Adv. Theor. Math. Phys. 5 (2001) 947–968

Spin foam quantization of SO(4) Plebanski's action

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Abstract

The goal of this work is two-fold. In the first part of this paper we regard classical Plebanski's action as a BF action supplemented by constraints (defined in the spirit of Barrett and Crane). We introduce a spin foam model for Riemannian general relativity by systematically implementing these constraints as restrictions on paths in the state-sum of the BF theory. The spin foam model obtained is closely related to —but not the same as— the Barrett-Crane model. More precisely, configurations satisfying our constraints correspond to a subset of the Barrett-Crane configurations. Surprisingly, all tetrahedra in the allowed configurations turn out to have zero volume.

In the second part of the paper we study the quantization of the effective action corresponding to the degenerate sectors of Plebanski's theory and obtain a very simple spin foam model. This model turns out to be precisely the one introduced by De Pietri et al. as an alternative to the one proposed by Barrett and Crane. This result establishes a clear-cut connection of the model with a classical action. The 4-simplex configurations of the model corresponding to the full Plebanski's action (obtained in the first part) turn out to be entirely contained in the set of configurations of the model of the degenerate sector.

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e-print archive: http://xxx.lanl.gov/abs/gr-qc/0203058

1 Introduction

In reference [1] Barrett and Crane introduced a very interesting model of Riemannian quantum gravity based on a constrained state-sum. The definition of the model can be nicely motivated by geometrical properties of the so-called 'quantum tetrahedron'. The definition of the quantum tetrahedron in 3 dimensions was originally introduced by Barbieri in [2]. Baez and Barrett showed that a generalization to 4 dimensions naturally leads to the the Barrett-Crane (BC) model [3, 4]. Evidence suggesting that the model corresponds to a discrete path integral for general relativity has been found in [5, 6]. The model turns out to be well defined on a finite (non-degenerate) triangulation once an appropriate normalization is chosen[7]. This normalization arises naturally in the so-called group field theory (GFT) formulation [8, 9], which in addition provides a prescription for summing over discretizations. The model has been extended to the Lorentzian sector in [10, 11, 12]. The finiteness properties are preserved in this extension[13].

The $SO(4)$ Plebanski action corresponds to the $SO(4)$ BF action plus certain Lagrange multiplier terms imposing constraints on the B field. Therefore, one can formally quantize the theory restricting the BF-path-integral to paths that satisfy the B-field constraints. In the literature, there is an implicit assumption that the BC model corresponds to a realization of this idea. In other words, the definition of the quantum tetrahedron in 4d (giving rise to the BC model) is sometimes regarded as an alternative way to impose the required restrictions on the B-configurations in the discretization. The purpose of this work is to analyze if this is the case by systematically carrying out this restrictions.

We will present a construction which defines the path integral of Plebanski's action on a fixed simplicial decomposition of space-time. As just mentioned, this is done by appropriately restricting the state-sum of the $SO(4)$ BF theory. The path-integral of the BF theory is defined on a triangulation using techniques similar to those in lattice-gauge theory. The spin-foam formulation —or state-sum— is obtained by performing the mode expansion of certain distributions on $SO(4)$. This is analogous to a Fourier transform where modes correspond to unitary irreducible representations of $SO(4)$ (Peter-Weyl theorem). The constraints on the B-field in the classical action can be naturally translated into restrictions on these modes. The definition of these constraints is not different in spirit from that of Barrett and Crane. However, we emphasize the requirement that the restrictions should be imposed on configurations of the BF theory. After making some natural definitions, a systematic derivation leads to a model that is closely related to the BC model but that does not agree with it. This new version has a puzzling feature: states of 3-geometries (boundary spin-network states) are annihilated by the volume operator. The point of view is related to that of Reisenberger and Freidel-Krasnov in [14, 15, 16]. No obvious modification of the prescription can lead to all the BC configurations.

In order to find a possible interpretation of this result we concentrate on one of the degenerate sectors of Plebanski's action described in [17]. It turns out that one can define a spin foam quantum model corresponding to this sector in a straightforward way. For this, one simply applies the same techniques used in the case of the BF theory. Surprisingly, the model obtained coincides with the one introduced by De Pietri et al. in [18]. This model was defined as an alternative to the BC model arising naturally in the context of the group field theory (GFT) framework. Our result provides a clear-cut interpretations of the De Pietri et al. formulation as a quantization of a classical action. An interesting result is that all the allowed configurations for a 4-simplex in the previous model (corresponding to generic theory) are special configurations of this model.

The article is organized in the following way. In the next section we recall essential facts about $SO(4)$ Plebanski formulation. In Section 3 we briefly review the spin foam quantization of the BF theory and introduce our basic definitions. In Section 4 we solve the constraints that lead one from the BF theory to general relativity and construct the corresponding state-sum model. We interpret the results and show that configurations have zero 3-volume. In Section 5 we quantize the effective action corresponding to the degenerate sectors of Plebanski's action and show that the previous model corresponds to a sub-set of the spin foam configurations obtained in the degenerate sector. We end with concluding remarks in Section 6.

2 Classical $SO(4)$ Plebanski action

Let us start by briefly reviewing Plebanski's formulation[19] at the classical level. Plebanski's Riemannian action depends on an $so(4)$ connection A, a Lie-algebra-valued 2-form B and Lagrange multiplier fields λ and μ. Writing explicitly the Lie-algebra indices, the action is given by

$$S[B, A, \lambda, \mu] = \int \left[B^{IJ} \wedge F_{IJ}(A) + \lambda_{IJKL}\, B^{IJ} \wedge B^{KL} + \mu \epsilon^{IJKL} \lambda_{IJKL}\right], \tag{1}$$

where μ is a 4-form and $\lambda_{IJKL} = -\lambda_{JLKI} = -\lambda_{IJLK} = \lambda_{KLIJ}$ is tensor in the internal space. Variation with respect to μ imposes the constraint

$\epsilon^{IJKL}\lambda_{IJKL} = 0$ on λ_{IJKL}. λ_{IJKL} has then 20 independent components. Variation with respect to the Lagrange multiplier λ imposes 20 algebraic equations on the 36 B. Solving for μ they are

$$B^{IJ} \wedge B^{KL} - \frac{1}{4!}\epsilon_{OPQR}B^{OP} \wedge B^{QR}\epsilon_{IJKL} = 0 \tag{2}$$

which is equivalent to

$$\epsilon_{IJKL}B^{IJ}_{\mu\nu}B^{KL}_{\rho\sigma} = e\epsilon_{\mu\nu\rho\sigma}, \tag{3}$$

for $e \neq 0$ where $e = \frac{1}{4!}\epsilon_{OPQR}B^{OP}_{\mu\nu}B^{QR}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$[20]. The solutions to these equations are

$$B = \pm^*(e \wedge e), \quad \text{and} \quad B = \pm e \wedge e, \tag{4}$$

in terms of the 16 remaining degrees of freedom of the tetrad field e^I_a. If one substitutes the first solution into the original action one obtains an effective action that is precisely that of general relativity in the Palatini formulation

$$S[e, A] = \int \mathrm{Tr}\left[e \wedge e \wedge {}^*F(A)\right]. \tag{5}$$

3 Quantum $SO(4)$ BF theory

Classical ($Spin(4)$) BF theory is defined by the action

$$S[B, A] = \int \mathrm{Tr}\left[B \wedge F(A)\right], \tag{6}$$

where B^{IJ}_{ab} is a $Spin(4)$ Lie-algebra valued 2-form, A^{IJ}_a is a connection on a $Spin(4)$ principal bundle over $\mathcal{M}$. The theory is rather trivial and all classical solutions are locally equivalent (up to gauge transformations). The theory has only global degrees of freedom.

One can quantize the theory à la Feynman introducing a path integral measure. This is easily done by replacing the manifold $\mathcal{M}$ by an arbitrary simplicial decomposition Δ [1]. Take a fixed triangulation Δ of $\mathcal{M}$. The 2-skeleton of the dual of the triangulation defines a cellular 2-complex Δ^*. Associate $B_f \in so(4)$ to each triangle in Δ (for convenience we use the face sub index f since triangles are in one-to-one correspondence to faces $f \in \Delta^*$), and a group element $g_e \in Spin(4)$ to each edge $e \in \Delta^*$. Consider

[1] More generally, the path integral for the BF theory can be defined on an arbitrary cellular decomposition of $\mathcal{M}$. See [21].

the holonomy around faces $U_f = g_{e_1} g_{e_2} \cdots g_{e_n}$, i.e., the product of group elements of the corresponding edges around one face (an arbitrary orientation of faces has been chosen). The discretized version of the partition function becomes

$$\mathcal{Z}(\Delta) = \int \prod_{f \in \Delta^*} dB_f^{(6)} \prod_{e \in \Delta^*} dg_e \; e^{i \mathrm{Tr}[B_f U_f]}. \tag{7}$$

The measure $dB_f^{(6)}$ is the Lebesgue measure on $\Re^6$, while dg corresponds to the normalized Haar measure of $Spin(4)$. Now the integration over the B_f's can be done explicitly [22], and the result is:

$$Z(\Delta) = \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} \delta(g_{e_1} \cdots g_{e_n}). \tag{8}$$

Expanding the delta distribution in unitary irreducible representations (Peter-Weyl decomposition [2]) we obtain

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{\rho\} \to \{f\}} \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} \Delta_{\rho_f} \; \mathrm{Tr} \left[\rho_f(g_e^1 \cdots g_e^N) \right], \tag{9}$$

where $\mathcal{C} : \{\rho\} \to \{f\}$ denotes the assignment of irreducible representations to faces in the dual 2-complex Δ^*. Each particular assignment is referred to as a *coloring*, $\mathcal{C}$.

Next step is to integrate over the connection g_e. Since edges $e \in \Delta^*$ bound four different faces, each group element g_e appears in the mode expansion of four delta functions in (8). The formula we need is that of the projection operator into the trivial component of the tensor product of four irreducible representations, namely

$$\int dg \; \rho_1(g) \otimes \rho_2(g) \otimes \rho_3(g) \otimes \rho_3(g) = \sum_{\iota} C^{\iota}_{\rho_1\rho_2\rho_3\rho_4} \; C^{*\iota}_{\rho_1\rho_2\rho_3\rho_4}, \tag{10}$$

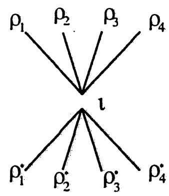
where $C^{\iota} \in \mathcal{H}_{\rho_1} \otimes \mathcal{H}_{\rho_2} \otimes \mathcal{H}_{\rho_3} \otimes \mathcal{H}_{\rho_4}$ represents an orthonormal basis of invariant vectors and the sum on the RHS ranges over all the basis elements ι.

[2] Peter-Weyl theorem implies that

$$\delta(g) = \sum_{\rho} \Delta_{\rho} \mathrm{Tr}[\rho(g)],$$

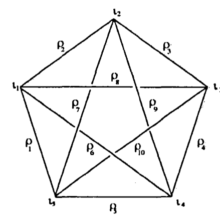
where $\rho(g)$ is the unitary irreducible representation of dimension Δ_{ρ}.

The RHS of equation (11) can be represented graphically as

$$\sum_{\iota} C^{\iota}_{\rho_1\rho_2\rho_3\rho_4}\, C^{*\iota}_{\rho_1\rho_2\rho_3\rho_4} = \sum_{\iota}$$ 

. (11)

Therefore, integrating over the g_e's using (11) and keeping track of indices we obtain

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f:\{f\}\to\rho_f}\ \sum_{\mathcal{C}_e:\{e\}\to\{\iota_e\}}\ \prod_{f\in\Delta^*}\Delta_\rho \prod_{v\in\Delta^*}$$ 
, (12)

where the pentagonal diagram representing the vertex amplitude denotes the trace of the product of five intertwiners $C^{\iota}_{\rho_1\rho_2\rho_3\rho_4}$ according to the graphical notation of (11). Vertices $v \in \Delta^*$ are in one-to-one correspondence to 4-simplexes in the triangulation Δ. In addition we also have $\mathcal{C}_e : \{e\} \to \{\iota_e\}$ representing the assignment of intertwiners to edges. The sum over the *coloring* of edges, $\mathcal{C}_e$, comes from (11) (for an extensive explanation of the construction of the state-sum for the BF theory and the notation used here see[23]).

What happened in going from equation (7) to (9)? We have replaced the continuous multiple integral over the B's by the sum over representations of $SO(4)$. Roughly speaking, the degrees of freedom of B are now encoded in the representation being summed over in (9). One can make a more precise definition of what 'B' is at the level of (9). In order to motivate our definition we isolate a single face contribution to the integrand in the partition function (7). Then we notice that the right invariant vector field $-i\mathcal{X}^{IJ}(U)$ has a well defined action at the level of equation (9) and acts as a 'quantum' B at the level of (7) since

$$-i\mathcal{X}^{IJ}(U)\left(e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1} = X^{IJ\ \mu}{}_{\nu} U^{\nu}{}_{\sigma} \frac{\partial}{\partial U^{\mu}{}_{\sigma}} e^{i\mathrm{Tr}[BU]}|_{U\sim 1} =$$
$$= \mathrm{Tr}[X^{IJ}UB]e^{i\mathrm{Tr}[BU]}|_{U\sim 1} \sim B^{IJ}e^{i\mathrm{Tr}[BU]}, \qquad (13)$$

where X^{IJ} are elements of an orthonormal basis in the $SO(4)$ Lie-algebra. The evaluation at $U = 1$ is motivated by the fact that configurations in the BF partition function (8) have support on flat connections.

The constraints in (3) are quadratic in the B's. We have then to worry about cross terms, more precisely the nontrivial case corresponds to:

$$\begin{aligned}&\epsilon_{IJKL}\mathcal{X}^{IJ}(U)\mathcal{X}^{KL}(U)\left(e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1}\\&=-\epsilon_{IJKL}\left(\mathrm{Tr}[X^{IJ}UB]\mathrm{Tr}[X^{KL}UB]e^{i\mathrm{Tr}[BU]}+i\mathrm{Tr}[X^{IJ}X^{KL}UB]e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1}\\&\sim\epsilon_{IJKL}B^{IJ}B^{KL}e^{i\mathrm{Tr}[BU]},\end{aligned} \tag{14}$$

where the second term on the second line can be dropped using that $\epsilon_{IJKL}X^{IJ}$ $X^{KL}\propto 1$ (one of the two $SO(4)$ Casimir operators) and $U\sim 1$. Therefore, we define the B_f field associated to a face at the level of equation (9) as the appropriate right invariant vector field $-i\mathcal{X}^{IJ}(U_f)$ acting on the corresponding discrete holonomy U_f, namely

$$B_f^{IJ}\to -i\mathcal{X}^{IJ}(U_f). \tag{15}$$

It is easy to verify that one can use left invariant vector fields instead in the previous definition without changing the following results.

4 Implementation of the constraints that reduce the BF theory to general relativity

4.1 Formulation of the problem

Now we describe the implementation of the constraints (3). The idea is to concentrate on a single 4-simplex amplitude using the locality of the BF theory state sum [3]. The 4-simplex wave function is obtained using (8) on the dual 2-complex with boundary defined by the intersection of the dual of a single 4-simplex with a 3-sphere, see Figure 1. We refer to this fundamental building block as 'atom' as in [14]. The boundary values of the discrete connection are held fixed. We denote as $h_{ij}\in Spin(4)$ ($i\neq j$, $i,j=1\cdots 5$ and $h_{ij}=h_{ji}^{-1}$) the corresponding 10 boundary variables (associated to thin boundary edges in Figure 1) [4] and $g_i\in Spin(4)$ ($i=1,\cdots,5$) the internal

[3]The term 'local' here is used as defined by Reisenberger in [14]. It means that the spin foam can be written as 4-simplex contributions that communicate with other 4-simplexes by boundary data (connection). The full amplitude is obtained by integrating out the boundary connections along the common boundary of the 4-simplexes that make up the simplicial complex.

[4]Strictly speaking, the boundary connections h_{ij} are defined as the product $h'_{ij}h''_{ij}$ where h' and h'' are associated to half paths as follows: take the edge ij for simplicity and assume it is oriented from i to j. Then h'_{ij} is the discrete holonomy from i to some

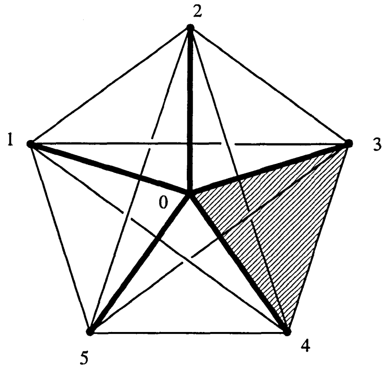


Figure 1: A fundamental *atom* is defined by the intersection of a dual vertex in Δ^* (corresponding to a 4-simplex in Δ) with a 3-sphere. The thick lines represent the internal edges while the thin lines the intersections of the internal faces with the boundary. One of the faces has been emphasized.

connection (corresponding to the thick edges in Figure 1). According to (8) the 4-simplex BF amplitude $4SIM_{BF}(h_{ij})$ is given by

$$4SIM_{BF}(h_{ij}) = \int \prod_i dg_i \prod_{i<j} \delta(g_i h_{ij} g_j). \tag{16}$$

With the definition of the B fields given in (15) the constrained amplitude, $4SIM_{const}(h_{ij})$, formally becomes

$$4SIM_{const}(h_{ij}) = \int \prod_i dg_i \delta\left[\text{Constraints}(\mathcal{X}(U_{ij}))\right] \prod_{i<j} \delta(g_i h_{ij} g_j), \tag{17}$$

where $U_{ij} = g_i h_{ij} g_j$ is the holonomy around the triangular face (wedge) $0ij$ according to Figure 1. It is easy to verify, using an equation analogous to (13), that one can define the B's by simply acting with the right invariant vector fields on the boundary connection h_{ij}. Therefore, the previous equation is equivalent to

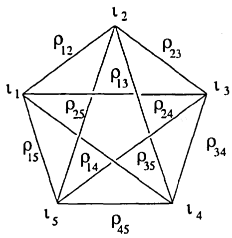
$$4SIM_{const}(h_{ij}) = \delta\left[\text{Constraints}(\mathcal{X}(h_{ij}))\right] \int \prod_i dg_i \prod_{i<j} \delta(g_i h_{ij} g_j), \tag{18}$$

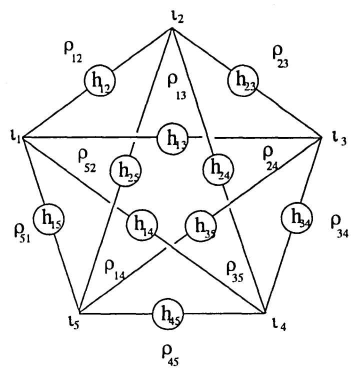
where we have taken the delta function out of the integral. The quantity on which the formal delta distribution acts is simply $4SIM_{BF}(h_{ij})$ (defined

point in the center of the path and h''_{ij} is the holonomy from that center point to j. This splitting of variables is necessary when matching different atoms to reconstruct the simplicial amplitude. The use of this variables (wedge variables) will be crucial in Section 5. For a more detailed description of wedge variables see [14, 15].

in (16)), which after integrating over the internal connection g_i, and using equation (11) becomes

$$4SIM_{BF}(h_{ij}) = \sum_{\rho_1\cdots\rho_{10}} \sum_{\iota_1\cdots\iota_5}$$





, (19)

where the circles represent the corresponding ρ-representation matrices evaluated on the the corresponding boundary connection h. The term on the left is a $15j$-symbol as in (12) while the term on the right is the trace of five intertwiners with the respective boundary connection insertions. Notice then that nodes on the two pentagonal diagrams are linked together by the value of their intertwiner.

The 4-simplex amplitude for the constraint spin foam model is then defined as the restriction of $4SIM_{BF}(h_{ij})$ imposed by the quantum version of the constraints (3). The latter are defined by the following set of differential equations

$$\epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ik})\ 4SIM_{const}(h_{ij}) = 0 \quad \forall\ j,k, \tag{20}$$

and where the index $i = 1,\cdots,5$ is held fixed. The translation of the continuum constraint (3) into discrete elements associated to faces in Δ^* is analogous to that given in [3, 20]. Notice that (20) is to be thought as a condition on BF amplitudes and is not a general equation to be imposed to any 4-simplex amplitude. Recall that the strategy is to constraint the BF theory to obtain a definition of the path integral for general relativity [5].

[5]We illustrate the general idea with the following simple example. Imagine that the analog of $4SIM_{BF}$ function (eq. (16)) is the integral

$$A = \int dkdpe^{ikx+ipy} = \delta(x)\delta(y), \tag{21}$$

where $x, y \in [0, 2\pi]$ represent the boundary 'connections'. The analog of the constraint (3) is defined to be $k - p = 0$ which in turn implies the constrained amplitude to be

$$A_{const} = \delta(x+y).$$

Let us now apply the prescription used in the BF theory. We can expand the un-

Equations closely related to (20) can also be obtained as the geometric restrictions on the B's to be simple bi-vectors coming from a dual cotetrad or to characterize the geometry of a tetrahedron in 4 dimensions [4, 3]. In this case the equivalent of B correspond to bivectors defined by the faces of a classical tetrahedron. Using geometric quantization one obtains the Hilbert space of states of the 'quantum tetrahedron' where the B's are promoted to operators. Notice the our B operator (13) is obtained directly from the BF path integral and one does not need to invoke any additional quantization principle. The procedure is completely analogous to the simple example of Footnote 5. A similar point of view has been taken by Reisenberger and Freidel-Krasnov in [14, 16].

4.2 Restricted BF paths

The following procedure is very similar in spirit to the BC prescription[3, 4]. The essential difference is that we now require the set of restricted configurations to be contained in the set of modes of the BF amplitude, $4SIM_{BF}$.

There are seven equations (20) for each value of $i = 1, \cdots, 5$. If we consider all the equations for the 4-simplex amplitude then some of them are redundant. The total number of independent conditions is 20, in agreement with the number of classical constraints (3). For a given i in (20) (i.e., a given tetrahedron) and for $j = k$ the equation becomes

$$\epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ij})\ 4SIM_{const}(h_{ij}) \\ = \left[j^{\ell}_{ij}(j^{\ell}_{ij}+1) - j^{r}_{ij}(j^{r}_{ij}+1)\right] 4SIM_{const}(h_{ij}) = 0, \quad (22)$$

where we have used $\rho = j^{\ell} \otimes j^{r}$ for $j^{\ell}, j^{r} \in \mathrm{Irrep}[SU(2)]$. The previous constraints are solved by requiring the corresponding representation ρ_{ij} to be simple, i.e., $\rho_{ij} = j_{ij} \otimes j_{ij}$.

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constrained function (21) in terms of 'spin foam' amplitudes

$$A = \frac{1}{4\pi^2}\sum_{n,m} e^{inx+imy}.$$

In this case this corresponds to Fourier expanding delta function on $S_1 \times S_1$ (Peter-Weyl decomposition for $U(1) \times U(1)$). The constraint is now represented by a combination C of right invariant vector fields on $U(1)$: $C = \partial_x - \partial_y$. So we can now impose the constraints by means of selecting those configurations (modes) in (21) that are annihilated by C. The equation analogous to (20) is

$$(\partial_x - \partial_y)e^{inx+imy} = (n-m)e^{inx+imy} = 0$$

which implies $n = m$ and $A_{const} = \delta(x+y)$.

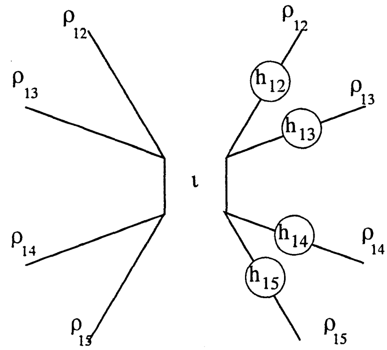


Figure 2: A tree decomposition of one of the nodes in (19). Any tree decomposition is equivalent.

This solves 10 of the 20 equations. The next non-trivial condition imposed by (20) is when $j \neq k$. In this case we have

$$\begin{aligned} & 2\epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ik})\ 4SIM_{const}(h_{ij}) \\ &= \epsilon_{IJKL}\left(\mathcal{X}^{IJ}(h_{ij}) + \mathcal{X}^{IJ}(h_{ik})\right)\left(\mathcal{X}^{KL}(h_{ij}) + \mathcal{X}^{KL}(h_{ik})\right) 4SIM_{const}(h_{ij}) \\ &= \left[\iota^{\ell}(\iota^{\ell}+1) - \iota^{r}(\iota^{r}+1)\right] 4SIM_{const}(h_{ij}) \\ &= 0, \end{aligned} \tag{23}$$

where we used the gauge invariance at the 3-valent node in the tree decomposition that pairs the representation ρ_{ij} with the ρ_{ik}[6], and that we have already solved (22). In the last line we assume that the internal color of the corresponding 4-intertwiner is $\iota = \iota^{\ell} \otimes \iota^{r}$. This choice of tree decomposition in the case $ij = 12$ and $ik = 13$ is illustrated in Figure 2. The solution is clearly $\iota = \iota \otimes \iota$.

What happens now with any of the other two remaining conditions, for example, $E(ij', ik')$ for $k \neq k'$, $j \neq j'$ and $j' \neq k'$? It seems that we ran out o possibilities of restricting the representations. Generically this equations will not be satisfied because an intertwiner that has simple ι in one tree decomposition has not only simple ι''s components in a different tree decomposition and the equation would be violated. However there is a case in which this happens trivially, namely when the dimension of the invariant part of the tensor product of the four corresponding representations is unity.

[6]The gauge invariance at the node allows us to express the sum of right-invariant vector fields acting on the external 'legs' (see Figure 2.) as a right-invariant vector field acting on the internal representation ι. Of course right versus left invariant vector field is a matter of convention which implies a choice of orientation. Since $Spin(4)$ representations are self dual we have $\mathcal{X}^{IJ}_R(h) = -\mathcal{X}^{IJ}_L(h^{-1})$.

Let us write this condition as an equation since this corresponds to the solution of the remaining 5 independent conditions, namely

$$\dim\left(\mathrm{Inv}\left[\rho_{ij}\otimes\rho_{ik}\otimes\rho_{im}\otimes\rho_{ip}\right]\right)=1 \tag{24}$$

In this case ι would be simple in any tree decomposition if it is simple in one particular one. Notice that this is the only solution to our constraints as a trivial consequence of the theorem proven by Reisenberger in [24]. Our set of solutions are contained in the Barret-Crane solutions since our intertwiners agree with the BC one every time that equation (24) is satisfied. Solutions to the previous equation can be characterized as follows. Since all the ρ's are simple we can concentrate on their right (or left) components. Assume $j_1 \le j_2 \le j_3 \le j_4$ then the condition is $j_1 + j_2 + j_3 = j_4$. Explicitly, a few examples of solutions are $(\frac{1}{2}, \frac{1}{2}, 2, 3)$, $(\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{1}{2})$, $(1, 5, 226, 220)$, etc. The intertwiner color is completely determined by the face colors and a choice of tree decomposition. If we pair j_1 and j_2 in our previous example then $\iota = j_1 + j_2 = j_4 - j_3$. The amplitude $4SIM_{const}$ is independent of the tree decomposition chosen.

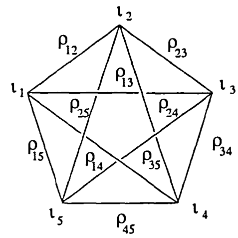
4.3 Gluing 4-simplexes

Once we have solved equations (20) for a single 4-simplex we can calculate the amplitude of any simplicial decomposition of $\mathcal{M}$, Δ. This is achieved by putting together 4-simplexes with consistent boundary connections and gluing them together by means of integrating over the boundary data in the standard way.

Let us point out that there is a potential ambiguity in this step. We have implemented constraints in the path integral and this generally should be supplemented with the appropriate modification of the measure. This could affect the values of lower dimensional simplexes such as face and edge amplitudes. Constraints (20) act on each edge (tetrahedron) separately, heuristically one would expect a Jacobian factor to modify the edge amplitude of the model since the constraints are non linear functions of the B's. A rigorous derivation of such factors from the path integral definition is lacking in our argument. We believe that this might shed light on the problem of the correct normalization of this type of spin foam models. A more detailed study of this issue is being explored in [25].

If we do so then we end up with

$$Z_{const}(\Delta) = \sum_{C_f:\{f\}\to\rho^s_f} \prod_{f\in\Delta^*} \Delta_\rho \prod_{e\in\Delta^*} A_e \prod_{v\in\Delta^*}$$



, (25)

where ρ^s_f denote the set of representations selected by conditions (20), ι^s_e are the corresponding colors of intertwiners, and A_e is the appropriate edge amplitude (undetermined in our prescription).

4.4 Volume

In this section we discuss a rather puzzling feature of the model we have defined above.

If we consider boundaries, then the spin-network states induced as boundaries of spin foams are four-valent and the representations of the corresponding edges satisfy (24). Now using the standard definition of the volume operator on this set of states we obtain an identically zero result, i.e, the 3-volume operator $V_{(3)}$ annihilates the states that solve (20). The reason is that the volume is given by[2]

$$V^2_{(3)} \propto \left[(\mathcal{X}_i + \mathcal{X}_j)^2, (\mathcal{X}_i + \mathcal{X}_k)^2\right], \quad (26)$$

where the square is taken using the internal metric δ_{IJ}. The solutions to the constraints happen to diagonalize both operators in the commutator which implies $V_{(3)} = 0$.

5 Degenerate sector

As shown in [15, 20], constraints (3) correspond to the non-degenerate phase of solutions of the general constraints (i.e., phase with $e \neq 0$). In [15] Reisenberger explicitly solved the constraints in the degenerate sectors and showed that, in these cases, the action reduces to

$$S^{\pm}_{deg} = \int B^r_i \wedge (F_i(A^r) \pm V^j_i F_j(A^\ell)), \quad (27)$$

where the upper index r (respectively ℓ) denotes the self-dual (respectively anti-self-dual) part of B and A in the internal space, and $V \in SO(3)$.

Let us concentrate in the sector with the minus sign in the previous expression. Then it is straightforward to define the discretized path integral along the same lines as BF theory in Section 3. The result is

$$\mathcal{Z}(\Delta) = \int \prod_{f \in \Delta^*} dB_f^{r(3)} dv_f \prod_{e \in \Delta^*} dg_e^\ell dg_e^r \; e^{i\mathrm{Tr}\left[B_f^r U_f^r v_f U_f^{\ell-1} v_f^{-1}\right]}. \tag{28}$$

Integrating over the B field we obtain

$$Z(\Delta) = \int \prod_{e \in \Delta^*} dg_e^\ell dg_e^r \prod_{f \in \Delta^*} dv_f \; \delta^{(3)}(g_{e_1}^r \cdots g_{e_n}^r v_f (g_{e_1}^\ell \cdots g_{e_n}^\ell)^{-1} v_f^{-1}), \tag{29}$$

where dg_e^ℓ, dg_e^r, and dv_f are defined in terms of the $SU(2)$ Haar measure and the delta function $\delta^{(3)}$ denotes an $SU(2)$ distribution.

In order to obtain the corresponding state-sum it is easier to concentrate on a single 4-simplex amplitude. Furthermore, we start by the wedge shown in Figure 3. In this figure we represent one of the 10 wedges that form a 4-simplex atom (see Figure 1). Both the internal connection g_{ij} ($g_{ij} = g_{ji}^{-1}$) and the boundary connection variables h_{ij} ($h_{ij} = h_{ji}^{-1}$) are in $Spin(4)$, while $u_{ljki} \in SU(2) \subset Spin(4)$ is an auxiliary variable. The $SU(2)$ subgroup is defined as the diagonal insertion $ug = (ug^\ell, ug^r)$. The wedge amplitude is defined as

$$w = \int du_{ljki} \; \delta^{(6)}(g_{ki} h_{il} u_{ljki} h_{lj} g_{jk}) \tag{30}$$

according to the notation in Figure 3 and where the $\delta^{(6)}$ denotes a $Spin(4)$ delta distribution. Any face in the 2-complex will be defined by as many such wedges as 4-simplexes share the corresponding face. Figure 4 illustrates the case for a triangular face. The vertices $1, 2$ and 3 correspond to the centers of the three 4-simplexes sharing the face. The dotted line denotes the region along which the boundary of the three atoms (Figure 1) join.

It is easy to check that integrating over all but one boundary variables h_{ij}, the contribution of a combination of wedges forming a face $f \in \Delta^*$ is given by

$$\int du_f dh \; \delta^{(6)}(U_f h u_f h^{-1}), \tag{31}$$

where $U_f \in Spin(4)$ is the discrete holonomy, $u_f \in SU(2) \subset Spin(4)$ is a product of the u_w associated to the corresponding wedges and h is the

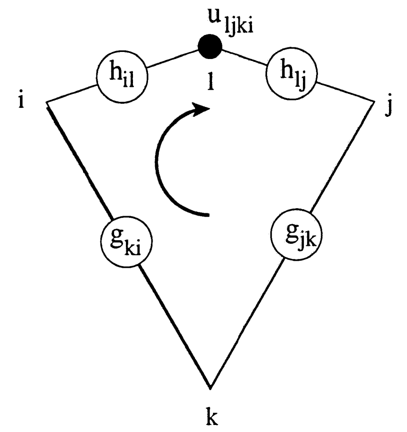


Figure 3: Diagramatic representation of the single wedge contribution (30). The group variables g_{ki} and g_{jk} correspond to the internal connections while h_{il} and h_{lj} to boundary data. u_{ljki} is an independent auxiliary variable in the $SU(2)$ subgroup.

remaining boundary connection. In the case shown in Figure 4, $U_f = g_{1i}g_{i2}g_{2j}g_{j3}g_{3k}g_{k1}$, $u_f = u_3u_2u_1$, and $h = h_{kl}$. Using that $Spin(4) = SU(2) \times SU(2)$ and the definition of the $SU(2)$ subgroup where u lives, the integral over u_f of the previous equation becomes

$$\int du_f dh^\ell dh^r \ \delta^{(3)}(U^\ell_f h^\ell u_f h^{\ell-1})\delta^{(3)}(U^r_f h^r u_f h^{r-1})$$
$$= \int d(h^\ell h^{r-1}) \ \delta^{(3)}(U^\ell_f h^\ell h^{r-1} U^{r-1}_f h^r h^{\ell-1}), \qquad (32)$$

where we have used that $dh = dh^\ell dh^r$ and $\delta^{(6)}(g) = \delta^{(3)}(g^\ell)\delta^{(3)}(g^r)$ as well as the invariance and normalization of the Haar measure. The previous face amplitude coincides with that in (29) if we define $v_f = h^\ell h^{r-1}$. Therefore, (30) defines the wedge amplitude of (29).

Now we can write the analog of equation (19) for the 4-simplex amplitude, $4SIM_{Deg}(h_{ij})$, putting together the 10 corresponding wedges and integrating

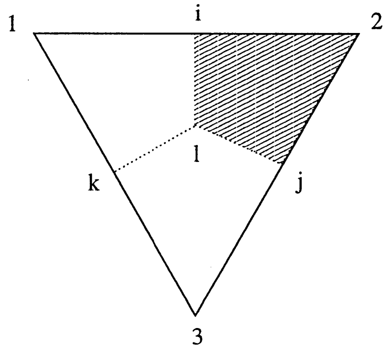
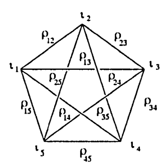
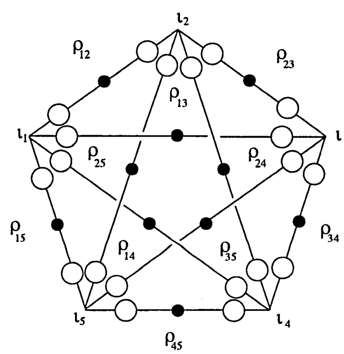


Figure 4: A triangular face made up of three wedges. The wedge $2ilj$ has been emphasized.

over the internal g's, namely

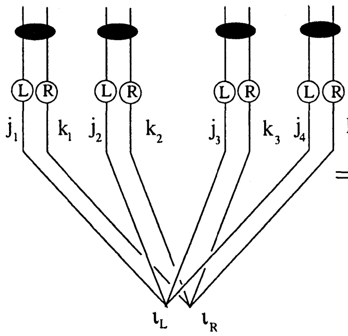
$$4SIM_{Deg}(h_{ij}) = \sum_{\rho_1 \cdots \rho_{10}} \sum_{\iota_1 \cdots \iota_5}$$



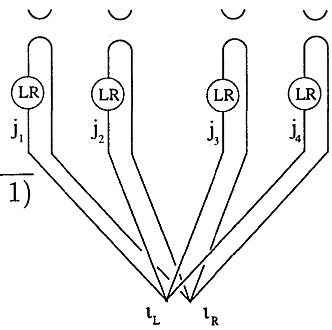


, (33)

where the dark dots denote integration over the $SU(2)$ diagonal subgroup, and the white circles represent the boundary connections. To keep the diagrammatic notation simple we have dropped some labels. The next step is to perform the integration over the u's. We concentrate on a single intertwiner in (33), i.e., a single node in the pentagonal diagram on the right of the previous equation. Using the orthogonality of $SU(2)$ unitary irreducible representations and the fact that the representations ρ of $Spin(4)$ are of the form $\rho = j \otimes k$ for j, k $SU(2)$ unitary irreducible representations we have



$$= \frac{\delta_{j_1,k_1} \cdots \delta_{j_4,k_4}}{(2j_1+1)\cdots(2j_4+1)}$$



, (34)

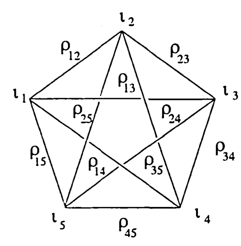
where we represent representations $\rho = j \otimes k$ as parallel lines, the symbol R, L in the circles on the left denotes h^r, h^ℓ $SU(2)$-representation matrices, RL on the right denotes the product $h^r h^{-1\ell}$, and the dark dots are subgroup integrations.

The Kronecker deltas in the previous equation implies that the ρ's labeling faces must be simple, i.e., $\rho = j \otimes j$.

Finally, it is easy to verify that when gluing various 4-simplex atoms together by means of integrating over matching boundary connections the integration simply set $\iota_R = \iota'_R$, $\iota_L = \iota'_L$ where ι, ι' are the intertwiners corresponding to the tetrahedron shared by the two 4-simplexes. Notice that no simplicity condition is imposed on ι. Now consider an arbitrary face bounded by n edges. Such a face is made up n wedges. Therefore there is a factor $\Delta_{jj}^{2n} = (2j+1)^{2n}$ coming from the delta function mode expansion in (recall Footnote (2).), a factor $(2j+1)^{-n}$ from the factors in (34), and finally a factor $(2j+1)^{-n}$ from the boundary connection integrations in the gluing. This results in a face amplitude equal to unity.

Putting all this together one gets a spin foam model were only face representations are constrained to be simple while intertwiners are arbitrary. Explicitly

$$Z_{deg}(\Delta) = \sum_{C_f:\{f\}\to\rho^s_f} \sum_{C_e:\{e\}\to\{\iota_e\}} \prod_{v\in\Delta^*}$$



. (35)

This is precisely the spin foam obtained in [18]! This model was obtained as a natural modification of the GFT that defines a variant of the BC model. Here we have rediscovered the model from the systematic quantization of S^{-}_{deg} defined in (27). This establishes the relation of the model with a classical action! It corresponds to spin foam quantization of the '$-$' degenerate sector of $SO(4)$ Plebanski's theory.

The $+$ sector action (27) can be treated in a similar way. The only modification is that of the subgroup. Instead of using the diagonal insertion defined above one has to define $u \in SU(2) \subset Spin(4)$ so that $ug = (ug^{\ell}, u^{-1}g^{r})$.

We have restricted to simplicial decompositions but all this should be generalizable along the lines of reference [21] for arbitrary cellular decompositions of $\mathcal{M}$. This generalization seems straightforward although it should be investigated in detail.

To conclude this section let us notice that the allowed 4-simplex configurations of the model of Section 4 are fully contained in the set of 4-simplex configurations of the model obtained here. We come back to this issue in the following section.

6 Discussion

The principal idea behind this work was to study the spin foam quantization of Plebanski formulation of gravity by restricting the paths that appear in the $SO(4)$ BF theory. This strategy is supported by the fact that Plebanski's action can be thought of as the $SO(4)$ BF theory, supplemented by certain constraints on the B field. Gravity in the Palatini formulation is obtained as one of the non-degenerate sectors of the solutions to the classical constraints. In the model introduced in section 4 we defined a prescription for implementing these constraints by restricting the set of histories of the BF theory to those satisfying the 'quantum analog' of (3). Solution configurations of a single 4-simplex in the model are special 4-simplex BF configurations. Even though the 4-simplex configurations appearing here are a sub-set of the Barrett-Crane configurations, the great majority of the Barrett-Crane configurations (independently of the normalization chosen) are excluded by the requirement that they be BF configurations. The nature of the constraints in the BC model is essentially the same as the ones defined here. The difference is that in the BC case constraints are implemented on a single intertwiner while here we keep track of the fact that intertwiners appear in pairs in the BF state sum (see (11)).

There are alternative ways to motivate the definition of the Barrett-

Crane model which are independent of the line of thought used here. There is also evidence that relate it to a theory of quantum gravity. However, we believe that this work shows that there is no obvious way to interpret it as the quantization of Plebanski's action. Using reasonable definitions we have shown that one obtains a more restrictive state sum.

In the context of the BC model, reference [4] shows how one can restrict the states of the 'quantum tetrahedron' so that fake tetrahedra are ruled out of the state sum. In our context this amounts to resolving the ambiguity between the $e \wedge e$ and $^*(e \wedge e)$ solutions of the constraints (see (4)) at the quantum level. In [4] it is shown that the 'correct' configurations are selected by imposing the so-called chirality constraint which is automatically satisfied at the quantum level because it can be written as the commutator of the simplicity constraints (20). It is also shown that the two spaces of solutions ($e \wedge e$ and $^*(e \wedge e)$ respectively) intersect on the set of configurations for which $V_{(3)} = 0$. It is easy to see that all this can be translated to our context. The vanishing of the volume operator implies that in our model one can not distinguish the two type of configurations and that the ambiguity (4) remains at the quantum level.

The model of Section 4 contains only degenerate configurations in the sense that spin-network states on the boundaries have zero volume. This shows that the model cannot reproduce any of the semi-classical states of general relativity. Somehow our definition of the constraints at the level of the state-sum are so strong that non-degenerate configurations have been eliminated. Some evidence supporting this view can be obtained considering the following argument. Constraints are implemented locally on each 4-simplex; therefore, we can concentrate on a single 4-simplex to analyze their action. If we do so, then we conclude that all the 4-simplex configurations in the model (25) are entirely contained in the set of 4-simplex configurations of the model found in Section 5.

Is there a way out? If we maintain the point of view of defining the model starting from Plebanski's action then the problem can be traced back to our definition of constraints. As it was pointed out in [20] there are two ways to write Plebanski's constraints in the non-degenerate sector (i.e., when $e \neq 0$ in (3)). If we stick with (3) then one can try to change the definition of the B operators. One possibility would be to change right-invariant by left invariant vector fields in the definition (15). One can do this consistently only if one changes the orientation of the 4-simplex in which case the final result remains the same. One can try to use the sum of right-invariant and left invariant vector fields. This is certainly a possibility, and actually converges faster to the value of B in the sense of equation (13) when $U \to 1$. However, since the constraints are quadratic in the B there will be cross-terms in (20). This terms cannot be expressed in terms of $Spin(4)$ Casimir operators, and consequently, the constraints cannot be solved in terms of simple restrictions on the set of representations involved in the state-sum. There seem to be no obvious way to use (3) and avoid the discouraging results of Section 4.

The other possibility is to use

$$\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \propto \epsilon^{IJKL}, \tag{36}$$

where the difference with (3) is in the fact that we have traded internal with space-time indices. However with this choice, the connection with the BC model becomes much more uncertain. Notice that in this form, the constraints have free Lie-algebra indices and therefore cannot be written as Casimir operators as they stand. This version of the constraints has been studied in the literature. Such constraints have been incorporated in a spin foam model of Riemannian general relativity in terms of self-dual variables by Reisenberger in [15]. In the context of $SO(4)$ Plebanski's action a model along this lines has been defined by Freidel et al. in [16]. But all these models are quite different from the BC model. We believe that this shows that there is no obvious means of interpreting the BC model as a spin foam quantization of Plebanski's theory.

Let us conclude by analyzing the results of the last section. In Section 5 we quantized the degenerate sectors of Plebanski's action in a fairly straightforward way. In this case we do not impose any constraints and the state sum follows directly from the discretized definition of the path integral of the theory. There are no ambiguities in lower dimensional simplex-amplitudes. The model turns out to be precisely the one introduced by De Pietri, Freidel, Krasnov and Rovelli in [18]. This work establishes a clear connection between that model and the effective action corresponding to one of the degenerate sectors of Plebanski's action.

Finally the model is well defined, is not topological and has a clear connection to a continuous action. It is somehow between the theory we want to define and the simpler theories we understand well but do not have local excitations (such as BF theory and gravity in lower dimensions). From this viewpoint we believe that it might be useful to explore its properties as a 'toy model' for understanding open issues in the spin foam approach to quantum gravity. Among these is the very important problem of the continuum limit (i.e., the issue of summing-over versus refining discretizations) and the interpretation of the path integral in the diffeomorphism invariant context (time evolution versus the projector/extractor operator on physical states).

7 Acknowledgments

I thank Abhay Ashtekar, Martin Bojowald and Amit Ghosh for stimulating discussion and insightful questions. I also thank John Baez, Rodolfo Gambini, Jorge Pullin, Michael Reisenberger and Carlo Rovelli for motivating remarks. I am grateful to Abhay for many suggestions that have improve the presentation of this manuscript. This work was supported in part by NSF Grant PHY-0090091, and Eberly Research Funds of Penn State.

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