may be argued along lines sketched above for E_6. Using (3.38), we learn that the three factors contribute $\mathrm{Pf}(\widehat{m})$, 0, and $\mathrm{Pf}(\widehat{m})/2$ to the instanton number, so for $E_7/\mathbf{Z}_2$,

$$\Delta'(\widehat{m}) = \frac{\mathrm{Pf}(\widehat{m})}{2} \bmod 1. \tag{3.54}$$

Hence there will be oblique confinement for E_7.

$Spin(2n+1)$

We consider next the case of $Spin(2n+1)$. The center is $\mathbf{Z}_2$, and the adjoint group is $G = SO(2n+1)$.

To find an unliftable $SO(2n+1)$ bundle with given $\widehat{m}$, we proceed as follows. Starting with a subgroup

$$SO(3) \times SO(2n-2) \subset SO(2n+1) \tag{3.55}$$

and observing that $SO(3) = SU(2)/\mathbf{Z}_2$, we take the $SU(2)$ bundle $V_2(\widehat{m})$ times a trivial $SO(2n-2)$ bundle. According to (3.38), the instanton number of $V_2(\widehat{m})$ as an $SO(3)$ instanton is $\mathrm{Pf}(\widehat{m})/2$. (In other words, its instanton number is $\mathrm{Pf}(\widehat{m})/2$ times the smallest instanton number of an $SO(3)$ instanton on $\mathbf{S}^4$, which is conventionally called 1.) However, an $SO(3)$ instanton of instanton number 1 has instanton number 2 when embedded in $SO(2n+1)$, $n \geq 2$ via the embedding (3.55).[19] Hence, the $SO(2n+1)$

[19] An $SO(2n+1)$ field of instanton number 1 on $\mathbf{S}^4$ is made via the embedding $SU(2) \times SU(2) = SO(4) \subset SO(2n+1)$ by placing a field of instanton number 1 in the first $SU(2)$. Placing instanton number 1 in the $SO(3)$ in (3.55) amounts to putting instanton number 1 in each of the two $SU(2)$'s, which gives instanton number 2 in $SO(2n+1)$. Hence, the "quantum" of instanton charge on $\mathbf{S}^4$ is half as big for $SO(n)$ with $n > 3$ as for $SO(3)$.

instanton number is $\mathrm{Pf}(\widehat{m})$, and vanishes modulo 1. So

$$\Delta'(\widehat{m}) = 0 \text{ modulo } 1 \tag{3.56}$$

for $Spin(2n+1)$, and $SO(2n+1)$ has no oblique confinement. In other words, if $SO(2n+1)$ $N=1$ super Yang-Mills theory is probed by external 't Hooft and Wilson loops in the spinor representation of the gauge group, one will detect ordinary rather than oblique confinement.

The unliftable bundle that we constructed above makes sense for $SO(2n)$ as well as for $SO(2n+1)$. However, in contrast to $Spin(2n+1)$, whose center is $\mathbf{Z}_2$, $Spin(2n)$ has center $\mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_4$ for even or odd n, so to give a complete answer for $Spin(2n)$ requires further consideration that we present below. But if one considers a $Spin(2n)$ theory with dynamical charges in the vector representation of $SO(2n)$, then the center is effectively reduced to $\mathbf{Z}_2$. We can accomplish this by taking the minimal $N=1$ super Yang-Mills theory with gauge group $SO(2n)$ and adding chiral superfields in the vector representation with a large (supersymmetric) bare mass. The above computation shows that the relevant spectral flow vanishes, so this theory should have ordinary rather than oblique confinement in all the vacua.

As has been pointed out by N. Seiberg, this assertion can be verified in another way using dynamical results in [14]. Consider an $N=1$ supersymmetric theory with gauge group $SO(k)$ and $k-2$ chiral superfields Q_i in the vector representation; let M_{ij} be the "meson fields" $Q_i \cdot Q_j$ and set $U = \det M$. The U plane parametrizes a Coulomb phase. (This statement depends on the fact that the number of chiral multiplets is exactly $k-2$.) We assume $k \geq 5$ to avoid some special phenomena at small k, but it will not matter whether k is odd or even. There are two points on the U plane with massless monopoles or dyons. At one point *(A)* there are massless "dyons" $q^i_\pm$, $i = 1, \ldots, k$ with a coupling $M_{ij} q^i_+ q^j_-$, and at a second locus *(B)* there is a single pair of massless "monopoles" $S_\pm$ with a coupling $S_+ S_- (\det M - 1)$. The reason that we have put the words "monopoles" and "dyon" in quotes in the last sentence is that it has not been completely clear which of these particles are monopoles and which are dyons, but the analysis in [14] shows that the nonperturbative massless particles in the *(A)* and *(B)* vacua are of opposite types. To perturb down to the pure $SO(k)$ theory, one adds a generic quark mass term, corresponding to a perturbation $\mathrm{Tr}\, mM$ of the superpotential. For large m, the theory reduces to the pure $SO(k)$ theory, but on the other hand, the value of m does not affect the vacuum structure. Upon turning on m, most of the U plane including the *(A)* vacuum disappears; the *(B)* vacuum splits into $k-2$ massive vacua, which (as $k-2$ equals the dual Coxeter number of $SO(k)$ for $k \geq 5$) is the correct number to saturate $\mathrm{Tr}\,(-1)^F$ for $SO(k)$. All vacua, as they arise from *(B)*, are of the same type, showing (in agreement with the direct computation above) that the minimal $N = 1$ theory with $SO(k)$ gauge group must have no spectral flow as long as we only consider bundles that are well-defined in the vector representation. Moreover, in view of our microscopic knowledge about this $SO(k)$ theory, this shows that the massless excitations at the point *(B)* really are monopoles, corresponding to ordinary confinement.

Spin$(4n + 2)$

We will now carry out more complete analyses for *Spin*$(4n + 2)$ and *Spin*$(4n)$, including the bundles that do not lift to the vector representation of $SO(4n + 2)$ or $SO(4n)$. First we consider *Spin*$(4n + 2)$.

The center of *Spin*$(4n + 2)$ is $\mathbf{Z}_4$; the adjoint group is $G =$ *Spin*$(4n + 2)/\mathbf{Z}_4$.

For $n = 1$, we have *Spin*$(6) = SU(4)$. For an $SU(4)/\mathbf{Z}_4$ bundle of magnetic flux $\widehat{m}$, we can use our friend $V_4(\widehat{m})$ (or more precisely its adjoint bundle). The instanton number of $V_4(\widehat{m})$ is $\mathrm{Pf}(\widehat{m})/4$.

To study *Spin*$(4n + 2)$, we use the subgroup

$$Spin(6) \times Spin(4) \times \cdots \times Spin(4), \tag{3.57}$$

with $n - 1$ factors of *Spin*(4). A *Spin*$(4n + 2)/\mathbf{Z}_4$ bundle of magnetic flux $\widehat{m}$ can be constructed using the *Spin*(6) bundle $V_4(\widehat{m})$, and the bundles $V_2(2\widehat{m}) \otimes V_2(0)$ for each of the *Spin*$(4) = SU(2) \times SU(2)$ factors in (3.57). (As $2\widehat{m}$ is of order 2, $V_2(2\widehat{m})$ is defined.) This is chosen so that the two spinor representations of *Spin*$(4n + 2)$ are obstructed by $\widehat{m}$ and $-\widehat{m}$, respectively, and the vector representation by $2\widehat{m}$. The instanton number of $V_2(2\widehat{m}) \otimes V_2(0)$ is integral, and that of $V_4(\widehat{m})$ is $\mathrm{Pf}(\widehat{m})/4$, so altogether we have

$$\Delta'(\widehat{m}) = \frac{\mathrm{Pf}(\widehat{m})}{4} \text{ modulo } 1 \tag{3.58}$$

for $SO(4n + 2)$. So *Spin*$(4n + 2)$ has oblique confinement.

Let us now reconcile this with the assertion that if massive particles in the vector representation are added, one does not see oblique confinement. A formal way to say things is that in a theory that contains particles in the vector representation of $SO(4n + 2)$, we must take $\widehat{m}$ to be divisible by 2, and then $\Delta'(\widehat{m})$ vanishes and we do not observe oblique confinement.

A more physical (and essentially standard) explanation is as follows. First we recall in detail the meaning of oblique confinement. Formulating the theory on a spatial manifold $\mathbf{R}^3$, we let, for C a loop in $\mathbf{R}^3$, $W(C)$ denote a Wilson loop operator for an external charge propagating around C in the positive chirality spinor representation of $Spin(4n+2)$. We also let $H(C)$ be an 't Hooft loop constructed with a gauge transformation on $\mathbf{R}^3 - C$ whose monodromy around C is a generator of the center $\mathbf{Z}_4$ of $Spin(4n+2)$. If C and C' are loops in $\mathbf{R}^3$ of linking number 1, one has the 't Hooft algebra

$$W(C)H(C') = \exp(2\pi i/4)H(C')W(C). \tag{3.59}$$

In a confining vacuum, there is an integer s such that the operator $H(C)W^s(C)$ has no area law (and H^rW^t does unless $t = sr \bmod 4$); oblique confinement is the case that $s \neq 0$. If we add massive charges in the vector representation, then $H(C)$ no longer makes sense as an observable, but $H(C)^2$ does. In addition, we should consider $W(C)^2$ to be trivial since it can be screened by a charge in the vector representation. The basic observables are thus $H(C)^2$ and $W(C)$, and they obey an 't Hooft algebra appropriate for a group $SO(4n+2)$ with fundamental group $\mathbf{Z}_2$:

$$H(C)^2W(C') = -W(C')H(C)^2. \tag{3.60}$$

The vacuum that formerly was described as having no area law for $H(C)W^s(C)$ must now (since only even powers of $H(C)$ are defined) be described as having no area law for $H(C)^2W^{2s}(C)$; since $W^{2s}(C)$ is trivial, this is equivalent to saying that $H(C)^2$ has no area law. The information about s is lost.

Spin(4*n*)

The last example is $Spin(4n)$, with center $\mathbf{Z}_2 \times \mathbf{Z}_2$. The adjoint group is $G = Spin(4n)/\mathbf{Z}_2 \times \mathbf{Z}_2$.

We select generators a_1, a_2 of the center of $Spin(4n)$ defined as follows. a_1 is $+1$ on the positive chirality spinor representation, -1 on the negative chirality spinor representation, and -1 on the vector representation; a_2 acts on those three representations as $-1, +1, -1$. Equivalently, we can regard a_1 and a_2 as generators of $\pi_1(G)$. Since $H^2(\mathbf{T}^4, \pi_1(G)) = H^2(\mathbf{T}^4, \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} \pi_1(G)$, we can write an arbitrary $\widehat{m} \in H^2(\mathbf{T}^4, \pi_1(G))$ as $\widehat{m} = \widehat{m}_1 a_1 + \widehat{m}_2 a_2$, with $\widehat{m}_1, \widehat{m}_2 \in H^2(\mathbf{T}^4, \mathbf{Z}_2)$.

The first example of $Spin(4n)$ is $Spin(4) = SU(2) \times SU(2)$. A $Spin(4)/\mathbf{Z}_2 \times \mathbf{Z}_2$ bundle of general $\widehat{m}$ is derived from $V_2(\widehat{m}_1) \otimes V_2(\widehat{m}_2)$. The instanton number is $\frac{1}{2}(\mathrm{Pf}(\widehat{m}_1) + \mathrm{Pf}(\widehat{m}_2))$.

For general n, we look at the subgroup

$$Spin(4) \times Spin(4) \times \cdots \times Spin(4) \subset Spin(4n), \tag{3.61}$$

with n factors. If n is odd, we use $V_2(\widehat{m}_1) \otimes V_2(\widehat{m}_2)$ in each factor, and get instanton number $(n/2)(\mathrm{Pf}(\widehat{m}_1) + \mathrm{Pf}(\widehat{m}_2))$, so

$$\Delta'(\widehat{m}) = \frac{\mathrm{Pf}(\widehat{m}_1) + \mathrm{Pf}(\widehat{m}_2)}{2} \text{ modulo } 1 \tag{3.62}$$

for $Spin(8k+4)$. The choice of bundle has been made to ensure that the total magnetic flux obstructing the existence of the positive or negative chirality spin bundles is $\widehat{m}_1$ or $\widehat{m}_2$, respectively. If n is even, to achieve the same end, we build the bundle using $V_2(\widehat{m}_1 + \widehat{m}_2) \otimes V_2(0)$ in the first factor and $V_2(\widehat{m}_1) \otimes V_2(\widehat{m}_2)$ in the others. Then we get

$$\Delta'(\widehat{m}) = \frac{(\widehat{m}_1, \widehat{m}_2)}{2} \text{ modulo } 1 \tag{3.63}$$

for $Spin(8k)$. (We have used the fact that in view of (3.34), $(\widehat{m}_1, \widehat{m}_2) = \mathrm{Pf}(\widehat{m}_1 + \widehat{m}_2) - \mathrm{Pf}(\widehat{m}_1) - \mathrm{Pf}(\widehat{m}_2)$.) Thus, $Spin(8k)$ and $Spin(8k+4)$ both have oblique confinement, with somewhat different details.

In a theory in which there are dynamical charges in the vector representation of $Spin(4n)$, we must set $\widehat{m}_1 = \widehat{m}_2$, and then $\Delta'(\widehat{m}) = 0$ so we do not observe oblique confinement. (For $Spin(8k)$, this depends on the fact that $(\widehat{m}, \widehat{m})$ is always even.) One can give a more physical explanation of this as we did for $Spin(4n+2)$.

3.5 Outer Automorphisms

We now consider the case that G admits a group C of outer automorphisms. In practice, for G a simple Lie group, C will be $\mathbf{Z}_2$, except in the case of $G = Spin(8)$, where C can be the triality group (the group of permutations of three elements) or a subgroup. For example, concretely, for $G = SU(n)$, $C = \mathbf{Z}_2$ is generated by the operation of complex conjugation or "charge conjugation" that exchanges the representations $\mathbf{n}$ and $\overline{\mathbf{n}}$ of $SU(n)$.

Such a group of outer automorphisms of the gauge group is realized in the quantum field theory as a global symmetry group, and one would like to know if C is spontaneously broken. We expect that it is not, since the usual logic of confinement does not distinguish representations of G that differ by the action of c, so we do not expect c to be spontaneously broken. For example, for $G = SU(n)$, we do not expect spontaneous breaking of charge conjugation symmetry.

There are two ways that we can formulate a criterion for C to be unbroken in terms of generalizations of $\mathrm{Tr}\,(-1)^F$. First of all, because C commutes with supersymmetry, we can pick any element $c \in C$ and define $I(c) = \mathrm{Tr}\, c(-1)^F$. This trace can be evaluated just in the space of zero energy states, as the states of nonzero energy cancel in bose-fermi pairs. If then the h vacua of the theory are all C-invariant in infinite volume, then we expect

$$I(c) = (-1)^r h, \tag{3.64}$$

just as at $c = 0$. We can generalize this to include the magnetic and electric flux m and e. The only subtlety is that they must be c-invariant in order for c to act in the Hilbert space with specified m and e. This condition can be very restrictive; for example, for $G = SU(n)/\mathbf{Z}_n$, the charge conjugation automorphism c acts by $m \to -m$, $e \to -e$. At any rate, for such m and e, we expect $I(e, m; c) = \mathrm{Tr}_{\mathcal{H}_{e,m}} c(-1)^F$ to be independent of c as it can be computed from C-invariant vacuum states.

The second approach to formulating a criterion for C to be unbroken is to consider the gauge theory formulated on a bundle that is twisted by some element $c \in C$. We can combine C and G to a disconnected gauge group G' that fits into an exact sequence

$$0 \to G \to G' \to C \to 0. \tag{3.65}$$

We think of G' as the gauge group and formulate the theory using a nontrivial G' bundle over $\mathbf{T}^3$. This is analogous to the way the magnetic flux was introduced when $\pi_1(G) \neq 0$, and just as in that case, in the quantization we want to require the quantum states to be invariant only under a restricted group of gauge transformations – the gauge transformations by elements of G.

This procedure is easy to carry out. If G is simply-connected, a G' bundle on $\mathbf{T}^3$ can be classified very simply. The holonomy around a circle $S \in \mathbf{T}^3$ is an element of G' whose image in C depends only on the class of S in $\pi_1(\mathbf{T}^3)$. This gives a homomorphism $\phi : \pi_1(\mathbf{T}^3) \to C$, which completely classifies the G-bundles. Since $\pi_1(\mathbf{T}^3)$ is commutative, the image of ϕ is a commutative and hence cyclic subgroup of C in all cases (even $G = Spin(8)$). For a suitable decomposition $\mathbf{T}^3 = \mathbf{S}^1 \times \mathbf{T}^2$, the bundle can be described as follows: there is some $c \in C$, such that in going around $\mathbf{S}^1$, the fields are conjugated by c. We let X_c denote the bundle constructed in this way, and we write I_c for the index $\mathrm{Tr}\,(-1)^F$ for the supersymmetric gauge theory quantized using the bundle I_c.

We evaluate I_c as usual by taking a large metric on $\mathbf{T}^3$ and saturating the trace by zero energy vacuum states. The only subtlety is that, when we quantize on X_c, a vacuum state of the infinite volume theory that is not c-invariant will not contribute, since in trying to glue in such a state on $\mathbf{T}^3$, one will need a domain wall (because of the twist by c) with a large cost in energy. If, however, all vacuum states of the infinite volume theory are C-invariant and massive, they should all contribute in the quantization on X_c, and so we expect

$$I_c = (-1)^r h \text{ for all } c. \tag{3.66}$$

We can extend this to the case that G is not simply-connected, so that the bundles are classified by a magnetic flux $m \in H^2(\mathbf{T}^3, \pi_1(G))$ as well as by c. As above, m must be c-invariant. For all such pairs c, m, we expect by the same reasoning.

$$I_c(m) = (-1)^r h. \tag{3.67}$$

Finally, we can include the electric flux e, now regarded as a character of the c-invariant subgroup Γ^c of Γ. The same logic leads us to predict that $I_c(e, m) = 0$ unless e is a multiple of the spectral flow. The spectral flow must be recomputed for $c \neq 0$.

We have given two criteria for testing whether C acts trivially on the vacuum states of the infinite volume theory. The two criteria differ by whether the twist by c is made in the "time" direction or in a "space" direction. They therefore are related to each other by $SL(4, \mathbf{Z})$ symmetry. In a path integral evaluation, the equivalence of the two criteria would be manifest. In section 4.5, we will test the above predictions in a microscopic computation that is made in a Hamiltonian approach. From this point of view, the equivalence of the two criteria is not obvious so we will check both. Agreement with the above predictions will give support for the conjecture that C is an unbroken symmetry of the infinite volume theory.

4 Microscopic Computations

In this section, we begin microscopic computations to verify the predictions made in sections 2 and 3. In $2+1$ dimensions, we repeat part of the analysis in [3] for completeness and as background to $3+1$ dimensions. In $3+1$ dimensions, the strategy is the same as in [1], but we will be more comprehensive. In fact, many arguments below are given in embryonic form in [1] and are presented here more fully. In this section, we illustrate some of the ideas with simple examples involving classical groups. A systematic survey of the more elaborate examples is reserved for section 5.

4.1 $2+1$ Dimensional Case

We begin in $2+1$ dimensions. We quantize on $\mathbf{T}^2 \times \mathbf{R}$ the minimal supersymmetric theory, whose Lagrangian was presented at the beginning of section 2. However, we specialize to $k = \pm h/2$, because we want to treat the low energy phase space as an orbifold; as explained in [3], this is only valid for $k = \pm h/2$.

We will analyze the low-lying states in a weak coupling approximation, valid if the gauge coupling is very small or equivalently if the radius of the $\mathbf{T}^2$ is very small. We begin with the case that the gauge group G is simply-connected (as well as being simple, connected, and compact). We let r denote the rank of G.

The first step is to find the classical states of zero energy. To minimize the energy, a configuration should be time-independent. In addition, the spatial part of the curvature should vanish on $\mathbf{T}^2$. This means that the restriction of the connection to $\mathbf{T}^2$ defines a flat connection, which can be described up to holonomy by the monodromies U_1, U_2 around the two directions in $\mathbf{T}^2$. Since the fundamental group of $\mathbf{T}^2$ is abelian, U_1 and U_2 commute.

The moduli space of flat connections on $\mathbf{T}^2$ is the moduli space of such commuting pairs, up to conjugation. One can always conjugate any one group element to lie in the maximal torus $\mathbf{T}_G$ of G. In fact, G being simply-connected, given any two commuting group elements U_1 and U_2, one can simultaneously conjugate them to lie in $\mathbf{T}_G$. After conjugating U_1 and U_2 into $\mathbf{T}_G$, one can still divide by the Weyl group W, so the moduli space of flat connections on $\mathbf{T}^2$ is in fact

$$\mathcal{M} = (\mathbf{T}_G \times \mathbf{T}_G)/W. \tag{4.1}$$

This is a nontrivial statement whose obvious analog for three or more commuting elements of G is in general false. To prove that $(\mathbf{T}_G \times \mathbf{T}_G)/W$ is a component of $\mathcal{M}$, it suffices to show that given $U_1, U_2 \in \mathbf{T}_G$, any deformation of them as a commuting pair is still (up to conjugacy) in $\mathbf{T}_G$. This can be proved by looking at the problem for first order deformations. Such an argument works for commuting n-tuples of any n, to show that $(\mathbf{T}_G \times \cdots \times \mathbf{T}_G)/W$ is always a component of the moduli space of commuting n-tuples.

But why is $(\mathbf{T}_G \times \mathbf{T}_G)/W$ the *only* component of $\mathcal{M}$? This is the assertion that fails in one dimension higher. To prove the assertion in two dimensions, note that if we pick a complex structure on $\mathbf{T}^2$, then $\mathcal{M}$ can be interpreted as the moduli space of holomorphic semistable $G_{\mathbf{C}}$ bundles on $\mathbf{T}^2$. Moduli spaces of semistable bundles on a Riemann surface are always connected and irreducible.[20] There is no analog of this argument for commuting triples.

If G is not simply-connected, we should refine this description of the moduli space slightly. Let again $\widehat{G}$ be the universal cover of G. $F = \widehat{G}/G$ is a finite abelian group that can be regarded as a quotient of the center of $\widehat{G}$. The most important example is really that G is the adjoint group (whose center is trivial) and F is the center of $\widehat{G}$; considering this example gives the most complete information. In quantizing the theory, according to the recipe in section 2, we impose invariance not under the group $\mathcal{W}$ of all gauge transformations, but under the restricted group $\mathcal{W}_0$ of gauge transformations that are single-valued when lifted to $\widehat{G}$. This means that we should consider the U_i to take values in $\mathbf{T}_{\widehat{G}}$, the maximal torus of $\widehat{G}$ (which of course is a finite cover of $\mathbf{T}_G$, the maximal torus of G). The phase space to be quantized is thus really

$$\widehat{\mathcal{M}} = (\mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}})/W. \tag{4.2}$$

As described in section 2, a residual group $\Gamma = \mathcal{W}/\mathcal{W}_0$ acts in the problem. An element of Γ is a pair $\gamma = (a_1, a_2)$, where the a_i are elements of F (regarded as elements of the center of $\widehat{G}$). γ acts on $\widehat{\mathcal{M}}$ by

$$U_i \to a_i U_i, \ i = 1, 2, \tag{4.3}$$

and $\mathcal{M} = \widehat{\mathcal{M}}/\Gamma$. As we have recalled in section 2, states of given "electric flux" are states in which Γ acts with a given character. Until we incorporate the electric flux, it will not matter whether the phase space is $\mathcal{M}$ or $\widehat{\mathcal{M}}$. The explicit description of the moduli space in (4.2) assumes that the magnetic flux m is zero; we discuss the generalization later.

Now let us find the zero modes of the fermions. A fermion zero mode must be annihilated by the Dirac operator on $\mathbf{T}^2$: $\sum_{i=1}^2 \Gamma^i D_i \psi = 0$. By squaring the Dirac operator and using the fact that the curvature vanishes, we get $\sum_{i=1}^2 D_i^2 \psi = 0$. Upon multiplying by the complex conjugate of ψ and integrating by parts, we learn that $\int_{\mathbf{T}^2} |D\psi|^2 = 0$, so finally a fermion zero mode obeys

$$0 = D_i \psi. \tag{4.4}$$

Being covariantly constant, ψ must be invariant under the monodromies U_1 and U_2. If we are at a generic point in $\mathcal{M}$, this means that ψ is a constant with values in the Lie algebra $\mathbf{t}$ of the maximal torus.

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[20]That is so because there is no integrability condition for a $\overline{\partial}$ operator in complex dimension one. If $\overline{\partial}_A$ and $\overline{\partial}_{A'}$ are any two $\overline{\partial}$ operators on the same bundle, one can literally interpolate between them by setting $\overline{\partial}_t = t\overline{\partial}_A + (1-t)\overline{\partial}_{A'}$ with a complex parameter t. Likewise, the moduli spaces are irreducible since the obstruction theory is trivial.

Now let us construct the quantum Hilbert space, in the approximation of considering only the classical states of zero energy. The first step is to quantize the fermion zero modes, setting the bosons to a specific point in $\mathcal{M}$. Let ψ_+ and ψ_- be the components of ψ of positive and negative chirality on $\mathbf{T}^2$. (They are of course complex conjugates of one another.) The canonical anticommutation relations are

$$\{\psi_+^a, \psi_-^b\} = \delta^{ab}, \quad a, b = 1, \ldots, r. \tag{4.5}$$

If we let $|\Omega\rangle$ be a state annihilated by the ψ_-^a, then the fermion Hilbert space is spanned by the states $|\Omega\rangle, |\Omega^a\rangle = \psi_+^a|\Omega\rangle, |\Omega^{ab}\rangle = \psi_+^a\psi_+^b|\Omega\rangle$, and so on. We let $|\widetilde{\Omega}\rangle$ denote the state $\psi_+^1\psi_+^2\ldots\psi_+^r|\Omega\rangle$ with all fermion states filled.

If we can treat $\mathcal{M}$ as an orbifold, then a zero energy state on $\mathcal{M}$ is the same as a zero energy state on $\mathbf{T}_G \times \mathbf{T}_G$ that is W-invariant. Taking the flat metric on $\mathbf{T}_G \times \mathbf{T}_G$, to get a state of zero energy, we must take the wave function to be invariant under translations on $\mathbf{T}_G \times \mathbf{T}_G$. The bosonic part of the wave-function is thus a constant, which is automatically W-invariant. Any state in the fermion Fock space, obtained by acting on $|\Omega\rangle$ with s of the ψ_+^a for any $s \leq r$, has zero energy. The space of zero energy states is simply the W-invariant subspace of the fermion Fock space (and therefore, in particular, is the same whether we have in the quantization imposed invariance under the restricted group $\mathcal{W}_0$ of gauge transformations or the full group $\mathcal{W}$ of all gauge transformations).

The fermion creation operators ψ_+^a transform in the representation $\mathbf{t}$ of W. The product of all r fermion creation operators $\psi_+^1\psi_+^2\ldots\psi_+^r$ transforms in a one-dimensional real representation Θ of W, in which each elementary reflection is represented by -1.

It is argued in [3] that this system can be treated as an orbifold precisely if $k = \pm h/2$, and that for those values of k, of the two states $|\Omega\rangle$ and $|\widetilde{\Omega}\rangle$, one is W-invariant and the other transforms in the representation Θ. Moreover, under $k \leftrightarrow -k$, the role of the two states is exchanged. We will suppose that $k = \pm h/2$ is such that $|\Omega\rangle$ is W-invariant.

Given a complex structure on $\mathbf{T}^2$, the torus $\mathbf{T}_G \times \mathbf{T}_G$ acquires a complex structure, with complex coordinates z^a, $a = 1, \ldots, r$. The $\overline{\partial}$ cohomology group $H^{0,q}(\mathbf{T}_G \times \mathbf{T}_G)$ has a basis of $(0, q)$-forms $d\overline{z}^{a_1} d\overline{z}^{a_2} \ldots d\overline{z}^{a_q}$. Such a form corresponds in a natural way to $\psi_+^{a_1}\psi_+^{a_2}\ldots\psi_+^{a_q}|\Omega\rangle$, which is one of the basis elements of the fermion Fock space, so we can think of the fermion Fock space as

$$\mathcal{F} = \oplus_{q=0}^r H^{0,q}(\mathbf{T}_G \times \mathbf{T}_G). \tag{4.6}$$

The space of zero energy states in the quantum theory is not $\mathcal{F}$ but $\mathcal{F}^W$, the W-invariant subspace of $\mathcal{F}$. Imposing W-invariance projects the cohomology of $\mathbf{T}_G \times \mathbf{T}_G$ to that of $\mathcal{M} = (\mathbf{T}_G \times \mathbf{T}_G)/W$, so

$$\mathcal{F}^W = \oplus_{q=0}^r H^{0,q}(\mathcal{M}). \tag{4.7}$$

This is the space of zero energy states.

On the other hand, by a theorem of Looijenga [15] and Bernshtein and Shvartsman [16] (another proof was given in [17]), $\mathcal{M}$ is a weighted projective space, $\mathcal{M} = \mathbf{WCP}^r_{s_0,s_1,\dots,s_r}$, where the weights s_i are 1 and the coefficients of the highest coroot of G. (For example, for $G = SU(N)$ the weights are all 1, and for $G = E_8$ the weights are 1,2,2,3,3,4,4,5, and 6.) The $\overline{\partial}$ cohomology of a weighted projective space is easily described. $H^{0,0}(\mathcal{M})$ is one-dimensional, being generated by the constant function 1, and $H^{0,q}(\mathcal{M}) = 0$ for $q > 0$.

So there is precisely one zero energy state, and the index (for $m = 0$) is $I(0) = 1$, as predicted on macroscopic grounds in section 2. Now let us include the electric flux. Here it is important to specify that the bosonic phase space is $(\mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}})/W$, acted on by the finite group Γ. Γ acts on the bosonic coordinates of the phase space according to (4.3), and acts trivially on the fermion zero modes. If we fix a nontrivial character e of Γ, then the bosonic wave-function cannot be constant and necessarily carries a strictly positive energy. So the one zero energy state found above has $e = 0$. Hence, the index is as expected $I(e, 0) = 0$ for $e \neq 0$, and $I(0, 0) = 1$.

4.2 $3 + 1$ Dimensions

Now we consider the problem of computing the index for the $N = 1$ super Yang-Mills theory in $3 + 1$ dimensions, formulated on a torus $\mathbf{T}^3$. In this section, we take the magnetic flux to vanish, so the gauge field is a connection on a trivial G-bundle X.

Just as in $2 + 1$ dimensions, the first step is to take g very small and reduce to the moduli space of flat connections on X. Such a connection has commuting holonomies U_1, U_2, U_3. It turns out, for general G, that the moduli space $\widehat{\mathcal{N}}$ of commuting triples has several components $\widehat{\mathcal{N}}_i$. The most obvious component, which we will call $\widehat{\mathcal{N}}_0$, is the one that contains the point $U_1 = U_2 = U_3 = 1$. We will call $\widehat{\mathcal{N}}_0$ the identity component of $\widehat{\mathcal{N}}$. Any U_i in $\widehat{\mathcal{N}}_0$ can be simultaneously conjugated to the maximal torus – to be precise, the maximal torus of $\widehat{G}$, since in quantization we divide only by the restricted gauge group $\mathcal{W}_0$. This component of the bosonic phase space is then

$$\widehat{\mathcal{N}}_0 = (\mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}})/W. \tag{4.8}$$

As in $2+1$ dimensions, a finite group $\Gamma = \mathcal{W}/\mathcal{W}_0$ acts on $\widehat{\mathcal{N}}$ by

$$U_i \to a_i U_i, \quad a_i \in \widehat{G}/G. \tag{4.9}$$

The quotient $\mathcal{N} = \widehat{\mathcal{N}}/\Gamma$ classifies flat connections modulo the action of the full group $\mathcal{W}$. Likewise, $\mathcal{N}_0 = \widehat{\mathcal{N}}_0/\Gamma$ is the identity component of $\mathcal{N}$.

We will review in section 5 a systematic construction of the components of $\widehat{\mathcal{N}}$ for any G, but to orient the reader we will briefly recall a direct construction (given in the appendix to [2]) for $G = Spin(n)$ with $n \geq 7$. We first construct a flat $Spin(7)$ bundle E on $\mathbf{T}^3$ that has no moduli. The holonomies are diagonal matrices U_i. We assume that, for $k = 1, \ldots, 7$, the k^{th} diagonal matrix elements of (U_1, U_2, U_3) are of the form $(\pm 1, \pm 1, \pm 1)$ with each sequence of three numbers ± 1 appearing precisely once except $(1,1,1)$. (The flat bundle with these holonomies is a sum of real line bundles; by computing Stieffel-Whitney classes, one can show that it is topologically trivial.) To embed this in $Spin(n)$, we simply take the U_i to be diagonal matrices whose first seven eigenvalues are as described and whose other $n-7$ diagonal elements are all 1. The subgroup of $Spin(n)$ commuting with the U_i has for its connected component $Spin(n-7)$. Let $\widehat{\mathcal{N}}_1$ be the component of $\widehat{\mathcal{N}}$ that contains the commuting triple just described.

To show that $\widehat{\mathcal{N}}_1$ is distinct from $\widehat{\mathcal{N}}_0$, it suffices to observe that the unbroken groups $Spin(n)$ and $Spin(n-7)$ have different ranks. Indeed, the rank of the unbroken subgroup is always conserved under continuous deformation of a commuting triple. For if $\vec{U} = (U_1, U_2, U_3)$ leaves fixed a group $H_{\vec{U}}$, then any small deformation of $\vec{U}$ as a commuting triple commutes with a group that contains a maximal torus of $H_{\vec{U}}$; the small deformation can in fact be accomplished by $U_i \to U_i V_i$ with V_i in a maximal torus of $H_{\vec{U}}$. It can be shown for $Spin(n)$ that $\widehat{\mathcal{N}}$ has precisely the two components $\widehat{\mathcal{N}}_0$ and $\widehat{\mathcal{N}}_1$.

For any component $\widehat{\mathcal{N}}_i$, we let r_i be the rank of the unbroken group; it is one of the most important invariants of $\widehat{\mathcal{N}}_i$. The dimension of $\widehat{\mathcal{N}}_i$ is $3r_i$, since locally U_1, U_2, and U_3 vary in the maximal torus of the unbroken group. In particular, locally each of U_1, U_2, and U_3 is specified by r_i parameters.

Computation

We will first describe the contribution to the index of $\widehat{\mathcal{N}}_0$, so we assume that the U_i can be simultaneously conjugated to a maximal torus. A fermion zero mode is, as in $2+1$ dimensions, a constant (invariant under translations on $\mathbf{T}^3$) with values in the Lie algebra $\mathbf{t}$ of the maximal torus. However, we get two copies of the space of fermion zero modes that we had in $2+1$ dimensions, because the four-dimensional gluino has two chiral components λ_α, $\alpha = 1,2$ and two antichiral components $\overline{\lambda}_{\dot\alpha}$, $\dot\alpha = 1,2$ (while the $(2+1)$-dimensional gluino has only two real components). The zero modes of $\overline{\lambda}_{\dot\alpha}$ are fermion "creation operators" $\overline{\psi}^a_{\dot\alpha}$, with $a = 1\dots r$ running over a basis of $\mathbf{t}$; the zero modes of λ_α are "annihilation operators" ψ^a_α. If $|\Omega\rangle$ is a "Fock vacuum," annihilated by the λ_α modes, then a basis of the fermion Fock space is given by the states

$$\overline{\psi}_1^{a_1}\dots\overline{\psi}_1^{a_s}\overline{\psi}_2^{b_1}\dots\overline{\psi}_2^{b_t}|\Omega\rangle,\quad 0\le s,t\le r. \tag{4.10}$$

Just as in $2+1$ dimensions, the bosonic wave function of a zero energy state must be constant, and the space of zero energy states is simply the Weyl-invariant part of the fermion Fock space. The fermion Fock space is now not simply the space $\mathcal{F}$ constructed in (4.6) for the $(2+1)$-dimensional problem, but rather, because we have two identical sets of fermion creation operators, it is

$$\widetilde{\mathcal{F}} = \mathcal{F}\otimes\mathcal{F}. \tag{4.11}$$

The space of zero energy states of the $(3+1)$-dimensional gauge theory is just the W-invariant part of $\widetilde{\mathcal{F}}$.

It is easy to identify $r+1$ Weyl-invariant states in $\widetilde{\mathcal{F}}$. Indeed, if

$$Z = \sum_{a,b}\delta_{ab}\overline{\psi}_1^a\overline{\psi}_2^b, \tag{4.12}$$

where δ_{ab} is the W-invariant metric on $\mathbf{t}$, then Z is W-invariant and hence so are the states $Z^p|\Omega\rangle$, for $p = 0,1,\dots,r$. We want to show that there are no other W-invariant states.

For this we use the following interpretation of $\widetilde{\mathcal{F}}$. As in the $(2+1)$-dimensional discussion, we consider the complex torus $\mathbf{T}_{\widehat{G}}\times\mathbf{T}_{\widehat{G}}$ with complex coordinates z^a. [21] Identifying $\overline{\psi}_1^a$ as dz^a and $\overline{\psi}_2^a$ as $d\overline{z}^a$, we see that the basis of $\widetilde{\mathcal{F}}$ given in (4.10) gives a basis for the translationally invariant differential forms on $\mathbf{T}_{\widehat{G}}\times\mathbf{T}_{\widehat{G}}$, or in other words a basis for the de Rham cohomology of $\mathbf{T}_{\widehat{G}}\times\mathbf{T}_{\widehat{G}}$. We thus can identify

$$\widetilde{\mathcal{F}} = \oplus_{p,q=0}^r H^{p,q}(\mathbf{T}_{\widehat{G}}\times\mathbf{T}_{\widehat{G}}). \tag{4.13}$$

[21] One significant difference from the $(2+1)$-dimensional analysis should be pointed out. The space $\mathbf{T}_{\widehat{G}}\times\mathbf{T}_{\widehat{G}}$ (with precisely two factors of $\mathbf{T}_{\widehat{G}}$) was a natural part of that problem, but here has been introduced as an auxiliary tool to get a simple description of the fermion Fock space.

In this interpretation, $Z^p|\Omega\rangle$ is obtained by acting on $|\Omega\rangle$ with p creation operators $\overline{\psi}_1^a$ and p creation operators $\overline{\psi}_2^a$ and hence is a form of type (p, p).

The W-invariant part of $\widetilde{\mathcal{F}}$ is hence the W-invariant part of the de Rham cohomology of $\mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}}$, or in other words it is the de Rham cohomology of $\widehat{\mathcal{N}} = (\mathbf{T}_{\widehat{G}} \times \mathbf{T}_{\widehat{G}})/W$. But $\widehat{\mathcal{N}}$ is a weighted projective space $\mathbf{WCP}^r_{s_0,s_1,\ldots,s_r}$. The de Rham cohomology of a weighted projective space of complex dimension r is $(r+1)$-dimensional. It is generated in fact by the (p,p) forms ω^p, $p = 0, \ldots, r$, where ω is the Kahler class. These correspond to $Z^p|\Omega\rangle$. So we have shown that the space of zero energy states that come by quantizing this component of the moduli space is $(r+1)$-dimensional. Moreover, these states all have the same eigenvalue of $(-1)^F$, since $(-1)^F$ commutes with Z. We will analyze later the overall sign.

We should not in general conclude that (for zero magnetic flux m) $\mathrm{Tr}(-1)^F$ is equal to $\pm(r+1)$. What we have worked out is just the contribution of the identity component $\widehat{\mathcal{N}}_0$ of the moduli space $\widehat{\mathcal{N}}$ of commuting triples. The index must be computed as a sum of the contributions of the $\widehat{\mathcal{N}}_i$.

The contributions of the other components can be worked out without doing any essentially new computation. Let $\vec{U} = (U_1, U_2, U_3)$, be any triple of commuting elements in a component $\widehat{\mathcal{N}}_i$, and let $H_{\vec{U}}$ be the identity component of the subgroup of G that commutes with the U_i. The rank r_i of $H_{\vec{U}}$ is an invariant of $\widehat{\mathcal{N}}_i$, as we explained earlier.

Now, it is always possible to find in each component $\widehat{\mathcal{N}}_i$ a $\vec{U}$ such that $H_{\vec{U}}$ is a simple Lie group. (For example, in the $Spin(n)$ example considered above, we described a point in $\mathcal{N}_1$ with $Spin(n-7)$ as the identity component of the unbroken symmetry group.) Let us call such a group H_i and call its Weyl group W_i. (For given i, there may be more than one possible H_i.) If $\mathcal{N}_i$ were the same as $(\mathbf{T}_{H_i} \times \mathbf{T}_{H_i} \times \mathbf{T}_{H_i})/W_i$, we would simply repeat the above computation and conclude that the contribution of $\mathcal{N}_i$ to the index is $\pm(r_i+1)$. This description of $\mathcal{N}_i$ is not quite correct. A precise version is given in part (4) of Theorem 1.4.1 in [7], and shows that $\mathbf{T}_{H_i}$ must be replaced by a slightly different torus (which does not matter since the zero energy wave functions are invariant under translation on the torus anyway) and W_i by a slightly larger group (which does not matter since the operator R introduced in (4.12) is actually $SO(r)$-invariant, not just W-invariant). So effectively the evaluation of the contribution of $\widehat{\mathcal{N}}_i$ to the index is equivalent to the evaluation of the contribution of the identity component for the group H_i. So indeed each contribution is $\pm(r_i+1)$. We will explain in section 4.3 that the signs are all equal (and we will give a precise framework in which it is natural to set the sign to $(-1)^r$). Hence, the index for zero magnetic flux

m is

$$I(0) = (-1)^r \sum_i (r_i + 1) \tag{4.14}$$

where the sum runs over all components of $\widehat{\mathcal{N}}_i$. According to [5–7], this sum always equals h, the dual Coxeter number of G, in agreement with the prediction described in section 3.

Now we want to include the electric flux e. We want to show that (still for $m = 0$), the index $I(e, 0)$ vanishes for $e \neq 0$. We recall that e is a character of a finite group Γ that acts as in (4.9). If we can show that the action of Γ maps each $\widehat{\mathcal{N}}_i$ to itself (rather than permuting the $\widehat{\mathcal{N}}_i$) then the argument will go through just as in $2 + 1$ dimensions: a state that transforms nontrivially under Γ must have a nonconstant wavefunction on $\widehat{\mathcal{N}}_i$, and hence a positive energy. If Γ acts by nontrivial permutations of the $\widehat{\mathcal{N}}_i$, this conclusion would not follow: in this case, suitable linear combinations of states supported on different components would give zero energy states transforming nontrivially under Γ, implying that $I(e, 0)$ would not always vanish for nonzero e.

That Γ maps each $\widehat{\mathcal{N}}_i$ to itself follows from the following facts:

(1) The commuting triple $\vec{U}$ defines a flat connection A; the Chern-Simons invariant $CS(A)$ of this connection is Γ-invariant mod 1.

(2) Each component of $\widehat{\mathcal{N}}_i$ has a different value of $CS(A)$ mod 1.

The first assertion is essentially a consequence of the computations in section 3.4. Given a flat bundle on $\mathbf{T}^3$ defined by $\vec{U}$ and a gauge transformation defined by $\gamma \in \Gamma$, we can, by using γ as a gluing function in the "time" direction, build up a flat bundle Y over $\mathbf{T}^3 \times \mathbf{S}^1$. The change in $CS(A)$ under γ is the same as the instanton number of Y mod 1. The characteristic class $\widehat{m} \in H^2(\mathbf{T}^3 \times \mathbf{S}^1, \pi_1(G))$ of Y is purely electric, since we started with a bundle on $\mathbf{T}^3$ of zero magnetic flux. The computations in section 3.4 show that the instanton number is an integer for any bundle whose characteristic class is purely electric. So $CS(A)$ is invariant under γ when $m = 0$.

The statement that the different $\widehat{\mathcal{N}}_i$ have different values of $CS(A)$ mod 1 is a theorem in [7] (part (3) of Theorem 1.8.1). As we will presently explain, a sharper version of this statement (also proved in [7]) is actually needed to reconcile the microscopic computation that we are presenting here with the macroscopic arguments of section 3 concerning chiral symmetry.

Though this involves jumping ahead of our story a bit, it is convenient to here explain the analog of the above statements with nonzero magnetic flux

$m \in H^2(\mathbf{T}^3, \pi_1(G))$. Statement (2) holds for arbitrary m [7] (indeed, as will be apparent, this is needed to recover from the microscopic computation our expectations concerning chiral symmetry). But the computations in section 3.4 make clear that statement (1) does not hold – in general, $CS(A)$ is not γ-invariant, since the instanton number on a bundle over $\mathbf{T}^4$ is not always an integer. Clearly, in order to map the individual $\widehat{\mathcal{N}}_i(m)$ to themselves (as opposed to permuting them non-trivially), an element $\gamma \in \Gamma$ must leave $CS(A)$ invariant for bundles of given m. Statement (2) implies that the converse is also true: γ maps the $\widehat{\mathcal{N}}_i(m)$ to themselves if and only if it leaves $CS(A)$ fixed. So the zero energy states are invariant not in general under Γ but under its subgroup Γ' that leaves fixed $CS(A)$. Any character of Γ that is trivial on Γ' is a multiple of the spectral flow character, so we conclude that $I(e, m) = 0$ unless e is a multiple of the spectral flow.

Singularities Of The Moduli Space

In our computations, we have treated the moduli spaces $\widehat{\mathcal{N}}$ as orbifolds. Actually, the description that we have given of the low energy effective action of the theory breaks down near singularities of $\widehat{\mathcal{N}}$. How do we know that there are not additional zero energy states supported near the singularities and hence invisible in our analysis? In what follows, we consider $3+1$ dimensions. (The $(2+1)$-dimensional case has been discussed in [3], and involves a few further details, revolving around the Chern-Simons coupling.)

Suppose first that $G = SU(2)$. In this case, the maximal torus $\mathbf{T}_{\widehat{G}}$ is just a circle $\mathbf{S}$, and the Weyl group is $\mathbf{Z}_2$, acting on a commuting triple (U_1, U_2, U_3) (with all $U_i \in \mathbf{S}$) by $U_i \to U_i^{-1}$. This transformation has eight fixed points where all U_i are ± 1. At these points in the moduli space, additional bose and fermi modes, which generically carry nonzero energy, move down to zero energy, and hence our description of the low energy effective theory breaks down. The fixed points are permuted by the action of the finite group Γ, so they behave in the same way, and it suffices to consider the fixed point at $U_i = 1$.

It is not difficult to give a description of the low energy effective theory that *is* valid near $U_i = 1$. $U_i = 1$ corresponds to the trivial flat connection, $A = 0$. Expanding around $A = 0$, the low energy modes are the modes that are, in a suitable gauge, constant on $\mathbf{T}^3$ (but not necessarily aligned in a Cartan subalgebra). The effective theory of these modes is the theory obtained by dimensional reduction (as opposed to compactification) of the $SU(2)$ super Yang-Mills theory from $3+1$ to $0+1$ dimensions. In other words, the effective theory is the $SU(2)$ "matrix quantum mechanics" in $0+1$ dimensions, with three matrices (coming from dimensional reduction of the three components of A) plus their superpartners, and with four supercharges.

Part of the string duality picture is that matrix quantum mechanics with 16 supercharges has zero-energy bound states, but matrix quantum mechanics with fewer than 16 supercharges lacks them. Absence of zero energy normalizable states with fewer than 16 supercharges has not been fully proved, but there are many partial results [18–21]. So in our case, for $G = SU(2)$, we expect no zero-energy states localized near the singularities.

If G has rank higher than one, we must argue inductively. Singularities of $\widehat{\mathcal{N}}$ are loci $\widehat{\mathcal{N}}_H$ on which various subgroups H of G are restored. If G has rank r and H has rank s, then the relevant part of the effective theory near $\widehat{\mathcal{N}}_H$ (but away from singularities involving groups of rank higher than s) is the matrix quantum mechanics with gauge group H and four supercharges. Given the presumed absence of zero-energy bound states in this matrix quantum mechanics for all H, one would argue inductively in s that none of the singular loci support bound states.

One would also like to show that the zero-energy states that we have seen in the orbifold quantization of $\widehat{\mathcal{N}}$ do not fail to be normalizable because of their behavior near the singularities. For this, one must show that in the matrix quantum mechanics, there are supersymmetric zero-energy states that fail to be normalizable because of their behavior at infinity on the Coulomb branch, and are in correspondence with the zero-energy states that we have seen on $\widehat{\mathcal{N}}$. (Infinity on the Coulomb branch in the matrix quantum mechanics matches on to the smooth part of $\widehat{\mathcal{N}}$ in the quantization of the orbifold.) This is really a consequence of absence of normalizable zero energy states. Let ψ_0 be a smooth wavefunction that does coincide near infinity with a supersymmetric zero-energy state constructed on $\widehat{\mathcal{N}}$ but is not necessarily a solution of the Schrodinger equation everywhere. Then, if Δ is the Hamiltonian of the matrix quantum mechanics, we do not necessarily have $\Delta\psi_0 = 0$, but at least $\Delta\psi_0$ is an $\mathbf{L}^2$ state (since ψ_0 is a zero-energy state near infinity). Then we try to obey $\Delta\psi = 0$ with $\psi = \psi_0 + \psi_1$ and ψ_1 a normalizable state (whose addition will not modify the behavior at infinity). The equation we have to obey is $\Delta\psi_1 = -\Delta\psi_0$, and as $-\Delta\psi_0$ is a vector in the Hilbert space Y of $\mathbf{L}^2$ wavefunctions, and Δ is invertible in this space, a unique $\psi_1 \in Y$ obeying the equation does exist.

4.3 Chiral Symmetry Breaking

Now we want to extend this analysis to compute the dimensions h^k of the cohomology groups $H^k(Q)$, as defined in section 3.

The key point is to compute the R-charge of the zero-energy vacuum states obtained by quantizing $\widehat{\mathcal{N}}_i$. Each mode of the $\overline{\lambda}$ operator has charge 1. We have to include the charge of the states in the filled Dirac sea. Let N_-, N_0, and N_+ be, formally, the number of negative, zero, and positive eigenvalues of the three-dimensional Dirac operator acting on $\overline{\lambda}$. Of course, N_- and N_+ are infinite, while, in expanding around a flat connection A in the component $\widehat{\mathcal{N}}_i$, one has $N_0 = 2r_i$.

The charge of the Dirac sea is formally N_-. To regularize this expression, we subtract a multiple of $N = N_- + N_0 + N_+$, which formally is a constant (the total number of modes of the $\overline{\lambda}$ field) and which is actually zero with zeta function regularization. We partly regularize the charge of the Dirac sea by subtracting $\frac{1}{2}N$ from N_-, to give $\frac{1}{2}(N_- - N_+) - \frac{1}{2}N_0 = \frac{1}{2}(N_- - N_+) - r_i$. A regularized version of $\frac{1}{2}(N_+ - N_-)$ is given by the Atiyah-Patodi-Singer η invariant of the flat connection A, and according to the APS theorem, this η invariant is equal to $2h\,CS(A)$. The dual Coxeter number h appears because the gluino fields take values in the adjoint representation. We therefore identify the R-charge of the state $|\Omega\rangle$ as $-2h\,CS(A) - r_i$. Since Z has R-charge 2, the zero-energy states $Z^s|\Omega\rangle$ have R-charges

$$-2h\,CS(A) - r_i, -2h\,CS(A) - r_i + 2, -2h\,CS(A) - r_i + 4, \ldots, -2h\,CS(A) + r_i. \tag{4.15}$$

In particular, for the identity component $\widehat{\mathcal{N}}_0$, we can take $A = 0$ and hence $CS(A) = 0$; also the rank of $\widehat{\mathcal{N}}_0$ is $r_0 = r$. So the R-charges of these states are congruent to r mod 2, and as the R-charge is the fermion number mod 2, these states have $(-1)^F = (-1)^r$. This is the reason for the sign choice that was claimed in section 3. (The values of the $CS(A)$ are such that the sign is the same for all components [7]; this is a special case of what we say below in comparing to the predictions of section 3 concerning chiral symmetry breaking.)

The use of the APS theorem to evaluate η assumes that there exists a four-manifold B of boundary $\mathbf{T}^3$ over which the connection A extends. B certainly exists if the G-bundle X on which A is a connection is topologically trivial, for then, writing $\mathbf{T}^3 = \mathbf{T}^2 \times \mathbf{S}^1$, we can take $B = \mathbf{T}^2 \times D$ with D a disc of boundary S^1. This covers the case that G is simply-connected. Even if G is not simply connected, as long as its fundamental group is cyclic, we can always assume that the magnetic flux $\widehat{m}$ is a pullback from $\mathbf{T}^2$ in some way of writing $\mathbf{T}^3 = \mathbf{T}^2 \times \mathbf{S}^1$, and then again we can take $B = \mathbf{T}^2 \times D$. With more care, B can be constructed also in the remaining case $G = Spin(4n)/\mathbf{Z}_2 \times \mathbf{Z}_2$.

For any group, the Chern-Simons invariant vanishes on the identity component $\widehat{\mathcal{N}}_0$, because a flat connection with trivial holonomies is invariant under a reflection of one axis in $\mathbf{T}^3$; such a reflection reverses the sign of the Chern-Simons invariant. A more general result along these lines is as follows. Suppose a component $\widehat{\mathcal{N}}_i$ of $\widehat{\mathcal{N}}$ contains a point that labels the commuting triple (U_1, U_2, U_3), where $U_3^k = 1$ for some integer k. (As we will review in section 5, this condition is always obeyed by each of the $\widehat{\mathcal{N}}_i$.) Then the Chern-Simons invariant of $\widehat{\mathcal{N}}_i$ is an integer multiple of $1/k$. To prove this, one considers the four-manifold $B = \mathbf{T}^2 \times W$ where W is a k-holed sphere. Because $U_3^k = 1$, there is a flat connection on B whose holonomies on $\mathbf{T}^2$ are U_1 and U_2 and whose holonomy around any of the boundary circles of W is U_3. The boundary of B is k copies of $\mathbf{T}^3$, with holonomies (U_1, U_2, U_3) on each, so we have

$$k\,CS(A) = \frac{1}{8\pi^2}\int_B \mathrm{tr} F \wedge F = 0 \bmod 1. \tag{4.16}$$

Examples

Our prediction in section 3 was that there is for every value of the R-charge k such that $k + r$ is even a unique zero-energy state. From (4.15), we see that this puts a severe restriction on the values of $CS(A)$ for the different $\widehat{\mathcal{N}}_i$ (and in particular, those values must be distinct or there would be at least two zero-energy states of the same R-charge). The fact that (4.15) does lead after evaluation of the $CS(A)$ to the expected spectrum of R-charges for the zero-energy states is Theorem 1.8.2 in [7]. Let us give some examples of how this works out for specific cases. (See section 5 for a more elaborate example with gauge group E_8.)

For $G = SU(n)$, the dual Coxeter number is $h = n$, so the R-charge is defined mod $2n$. The only component of $\widehat{\mathcal{N}}$ is the identity component $\widehat{\mathcal{N}}_0$. For this component, $r_0 = n - 1$, so according to (4.15), the values of the R-charge for the zero-energy states for $SU(n)$ are

$$-n+1, -n+3, -n+5, \ldots, n-1. \tag{4.17}$$

Thus, $h^k(SU(n)) = 1$ if k is congruent to $n-1$ modulo 2, and zero otherwise. This is in agreement with expectations.

Similarly for $G = Sp(n)$, the dual Coxeter number is $h = n + 1$, so the R-charge is defined mod $2n + 2$. $\widehat{\mathcal{N}}_0$ is the only component, with $r_0 = n$. So the R-charges of zero-energy states are

$$-n, -n+2, -n+4, \ldots, n. \tag{4.18}$$

This agrees with expectations. These examples were described in [1].

To give an example with more than one component, let us take $G = Spin(t)$. The dual Coxeter number is $h = t - 2$, so the R-charge is defined mod $2h = 2t - 4$. If, for example, $t = 2n$ is even, then $\mathcal{N}_0$ has $r_0 = n$ and contributes zero-energy states of charges

$$-n, -n+2, -n+4, \ldots, n. \tag{4.19}$$

There is one additional component $\mathcal{N}_1$. A commuting triple (U_1, U_2, U_3) parametrized by a point in $\mathcal{N}_1$ was described in section 4.2; it has $U_3^2 = 1$, so the Chern-Simons invariant for $\mathcal{N}_1$ is 0 or 1/2. In fact, it is 1/2; this will be explained in section 5. The rank r_1 is (for $t = 2n$) $n - 4$. So setting $-2h\,CS(A) = -\frac{1}{2}(2t-4) = -2n+2$, the charges of zero-energy states obtained by quantizing $\mathcal{N}_1$ are

$$-3n+6, -3n+8, \ldots, -n-2. \tag{4.20}$$

If we combine (4.19) and (4.20), we see (since $-3n+6$ is congruent mod $2h = 2t - 4 = 4n - 4$ to $n + 2$) that there is one zero-energy state for every even integer mod $2h$, as expected based on chiral symmetry breaking from the analysis in section 3.

The case that $t = 2n + 1$ is similar. We still have $r_0 = n$, so the charges of zero-energy states from quantization of $\mathcal{N}_0$ are still given by (4.19). But we now have $2h = 2t - 4 = 4n - 2$ and $r_1 = n - 3$, so the formula (4.20) for charges of zero-energy states from quantization of $\mathcal{N}_1$ is replaced by

$$-3n+4, -3n+6, \ldots, -n-2. \tag{4.21}$$

Again, there is precisely one zero-energy state for every even charge mod $2h$.

4.4 Incorporation Of The Magnetic Flux

We will now discuss the incorporation in this discussion of the magnetic flux $m \in H^2(\mathbf{T}^2, \pi_1(G))$ or $H^2(\mathbf{T}^3, \pi_1(G))$.

In $2+1$ dimensions, for any m, the moduli space $\widehat{\mathcal{M}}(m)$ of flat bundles is always a weighted projective space [15–17]. Hence, $H^{0,i}(\widehat{\mathcal{M}}(m))$ vanishes except for $i = 0$, and $H^{0,0}(\widehat{\mathcal{M}}(m))$ is one-dimensional. So $\mathrm{Tr}(-1)^F = 1$ for all m.

To approach this in a somewhat more intuitive way, observe that for any m, it is possible to find a flat connection that breaks G down to a simple Lie group H [22]. We give a simple example below. Then the construction of zero-energy states for G and m is equivalent to the analysis we have already performed for gauge group H and $m = 0$. [22] So since $\mathrm{Tr}\,(-1)^F = 1$ for all H at $\widehat{m} = 0$, it equals 1 for all G and all m.

Now, to complete the verification of the predictions in section 2, let us show, still in $2+1$ dimensions, that $I(e,m) = 0$ for all nonzero e. First of all, a flat connection on a G-bundle with $m \neq 0$ can be explicitly described by a pair of monodromies U_1, U_2 that commute in G but which, if lifted to the universal cover $\widehat{G}$ of G, do not quite commute. Instead, as in (2.8), they obey

$$U_1 U_2 = U_2 U_1 \widehat{f}, \tag{4.22}$$

where $\widehat{f}$ is a central element of $\widehat{G}$ that is determined by m. Thus U_1 and U_2 are "almost commuting." The residual symmetry group Γ, of which the electric flux e is a character, acts as in (4.3) by

$$U_i \to a_i U_i \tag{4.23}$$

with a_i central elements of $\widehat{G}$. In particular, Γ always acts by translations on the appropriate torus and hence leaves invariant the only zero energy state. This leads to our claimed result that $I(e,m) = 0$ for $e \neq 0$, and $I(0,m) = 1$.

$3+1$ *Dimensions*

In $3+1$ dimensions, things are slightly more elaborate. First of all, for $m \neq 0$, just as for $m = 0$, there are several components $\widehat{\mathcal{N}}_i(m)$ of the moduli space $\widehat{\mathcal{N}}(m)$. On each of these, the unbroken subgroup of G is of rank r_i for some r_i. At some point on the moduli space, the unbroken group is a simple group H_i of rank r_i (more than one H_i may appear at different points on the same $\widehat{\mathcal{N}}_i(m)$). The contribution of $\widehat{\mathcal{N}}_i(m)$ to $\mathrm{Tr}\,(-1)^F$ is hence $\pm(r_i+1)$. To get the expected result $\mathrm{Tr}\,(-1)^F = (-1)^r h$, we hence need

$$\sum_i (r_i + 1) = h, \tag{4.24}$$

and moreover the signs should agree. The assertion (4.24) has been proved in Theorem 1.5.1 of [7], and the assertion about the signs is a special case of Theorem 1.8.2 of that paper, which asserts that the values of the Chern-Simons invariants of the various components of $\widehat{\mathcal{N}}_i(m)$ are such that there is precisely one zero-energy state for every even or every odd integer mod $2h$.

[22] Actually, the moduli space $\widehat{\mathcal{M}}(m)$ for G with magnetic flux m is $\mathbf{T}'/W'$ where $\mathbf{T}'$ is a slightly different torus from $\mathbf{T}_{\widehat{H}} \times \mathbf{T}_{\widehat{H}}$ and W' is a slightly larger group than W_H. See Theorem 1.3.1 of [7] for a precise statement. Neither of these points affects the identification of zero-energy states since the zero-energy states are constants on $\mathbf{T}'$ anyway, and the only W_H-invariant state in the quantization has full $SO(r)$ symmetry.

As for the further prediction that $I(e,m) = 0$ unless e is a multiple of the spectral flow, the reason for this has already been explained in section 4.2: an element of γ leaves fixed the individual $\widehat{\mathcal{N}}_i(m)$ if and only if it leaves fixed the Chern-Simons invariant of a bundle of magnetic flux m.

Specialization To $SU(n)/\mathbf{Z}_n$

In the rest of this section, we examine in detail how these predictions work out for $G = SU(N)/\mathbf{Z}_n$. For details concerning the other classical groups, see section 5.3.

Since the fundamental group of G is the cyclic group $\mathbf{Z}_n$, we can pick a way of writing $\mathbf{T}^3 = \mathbf{T}^2 \times \mathbf{S}^1$ such that m is a pullback from the first factor. Let us first look at the basic example that the magnetic flux on $\mathbf{T}^2$ is "1." This means that the holonomies U_1, U_2, U_3 of a flat connection, if lifted to $SU(n)$, obey

$$U_1 U_2 = U_2 U_1 \exp(2\pi i/n) \tag{4.25}$$

and

$$U_i U_3 = U_3 U_i \text{ for } i = 1, 2. \tag{4.26}$$

$SU(n)$ elements U_1, U_2 obeying (4.25) exist and are unique up to conjugation. Moreover, such a U_1 and U_2 commute only with the center of $SU(n)$ so

$$U_3 = \exp(2\pi i p/n) \tag{4.27}$$

for some integer p. The moduli space $\widehat{\mathcal{N}}(m)$, for this m, thus consists of n points, with U_1 and U_2 given by some fixed solution of (4.25) and U_3 as in (4.27).

Each such point is a connected component $\widehat{\mathcal{N}}_p(m)$ of $\widehat{\mathcal{N}}(m)$, for $p = 1, \ldots, n$. The expected formula $\sum_p (r_p + 1) = h$ is obeyed with p taking n possible values, $r_p = 0$ for all p, and $h = n$.

Now let us study the action of Γ. An element (a_1, a_2, a_3) of Γ, with each a_i being an element of the center of $SU(n)$, say

$$a_i = \exp(2\pi i \alpha_i/n), \quad \alpha_i \in \{0, 1, \ldots, n-1\}, \tag{4.28}$$

acts on the U_i by

$$U_i \to a_i U_i. \tag{4.29}$$

The action of a_1 and a_2 is trivial; U_1 and U_2 are mapped to a different solution of (4.25), but any two such solutions are equivalent up to conjugation. The action of a_3 permutes the $\widehat{\mathcal{N}}_p$ by $p \to p + \alpha_3$. The condition for γ to leave fixed the individual $\widehat{\mathcal{N}}_p$ is thus precisely that $\alpha_3 = 0$. Next, let us find the condition for γ to leave fixed the Chern-Simons invariant of a connection on $\mathbf{T}^3$. Given a connection A on a G-bundle X over $\mathbf{T}^3$, we use γ as a gluing function in the time direction to build a G-bundle $\widehat{X}$ over $\mathbf{T}^4 = \mathbf{T}^3 \times \mathbf{S}^1$. The change under γ in the Chern-Simons invariant of A is the instanton number of $\widehat{X}$ mod 1. The bundle $\widehat{X}$ has a characteristic class $\widehat{m}$ whose magnetic part (the restriction to $\mathbf{T}^3$) is the original m, while the electric part is determined by γ. As we have computed in (3.38), the instanton number mod 1 of $\widehat{X}$ is

$$\Delta'(m,\gamma) = \frac{\mathrm{Pf}(\widehat{m})}{n} = \frac{\widehat{m}_{12}\widehat{m}_{34} + \widehat{m}_{23}\widehat{m}_{14} + \widehat{m}_{31}\widehat{m}_{24}}{n}. \tag{4.30}$$

We have $\widehat{m}_{i4} = \alpha_i$ for $i = 1, 2, 3$, and $\widehat{m}_{ij} = m_{ij}$ for $i, j = 1, 2, 3$. Finally, the only nonzero m_{ij} is $m_{12} = 1$. So we get

$$\Delta'(m,\gamma) = \frac{\alpha_3}{n}. \tag{4.31}$$

So the condition to have $\Delta'(m,\gamma) = 0$ is $\alpha_3 = 0$, which, as we aimed to prove, is the same as the condition for γ to leave fixed the individual $\widehat{\mathcal{N}}_p$.

Since the Chern-Simons invariant of $\widehat{\mathcal{N}}_0(m)$ is zero, the Chern-Simons invariant of $\widehat{\mathcal{N}}_p(m)$ is equal to the change in Chern-Simons invariant under a gauge transformation with $\alpha_3 = p$ and hence is

$$CS(A) = \frac{p}{n}. \tag{4.32}$$

The state obtained by quantizing $\widehat{\mathcal{N}}_p(m)$ hence has R-charge $-2h\, CS(A) = -2n(p/n) = -2p$, and so, taking all values of p, there is precisely one zero-energy state of every even charge mod $2n$, as expected.

It is not difficult to extend this to an arbitrary m, still keeping $G = SU(n)/\mathbf{Z}_n$. There is no essential loss in assuming that the only nonzero component of m is m_{12}, which we take to be $q \in \mathbf{Z}_n$. The conditions for a flat connection are now

$$U_1U_2 = U_2U_1 \exp(2\pi i q/n) \tag{4.33}$$

as well as

$$U_iU_3 = U_3U_i, \text{ for } i = 1, 2. \tag{4.34}$$

Let u be the greatest common divisor of n and q, and $v = n/u$. Then (4.33) has an irreducible solution in $v \times v$ matrices A_1, A_2. We embed $SU(u) \times SU(v)$

in $SU(n)$ such that the $\mathbf{n}$ of $SU(n)$ decomposes under $SU(u) \times SU(v)$ as $\mathbf{u} \otimes \mathbf{v}$. The general solution of (4.33) and (4.34) is up to conjugacy

$$\begin{aligned} U_1 &= M_1 \times A_1 \\ U_2 &= M_2 \times A_2 \\ U_3 &= M_3 \times \exp(2\pi i p/n), \end{aligned} \tag{4.35}$$

where the M_i are commuting elements of $SU(u)$ and $p \in \{0, 1, \ldots, v-1\}$; we identify $p \cong p + v$ since a factor $\exp(2\pi i/u)$ can be absorbed in M_3. The moduli space $\widehat{\mathcal{N}}(m)$ has v components $\widehat{\mathcal{N}}_p(m)$, labeled by p which appears in (4.35), and each component has unbroken subgroup of rank $r = u - 1$. The sum $\sum_i (r_i + 1)$ thus equals $uv = n$, as expected.

By repeating the above computation, we see that $\gamma = (a_1, a_2, a_3)$ changes the instanton number by

$$\Delta'(m, \gamma) = \frac{q\alpha_3}{n} \bmod 1. \tag{4.36}$$

The condition for this to vanish is that α_3 should be divisible by v. γ acts on p by $p \to p + \alpha_3$, so likewise, $\widehat{\mathcal{N}}_p(m)$ is mapped to itself if and only if α_3 is divisible by v. (4.36) also means, since $\widehat{\mathcal{N}}_0(m)$ has Chern-Simons invariant zero, that $\widehat{\mathcal{N}}_p(m)$ has Chern-Simons invariant

$$CS(A) = qp/n, \tag{4.37}$$

which, as p varies, ranges over all integer multiples of $1/v$. From this, one can again show that there is one zero-energy state of every even or every odd R-charge.

4.5 Outer Automorphisms

Finally, we wish to give some examples of verifying the predictions given in section 3.5 for gauge groups G with a group C of outer automorphisms. As we recall, C is realized as a global symmetry group, and we want to check the claim that the vacuum states of the infinite volume theory are all C-invariant.

We gave two criteria in section 3.5. One is to compute $I(c) = \operatorname{Tr} c(-1)^F$. The prediction is that it is independent of c. To verify this, it suffices to check that the vacuum states found in the above computation are all c-invariant. Then, computing the trace as a sum over these states, we get a c-invariant result.

To prove that the vacua are C-invariant, the key point (rather as in the analysis of electric flux) is to show that each component of $\mathcal{N}$ is mapped to itself by C (as opposed to C permuting the components). Indeed, the Chern-Simons three-form of a connection is defined using a quadratic form on the Lie algebra that is both G-invariant and C-invariant, so in particular the Chern-Simons invariant of a flat connection is C-invariant. Since the different components $\mathcal{N}_i$ of $\mathcal{N}$ have different Chern-Simons invariants, each is mapped to itself by C.

C may act nontrivially on both the boson and fermion zero modes that are encountered in quantizing $\mathcal{N}_i$. The action on the boson zero modes is irrelevant, since the zero energy states are independent of the bosons. The action on the fermions comes from an orthogonal transformation on the Lie algebra index of the fermion zero modes λ^a_α, $\overline{\lambda}^a_{\dot\alpha}$. The fermion bilinear Z introduced in (4.12) is not just Weyl-invariant but invariant under all orthogonal transformations of the Lie algebra of the maximal torus, and hence in particular is C-invariant. So the zero energy states are all C-invariant, as expected.

The Second Criterion

The second criterion for C to be unbroken involved introducing a bundle X_c twisted by an element of c and computing the index I_c for the supersymmetric theory quantized on such a bundle. We will verify the predictions in several representative cases.

First we consider the case that $G = SU(n)$ and G' is an extension of G by the complex conjugation automorphism c. Thus, there is an exact sequence

$$1 \to G \to G' \to C \to 1, \tag{4.38}$$

where $C = \mathbf{Z}_2$ is generated by c.

We work on the G' bundle X_c over $\mathbf{T}^3$ described in section 3.5. This means that, writing $\mathbf{T}^3 = \mathbf{S}^1 \times \mathbf{T}^2$, the fields are conjugated by c in going around the $\mathbf{S}^1$. A flat connection on X_c has commuting holonomies $(U_1, U_2, U_3) \in G'$ whose images in C are $(c, 1, 1)$. There are two components of the moduli space $\widehat{\mathcal{N}}$ of commuting triples. One component $\widehat{\mathcal{N}}_0$ has a representative with $(U_1, U_2, U_3) = (c, 1, 1)$; the unbroken group H_0 is $SO(n)$. The second component $\widehat{\mathcal{N}}_1$ has a representative with $U_1 = c$, $U_2 = \mathrm{diag}(-1, -1, 1, 1, \dots, 1)$, $U_3 = \mathrm{diag}(1, -1, -1, 1, 1, \dots, 1)$. The unbroken subgroup is $H_1 = SO(n-3)$. If we let r_i denote the rank of H_i for $i = 0, 1$, then the expected formula $\sum_i (r_i + 1) = h$ becomes $(r_0 + 1) + (r_1 + 1) = n$. This holds for all n, even or odd.

Now let us consider $G = Spin(n)$, and let G' be the extension of G by a reflection c of one of the coordinates in the n-dimensional vector representation of $Spin(n)$. Thus, in that representation, we can think of c as the diagonal matrix $c = \mathrm{diag}(-1, 1, 1, \dots, 1)$. We again work on a bundle twisted by c in going around the first factor in $\mathbf{T}^3 = \mathbf{S}^1 \times \mathbf{T}^2$. The moduli space $\widehat{\mathcal{N}}$ of commuting triples (U_1, U_2, U_3) in G' that map to $(c, 1, 1)$ in C has again two components. One component $\widehat{\mathcal{N}}_0$ has a representative with $U_1 = c$, $U_2 = U_3 = 1$, and unbroken group $H_0 = SO(n-1)$. To describe the second component, note that for $n = 6$, one can make a commuting triple that has the desired image in C by taking the U_i to equal diagonal matrices V_i whose diagonal elements are all triples of the numbers ± 1 except $(1, 1, 1)$ and $(-1, 1, 1)$. Then if we set $U_i = V_i \oplus 1$, with 1 the identity element of $SO(n-6)$, we get a commuting triple in a component $\widehat{\mathcal{N}}_1$ with $H_1 = SO(n-6)$. The relation $\sum_i (r_i + 1) = h$ becomes $(r_0 + 1) + (r_1 + 1) = n - 2$, and this holds for all n, even or odd.

5 More On Commuting Triples

In this section, we will give more information about commuting triples in a simple Lie group G. Following some of the ideas in [5–7] as well as comments by A. Borel, R. Friedman and J. Morgan, we will explain the formula $\sum_i (r_i + 1) = h$, where the sum runs over the components of the moduli space of commuting triples. The goal is not to give proofs (which can be found in the references) but to present some facts that some readers may find helpful. We will also explain how to concretely construct a commuting triple in each component, and how to compute its Chern-Simons invariant.

5.1 Simply Connected Case

We assume first that the compact, simple Lie group G is also connected and simply-connected.

The moduli space $\mathcal{M}$ of commuting pairs U_1, U_2 in G is connected. For a generic point $\vec{U} = (U_1, U_2)$ in $\mathcal{M}$, the subgroup $H_{\vec{U}}$ of G consisting of elements that commute with the U_i is connected.

Consider a commuting triple (U_1, U_2, U_3), and let $\vec{U}$ still denote the pair (U_1, U_2). Obviously, $U_3 \in H_{\vec{U}}$, so if $H_{\vec{U}}$ is connected, we can continuously deform U_3 to the identity while preserving the fact that it commutes with U_1 and U_2. Then, since $\mathcal{M}$ is connected, we can continuously deform U_1 and

U_2 to the identity while preserving the fact that they commute.

So in short, a commuting triple (U_1, U_2, U_3) can represent a point in a non-identity component of the moduli space $\mathcal{N}$ of commuting triples only if $H_{\vec{U}}$ is disconnected and U_3 is in a non-identity component of $H_{\vec{U}}$.

It turns out that the moduli space $\mathcal{M}$ of commuting pairs is an orbifold, and has orbifold singularities precisely at those points at which $H_{\vec{U}}$ is disconnected. This is somewhat surprising, as one might expect a singularity whenever $H_{\vec{U}}$ (which generically is abelian) becomes nonabelian, whether it is connected or not. But it turns out that a singularity in the complex structure of $\mathcal{M}$ only arises if $H_{\vec{U}}$ is disconnected. The group of components of $H_{\vec{U}}$ is always a cyclic group $\mathbf{Z}_k$ for some k, and the singularity of $\mathcal{M}$ is an orbifold singularity $\mathbf{C}^r/\mathbf{Z}_k$, with r the rank of G and some linear action of $\mathbf{Z}_k$ on $\mathbf{C}^r$.

In fact, $\mathcal{M}$ is always a weighted projective space $\mathbf{WCP}^r_{s_0,s_1,\ldots,s_r}$ with the weights being 1 and the coefficients of the highest coroot of G [15–17]. In particular, one always has

$$\sum_i s_i = h. \tag{5.1}$$

We can describe $\mathbf{WCP}^r_{s_0,s_1,\ldots,s_r}$ via homogeneous coordinates z_i of weight s_i. The moduli space is the quotient of $\mathbf{C}^{r+1}$ minus the origin by $\mathbf{C}^*$, where $\mathbf{C}^*$ acts by

$$z_i \to \lambda^{s_i} z_i, \quad \text{for } \lambda \in \mathbf{C}^*. \tag{5.2}$$

An orbifold singularity arises if for some set of z_i, not all zero, one has $z_i = \lambda^{s_i} z_i$. This happens for $\lambda = \exp(2\pi i u/k)$ (with relatively prime integers u and k) precisely if $z_i = 0$ unless k is a divisor of s_i.

Before explaining the general picture, let us work out what happens for the classical groups. For $G = SU(n)$ or $Sp(n)$, the weights are all $s_i = 1$. $\mathcal{M}$ is an ordinary projective space, and in particular is smooth. So $H_{\vec{U}}$ is always connected, and $\mathcal{N}$ has only the identity component $\mathcal{N}_0$.

Now let us consider $G = SO(2n)$. ($SO(2n+1)$ behaves quite similarly.) The weights are $1, 1, 1, 1, 2, 2, \ldots, 2$ with four 1's and the rest 2's. An orbifold singularity arises precisely if $z_1 = \cdots = z_4 = 0$. It is a $\mathbf{Z}_2$ orbifold singularity. On this locus, U_1 and U_2 are parametrized by $n-4$ complex parameters (the remaining z_i modulo scaling by $\mathbf{C}^*$), so the unbroken group has rank $n-4$. So for $SO(2n)$, $\mathcal{N}$ has the identity component $\mathcal{N}_0$, of rank $r_0 = n$, and a second component $\mathcal{N}_1$, of rank $r_1 = n-4$, confirming that the components considered in section 4.2 are the only ones.

Now let us work out the case of a general G. We fix an integer k which is a divisor of some of the s_i, and we let $\mu(k)$ be the number of s_i that are divisible by k. Then $\mu(k)-1$ is the complex dimension of the subspace $\mathcal{M}_k$ of $\mathcal{M}$ on which there is a $\mathbf{Z}_k$ orbifold singularity. Hence the rank $r(k)$ of a component of $\mathcal{N}$ for which $U_1, U_2 \in \mathcal{M}_k$ is $r(k)=\mu(k)-1$. For $U_1, U_2 \in \mathcal{M}_k$, the unbroken group $H_{\vec{U}}$ has a group of components that is $\mathbf{Z}_k$. U_3 could lie in any component of $H_{\vec{U}}$, but to avoid multiple-counting, we want to consider only components of $\mathcal{N}$ that could not be constructed using a smaller value of k. For this, we assume that U_3 lies in a component of $H_{\vec{U}}$ whose image in $\mathbf{Z}_k$ is a generator of $\mathbf{Z}_k$; the number of such components is $\phi(k)$, the number of integers mod k that are prime to k.

The contribution to $\sum_i(r_i+1)$ from a component of $\mathcal{N}$ associated with a $\mathbf{Z}_k$ orbifold singularity in $\mathcal{M}$ is hence $\phi(k)(r(k)+1)=\phi(k)\mu(k)$. Since $h=\sum_i s_i$, to prove that $\sum_i(r_i+1)=h$, we need

$$\sum_k \phi(k)\mu(k)=\sum_i s_i. \tag{5.3}$$

In fact, for each s_i, one has the elementary number theory formula

$$\sum_{k|s_i} \phi(k)=s_i. \tag{5.4}$$

Summing (5.4) over i and using the definition of $\mu(k)$ as the number of i such that $k|s_i$, we arrive at (5.3).

Explicit Construction

Now we want to give some indications of how to explicitly describe a commuting triple in each component and compute the Chern-Simons invariants.

If one selects an integer k and omits from the extended Dynkin diagram of G all nodes whose label s_i is not divisible by k, one is left with the Dynkin diagram of a Lie group W of rank $r-\mu(k)$. A fact that is not obvious but is easy to check by examining all the examples (using the classification of simple Lie groups and their extended Dynkin diagrams) is that this group is always a product $W=SU(w_1)\times SU(w_2)\times\cdots\times SU(w_s)$ of SU groups of various ranks. (A more conceptual explanation is in [7].)

This does not mean that W is a subgroup of G. Rather, G contains a subgroup W/D, where D is a cyclic subgroup of the center $Q=\mathbf{Z}_{w_1}\times\mathbf{Z}_{w_2}\times\cdots\times\mathbf{Z}_{w_s}$ of W. D always has the property that, if one omits all factors in Q but one, D projects surjectively onto that factor.

Hence if f is a generator of D, and $U_1, U_2 \in W$ are such that

$$U_1 U_2 = U_2 U_1 f, \tag{5.5}$$

then the subgroup of W that commutes with U_1 and U_2 is only the center.[23] In G, U_1 and U_2 commute. For any $f' \in Q$, (U_1, U_2, f') is a commuting triple. For suitable f', we get the commuting triples described above.

Since (U_1, U_2, f') can be regarded as elements of W, which is a product of SU groups, the Chern-Simons invariant of this commuting triple can be evaluated using the explicit computation in section 4.4 for $SU(n)$. This is analogous to what was done in section 3.4, where the evaluation of the spectral flow for any G was reduced to a computation for $SU(n)$.

Let us carry out this procedure for an illustrative example. We set $G = E_8$ and $k = 5$. There is a single node on the E_8 Dynkin diagram of weight divisible by (and in fact equal to) 5; if we omit it, we are left with the Dynkin diagram of $W = SU(5) \times SU(5)$, whose center is $Q = \mathbf{Z}_5 \times \mathbf{Z}_5$. The decomposition of the adjoint representation of E_8 under W is

$$\mathbf{248} = (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}). \tag{5.6}$$

The subgroup D of Q that acts trivially on the $\mathbf{248}$ is generated by $f = \exp(2\pi i/5) \times \exp(4\pi i/5) \in SU(5) \times SU(5)$. The solution U_1, U_2 of (5.5) is unique up to conjugation. Upon setting $f' = \exp(2\pi i b/5) \times 1$ with $b \in \{1, 2, 3, 4\}$, we get a commuting triple (U_1, U_2, f'). This triple completely breaks E_8 and hence has rank zero, as expected since only a single weight on the E_8 Dynkin diagram is divisible by 5. To verify that the triple completely breaks E_8, it suffices to note that f' breaks E_8 to $SU(5) \times SU(5)$ (as is clear from (5.6)) and that U_1 and U_2 completely break $SU(5) \times SU(5)$.

To compute the Chern-Simons invariant, we just have to compute the Chern-Simons invariant of the triple $(U_1, U_2, f') \in SU(5) \times SU(5)$. Since f' is in the first factor, this is the same as the Chern-Simons invariant of $(U_1, U_2, \exp(2\pi i b/5))$ in $SU(5)$, and according to (4.32) is $b/5$. While we have explained this for a particular G and a particular k, in general, the Chern-Simons invariants of all of the components of $\widehat{\mathcal{N}}$ can be computed in this way.

For components of $\widehat{\mathcal{N}}$ built starting with a divisor k, the Chern-Simons invariants turn out to be, by a computation similar to the case that was just explained, b/k where b ranges over the integers prime to k mod k. With this

[23] In any given $SU(w)$ factor of W, the equation reads $U_1 U_2 = U_2 U_1 \exp(2\pi i p/w)$, where p is prime to w. This follows from the fact that D projects surjectively onto the center of $SU(w)$.

information and the ranks $r(k) = \mu(k) - 1$, the R-charges of the zero-energy states can be computed. As stated in Theorem 1.8.2 of [7], this leads to a spectrum of R-charges that agrees with the expectations based on chiral symmetry breaking. The results for E_8 are summarized in the table. For other simply-connected groups, the results are similar though (since fewer values of k enter) perhaps less elaborate. The results for classical groups were described in section 4.3.

Table 1: This table shows the divisors k for E_8, the ranks of the associated components of $\mathcal{N}$, their Chern–Simons (CS) invariants, and the R-charges of zero energy vacuum states obtained by quantizing them. Every even integer mod $2h = 60$ appears precisely once in the list of R-charges.

k	Rank	CS Invariants	R-Charges of Vacua
1	8	0	−8, −6, −4, −2, 0, 2, 4, 6, 8
2	4	1/2	26, 28, 30, 32, 34
3	2	1/3, 2/3	18, 20, 22, 38, 40, 42
4	1	1/4, 3/4	14, 16, 44, 46
5	0	1/5, 2/5, 3/5, 4/5	12, 24, 36, 48
6	0	1/6, 5/6	10, 50

For one more example, take $G = Spin(2n)$ and $k = 2$. Omitting all nodes of the extended Dynkin diagram that have weight divisible by (and in fact equal to) 2, we are left with the Dynkin diagram of $W = SU(2)^4$. The subgroup of G is in fact $W/\mathbf{Z}_2$. (The nontrivial element of $\mathbf{Z}_2$ is the product of the elements -1 in each of the four factors of W.) In the above construction, one can take f' to be the element -1 of the first $SU(2)$ factor in W. The embedding of $W/\mathbf{Z}_2$ in $Spin(8)$ uses the embeddings (here stated at the Lie algebra level) $SU(2) \times SU(2) = Spin(4)$ and $Spin(4) \times Spin(4) \subset Spin(8)$. Via this chain, the elements (U_1, U_2, U_3) constructed as above can be regarded as elements of $Spin(8)$. Indeed, following through the definitions, one sees that the U_i are diagonal matrices whose eigenvalues are all possible triples of numbers ± 1. When this is embedded in $Spin(2n)$, we get the description given in section 4.2. The Chern-Simons invariant is 1/2; the computation can be done in the first $SU(2)$ factor of W (since f' is in this factor) and equals 1/2 by virtue of (4.32).

5.2 Inclusion of Magnetic Flux

Now we want to repeat this for G that is connected but not necessarily simply-connected. The novelty now is that one can include the magnetic flux $m \in H^2(\mathbf{T}^3, \pi_1(G))$ and the dual electric flux e. We may as well assume that G is the adjoint group (with trivial center); picking a different G amounts to restricting the choices of m and e.

As in section 4, we let $\widehat{G}$ be the universal cover of G, and $\widehat{\mathcal{N}}$ the moduli space of flat G connections modulo gauge transformations that are single-valued if lifted to $\widehat{G}$. On $\widehat{\mathcal{N}}$ acts a finite group $\Gamma = \mathrm{Hom}(\pi_1(\mathbf{T}^3), \pi_1(G))$; the quotient is $\mathcal{N} = \widehat{\mathcal{N}}/\Gamma$. Let Γ_0 be the subgroup of Γ that leaves fixed the Chern-Simons invariant.

As we have discussed in section 3, when $\pi_1(G) \neq 0$, we can formulate the idea of confinement and oblique confinement. It is believed that for all G, the $N = 1$ supersymmetric Yang-Mills theory with gauge group G is confining. As for whether the confining vacua have "ordinary" or "oblique" confinement, we have surveyed the possibilities in section 3.4. For each G and m, there is a "spectral flow" character $\Delta'(m, \gamma)$ computed in section 3.4. It is defined as the change in the Chern-Simons invariant of a bundle of flux m under a gauge transformation γ. Let $w(G, m)$ be the order of Δ' (in the finite group E of characters of Γ or electric fluxes). $\gamma \in \Gamma$ lies in Γ_0 if and only if $\Delta'(m, \gamma) = 0$. So $w(G, m)$ is the index of Γ_0 in Γ.

For simplicity, we assume that $\pi_1(G)$ is cyclic (this omits only the case $G = SO(4n)/\mathbf{Z}_2$). In that case, as we have discussed in section 3, for some way of writing $\mathbf{T}^3 = \mathbf{T}^2 \times \mathbf{S}^1$, m is a pullback from the first factor.

A flat connection on $\mathbf{T}^3$ with flux m can be described by commuting holonomies U_1, U_2, U_3 that if lifted to the universal cover $\widehat{G}$ obey

$$\begin{aligned} U_1U_2 &= U_2U_1f \\ U_iU_3 &= U_3U_i \quad \text{for } i = 1, 2. \end{aligned} \tag{5.7}$$

Here, as explained in section 2.1, f is an element of the center of $\widehat{G}$ determined by m.

For given $\vec{U} = (U_1, U_2)$, U_3 must lie in the subgroup $H_{\vec{U}}$ consisting of elements of $\widehat{G}$ that commute with U_1, U_2. The difference from the previous case is that in general, $H_{\vec{U}}$ is not connected for generic U_1, U_2. The number of components is precisely $w(G, m)$. In fact, consider an element of Γ of the form $\gamma = (1, 1, a)$, with a an element of the center of $\widehat{G}$. If a is in the identity component of $H_{\vec{U}}$, then the triple (U_1, U_2, a) can be continuously deformed (preserving (5.10)) to $(U_1, U_2, 1)$ and hence γ leaves fixed the Chern-Simons invariant. So the subgroup of the center of $\widehat{G}$ that is in the identity component of $H_{\vec{U}}$ is of index at least $w(G,m)$, and hence $H_{\vec{U}}$ has at least $w(G,m)$ components. This is the actual number of components, for given U_1, U_2. For instance, in an example considered in section 4.4, with $G = SU(n)/\mathbf{Z}_n$ and m equal to "1," we have $w(G,m) = n$, and $H_{\vec{U}}$ consists precisely of the center of $SU(n)$, which consists of n points, each of which is a connected component.

Even when G is not simply-connected, the moduli space $\widehat{\mathcal{M}}$ of flat G-bundles on $\mathbf{T}^2$ with magnetic flux m is still a weighted projective space $\mathbf{WCP}^t_{u_0,u_1,\ldots,u_t}$ for some t and some weights u_i. It is no longer true that t equals the rank of G, or that $\sum_i u_i = h$. Rather,

$$h = w(G,m) \sum_i u_i. \tag{5.8}$$

(For instance, in the above-cited example for $SU(n)/\mathbf{Z}_n$, we have $h = w(G,m) = n$, $t = 0$, and $u_0 = 1$.)

Just as before, if we set to zero all homogeneous coordinates whose weights are not divisible by some integer k, we get a locus of orbifold singularities in $\widehat{\mathcal{M}}(m)$. On this locus, the number of components of $H_{\vec{U}}$ is $w(G,m)k$. Repeating the computation that led to (5.3), the contribution to $\sum_i (r_i + 1)$ from orbifold singularities of order k in $\widehat{\mathcal{M}}(m)$ is not $\phi(k)\mu(k)$, as we had before, but $w(G,m)\phi(k)\mu(k)$. Using (5.4) and (5.8), we hence get $\sum_k w(G,m)\phi(k)\mu(k) = w(G,m)\sum_i u_i = h$, as desired.

5.3 Examples With Nonzero m

We conclude with a survey of some examples with nonzero m. The discussion is in rough parallel with the evaluation of the spectral flow for various examples in section 3.4. We have already analyzed $SU(n)$ in section 4.4. We consider here the other classical groups. Except for $Spin(4n)$, we can assume (as shown in section 3) that the magnetic flux has only one nonzero component m_{12}.

$Sp(n)$

The center of $Sp(n)$ is $\mathbf{Z}_2$, so there is only one possible nonzero value of m_{12}. A flat $Sp(n)/\mathbf{Z}_2$ bundle with $m_{12} \neq 0$ has holonomies that if lifted to

$Sp(n)$ obey

$$\begin{aligned} U_1U_2 &= -U_2U_1 \\ U_iU_3 &= -U_3U_i, \text{ for } i = 1,2. \end{aligned} \tag{5.9}$$

We use the embedding $Sp(1)\times O(n) \subset Sp(n)$, under which the fundamental representation of $Sp(n)$ decomposes as the tensor product of the fundamental representations of $Sp(1)$ and $O(n)$. In $Sp(1) = SU(2)$, the equation $U_1U_2 = -U_2U_1$ has a unique solution up to conjugation. Let $U_1 = A$, $U_2 = B$ be such a solution. Then in $Sp(n)$, we take $U_1 = A \times 1$, $U_2 = B \times 1$, where 1 is the identity element of $O(n)$.

The unbroken subgroup is $H = O(n)$, which has two components consisting of elements of determinant 1 or -1. The moduli space $\mathcal{N}$ of commuting triples has two components, depending on whether U_3 is in the identity or non-identity component of H. If we set $U_3 = 1$, we get the identity component $\mathcal{N}_0$ of $\mathcal{N}$. The unbroken group is $SO(n)$, of rank $r_0 = n/2$ or $(n-1)/2$ for n even or odd. Setting $U_3 = \mathrm{diag}(-1,1,1,\dots,1)$ we get a second component $\mathcal{N}_1$, with unbroken $SO(n-1)$ and rank $r_1 = n/2 - 1$ or $(n-1)/2$ for even or odd n. Either way, the relation $(r_0 + 1) + (r_1 + 1) = h$ is obeyed, with $h = n + 1$.

The nontrivial element of the center of $Sp(n)$ is the element -1, which is in the identity or non-identity component of H for even or odd n. Hence, in accord with what we found in section 3.4, there is oblique confinement precisely if n is odd. For odd n, the two components of $\mathcal{N}$ have the same rank and are permuted by the action of the group Γ of discrete gauge transformations, so that zero energy states can carry nonzero electric flux. For even n, the two components of $\mathcal{N}$ have different rank and are each mapped to themselves by Γ, so that zero energy states have vanishing electric flux.

$Spin(2k+1)$

The center of $Spin(2k+1)$ is again $\mathbf{Z}_2$. A flat $Spin(2k+1)/\mathbf{Z}_2 = SO(2k+1)$ bundle with m_{12} nonzero (and other components of the magnetic flux vanishing) has holonomies U_i that commute in $SO(2k+1)$, but such that upon lifting to $Spin(2k+1)$, precisely the same relations as (5.9) are obeyed.

We can take $U_1 = \mathrm{diag}(-1,-1,1,\dots,1)$, $U_2 = \mathrm{diag}(-1,1,-1,\dots,1)$ (each with precisely two -1's). The unbroken subgroup of $SO(2k+1)$ is $H = O(2k-2)$, with two components. U_3 may be placed in either component of H, giving representatives of the two components of $\mathcal{N}$. For one component $\mathcal{N}_0$, we can pick $U_3 = 1$, and the unbroken group has connected component $SO(2k-2)$; for the second component $\mathcal{N}_1$, we can pick $U_3 = \text{diag}(-1,-1,-1,-1,1,\dots,1)$ (with precisely four -1's, ensuring that U_3 commutes with U_1, U_2 even when lifted to $Spin(2k+1)$); the unbroken group has connected component $SO(2k-3)$. The familiar relation $(r_0+1)+(r_1+1)=h$ is obeyed, with $h = 2k-1$.

The nontrivial central element of $Spin(2k+1)$ is a 2π rotation, which can be carried out in H and is connected to the identity in H, for all k. Hence, as claimed in section 3.4, there is no oblique confinement for $Spin(2k+1)$. The group Γ maps each component of $\mathcal{N}$ to itself, and the zero energy states have vanishing electric flux.

$Spin(4n+2)$

The center of $Spin(4n+2)$ is $\mathbf{Z}_4$. $Spin(4n+2)$ has an outer automorphism that exchanges the two spinor representations and acts on m by $m \to -m$. Modulo this automorphism, we may assume that m_{12}, if nonzero, equals 1 or 2 mod 4. For $m_{12} = 2$, the moduli space has two components that are constructed just as for $Spin(2k+1)$, with the same U_1 and U_2 and the same two choices for U_3. There is one novelty compared to $Spin(2k+1)$: the center is $\mathbf{Z}_4$ instead of $\mathbf{Z}_2$, so the analysis of oblique confinement is more elaborate. The generator of $\mathbf{Z}_4$ is an element a that acts as -1 in the vector representation and $\pm i$ on the two spinor representations. a is not contained in the connected component of the unbroken group H, but a^2 (which is a 2π rotation in $SO(4n+2)$) is. So $w(G,m) = 2$ for $m = 2$. The spectral flow character is of order 2.

The more novel case is $m_{12} = 1$. A flat $Spin(4n+2)$ bundle with $m_{12} = 1$ can be constructed as in section 3.4 using a subgroup $Spin(6) \times Spin(4) \times \cdots \times Spin(4)$ of $Spin(4n+2)$, with $n-1$ factors of $Spin(4)$. In $Spin(6) = SU(4)$, we take $U_1U_2 = iU_2U_1$, and in each factor of $Spin(4) = SU(2) \times SU(2)$ we take $U_1 = A \times 1$, $U_2 = B \times 1$, where $AB = -BA$. The unbroken subgroup H has $Sp(n-1)$, of rank $n-1$, for its connected component. This connected component does not contain any element of the center of $Spin(4n+2)$. The moduli space $\mathcal{N}$ of commuting triples has four components, one for each element of the center of $Spin(4n+2)$; each component has a representative in which U_1 and U_2 are as above and U_3 is an element of the center. The relation $\sum_i (r_i+1) = h$ is obeyed with $h = 4n$ and $r_i = n-1$ for $i = 0,\dots,3$.

$Spin(4n)$

The center of $Spin(4n)$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$, with generators a_1, a_2 where a_1 is

1 on positive chirality spinors, -1 on negative chirality spinors, and -1 on vectors, while a_2 acts on those representations as $-1, 1, -1$. For $m = a_1 + a_2$, the construction of the components of $\mathcal{N}$ is the same as for $Spin(2k+1)$. The subgroup of the center of $Spin(4n)$ that is contained in the connected component of H is generated by $a_1 + a_2$, so as for $Spin(4n+2)$, $w(G, m) = 2$ and there are zero energy states carrying electric flux.

Assuming that m_{12} is the only nonzero component, the remaining case (modulo an outer automorphism that exchanges a_1 and a_2) is $m = a_1$. To analyze this case, we use the embedding $O(2) \times O(2n) \subset SO(4n)$, with the vector representation decomposing under the subgroup as $\mathbf{2} \otimes \mathbf{2n}$. We set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.10}$$

and $U_1 = A \times 1$, $U_2 = B \times 1$. The unbroken subgroup H is a spin (or "pin") double cover of $O(2n)$; the subgroup of the center of $Spin(4n)$ contained in the identity component of H is $\mathbf{Z}_2$, generated by a_1 or a_2 depending on whether n is even or odd. So again $w(G, m) = 2$ and there are zero energy states carrying electric flux. The moduli space $\mathcal{N}$ of commuting triples has four components, two of rank n and two of rank $n-1$, leading to the formula $\sum_i (r_1 + 1) = h$ with $h = 4n - 2$. Two components contain representatives in which U_3, if projected to $SO(4n)$, equals 1 (the actual U_3's differ in $Spin(4n)$ by a central element that projects to 1 in $SO(4n)$), and two contain representatives in which U projects to the element $\mathrm{diag}(-1, 1, 1, \ldots, 1) \in O(2n) \subset SO(4n)$. In each case, U_1 and U_2 are as above.

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