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Positive Mass Theorem on Manifolds admitting Corners along a Hypersurface

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Abstract

We study a class of non-smooth asymptotically flat manifolds on which metrics fail to be C^1 across a hypersurface Σ. We first give an approximation scheme to mollify the metric, then we show that the Positive Mass Theorem [8] still holds on these manifolds if a geometric boundary condition is satisfied by metrics separated by Σ.

1 Introduction and Statement of Results

The well-known Positive Mass Theorem in general relativity was first proved by R. Schoen and S.T. Yau in [8] for smooth asymptotically flat manifolds with non-negative scalar curvature. It is interesting to know on what kind of non-smooth Riemannian manifolds their techniques and results can be generalized. In this paper we study this question in a special setting where the metric fails to be C^1 across a hypersurface.

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Let $n \geq 3$ be a dimension for which the classical PMT [8] holds. Let $\alpha \in (0,1)$ be a fixed number and M be an oriented n-dimensional smooth differentiable manifold with no boundary. We assume that there exists a compact domain $\Omega \subset M$ so that $M \setminus \Omega$ is diffeomorphic to $\mathbb{R}^n$ minus a ball and $\Sigma = \partial\Omega$ is a smooth hypersurface in M.

Definition 1. **A metric $\mathcal{G}$ admitting corners along** Σ *is defined to be a pair of* (g_-, g_+), *where* g_- *and* g_+ *are* $C^{2,\alpha}_{loc}$ *metrics on* Ω *and* $M \setminus \overline{\Omega}$ *so that they are* C^2 *up to the boundary and they induce the same metric on* Σ.

Definition 2. *Given* $\mathcal{G} = (g_-, g_+)$, *we say* $\mathcal{G}$ *is* **asymptotically flat** *if the manifold* $(M \setminus \Omega, g_+)$ *is asymptotically flat in the usual sense* (see [7]).

Definition 3. *The* **mass** *of* $\mathcal{G} = (g_-, g_+)$ *is defined to be the mass of* g_+ (see [7]) *whenever the later exits.*

One of our main motivation to study such a pair $\mathcal{G} = (g_-, g_+)$ is its implicit relation with Bartnik's quasi-local mass of the bounded Riemannian domain $(\overline{\Omega}, g_-)$. It is generally conjectured that there exists a g_+ on $M \setminus \Omega$ so that $\mathcal{G} = (g_-, g_+)$ is a minimal mass extension of $(\overline{\Omega}, g_-)$ in the sense of [1].

Under a geometric boundary condition which originated in [2], we prove the following Positive Mass Theorem for $\mathcal{G}$.

Theorem 1. *Let* $\mathcal{G} = (g_-, g_+)$ *be an asymptotically flat metric admitting corners along* Σ. *Suppose that the scalar curvature of* g_-, g_+ *is non-negative in* Ω, $M \setminus \overline{\Omega}$, *and*

$$H(\Sigma, g_-) \geq H(\Sigma, g_+), \tag{H}$$

where $H(\Sigma, g_-)$ *and* $H(\Sigma, g_+)$ *represent the mean curvature of* Σ *in* $(\overline{\Omega}, g_-)$ *and* $(M \setminus \Omega, g_+)$ *with respect to unit normal vectors pointing to the unbounded region. Then the mass of* $\mathcal{G}$ *is non-negative. Furthermore, if* $H(\Sigma, g_-) > H(\Sigma, g_+)$ *at some point on* Σ, $\mathcal{G}$ *has a strict positive mass.*

Remark 1. *Under our sign convention for the mean curvature, we have that* $H(S^{n-1}, g_o) = n - 1$, *where* S^{n-1} *is the unit sphere in* $\mathbb{R}^n$ *and* g_o *is the Euclidean metric.*

One direct corollary of this theorem is that the boundary behavior of a metric g on $\overline{\Omega}$ imposes subtle restriction on the scalar curvature of g inside Ω. For instance, we have that

Corollary 1.1. *There does not exist a metric* g *with non-negative scalar curvature on a standard ball* $\overline{B} \subset \mathbb{R}^n$ *so that* ∂B *is isometric to* S^{n-1} *and the mean curvature of* ∂B *in* $(\overline{B}, g)$ *is greater but not equal to* $n - 1$.

Based on the work of H. Bray and F. Finster [4], we have a rigidity characterization of $\mathcal{G}$ when its mass is zero.

Theorem 2. *Let $n = 3$ and g_-, g_+ satisfy all the assumptions in Theorem 1. If g_- and g_+ are at least $C^{3,\alpha}_{loc}$, then the mass of g_+ being zero implies that g_- and g_+ are flat away from Σ and they induce the same second fundamental form on Σ. Hence, (Ω, g_-) and $(M \setminus \overline{\Omega}, g_+)$ together can be isometrically identified with the Euclidean space $(\mathbb{R}^3, g_o)$.*

To illustrate the relevance of Theorem 2 to the quasi-local mass of a bounded Riemannian domain, we mention the following corollary.

Corollary 1.2. *Let (M^3, g) be a manifold with non-negative scalar curvature, possibly with boundary. Let g_σ be a metric on S^2 so that there exist two isometric embeddings $\phi_1 : (S^2, g_\sigma) \to (\mathbb{R}^3, g_o)$ and $\phi_2 : (S^2, g_\sigma) \to (M^3, g)$, where $\phi_1(S^2), \phi_2(S^2)$ each bounds a compact region Ω_1, Ω_2 in $\mathbb{R}^3$, M^3 that has connected boundary. Then, if*

$$H(\phi_2(S^2), g) \geq H(\phi_1(S^2), g_o),$$

Ω_2 is isometric to Ω_1. In particular, Ω_2 has trivial topology.

Remark 2. *If we replace S^2 by an arbitrary compact surface Σ_g with genus $g \geq 1$, under the same assumption, our argument still works to show that the region bounded by $\phi_2(\Sigma_g)$ is* **flat**.

2 Explanation of condition (H)

In this section we give a motivation for the geometric boundary condition (**H**). One will see that it can be interpreted as a statement that the scalar curvature of $\mathcal{G}$ is distributionally non-negative across Σ.

Let g be a C^2 metric in a tubular neighborhood of Σ and ν be a unit normal vector field to Σ. Let K be the Gaussian curvature of Σ with respect to the induced metric $g|_\Sigma$ and R be the scalar curvature of g. Taking trace of the Gauss equation, we have that

$$2K = R - 2Ric(\nu, \nu) + H^2 - |A|^2, \tag{1}$$

where $Ric(\nu, \nu)$ is the Ricci curvature of g along ν, H and A are the mean curvature and the second fundamental form of Σ.

Assuming that Σ evolves with speed ν, we have the following evolution formula of the mean curvature

$$D_\nu H = -Ric(\nu,\nu) - |A|^2. \tag{2}$$

It follows from (1) and (2) that

$$R = 2K - (|A|^2 + H^2) - 2D_\nu H, \tag{3}$$

which suggests that $D_\nu H$ plays a dominant role in determining the sign of R if K, H and A are known to be bounded. In particular, for a metric $\mathcal{G} = (g_-, g_+)$ with $H(\Sigma, g_-) > H(\Sigma, g_+)$, the scalar curvature of $\mathcal{G}$ across Σ looks like a positive Dirac-Delta function with support in Σ. Hence, the spirit of Theorem 1 is that PMT still holds even if the scalar curvature is only assumed to be distributionally non-negative across Σ.

Remark 3. *The geometric boundary condition* (**H**) *was first introduced by R. Bartnik in* [2], *where he suggested the static metric extension conjecture for a bounded domain in a time-symmetric initial data set.*

3 Smoothing $\mathcal{G}$ across Σ

Given $\mathcal{G} = (g_-, g_+)$ on M, we want to approximate $\mathcal{G}$ by metrics which are C^2 across Σ.

First, we use the Gaussian coordinates of Σ to modify the differential structure on M so that $\mathcal{G}$ becomes a continuous metric across Σ. Let $U_-^{2\epsilon}$ be a 2ϵ-tubular neighborhood of Σ in $(\overline{\Omega}, g_-)$ for some $\epsilon > 0$. Let

$$\Phi_- : \Sigma \times (-2\epsilon, 0] \longrightarrow U_-^{2\epsilon}$$

be a diffeomorphism so that the pull back metric $\Phi_-^*(g_-)$ has the form $\Phi_-^*(g_-) = g_{-ij}(x,t)dx^i dx^j + dt^2$, where t is the coordinate for $(-2\epsilon, 0]$, $(x^1, \ldots, x^{n-1})$ are local coordinates for Σ and i, j runs through $1, \ldots, n-1$. Similarly, we have $\Phi_+ : \Sigma \times [0, 2\epsilon) \longrightarrow U_+^{2\epsilon}$, where $U_+^{2\epsilon}$ is a 2ϵ-tubular neighborhood of Σ in $(M \setminus \Omega, g_+)$ and $\Phi_+^*(g_+) = g_{+ij}(x,t)dx^i dx^j + dt^2$. Identifying $U = U_-^{2\epsilon} \cup U_+^{2\epsilon}$ with $\Sigma \times (-2\epsilon, 2\epsilon)$, we define $\tilde{M}$ to be a possibly new differentiable manifold with the background topological space M and the differential structure determined by the open covering $\{\Omega, M\setminus\overline{\Omega}, U\}$. Since $g_-|_\Sigma = g_+|_\Sigma$, $\mathcal{G}$ becomes a continuous metric g on $\tilde{M}$. Inside $U = \Sigma \times (-2\epsilon, 2\epsilon)$, we have that

$$g = g_{ij}(x,t)dx^i dx^j + dt^2, \tag{4}$$

where $g_{ij}(x,t) = g_{-ij}(x,t)$ when $t \leq 0$ and $g_{ij}(x,t) = g_{+ij}(x,t)$ when $t \geq 0$.

Second, we mollify the metric g inside U. Let $i \in \{0,1,2\}$, we define $\mathcal{S}^i(\Sigma)$ to be the Banach space of C^i symmetric $(0,2)$ tensors on Σ equipped with the usual C^i norm and $\mathcal{M}^i(\Sigma)$ to be the open and convex subset of $\mathcal{S}^i(\Sigma)$ consisting of C^i metrics. By (4) we have a well defined path in each $\mathcal{M}^i(\Sigma)$,

$$\gamma : (-2\epsilon, 2\epsilon) \longrightarrow \mathcal{M}^2(\Sigma) \hookrightarrow \mathcal{M}^1(\Sigma) \hookrightarrow \mathcal{M}^0(\Sigma) \tag{5}$$

where

$$\gamma(t) = g_{ij}(x,t)dx^i dx^j. \tag{6}$$

By ass umption, γ is a continuous path in $\mathcal{M}^2(\Sigma)$ and a piecewise C^1 path in $\mathcal{M}^1(\Sigma)$. Hence, there exists $L > 0$ depending only on $\mathcal{G}$ such that

$$\|\gamma(t) - \gamma(s)\|_{\mathcal{M}^1(\Sigma)} \leq L|t-s|, \quad \forall s,t \in [-\frac{3}{2}\epsilon, \frac{3}{2}\epsilon]. \tag{7}$$

We choose $\phi(t) \in {C_c}^\infty([-1,1])$ to be a standard mollifier on $\mathbb{R}^1$ such that

$$0 \leq \phi \leq 1 \text{ and } \int_{-1}^{1} \phi(t)dt = 1. \tag{8}$$

Let $\sigma(t) \in {C_c}^\infty([-\frac{1}{2}, \frac{1}{2}])$ be anther cut-off function such that

$$\begin{cases} 0 \leq \sigma(t) \leq \frac{1}{100} & t \in \mathbb{R}^1 \\ \sigma(t) = \frac{1}{100} & |t| < \frac{1}{4} \\ 0 < \sigma(t) \leq \frac{1}{100} & \frac{1}{4} < |t| < \frac{1}{2}. \end{cases} \tag{9}$$

Given any $0 < \delta \ll \epsilon$, let

$$\sigma_\delta(t) = \delta^2 \sigma(\frac{t}{\delta}) \tag{10}$$

and we define

$$\begin{aligned} \gamma_\delta(s) &= \int_{\mathbb{R}^1} \gamma(s - \sigma_\delta(s)t)\phi(t)dt, \qquad s \in (-\epsilon, \epsilon) \\ &= \begin{cases} \int_{\mathbb{R}^1} \gamma(t) \left(\frac{1}{\sigma_\delta(s)} \phi(\frac{s-t}{\sigma_\delta(s)}) \right) dt, & \sigma_\delta(s) > 0 \\ \gamma(s), & \sigma_\delta(s) = 0, \end{cases} \end{aligned} \tag{11}$$

where the integral takes place in $\mathcal{S}^0(\Sigma)$. By the convexity of $\mathcal{M}^0(\Sigma)$ in $\mathcal{S}^0(\Sigma)$, γ_δ is a path in $\mathcal{M}^0(\Sigma)$. We have the following elementary lemmas concerning the property of γ_δ and its relation with γ.

Lemma 3.1. *$\gamma_\delta(s)$ is a C^2 path in $\mathcal{M}^0(\Sigma)$ and a C^1 path in $\mathcal{M}^1(\Sigma)$.*

Proof: Let $i \in \{1,2\}$. The fact that $\gamma : (-\epsilon,\epsilon) \longrightarrow \mathcal{M}^{2-i}(\Sigma)$ is C^i away from 0 implies that $\gamma_\delta : (-\epsilon,\epsilon) \longrightarrow \mathcal{M}^{2-i}(\Sigma)$ is C^i away from $[-\frac{\delta^2}{100}, \frac{\delta^2}{100}]$. For $s \in (-\frac{\delta}{4}, \frac{\delta}{4})$, $\sigma_\delta(s) = \frac{\delta^2}{100}$ and (11) turns into

$$\gamma_\delta(s) = \int_{\mathbb{R}^1} \gamma(t) \left(\frac{1}{(\frac{\delta^2}{100})} \phi\left(\frac{s-t}{(\frac{\delta^2}{100})}\right) \right) dt, \tag{12}$$

which is the standard mollification of γ by ϕ with a constant scaling factor $\frac{\delta^2}{100}$. Hence, $\gamma_\delta(s) : (-\epsilon,\epsilon) \longrightarrow \mathcal{M}^{2-i}(\Sigma)$ is smooth in $(-\frac{\delta}{4}, \frac{\delta}{4})$. □

Lemma 3.2. *$\gamma_\delta(s)$ is a C^0 path in $\mathcal{M}^2(\Sigma)$ which is uniformly close to γ and agrees with γ outside $(-\frac{\delta}{2}, \frac{\delta}{2})$.*

Proof: The continuity of $\gamma_\delta : (-\epsilon,\epsilon) \longrightarrow \mathcal{M}^2(\Sigma)$ follows directly from that of γ. The estimate

$$\begin{aligned} \|\gamma_\delta(s) - \gamma(s)\|_{\mathcal{M}^2(\Sigma)} &= \left\| \int_{\mathbb{R}^1} (\gamma(s - \sigma_\delta(s)t) - \gamma(s))\, \phi(t)dt \right\|_{\mathcal{M}^2(\Sigma)} \\ &\leq \int_{\mathbb{R}^1} \|\gamma(s - \sigma_\delta(s)t) - \gamma(s)\|_{\mathcal{M}^2(\Sigma)} \phi(t)dt \end{aligned} \tag{13}$$

shows it is uniformly close to γ. Finally, $\sigma_\delta(s) = 0$ for $|s| > \frac{\delta}{2}$ implies

$$\gamma_\delta(s) = \int_{\mathbb{R}^1} \gamma(s)\phi(t)dt = \gamma(s). \tag{14}$$

□

Lemma 3.3. $\|\gamma_\delta(s) - \gamma(s)\|_{\mathcal{M}^0(\Sigma)} \leq L\delta^2$, for $s \in (-\epsilon,\epsilon)$.

Proof: It follows from (7) that

$$\begin{aligned} \|\gamma_\delta(s) - \gamma(s)\|_{\mathcal{M}^0(\Sigma)} &\leq \int_{\mathbb{R}^1} \|\gamma(s - \sigma_\delta(s)t) - \gamma(s)\|_{\mathcal{M}^0(\Sigma)} \phi(t)dt \\ &\leq L\delta^2. \end{aligned} \tag{15}$$

□

Now we define

$$g_\delta = \begin{cases} \gamma_\delta(t) + dt^2, & (x,t) \in \Sigma \times (-\epsilon,\epsilon) \\ g, & (x,t) \notin \Sigma \times (-\epsilon,\epsilon). \end{cases} \tag{16}$$

Lemma 3.1, 3.2 and 3.3 imply that g_δ is a globally C^2 metric on $\tilde{M}$ which agrees with g outside a strip region $\Sigma \times (-\frac{\delta}{2}, \frac{\delta}{2})$ and is uniformly close to g in C^0 topology.

Next, we proceed to estimate the scalar curvature of g_δ. We use the notations defined in section 2 with a lower index δ to denote the corresponding quantities of g_δ. By (16), the vector field $\frac{\partial}{\partial t}$ is perpendicular to the slice $\Sigma \times \{t\}$ for each $t \in (-\epsilon, \epsilon)$. Hence, inside $\Sigma \times (-\epsilon, \epsilon)$, (3) shows that

$$R_\delta(x,t) = 2K_\delta(x,t) - (|A_\delta(x,t)|^2 + H_\delta(x,t)^2) - 2\frac{\partial}{\partial t}H_\delta(x,t). \tag{17}$$

We will estimate each term on the right of (17). First we note that $K_\delta(x,t)$ is determined only by $\gamma_\delta(t)$, Lemma 3.2 then implies that $K_\delta(x,t)$ is bounded by constants depending only on $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{M}^2(\Sigma)$.

To estimate $A_\delta(x,t)$ and $H_\delta(x,t)$, we compute the first order derivative of $\gamma_\delta : (-\epsilon, \epsilon) \longrightarrow \mathcal{M}^0(\Sigma)$ because of the definition

$$A_{\delta ij}(x,t) =< \frac{\partial}{\partial x^j}, \nabla^\delta_{\frac{\partial}{\partial x^i}} \partial_t >= \frac{1}{2}\frac{\partial}{\partial t}\gamma_{\delta ij}(x,t). \tag{18}$$

By (11) we have that

$$\frac{\partial}{\partial t}\gamma_{\delta ij}(x,t) = \frac{\partial}{\partial t}\int_{\mathbb{R}^1} \gamma_{ij}(t - \sigma_\delta(t)s)\phi(s)ds. \tag{19}$$

When $|t| > \frac{\delta^2}{100}$, (19) gives that

$$\begin{aligned} \frac{\partial}{\partial t}\gamma_{\delta ij}(x,t) &= \int_{\mathbb{R}^1} \frac{d}{dt}\{\gamma_{ij}(t - \sigma_\delta(t)s)\}\phi(s)ds \\ &= \int_{\mathbb{R}^1} \gamma'_{ij}(t - \sigma_\delta(t)s)\{1 - s\delta\sigma'(\frac{t}{\delta})\}\phi(s)ds. \end{aligned} \tag{20}$$

When $|t| < \frac{\delta}{4}$, (12) implies that

$$\begin{aligned} \frac{\partial}{\partial t}\gamma_{\delta ij}(x,t) &= \frac{\partial}{\partial t}\int_{\mathbb{R}^1} \gamma_{ij}(s)\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds \\ &= \int_{\mathbb{R}^1} \gamma_{ij}(s)\frac{d}{dt}\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds \\ &= (-1)\int_{\mathbb{R}^1} \gamma_{ij}(s)\frac{d}{ds}\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds \\ &= (-1)\int_{-\infty}^0 \gamma_{ij}(s)\frac{d}{ds}\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds \\ &+ (-1)\int_0^\infty \gamma_{ij}(s)\frac{d}{ds}\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds. \end{aligned} \tag{21}$$

Integrating by parts and considering the fact $\gamma(t)$ is continuous at 0, we have that

$$\begin{aligned}\frac{\partial}{\partial t}\gamma_{\delta ij}(x,t) &= \int_{\mathbb{R}^1}\gamma'_{ij}(s)\left\{\frac{100}{\delta^2}\phi(\frac{100(t-s)}{\delta^2})\right\}ds \\ &= \int_{\mathbb{R}^1}\gamma'_{ij}(t-\sigma_\delta(t)s)\phi(s)ds. \qquad (22)\end{aligned}$$

Therefore, for every $t\in(-\epsilon,\epsilon)$, we have that

$$\frac{\partial}{\partial t}\gamma_{\delta ij}(x,t)=\int_{\mathbb{R}^1}\gamma'_{ij}(t-\sigma_\delta(t)s)\left\{1-s\delta\sigma'(\frac{t}{\delta})\right\}\phi(s)ds, \qquad (23)$$

which shows that $A_\delta(x,t)$ is bounded by constants depending only on $\gamma:(-\epsilon,\epsilon)\longrightarrow\mathcal{M}^0(\Sigma)$. Since $H_\delta(x,t)=g_\delta^{ij}A_{\delta ij}$, Lemma 3.2 and (23) also imply that $H_\delta(x,t)$ is bounded by constants depending only on $\gamma:(-\epsilon,\epsilon)\longrightarrow\mathcal{M}^0(\Sigma)$.

To estimate $\frac{\partial}{\partial t}H_\delta(x,t)$, we need to compute the second order derivative of $\gamma_\delta:(-\epsilon,\epsilon)\longrightarrow\mathcal{M}^0(\Sigma)$. A similar calculation as above gives that, for $|t|>\frac{\delta^2}{100}$,

$$\begin{aligned}\frac{\partial^2}{\partial^2 t}\gamma_{\delta ij}(x,t) &= \int_{\mathbb{R}^1}\gamma''_{ij}(t-\sigma_\delta(t)s)\left\{1-s\delta\sigma'(\frac{t}{\delta})\right\}^2\phi(s)ds+ \\ &\quad \int_{\mathbb{R}^1}\gamma'_{ij}(t-\sigma_\delta(t)s)\left\{-s\sigma''(\frac{t}{\delta})\right\}\phi(s)ds \qquad (24)\end{aligned}$$

and, for $|t|<\frac{\delta}{4}$,

$$\begin{aligned}\frac{\partial^2}{\partial^2 t}\gamma_{\delta ij}(x,t) &= \int_{\mathbb{R}^1}\gamma''_{ij}(t-\sigma_\delta(t)s)\phi(s)ds+ \\ &\quad \{g_{+}{}'_{ij}(0)-g_{-ij}{}'(0)\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\}. \qquad (25)\end{aligned}$$

Since

$$\frac{\partial}{\partial t}H_\delta(x,t)=\frac{\partial}{\partial t}\left\{g_\delta{}^{ij}(x,t)\right\}A_{\delta ij}(x,t)+g_\delta{}^{ij}(x,t)\frac{\partial}{\partial t}A_{\delta ij}(x,t), \qquad (26)$$

(24), (23) and Lemma 3.2 imply that, outside $\Sigma\times[-\frac{100}{\delta^2},\frac{100}{\delta^2}]$, $\frac{\partial}{\partial t}H_\delta(x,t)$ is bounded by constants only depending on $\gamma:(-\epsilon,\epsilon)\longrightarrow\mathcal{M}^0(\Sigma)$. On the other hand, inside $\Sigma\times[-\frac{\delta^2}{100},\frac{\delta^2}{100}]$, (25) and (26) show that

$$\begin{aligned}\frac{\partial}{\partial t}H_\delta(x,t) &= \frac{\partial}{\partial t}\left\{g_\delta{}^{ij}(x,t)\right\}A_{\delta ij}(x,t)+ \\ &\quad \frac{1}{2}g_\delta{}^{ij}(x,t)\left\{\int_{\mathbb{R}^1}\gamma''_{ij}(t-\sigma_\delta(t)s)\phi(s)ds\right\}+ \\ &\quad \frac{1}{2}g_\delta{}^{ij}(x,t)\left\{g_{+}{}'_{ij}(0)-g_{-}{}'_{ij}(0)\right\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\}. \qquad (27)\end{aligned}$$

The first two terms on the right are bounded by constants depending only on $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{M}^0(\Sigma)$. For the third one, we rewrite it as

$$\frac{1}{2}\left\{{g_\delta}^{ij}(x,t) - g^{ij}(x,0)\right\}\left\{{g_+}'_{ij}(0) - {g_-}'_{ij}(0)\right\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\}$$
$$+\left\{H(\Sigma, g_+)(x) - H(\Sigma, g_-)(x)\right\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\}. \quad (28)$$

By (7), Lemma 3.3 and the fact $|t| \leq \frac{\delta^2}{100}$, we have that

$$\begin{aligned} |{g_\delta}^{ij}(x,t) - g^{ij}(x,0)| &\leq |{g_\delta}^{ij}(x,t) - g^{ij}(x,t)| + |g^{ij}(x,t) - g^{ij}(x,0)| \\ &\leq CL\delta^2 + CL\delta^2, \end{aligned} \quad (29)$$

where $C > 0$ only depends on $\mathcal{G}$. Hence, we conclude that

$$\frac{\partial}{\partial t}H_\delta(x,t) = O(1) + \left\{H(\Sigma, g_+)(x) - H(\Sigma, g_-)(x)\right\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\} \quad (30)$$

inside $\Sigma \times [-\frac{\delta^2}{100}, \frac{\delta^2}{100}]$, where $O(1)$ represents bounded quantities with bounds depending only on $\mathcal{G}$.

We summarize the features of $\{g_\delta\}$ in the following proposition.

Proposition 3.1. *Let $\mathcal{G} = (g_-, g_+)$ be a metric admitting corners along Σ. Then $\exists$ a family of C^2 metrics $\{g_\delta\}_{0<\delta\leq\delta_0}$ on $\tilde{M}$ so that g_δ is uniformly close to g on $\tilde{M}$, $g_\delta = g$ outside $\Sigma \times (-\frac{\delta}{2}, \frac{\delta}{2})$ and the scalar curvature of g_δ satisfies*

$$R_\delta(x,t) = O(1), \quad for\ (x,t) \in \Sigma \times \{\frac{\delta^2}{100} < |t| \leq \frac{\delta}{2}\} \quad (31)$$

$$R_\delta(x,t) = O(1) + \left\{H(\Sigma, g_-)(x) - H(\Sigma, g_+)(x)\right\}\left\{\frac{100}{\delta^2}\phi(\frac{100t}{\delta^2})\right\},$$
$$for\ (x,t) \in \Sigma \times [-\frac{\delta^2}{100}, \frac{\delta^2}{100}], \quad (32)$$

where $O(1)$ represents quantities that are bounded by constants depending only on $\mathcal{G}$, but not on δ.

In case $H(\Sigma, g_-) \equiv H(\Sigma, g_+)$, the following corollary generalizes a reflecting argument used by H. Bray in his proof of the Riemannian Penrose Inequality [3].

Corollary 3.1. *Given $\mathcal{G} = (g_-, g_+)$, if $H(\Sigma, g_-) \equiv H(\Sigma, g_+)$, then $\exists$ a family of C^2 metrics $\{g_\delta\}_{0<\delta\leq\delta_0}$ on $\tilde{M}$ so that g_δ is uniformly close to g on $\tilde{M}$, $g_\delta = g$ outside $\Sigma \times (-\frac{\delta}{2}, \frac{\delta}{2})$ and the scalar curvature of g_δ is uniformly bounded inside $\Sigma \times [-\frac{\delta}{2}, \frac{\delta}{2}]$ with bounds depending only on $\mathcal{G}$, but not on δ.*

4 Proof of Theorem 1

We fix the following notations. Given a function f, we let f_+ and f_- denote the positive and negative part of f, i.e. $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Given a metric g, we define the conformal Laplacian of g to be $L_g(u) = \triangle_g u - c_n R(g) u$, where $c_n = \frac{n-2}{4(n-1)}$ and $R(g)$ is the scalar curvature of g. The mass of g will be denoted by $m(g)$ if it exists. Finally, we let $C_0, C_1, C_2, \ldots$ represent constants depending only on $\mathcal{G}$.

Throughout this section, we assume that $R(g_-), R(g_+) \geq 0$ in Ω, $M \setminus \overline{\Omega}$, and $H(\Sigma, g_-)(x) \geq H(\Sigma, g_+)(x)$ for all $x \in \Sigma$.

4.1 Conformal Deformation

We want to modify $\{g_\delta\}$ on $\tilde{M}$ to get C^2 metrics with non-negative scalar curvature. For that purpose we use conformal deformation. The following fundamental lemma is due to Schoen and Yau. Interested readers may refer to [8] for a detailed proof.

Lemma 4.1. [8] *Let g be a C^2 asymptotically flat metric on $\tilde{M}$ and f be a function that has the same decay rate at ∞ as $R(g)$, then $\exists$ a number $\epsilon_0 > 0$ depending only on the C^0 norm of g and the decay rate of g, ∂g and $\partial\partial g$ at ∞ so that if*

$$\left\{ \int_{\tilde{M}} |f_-|^{\frac{n}{2}} \, dg \right\}^{\frac{2}{n}} < \epsilon_0, \tag{33}$$

then

$$\left\{ \begin{array}{rcl} \triangle_g u - c_n f u & = & 0 \\ \lim_{x \to \infty} u & = & 1 \end{array} \right. \tag{34}$$

has a C^2 positive solution u defined on $\tilde{M}$ so that $u = 1 + \frac{A}{|x|^{n-2}} + \omega$ for some constant A and some function ω, where $\omega = O(|x|^{1-n})$ and $\partial\omega = O(|x|^{-n})$.

For each δ, we consider the following equation

$$\left\{ \begin{array}{rcl} \triangle_{g_\delta} u_\delta + c_n R_{\delta-} u_\delta & = & 0 \\ \lim_{x \to \infty} u_\delta & = & 1\,. \end{array} \right. \tag{35}$$

It follows from Proposition 3.1 and assumptions on $R(g_-)$ and $R(g_+)$ that

$$\left\{ \begin{array}{ll} R_{\delta-} = 0, & \text{outside } \Sigma \times [-\frac{\delta}{2}, \frac{\delta}{2}] \\ |R_{\delta-}| \leq C_0, & \text{inside } \Sigma \times [-\frac{\delta}{2}, \frac{\delta}{2}]\,. \end{array} \right. \tag{36}$$

Therefore, (33) holds with f and g replaced by $-R_{\delta-}$ and g_δ, for sufficiently small δ. We note that ϵ_0 can be chosen to be independent on δ because of Proposition 3.1. Hence the solution to (35) exists by Lemma 4.1. We have the following L^∞ and $C^{2,\alpha}$ estimate for $\{u_\delta\}$.

Proposition 4.1. $\lim_{\delta\to 0}\|u_\delta - 1\|_{L^\infty(\tilde{M})} = 0$ *and* $\|u_\delta\|_{C^{2,\alpha}(K)} \leq C_K$. *Here* K *is any compact set in* $\tilde{M}\setminus\Sigma$ *and* C_K *only depends on* g *and* K.

Proof: It suffices to obtain the L^∞ estimate of $|u_\delta - 1|$ because, once it is established, the $C^{2,\alpha}$ estimate will follow directly from the fact $\triangle_g u_\delta = 0$ outside $\Sigma \times [-\frac{\delta}{2}, \frac{\delta}{2}]$ and the standard Schauder theory. Let $w_\delta = u_\delta - 1$, we have that

$$\triangle_{g_\delta} w_\delta + c_n R_{\delta-} w_\delta = -c_n R_{\delta-} \tag{37}$$

where $w_\delta = \frac{A_\delta}{|x|^{n-2}} + \omega_\delta$ for some constant A_δ and some function ω_δ with the decay rate in Lemma 4.1. Multiply (37) by w_δ and integrate over $\tilde{M}$,

$$\int_{\tilde{M}} (w_\delta \triangle_{g_\delta} w_\delta + c_n R_{\delta-} {w_\delta}^2)\, dg_\delta = \int_{\tilde{M}} -c_n R_{\delta-} w_\delta \, dg_\delta. \tag{38}$$

Integrating by parts and using Hölder Inequality, we have that

$$\begin{aligned}\int_{\tilde{M}} |\nabla_{g_\delta} w_\delta|^2 dg_\delta \;\; &\leq \;\; c_n\left(\int_{\tilde{M}} |R_{\delta-}|^{\frac{n}{2}}\, dg_\delta\right)^{\frac{2}{n}} \left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}} \\ &\quad + c_n\left(\int_{\tilde{M}} |R_{\delta-}|^{\frac{2n}{n+2}}\, dg_\delta\right)^{\frac{n+2}{2n}} \left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{2n}} \end{aligned} \tag{39}$$

On the other hand, the Sobolev Inequality gives that

$$\left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}} \leq C_\delta \int_{\tilde{M}} |\nabla_{g_\delta} w_\delta|^2\, dg_\delta, \tag{40}$$

where C_δ denotes the Sobolev Constant of the metric g_δ. It follows from (39), (40) and the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ that

$$\begin{aligned}\left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}} \;\; &\leq \;\; C_\delta c_n \left(\int_{\tilde{M}} |R_{\delta-}|^{\frac{n}{2}}\, dg_\delta\right)^{\frac{2}{n}} \left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}} \\ &\quad + \frac{1}{2}{C_\delta}^2 {c_n}^2 \left(\int_{\tilde{M}} |R_{\delta-}|^{\frac{2n}{n+2}}\, dg_\delta\right)^{\frac{n+2}{n}} \\ &\quad + \frac{1}{2}\left(\int_{\tilde{M}} {w_\delta}^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}}. \end{aligned} \tag{41}$$

We note that Proposition 3.1 implies that C_δ is uniformly close to the Sobolev Constant of g. Hence, for sufficiently small δ, (41) gives that

$$\left(\int_{\tilde{M}} w_\delta^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{n}} \leq C\left(\int_{\tilde{M}} |R_{\delta -}|^{\frac{2n}{n+2}}\, dg_\delta\right)^{\frac{n+2}{n}} = o(1), \quad \text{as } \delta \to 0. \quad (42)$$

This $L^{\frac{2n}{n-2}}$ estimate and (37) imply the supremum estimate for w_δ

$$\begin{aligned} \sup_{\tilde{M}} |w_\delta| &\leq C\left\{\left(\int_{\tilde{M}} w_\delta^{\frac{2n}{n-2}}\, dg_\delta\right)^{\frac{n-2}{2n}} + \left(\int_{\tilde{M}} |R_{\delta -}|^{\frac{n}{n-2}} dg_\delta\right)^{\frac{n-2}{n}}\right\} \\ &= o(1) \ \text{ as } \delta \to 0 \end{aligned} \quad (43)$$

by the standard linear theory(Theorem 8.17 in[5]). □

Now we define

$$\tilde{g}_\delta = u_\delta^{\frac{4}{n-2}} g_\delta. \quad (44)$$

It follows from Proposition 4.1 that, passing to a subsequence, $\{\tilde{g}_\delta\}$ converges to g in C^0 topology on $\tilde{M}$ and in C^2 topology on compact sets away from Σ. By the conformal transformation formulae of scalar curvature [7], we also have that

$$\tilde{R}_\delta = -c_n^{-1} u_\delta^{-(\frac{n+2}{n-2})} L_{g_\delta}(u_\delta) = u_\delta^{\frac{4}{2-n}} R_{\delta +} \geq 0, \quad (45)$$

where $\tilde{R}_\delta$ represents the scalar curvature of $\tilde{g}_\delta$.

Lemma 4.2. *The mass of $\tilde{g}_\delta$ converges to the mass of $\mathcal{G}$.*

Proof: A straightforward calculation using the definition of mass reveals that

$$m(\tilde{g}_\delta) = m(g_\delta) + (n-1)A_\delta, \quad (46)$$

where A_δ is given by the expansion $u_\delta(x) = 1 + A_\delta |x|^{2-n} + O(|x|^{1-n})$. Applying integration by parts to (35) multiplied by u_δ, we have that

$$(2-n)\omega_n A_\delta = \int_{\tilde{M}} \left[|\nabla_{g_\delta} u_\delta|^2 - c_n R_{\delta -} u_\delta^2\right] dg_\delta, \quad (47)$$

where ω_n is the volume of the $n-1$ dimensional unit sphere in $\mathbb{R}^n$. It follows from that (46) and (47) that

$$m(g_\delta) = m(\tilde{g}_\delta) + \frac{n-1}{n-2}\omega_n \int_{\tilde{M}} \left[|\nabla_{g_\delta} u_\delta|^2 - c_n R_{\delta -} u_\delta^2\right] dg_\delta. \quad (48)$$

We note that the integral term above approaches 0 by Proposition 4.1, (36) and (39). Hence, we have that

$$\lim_{\delta \to 0} m(\tilde{g}_\delta) = \lim_{\delta \to 0} m(g_\delta) = m(\mathcal{G}).$$

□

Applying the classical PMT [8] to each $\tilde{g}_\delta$, we have that $m(\tilde{g}_\delta) \geq 0$. Thus, the non-negativity of $m(\mathcal{G})$ follows directly from Lemma 4.2.

4.2 Scalar Curvature Concentration

In this subsection, we assume that there exists strict jump of mean curvature across Σ, i.e.

$$H(\Sigma, g_-)(x) > H(\Sigma, g_+)(x) \quad \text{for some } x \in \Sigma.$$

We will prove that $\mathcal{G}$ has a strict positive mass.

Since $H(\Sigma, g_-)$ and $H(\Sigma, g_+)$ both are continuous functions on Σ, we can choose a compact set $K \subset \Sigma$ so that

$$H(\Sigma, g_-)(x) - H(\Sigma, g_+)(x) \geq \eta, \quad \forall x \in K \tag{49}$$

for some fixed $\eta > 0$. By Proposition 3.1, we have that

$$R_{\delta +}(x,t) \geq \eta \left\{ \frac{100}{\delta^2} \phi(\frac{100t}{\delta^2}) \right\} - C_0, \quad \forall (x,t) \in K \times [-\tfrac{\delta^2}{100}, \tfrac{\delta^2}{100}], \tag{50}$$

which suggests that the scalar curvature of g_δ and $\tilde{g}_\delta$ has a fixed amount of concentration on K.

To exploit this fact, we use conformal deformation again to make $\tilde{g}_\delta$ even scalar flat. Since $\tilde{R}_\delta = u_\delta^{\frac{4}{2-n}} R_{\delta +} \geq 0$, $\exists$ a C^2 positive solution to the following equations

$$\begin{cases} \triangle_{\tilde{g}_\delta} v_\delta - c_n \tilde{R}_\delta v_\delta &= 0 \\ \lim_{x \to \infty} v_\delta &= 1. \end{cases} \tag{51}$$

By the maximum principle, we have that

$$0 < v_\delta \leq 1. \tag{52}$$

Now define

$$\hat{g}_\delta = v_\delta^{\frac{4}{n-2}} \tilde{g}_\delta. \tag{53}$$

Similar to previous discussion, we know that $\hat{g}_\delta$ is an asymptotically flat metric and the scalar curvature of $\hat{g}_\delta$ is identically zero. Furthermore, $m(\hat{g}_\delta)$ and $m(\tilde{g}_\delta)$ are related by

$$m(\tilde{g}_\delta) = m(\hat{g}_\delta) + \frac{n-1}{n-2}\omega_n \int_{\tilde{M}} \left[|\nabla_{\tilde{g}_\delta} v_\delta|^2 + c_n \tilde{R}_\delta v_\delta^2 \right] \, d\tilde{g}_\delta, \tag{54}$$

where $m(\hat{g}_\delta) \geq 0$ by the classical PMT. Hence, to prove $m(\mathcal{G}) > 0$, it suffices to show the integral term in (54) has a strict positive lower bound.

Proposition 4.2.

$$\inf_{\delta > 0} \left\{ \int_{\tilde{M}} \left[|\nabla_{\tilde{g}_\delta} v_\delta|^2 + c_n \tilde{R}_\delta {v_\delta}^2 \right] \, d\tilde{g}_\delta \right\} > 0 \tag{55}$$

Proof: Assume (55) is not true, passing to a subsequence, we may assume that

$$\lim_{\delta \to 0} \int_{\tilde{M}} \left[|\nabla_{\tilde{g}_\delta} v_\delta|^2 + c_n \tilde{R}_\delta {v_\delta}^2 \right] \, d\tilde{g}_\delta = 0. \tag{56}$$

Since $\tilde{R}_\delta \geq 0$, (56) is equivalent to

$$\lim_{\delta \to 0} \int_{\tilde{M}} |\nabla_{\tilde{g}_\delta} v_\delta|^2 \, d\tilde{g}_\delta = 0 \quad \text{and} \quad \lim_{\delta \to 0} \int_{\tilde{M}} \tilde{R}_\delta {v_\delta}^2 \, d\tilde{g}_\delta = 0. \tag{57}$$

Outside $\Sigma \times [-\frac{\delta}{2}, \frac{\delta}{2}]$, we have $g_\delta = g$. Hence, (51) becomes

$$\triangle_{\tilde{g}_\delta} v_\delta - c_n \left(u_\delta^{\frac{4}{2-n}} R(g)_+ \right) v_\delta = 0. \tag{58}$$

It follows from Proposition 4.1, (52) and Schauder Estimates that, passing to a subsequence, v_δ converges to a function v in C^2 topology on compact sets away from Σ. By (57), we have that

$$\int_{\tilde{M} \setminus \Sigma} |\nabla_g v|^2 \, dg = 0, \tag{59}$$

which shows that v is a constant on Ω and $\tilde{M} \setminus \overline{\Omega}$.

We claim that $v = 1$ on $\tilde{M} \setminus \overline{\Omega}$. If not, we may assume $v = \beta < 1$ by (52). We fix a $\delta_0 \in (0, \epsilon)$ and denote the region inside $\Sigma \times \{\delta_0\}$ by Ω_{δ_0}. For each $\delta < \delta_0$, we let w_δ be the solution to the following equations

$$\begin{cases} \triangle_{\tilde{g}_\delta} w_\delta = 0 & \text{on } \tilde{M} \setminus \overline{\Omega}_{\delta_0} \\ w_\delta = v_\delta & \text{on } \Sigma \times \{\delta_0\} \\ w_\delta(x) \to 1 & \text{at } \infty. \end{cases} \tag{60}$$

Since w_δ minimizes the Dirichlet energy among all functions with the same boundary values, we have that

$$\int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_{\tilde{g}_\delta} w_\delta|^2 \, d\tilde{g}_\delta \le \int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_{\tilde{g}_\delta} v_\delta|^2 \, d\tilde{g}_\delta. \tag{61}$$

On the other hand, if we choose w to solve

$$\begin{cases} \triangle_g w = 0 & \text{on } \tilde{M}\setminus\overline{\Omega}_{\delta_0} \\ w = \beta & \text{on } \Sigma\times\{\delta_0\} \\ w(x)\to 1 & \text{at } \infty, \end{cases} \tag{62}$$

we have that

$$\int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_g w|^2 \, dg = \lim_{\delta\to 0}\int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_{\tilde{g}_\delta} w_\delta|^2 \, d\tilde{g}_\delta, \tag{63}$$

because $\tilde{g}_\delta \to g$ uniformly on $\tilde{M}$ and $v_\delta \to \beta$ uniformly on $\Sigma\times\{\delta_0\}$. Hence, it follows from (57), (61) and (63) that

$$\int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_g w|^2 dg \le \lim_{\delta\to 0}\int_{\tilde{M}\setminus\overline{\Omega}_{\delta_0}} |\nabla_{\tilde{g}_\delta} v_\delta|^2 d\tilde{g}_\delta = 0, \tag{64}$$

which implies that w must be a constant. Since $\beta < 1$, we get a contradiction. Therefore, $v = 1$ on $\tilde{M}\setminus\overline{\Omega}$.

Next, we let μ, μ_δ denote the $(n-1)$-dimensional volume measure induced by g, $\tilde{g}_\delta$ on Σ and let e_δ denote the energy $\int_{\tilde{M}} |\nabla_{\tilde{g}_\delta} v_\delta|^2 \, d\tilde{g}_\delta$.

We fix $0<\theta<1$ and $0<\sigma<\epsilon$. Since $v_\delta\to 1$ uniformly on compact set away from Σ, we have that

$$v_\delta > \theta \text{ on } \Sigma_\sigma, \quad \text{for } \delta \ll 1, \tag{65}$$

where Σ_t is the slice $\Sigma\times\{t\}$. We do all the estimates inside the strip $N_\sigma = \Sigma\times[-\sigma,\sigma]$. First, we have that

$$\int_\Sigma \left\{ \int_{-\sigma}^{\sigma} |\nabla_{\tilde{g}_\delta} v_\delta(x,t)|^2 dt \right\} \, d\mu_\delta(x) \le C_1 \int_{N_\sigma} |\nabla_{\tilde{g}_\delta} v_\delta|^2 \, d\tilde{g}_\delta \le C_1 e_\delta. \tag{66}$$

Let $l_\delta(x) = \int_{-\sigma}^{\sigma} |\nabla_{\tilde{g}_\delta} v_\delta(x,t)|^2 dt$, (66) becomes

$$\int_\Sigma l_\delta(x) \, d\mu_\delta(x) \le C_1 e_\delta. \tag{67}$$

For any $k > 1, \delta > 0$, we define

$$A_{\delta,k} = \left\{ x \in \Sigma \,\middle|\, l_\delta(x) \leq k \frac{C_1 e_\delta}{\mu_\delta(\Sigma)} \right\} \tag{68}$$

$$A^K_{\delta,k} = A_{\delta,k} \cap K \tag{69}$$

$$A^K_{\delta,k,\sigma} = A^K_{\delta,k} \times [-\sigma, \sigma]. \tag{70}$$

By (67) we have that

$$\mu_\delta(A_{\delta,k}) \geq (1 - \frac{1}{k})\mu_\delta(\Sigma). \tag{71}$$

Since μ_δ is uniformly close to μ, (71) implies that

$$\mu_\delta(A^K_{\delta,k}) \geq \frac{1}{2}\mu_\delta(K) \tag{72}$$

for some fixed large k and any $\delta \ll 1$.

For any $(x,t) \in A^K_{\delta,k,\sigma}$, we have that

$$\begin{aligned} |v_\delta(x,\sigma) - v_\delta(x,t)| &\leq C_2 \int_{-\sigma}^{\sigma} |\nabla_{\tilde{g}_\delta} v_\delta|(x,t)dt \\ &\leq C_2 (2\sigma)^{\frac{1}{2}} \left\{ \int_{-\sigma}^{\sigma} |\nabla_{\tilde{g}_\delta} v_\delta|^2(x,t)dt \right\}^{\frac{1}{2}} \\ &= C_2 (2\sigma)^{\frac{1}{2}} l_\delta(x)^{\frac{1}{2}} \\ &\leq C_2 (2\sigma)^{\frac{1}{2}} \left\{ k \frac{C_1 e_\delta}{\mu_\delta(\Sigma)} \right\}^{\frac{1}{2}}. \end{aligned} \tag{73}$$

It follows from (65) that

$$v_\delta(x,t) \geq \theta - C_2(2\sigma)^{\frac{1}{2}} \left\{ k \frac{C_1 e_\delta}{\mu_\delta(\Sigma)} \right\}^{\frac{1}{2}}. \tag{74}$$

On the other hand, for $x \in A^K_{\delta,k}$, we have that

$$\int_{-\delta}^{\delta} \tilde{R}_\delta(x,t)dt \geq u_\delta^{\frac{4}{2-n}}(x) \int_{-\frac{\delta^2}{100}}^{\frac{\delta^2}{100}} \left\{ \eta \left\{ \frac{100}{\delta^2} \phi(\frac{100t}{\delta^2}) \right\} - C_0 \right\} dt. \tag{75}$$

Therefore, we have the following estimate

$$
\begin{aligned}
\liminf_{\delta\to 0} \int_{A^K_{\delta,k,\sigma}} \tilde{R}_\delta {v_\delta}^2 \, d\tilde{g}_\delta &\geq \\
\liminf_{\delta\to 0} \left\{ \left\{ \theta - C_2 \left\{ (2\sigma) k \frac{C_1 e_\delta}{\mu_\delta(\Sigma)} \right\}^{\frac{1}{2}} \right\}^2 \int_{A^K_{\delta,k,\sigma}} \tilde{R}_\delta \, d\tilde{g}_\delta \right\} &\geq \\
\theta^2 C_3 \liminf_{\delta\to 0} \left\{ \int_{A^K_{\delta,k}} \left\{ \int_{-\delta}^{\delta} \tilde{R}_\delta(x,t) dt \right\} d\mu_\delta \right\} &\geq \\
C_3 \theta^2 \eta \liminf_{\delta\to 0} \mu_\delta(A^K_{\delta,k}) &\geq \\
\frac{1}{2} C_3 \theta^2 \eta \mu(K) > 0 & \qquad (76)
\end{aligned}
$$

which is a contradiction to (57). □

We conclude that $\mathcal{G}$ has a strict positive mass in case there exists strict jump of mean curvature across Σ.

5 Zero Mass Case

Let $\mathcal{G} = (g_-, g_+)$ satisfy all the assumptions in Theorem 1 and $m(g_+) = 0$. The following corollary on $R(g_+), R(g_-)$ follows directly from Theorem 1.

Corollary 5.1. *Under the above assumptions, g_- and g_+ both have zero scalar curvature in Ω and $M \setminus \overline{\Omega}$.*

Proof: First, we assume that $R(g_-)$ is not identically zero in Ω. Let u be a positive solution to the equation

$$
\begin{cases} \triangle_{g_-} u - c_n R(g_-) u = 0 & \text{on } \Omega \\ u = 1 & \text{on } \Sigma. \end{cases} \qquad (77)
$$

Consider $\tilde{\mathcal{G}} = (\tilde{g}_-, g_+)$, where $\tilde{g}_- = u^{\frac{4}{n-2}} g_-$. Since u solves the conformal Laplacian of g_-, $\tilde{g}_-$ has zero scalar curvature. By the strong maximum principle, we have $\frac{\partial u}{\partial \nu} > 0$, where ν is the unit outward normal to Σ. A direct computation shows that

$$
H(\Sigma, \tilde{g}_-)(x) = H(\Sigma, g_-)(x) + \frac{2}{n-2} \frac{\partial u}{\partial \nu}(x). \qquad (78)
$$

Hence, $H(\Sigma, \tilde{g}_-) > H(\Sigma, g_-) \geq H(\Sigma, g_+)$. Applying Theorem 1 to $\tilde{\mathcal{G}}$, we see that $m(\tilde{\mathcal{G}}) > 0$, which is a contradiction.

Second, we assume that $R(g_+)$ is not identically zero in $M \setminus \overline{\Omega}$. Let v be a positive solution to

$$\begin{cases} \triangle_{g_+} v - c_n R(g_+) v = 0 & \text{on } M \setminus \overline{\Omega} \\ v = 1 & \text{on } \Sigma \\ v \to 1 & \text{at } \infty. \end{cases} \tag{79}$$

Consider $\hat{\mathcal{G}} = (g_-, \hat{g}_+)$, where $\hat{g}_+ = v^{\frac{4}{n-2}} g_+$. A similar argument shows that $\hat{g}_+$ is scalar flat in $M \setminus \overline{\Omega}$ and $H(\Sigma, \hat{g}_+) < H(\Sigma, g_+) \leq H(\Sigma, g_-)$. Therefore, Theorem 1 implies that $m(\hat{\mathcal{G}}) > 0$. On the other hand, we have that

$$m(\hat{\mathcal{G}}) = m(\mathcal{G}) + A, \tag{80}$$

where $v = 1 + A|x|^{2-n} + O(|x|^{1-n})$. By the maximum principle, $A \leq 0$. Hence, $m(\mathcal{G}) \geq m(\hat{\mathcal{G}}) > 0$, which is again a contradiction to the assumption that $m(\mathcal{G}) = 0$. □

Corollary 5.1 only reveals information on the scalar curvature, it would be more interesting to know if $m(\mathcal{G}) = 0$ implies that $\mathcal{G}$ is flat away from Σ. Such a type of questions has been studied by H. Bray and F. Finster in [4]. In particular, they obtained the following result concerning the mass and the curvature of a metric which can be approximated by smooth metrics in their sense.

Proposition 5.1. [4] *Suppose $\{g_i\}$ is a sequence of C^3, complete, asymptotically flat metrics on M^3 with non-negative scalar curvature and the total masses $\{m_i\}$ which converge to a possibly non-smooth limit metric g in the C^0 sense. Let U be the interior of the sets of points where this convergence of metrics is locally C^3.*

Then if the metrics $\{g_i\}$ have uniformly positive isoperimetric constants and their masses $\{m_i\}$ converges to zero, then g is flat in U.

Now we are in a position to show that, in case $n = 3$, $\mathcal{G}$ is regular cross Σ and $(M, \mathcal{G})$ is isometric to $(\mathbb{R}^3, g_o)$.

Proof of Theorem 2: First, we show that g_- and g_+ are flat in Ω and $M \setminus \overline{\Omega}$. Since g_- and g_+ are $C^{3,\alpha}_{loc}$, it follows from the proof of Proposition 4.1 that $\{\tilde{g}_\delta\}$ converges to g locally in C^3 away from Σ. By Proposition 3.1, we know that $\tilde{g}_\delta$ and g are uniformly close on $\tilde{M}$, hence $\{\tilde{g}_\delta\}$ has uniformly positive isoperimetric constants. By Lemma 4.2, we know that $\lim_{\delta \to 0} m(\tilde{g}_\delta) = 0$. Therefore, g_- and g_+ are flat by Proposition 5.1.

Second, we show that $A_- = A_+$, where A_- and A_+ are the second fundamental forms of Σ in $(\overline{\Omega}, g_-)$ and $(M \setminus \Omega, g_+)$. Taking trace of the

Codazzi equation and using the fact that g_-, g_+ is flat, we have that

$$\begin{cases} div_{g_\sigma} A_- &= \nabla H(\Sigma, g_-) \\ div_{g_\sigma} A_+ &= \nabla H(\Sigma, g_+), \end{cases} \tag{81}$$

where g_σ is the induced metric $g_-|_\Sigma = g_+|_\Sigma$. On the other hand, Theorem 1 implies that $H(\Sigma, g_-) \equiv H(\Sigma, g_+)$ on Σ. Hence,

$$div_{g_\sigma}(A_- - A_+) = 0 \text{ and } tr_{g_\sigma}(A_- - A_+) = 0. \tag{82}$$

We recall the fact that any divergence free and trace free $(0,2)$ symmetric tensor on (S^2, g_σ) must vanish identically [6], thus we conclude that $A_- = A_+$. Now it follows from the fundamental theorem of surface theory in $\mathbb{R}^3$ that $\mathcal{G}$ is actually C^2 across Σ. The classical PMT [8] then implies that $(M, \mathcal{G})$ is isometric to $\mathbb{R}^3$ with the standard metric. □

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