Heterotic compactifications with principal bundles for general groups and general levels

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Abstract

We examine to what extent heterotic string worldsheets can describe arbitrary $E_8 \times E_8$ gauge fields. The traditional construction of heterotic strings builds each E_8 via a $\mathrm{Spin}(16)/\mathbb{Z}_2$ subgroup, typically realized as a current algebra by left-moving fermions, and as a result, only E_8 gauge fields reducible to $\mathrm{Spin}(16)/\mathbb{Z}_2$ gauge fields are directly realizable in standard constructions. However, there exist perturbatively consistent E_8 gauge fields which cannot be reduced to $\mathrm{Spin}(16)/\mathbb{Z}_2$ and so cannot be described within standard heterotic worldsheet constructions. A natural question to then ask is whether there exists any (0,2) super-conformal field theory (SCFT) that can describe such E_8 gauge fields. To answer this question, we first show how each 10-dimensional E_8 partition

function can be built up using other subgroups than $\text{Spin}(16)/\mathbb{Z}_2$, then construct "fibered WZW models" which allow us to explicitly couple current algebras for general groups and general levels to heterotic strings. This technology gives us a very general approach to handling heterotic compactifications with arbitrary principal bundles. It also gives us a physical realization of some elliptic genera constructed recently by Ando and Liu.

CONTENTS

1 Introduction

In the last few years there has been a great deal of interest in the "landscape" program as a mechanism for extracting phenomenological predictions from string theory by doing statistics on sets of potential vacua. One of the potential problems with this program is that the potential vacua are classified by low-energy effective supergravity theories, and it is not clear to what extent all possible supergravity theories can be described within string theory [1,2].

In this paper we will analyze examples potentially lacking UV completions, in heterotic strings. Specifically, we begin by observing that not all principal E_8 bundles with connection that satisfy the conditions for a supergravity vacuum can be described within traditional formulations of perturbative heterotic string theory. The basic problem is that traditional heterotic string constructions build each E_8 from a $\mathrm{Spin}(16)/\mathbb{Z}_2$ subgroup and so can only describe those E_8 bundles with connection reducible to $\mathrm{Spin}(16)/\mathbb{Z}_2$. However, not all principal E_8 bundles with connection are reducible to $\mathrm{Spin}(16)/\mathbb{Z}_2$, and those which cannot be so reduced, cannot be described within traditional heterotic string constructions.

That lack of reducibility suggests that there may be a problem with the existence of UV completions for such heterotic supergravity theories. However, we point out that there exists evidence from string duality that suggests UV completions for these exotic heterotic supergravities should still exist, and in the rest of the paper we go on to build new worldsheet theories which can be used to describe more general E_8 bundles with connection than the traditional constructions.

This paper can be broken into three main sections:

1. After initially reviewing the construction of E_8 bundles via $\mathrm{Spin}(16)/\mathbb{Z}_2$ subbundles in section 2, in sections 3 and 4 we analyze the extent to which E_8 bundles with connection can be described by the usual fermionic realization of the heterotic string. We find that there is a topological obstruction to describing certain E_8 bundles in dimension 10, but more alarmingly, in lower dimensions there is an obstruction to describing all gauge fields. In particular, we describe some examples of E_8 bundles with connection in dimension less than 10, which satisfy the usual constraints for a perturbative string vacuum but which cannot be described by traditional worldsheet realizations of the heterotic string. This seems to suggest that not all E_8 bundles with connection can be realized perturbatively. However, in section 5 we observe that other evidence such as F theory calculations suggests that, in fact, the other E_8 bundles with connection *can* be realized perturbatively, just not with traditional constructions. In the rest of the paper, we describe alternative constructions of heterotic strings which can be used to describe the "exceptional" gauge fields above.

2. The next part of this paper, section 6, is a discussion of alternative constructions of each E_8 in a 10-dimensional theory. The usual fermionic construction builds each E_8 using a $\mathrm{Spin}(16)/\mathbb{Z}_2$ subgroup — the left-moving worldsheet fermions realize a $\mathrm{Spin}(16)$ and a left-moving $\mathbb{Z}_2$ orbifold realizes the $/\mathbb{Z}_2$. However, there are other subgroups of E_8 that can also be used instead, such as $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ and

$SU(9)/\mathbb{Z}_3$. At the level of characters of affine algebras, such constructions have previously been described in e.g., [3]. We check that the 10-dimensional partition function of current algebras realizing other[1] E_8 subgroups correctly reproduces the usual self-dual modular invariant partition function.

3. To make this useful we need to understand how more general current algebras can be fibered nontrivially over a base, and so in the third part of this paper, sections 7 and 8, we develop[2] and analyze "fibered WZW models," which allow us to work with heterotic $(0, 2)$ supersymmetric SCFTs in which the left-movers couple to some general G-current algebra at level k, for general G and k, fibered nontrivially over the target space. Only for certain G and k can these conformal field theory (CFTs) be used in critical heterotic string compactifications, but the general result is of interest to the study of heterotic CFTs. The construction of these theories is interesting: bosonizing the left-movers into a WZW model turns quantum features of fermionic realizations into classical features, and so to understand the resulting theory requires mixing such classical features against quantum effects such as the chiral anomaly of the right-moving fermions. This construction also gives us a physical realization of some elliptic genera constructed in the mathematics community previously. The generalization of the anomaly cancelation condition that we derive in our model, for example, was independently derived by mathematicians thinking about generalizations of elliptic genera.

To a large extent, the three parts of this paper can nearly be read independently of one another. For example, readers who only wish to learn about fibered WZW model constructions should be able to read sections 7 and 8 without having mastered the earlier material.

Higher-level Kac–Moody algebras in heterotic compactifications have been considered previously in the context of free fermion models, see, for example, [9,10] which discuss their phenomenological virtues. In [10], for example, the higher-level Kac–Moody algebras are constructed by starting with critical heterotic strings realized in the usual fashion and then orbifolding in such a way as to realize higher-level Kac–Moody algebras from within the original

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[1]For example, the $(SU(5) \times SU(5))/\mathbb{Z}_5$ subgroup describes a $\mathbb{Z}_5$ orbifold of an $SU(5) \times SU(5)$ current algebra, just as the traditional $Spin(16)/\mathbb{Z}_2$ subgroup describes a $\mathbb{Z}_2$ orbifold of a $Spin(16)$ current algebra (realized by free fermions).

[2]After the initial publication of this paper it was pointed out to us that chiral fibered WZW models with $(0, 1)$ supersymmetry have been previously considered, under the name "lefton, righton Thirring models," see, for example, [4–8]. We develop the notion further, by studying anomaly cancelation, spectra, elliptic genera, and so forth in chiral fibered WZW models with $(0, 2)$ supersymmetry.

level 1 structure. However, in each of those previous works, the higher-level Kac–Moody algebras were all essentially embedded in an ambient level 1 algebra, the ordinary E_8 algebra. We are not aware of any previous work discussing heterotic compactifications with higher-level Kac–Moody algebras that realize those algebras directly, without an embedding into some ambient algebra, as we do in this paper with "fibered WZW" models.

2   Worldsheet obstruction in standard constructions

How does one describe an E_8 bundle on the worldsheet? It is well known how to construct the E_8 current algebra, and bundles with structure groups of the form $SU(n) \times U(1)^m$ are also understood in this language, but to understand more exotic cases, let us carefully work through the details for general nontrivial bundles.

For each E_8, there are[3] 16 left-moving fermions which couple to the pullback of a real vector bundle on the target space associated to a principal Spin(16) bundle. The worldsheet left-moving fermion kinetic terms have the form

$$h_{\alpha\beta}\lambda_-^\alpha D\lambda_-^\beta$$

where $h_{\alpha\beta}$ is a fiber metric on a real rank 16 vector bundle, and D is a covariant derivative which implicitly includes the pullback of a connection on such a bundle, so we see that we can describe only Spin(16) gauge fields. The worldsheet GSO projection is equivalent to a $\mathbb{Z}_2$ orbifold in which each of those fermions is acted upon by a sign. Performing the GSO projection is therefore equivalent to projecting the Spin(16) bundle to a Spin(16)/$\mathbb{Z}_2$ bundle, and the surviving adjoint and spinor representations of Spin(16)/$\mathbb{Z}_2$ are built into an E_8 bundle, into which the Spin(16)/$\mathbb{Z}_2$ bundle injects. (The Spin(32)/$\mathbb{Z}_2$ heterotic string is much simpler; the 32 left-moving spinors couple to a vector bundle associated to a principal Spin(32) bundle, and the GSO projection projects to Spin(32)/$\mathbb{Z}_2$.)

Factors of $\mathbb{Z}_2$ will play an important role in what follows, so let us take a moment to carefully check the statement above. Of the groups $O(16)$, $SO(16)$, Spin(16), and Spin(16)/$\mathbb{Z}_2$, only Spin(16)/$\mathbb{Z}_2$ is a subgroup of E_8

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[3]There is, of course, also a representation of the bosonic string in terms of chiral abelian bosons. However, that abelian bosonic representation can describe even fewer bundles with connection than the fermionic representation — essentially, only those in which the bundle with connection is reducible to a maximal torus — and so we shall focus on the fermionic presentation.

[11, 12], so after performing the GSO projection we had better recover a $\text{Spin}(16)/\mathbb{Z}_2$ bundle. Also, the fact that the adjoint representation of E_8 decomposes into the adjoint representation of so(16) plus one chiral spinor gives us another clue — if the subgroup were SO(16), then no spinors could appear in the decomposition. The $\mathbb{Z}_2$ quotient in $\text{Spin}(16)/\mathbb{Z}_2$ projects out one of the chiral spinors but not the other, giving us precisely the matter that we see perturbatively. Furthermore, $\text{Spin}(16)/\mathbb{Z}_2$ does not have a 16-dimensional representation, so the left-moving fermions cannot be in a vector bundle associated to a principal $\text{Spin}(16)/\mathbb{Z}_2$ bundle. Instead, they couple to a $\text{Spin}(16)$ bundle, and the GSO projection plays a crucial role.

Any data about a bundle with connection on the target space must be encoded in the fermion kinetic terms

$$h_{\alpha\beta}\lambda_-^{\alpha} D\lambda_-^{\beta}.$$

Since the only data encoded concerns $\text{Spin}(16)$ bundles, if we had an E_8 bundle with connection that could not be reduced to $\text{Spin}(16)/\mathbb{Z}_2$ and then lifted to $\text{Spin}(16)$, we would not be able to describe it on the worldsheet using the conventional fermionic realization of the heterotic string.

So far we have described what worldsheet structures define the E_8 bundle on the target space. Let us now think about the reverse operation. Given an E_8 bundle, what does one do to construct the corresponding heterotic string? First, one reduces the structure group from E_8 to $\text{Spin}(16)/\mathbb{Z}_2$, if possible, and then lifts from $\text{Spin}(16)/\mathbb{Z}_2$ to $\text{Spin}(16)$, if possible. The resulting $\text{Spin}(16)$ bundle defines the left-moving worldsheet fermions.

The catch is that not all E_8 bundles are reducible to $\text{Spin}(16)/\mathbb{Z}_2$ and not all $\text{Spin}(16)/\mathbb{Z}_2$ bundles can be lifted to $\text{Spin}(16)$ bundles. The second obstruction is defined by an analog of a Stiefel–Whitney class, which is more or less reasonably well understood. We will be primarily concerned in this paper with the first obstruction, which to our knowledge has not been discussed in the physics literature previously.

3 Principal E_8 bundles

3.1 Reducibility of principal E_8 bundles

In this section, we shall briefly outline[4] the technical issues involved in computing the obstruction to reducing an E_8 bundle to a $\text{Spin}(16)/\mathbb{Z}_2$ bundle.

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[4]We are indebted to A. Henriques for a lengthy discussion in which he explained the points of this section, and for giving us permission to repeat his homotopy analysis here.

We shall find that the only obstruction is an element of $H^{10}(M, \mathbb{Z}_2)$, where M is the spacetime 10-manifold on which the E_8 bundle lives.

An E_8 bundle is the same thing as a map $M \to BE_8$. In order to reduce the structure group of the bundle to $\mathrm{Spin}(16)/\mathbb{Z}_2$, we want to lift the map above to a map $M \to B\,\mathrm{Spin}(16)/\mathbb{Z}_2$. In fact, for our purposes, we can equivalently consider $\mathrm{BSO}(16)$, which is technically somewhat simpler.

In general, if M is simply connected (which we shall assume throughout this section), then the obstructions to reducing a principal G bundle on M to a principal H bundle for $H \subset G$ live in $H^k(M, \pi_{k-1}(G/H))$, which can be proven with Postnikov towers. Since this technology is not widely used in the physics community, let us expound upon this method for $H = 1$, and study the obstructions to trivializing a principal G bundle which, from the general statement above, live in $H^k(M, \pi_{k-1}(G))$. It is well known that a principal G bundle can be trivialized if its characteristic classes vanish, and so one would be tempted to believe that the group $H^k(M, \pi_{k-1}(G))$ corresponds to characteristic classes, but the correct relationship[5] is more complicated. In the case of E_8 bundles and $U(n)$ bundles, it is straightforward to check that the groups in which the obstructions live are the same as the ones the characteristic classes live in, making the distinction obscure: for E_8, since $\pi_3(E_8) = \mathbb{Z}$ is the only nonzero homotopy group in dimension 10 or less, the obstructions to trivialing a principal E_8 bundle on a manifold of dimension 10 or less live in $H^4(M, \mathbb{Z})$, same as the characteristic class, and, for $U(n)$ bundles, $\pi_i(U(n))$ is $\mathbb{Z}$ for i odd and less than $2n$, so the obstructions to trivializing $U(n)$ bundles live in $H^{even}(M, \mathbb{Z})$, the same groups as the Chern classes. Principal $O(n)$ bundles are more confusing and better illustrate the distinction between obstructions and characteristic classes. The homotopy groups

$$\pi_{3+8k}(O(n)) = \mathbb{Z} = \pi_{7+8k}(O(n))$$

(for n sufficiently large) and the corresponding obstructions correspond to the Pontryagin classes in degrees any multiple of four. However, there are additional $\mathbb{Z}_2$-valued characteristic classes of $O(n)$ bundles, known as the Stiefel–Whitney classes, and

$$\pi_{0+8k}(O(n)) = \mathbb{Z}_2 = \pi_{1+8k}(O(n))$$

(for n sufficiently large) corresponding to the first two Stiefel–Whitney classes w_1, w_2. However, other homotopy groups vanish

$$\pi_{2+8k}(O(n)) = 0 = \pi_{4+8k}(O(n)) = \pi_{5+8k}(O(n))$$

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[5]We would like to thank M. Ando for a patient explanation of this point.

and so there are no obstructions living in $H^3(M, \mathbb{Z}_2)$, $H^5(M, \mathbb{Z}_2)$, or $H^6(M, \mathbb{Z}_2)$, for example, despite the fact that there are Stiefel–Whitney classes in those degrees. An $O(n)$ bundle can be trivialized only if its characteristic classes all vanish, and yet we have found no obstructions corresponding to many Stiefel–Whitney classes, which appears to be a contradiction. Part of the resolution is that the relationship between characteristic classes and obstructions is complicated: for example, the degree four obstruction is $p_1/2$ and is only defined if the lower-order obstructions vanish (so that p_1 is even). Higher-order obstructions have an even more complicated relationship. At the same time, one can use Steenrod square operations and the Wu formula to determine many higher-order Stiefel–Whitney classes from lower ones — for example, if $w_1 = w_2 = 0$ then necessarily $w_3 = 0$. The upshot of all this is that if the obstructions all vanish, then the characteristic classes will all vanish, and so the bundle is trivializable, and there is no contradiction.

In any event, the obstructions to reducing a principal E_8 bundle to a principal $\mathrm{Spin}(16)/\mathbb{Z}_2$ bundle live in $H^k(M, \pi_{k-1}(F))$, where $F = E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ denotes the fiber of

$$B\,\mathrm{Spin}(16)/\mathbb{Z}_2 \longrightarrow BE_8.$$

We can compute the homotopy groups of that quotient using the long exact sequence in homotopy induced by the fiber sequence

$$E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2) \longrightarrow B\,\mathrm{Spin}(16)/\mathbb{Z}_2 \longrightarrow BE_8.$$

One can compute the following:

π_i for $i =$		1	2	3	4	5	6	7	8	9	10	11	12
$E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$		0	$\mathbb{Z}_2$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$B\mathrm{Spin}(16)/\mathbb{Z}_2$		0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
BE_8		0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0

We used the following facts to compute this table.

First, we know that E_8 looks like a $K(\mathbb{Z}, 3)$ up to dimension 14, and we also know $\pi_*(\mathrm{BSO})$ by Bott periodicity (see, for example, [13, Section 4.2]). So, to determine the long exact sequence in the relevant range, we only need to compute $\pi_4(B\mathrm{Spin}(16)/\mathbb{Z}_2) \to \pi_4(BE_8)$.

It turns out that $\pi_4(B\mathrm{Spin}(16)/\mathbb{Z}_2) \to \pi_4(BE_8)$ is an isomorphism. This is the case since $\mathrm{Spin}(16)/\mathbb{Z}_2 \to E_8$ comes from an inclusion of simply laced root systems and the SU(2)s coming from the roots are the generators of π_3.

The obstructions in $H^k(M, \pi_{k-1}(F))$ are pulled back from universal obstructions

$$H^k(BE_8, \pi_{k-1}(F)).$$

By the previous observation, this is isomorphic to $H^k(K(\mathbb{Z}, 4), \pi_{k-1}(F))$ in the relevant range.

From the table above, there are three possible obstructions, living in the groups

$$H^3(M, \mathbb{Z}_2), \quad H^9(M, \mathbb{Z}), \quad H^{10}(M, \mathbb{Z}_2).$$

The first of these we can eliminate immediately, since it is a pullback from $H^3(BE_8, \mathbb{Z}_2)$ but that group vanishes.

Next we check $H^9(K(\mathbb{Z}, 4), \mathbb{Z}) = \mathbb{Z}_3$ and $H^{10}(K(\mathbb{Z}, 4), \mathbb{Z}_2) = \mathbb{Z}_2 + \mathbb{Z}_2$. These groups will yield two potential obstructions: an element of $H^9(M, \mathbb{Z})$, pulled back from a class in $H^9(K(\mathbb{Z}, 4), \mathbb{Z})$, and an element of $H^{10}(M, \mathbb{Z}_2)$, pulled back from a class in $H^{10}(K(\mathbb{Z}, 4), \mathbb{Z}_2)$.

In principle, the universal obstruction in $H^9(K(\mathbb{Z}, 4), \mathbb{Z})$ can be nonzero because it agrees with the k-invariant of KO at $p = 3$. Its name is "Milnor's Q_1." It is a cohomology operation $Q_1 : H^n(-, \mathbb{Z}) \to H^{n+5}(-, \mathbb{Z})$.

So, let us concentrate at $p = 3$ for a moment. The question is, does there exist a 10-dimensional manifold M with a four-dimensional cohomology class x on which Q_1 is nonzero? It can be shown by a cobordism invariance argument [14] that on any oriented 10-manifold M, there is no such cohomology class.

Thus, so long as our 10-manifold M is oriented, the potential obstruction in $H^9(M, \mathbb{Z})$ always vanishes, leaving us with only one potential obstruction to reducibility of the structure group of the E_8 bundle, living in $H^{10}(M, \mathbb{Z}_2)$. Unfortunately, this obstruction can sometimes be nonzero. (Examples of oriented 10-manifolds with nonreducible E_8 bundles are described in [15], albeit to different ends.)

Although we have been unable to find any prior references discussing this obstruction, we have found some that came close to uncovering it. For example, Witten [16] points out the necessity of reducing E_8 to $\mathrm{Spin}(16)/\mathbb{Z}_2$ and also looks for obstructions, but only up to degree six: he observes that for compactifications to four dimensions, such a reduction is always possible.

3.2 Target space interpretation

So far we have discussed a technical issue that arises when trying to understand certain "exotic" E_8 bundles on a heterotic string worldsheet. Next, we shall discuss the interpretation of this obstruction in the 10-dimensional supergravity.

For chiral fermions in dimension $8k + 2$, it is known [17, p. 206] that the number of zero modes of the chiral Dirac operator is a topological invariant mod 2. (The number of zero modes of the nonchiral Dirac operator is a topological invariant mod 4.) In particular, since the 10-dimensional gaugino is a Majorana–Weyl spinor, the number of positive chirality gaugino zero modes is a topological invariant mod 2. For E_8 bundles, this topological invariant was discussed in [15, Section 3], where it was labeled $f(a)$ (where a is the analog of the Pontryagin invariant for E_8 bundles).

Curiously, the element of $H^{10}(X, \mathbb{Z}_2)$ that defines the obstruction to reducing an E_8 bundle to a $\mathrm{Spin}(16)/\mathbb{Z}_2$ bundle is that same invariant [18]. In other words, the number of chiral gaugino zero modes of the 10-dimensional Dirac operator is odd precisely when the E_8 bundle cannot be reduced to $\mathrm{Spin}(16)/\mathbb{Z}_2$ and hence cannot be described perturbatively on a heterotic string worldsheet.

This makes the current phenomenon sound analogous to the anomaly in four-dimensional SU(2) gauge theories with an odd number of left-handed fermion doublets, described in [19]. There, the anomaly could be traced to the statement that the five-dimensional Dirac operator had an odd number of zero modes, which translated into the statement that the relevant operator determinant in the four-dimensional theory was not well behaved under families of gauge transformations. There, however, it was the Dirac operator in one higher dimension that had an odd number of zero modes, whereas in the case being studied in this paper it is the Dirac operator in 10 dimensions, not 11 dimensions, that has an odd number of zero modes. Also, in the anomaly studied in [19], the fact that $\pi_4(\mathrm{SU}(2))$ is nonzero was crucial, whereas by contrast $\pi_{10}(E_8)$ vanishes. In fact that last fact was used in [17, p. 198] to argue that there should not be any global gauge anomalies in heterotic $E_8 \times E_8$ strings.

4 Connections

So far we have discussed reducibility of topological E_8 bundles to $\mathrm{Spin}(16)/$ $\mathbb{Z}_2$ bundles, but to realize a given E_8 gauge field in standard heterotic string

constructions, we must also reduce the connection on the bundle, not just the bundle itself.

In particular, on a principal G bundle, even a trivial principal G bundle, one can find connections with holonomy that fill out all of G and so cannot be understood as coming from connections on any principal H bundle for H a subgroup of G. It is easy to see this statement locally [20]: one can pick a connection whose curvatures at points in a small open set generate the Lie algebra of G and then the local holonomy will generate (the identity component of) G, and since our bundles are reducible (in fact, trivial) locally, one gets the desired result.

However, for our purposes it does not suffice to consider reducibility of generic connections. After all, for a perturbative vacuum of heterotic string theory, the connection must satisfy some stronger conditions: it must satisfy the Donaldson–Uhlenbeck–Yau equation, the curvature must be of type $(1, 1)$, and it must satisfy anomaly cancelation.

However, even when the Donaldson–Uhlenbeck–Yau condition is satisfied, it is still possible to have bundles with connection such that the bundle is reducible but not the connection. Examples of this were implicit in [21], which discussed how stability of bundles depends upon the metric. Briefly, the Kähler cone breaks up into subcones, with a different moduli space of bundles on each subcone. Some stable irreducible bundles will, on the subcone wall, become reducible. This means that the holomorphic structure (and also the holonomy of the connection) was generically irreducible but becomes reducible at one point. For this to be possible at the level of holomorphic structures means that the bundle was always topologically reducible. Thus, implicitly in [21] there were examples of topologically reducible bundles with irreducible connections satisfying the Donaldson–Uhlenbeck–Yau condition.

We shall construct some examples on K3 surfaces of E_8 gauge fields which satisfy all the conditions above for a perturbative heterotic string vacuum but which cannot be reduced to $\mathrm{Spin}(16)/\mathbb{Z}_2$.

4.1 Moduli spaces of flat connections

As a quick warm-up, let us briefly study how the moduli space of flat E_8 connections on T^2 arises in a heterotic compactification on T^2. The moduli space of flat E_8 connections on T^2 and one component of the moduli space of flat $\mathrm{Spin}(16)/\mathbb{Z}_2$ connections both have the form $(T^2)^8/W$, where W is the respective Weyl group. However, $W(D_8) \subset W(E_8)$ and

in fact $|W(E_8)/W(D_8)| = 135$, so the component of the moduli space of flat $\text{Spin}(16)/\mathbb{Z}_2$ connections is a 135-fold cover of the moduli space of flat E_8 connections.

The projection to the moduli space of flat E_8 connections is induced by T-dualities. The discrete automorphism group (T-dualities) of the heterotic moduli space includes a $O(\Gamma_8)$ factor, which acts as the E_8 Weyl group action above. When forming the moduli space, we mod out by this factor, and so we get the moduli space of flat E_8 connections, rather than that of $\text{Spin}(16)/\mathbb{Z}_2$ connections.

4.2 Analysis of connections

In this section we will construct an example[6] of an E_8 gauge field on a Calabi–Yau X, which cannot be reduced to $\text{Spin}(16)/\mathbb{Z}_2$ but which does satisfy the conditions for a consistent perturbative vacuum, namely

$$F_{0,2} = F_{2,0} = g^{i\bar{j}}F_{i\bar{j}} = 0$$

and that

$$\text{Tr } F^2 - \text{Tr } R^2$$

is cohomologous to zero.

To build this example, we use the fact that E_8 contains a subgroup $(\text{SU}(5) \times \text{SU}(5))/\mathbb{Z}_5$. This subgroup is not a subgroup of $\text{Spin}(16)/\mathbb{Z}_2$, and so an $\text{SU}(5) \times \text{SU}(5)/\mathbb{Z}_5$ gauge field whose holonomy is all of the group is an example of an E_8 gauge field that cannot be reduced to $\text{Spin}(16)/\mathbb{Z}_2$. To construct such an $(\text{SU}(5) \times \text{SU}(5))/\mathbb{Z}_5$ gauge field, it suffices to construct an $\text{SU}(5) \times \text{SU}(5)$ gauge field, then take the image under a $\mathbb{Z}_5$ action (whose existence is always guaranteed).

The perturbative anomaly cancelation condition is stated simply as a matching of $\text{Tr } F^2$ and $\text{Tr } R^2$ in cohomology, but for general groups the precise interpretation of that statement in terms of degree four characteristic classes. For an $\text{SU}(5) \times \text{SU}(5)$ bundle, anomaly cancelation should be

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[6]We would like to thank R. Thomas for an extensive discussion of this matter in late March and April 2006.

interpreted as the statement

$$c_2(\mathcal{E}_1) + c_2(\mathcal{E}_2) = c_2(TX)$$

where $\mathcal{E}_1$, $\mathcal{E}_2$ are principal SU(5) bundles.

As a check of anomaly cancelation in this context, suppose that SU(n) is a subgroup of SU(5). We can either embed the SU(n) in Spin(16)/$\mathbb{Z}_2$, and then build up a standard perturbative worldsheet, or we can embed it in SU(5) × SU(5)/$\mathbb{Z}_5$, which does not admit a perturbative description. This gives two paths to E_8, but these two paths commute.[7]

A careful reader might point out another subtlety in the statement of anomaly cancelation. For example, the degree four characteristic class of an SU(n)/$\mathbb{Z}_n$ bundle obtained from an SU(n) bundle $\mathcal{E}$ can be naturally taken to be[8] $c_2(\mathrm{End}_0\mathcal{E}) = 2nc_2(\mathcal{E})$, so in the case above there could plausibly be extra numerical factors. In any event, our methods are sufficiently robust that such modifications of the anomaly cancelation condition will not change the fact that there exist families of examples.[9] Put another way, nonreducible connections are common, not rare or unusual.

We need to find a bundle with connection that not only satisfies anomaly cancelation, but also the Donaldson–Uhlenbeck–Yau condition. By working with SU(n) gauge fields, we can translate such questions about connections into algebraic geometry questions. In particular, the requirement that the gauge field satisfy the Donaldson–Uhlenbeck–Yau equation becomes the requirement that the corresponding holomorphic rank 5 vector bundle be stable.

Ordinarily, checking stability can be rather cumbersome, but there is an easy way to build examples sufficient for our purposes. We can build holomorphic vector bundles on elliptic fibrations with section using the techniques of [23, 24]. (See also, e.g., [25–30] for some more modern applications

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[7]We would like to thank A. Knutson for a helpful discussion of this matter at the end of March 2006. Also, note that the automorphism exchanging the two SU(5)'s does not extend to E_8, which can also be seen from the asymmetry of the decomposition of the adjoint representation of E_8 under the subgroup above. Another way to see this is from the fact that the $\mathbb{Z}_5$ one quotients by is not symmetric under such a switch.

[8]Alternatively, we can get the same result from the fact that the trace in the adjoint rep of SU(n) is $2n$ times the trace in the fundamental rep [22], which is also twice the dual Coxeter number.

[9]If the reader objects that a wandering factor of 5 or 10, as might be expected in some interpretations of SU(5)^2/$\mathbb{Z}_5$, would make examples on K3's difficult, the quintic three-fold has c_2 divisible by 5, in fact $c_2 = 10H^2$, and there exist further examples there.

of the same technology.) Furthermore, these bundles are automatically stable (for metrics in the right part of the Kähler cone). One must specify a (spectral) cover of the base of the fibration, plus a line bundle on that cover.

Following the conventions of [24], to describe an $SU(r)$ bundle on an elliptic K3 with section we use a spectral cover describing an r-fold cover of the base of the fibration. The spectral cover will be in the class $|r\sigma + kf|$ where σ is the class of the section and f is the class of the fiber, and k is the second Chern class of the bundle [24, p. 5].

Furthermore, there is a line bundle that must be specified on that cover, and it can be shown [24] that that line bundle must have degree $-(r + g - 1)$, where $g = rk - r^2 + 1$ is the genus of the spectral cover (as it is a cover of $\mathbf{P}^1$, it is some Riemann surface). If the spectral curve is reduced and irreducible then the corresponding bundle will be stable; Bertini's theorem implies that such curves exist in the linear system.

In the present case, we want a holomorphic vector bundle of rank 5, $c_1 = 0$, $c_2 = 12$. The spectral cover that will produce such a result is in the linear system $|5\sigma + 12f|$. The genus of such a curve is 36, and the line bundle has degree -40. The dimension of the moduli space of spectral data is then $2 \times 36 = 72$.

So far we have established the existence of stable $SU(5) \times SU(5)$ bundles satisfying all the conditions for a consistent perturbative vacuum; we still need to demonstrate that the holonomy of the connection cannot be reduced below $SU(5) \times SU(5)$. To do this we can apply the recent work [32], which says that it is sufficient for each factor to be irreducible and to have irreducible second symmetric power. As this will be generically true [31], we see that the holonomy cannot be reduced below $SU(5) \times SU(5)$, and so by projecting along a $\mathbb{Z}_5$ automorphism we have a family of $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundles with the desired properties.

Thus, using the embedding of $(SU(5) \times SU(5))/\mathbb{Z}_5$ in E_8, we now have a family of E_8 bundles with connection on K3's which satisfy all the requirements for a consistent perturbative vacuum, but which cannot be reduced to $Spin(16)/\mathbb{Z}_2$ and so cannot be described with standard constructions of heterotic strings.

4.3 Low-energy theory

Compactification on a bundle with structure group $(SU(5) \times SU(5))/\mathbb{Z}_5$ breaks the E_8 to a mere $\mathbb{Z}_5$ — the commutant in E_8 is $\mathbb{Z}_5$. Similarly, if

one were to compactify on a bundle with structure group $\mathrm{Spin}(16)/\mathbb{Z}_2$, the commutant inside E_8 is $\mathbb{Z}_2$.

If it were the case that the low-energy theory in any E_8 bundle not describable on the worldsheet had gauge group only a finite group, then this might not be considered very interesting. However, there are other examples of subgroups of E_8 whose commutant has rank at least 1, and which cannot be embedded in $\mathrm{Spin}(16)/\mathbb{Z}_2$.

For example, the group $(E_7 \times U(1))/\mathbb{Z}_2$ is a subgroup of E_8 (that sits inside the $(E_7 \times \mathrm{SU}(2))/\mathbb{Z}_2$ subgroup of E_8) which has commutant $U(1)$ and is not a subgroup of $\mathrm{Spin}(16)/\mathbb{Z}_2$.

For another example, $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$ is a subgroup of E_8 and so its E_6 subgroup has commutant $\mathrm{SU}(3)$, but E_6 cannot be embedded in $\mathrm{Spin}(16)/\mathbb{Z}_2$. To see this, note that if E_6 could be embedded in $\mathrm{Spin}(16)/\mathbb{Z}_2$, then the Lie algebra $so(16)$ would have an e_6 subalgebra, and since there is a 16-dimensional representation of $so(16)$, that means e_6 would have a possibly reducible nontrivial 16-dimensional representation as well, just from taking the subalgebra described by some of the 16×16 matrices describing $so(16)$. However, the smallest nontrivial representation of e_6 is 27 dimensional, a contradiction. (Note that this is closely related to but distinct from the standard embedding for Calabi–Yau three-folds: the $\mathrm{SU}(3)$ subgroup of $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$ *does* sit inside $\mathrm{Spin}(16)/\mathbb{Z}_2$, unlike the E_6.)

5   F theory duals and the existence of perturbative realizations

So far we have argued that there exist some bundles with connection that cannot be realized using the standard description of heterotic $E_8 \times E_8$ strings. Does that mean that they do not arise in string theory? Such questions are important to the landscape program, for example, where one of the current issues involves understanding which backgrounds admit UV completions [1, 2].

Some insight into this question can be made with F theory duals. For example, Aspinwall [33, Section 2.3] describes an F theory dual to a heterotic compactification in which the bundle with connection has structure group $(E_7 \times U(1))/\mathbb{Z}_2$ and so cannot be realized with the standard construction of heterotic strings.

Such examples tell us that at least some of these bundles with connection can nevertheless be realized within string theory.

More abstract considerations lead one to the same conclusion. Imagine starting with a bundle with connection reducible to $\mathrm{Spin}(16)/\mathbb{Z}_2$ and deforming to an E_8 bundle with connection that is not reducible. Since the adjoint representation of E_8 decomposes into the adjoint and a chiral spinor representation of $\mathrm{Spin}(16)/\mathbb{Z}_2$, the deformation described would involve giving a vacuum expectation value to a spinor. This sounds reminiscent of describing Ramond–Ramond fields in type II strings with nonzero vacuum expectation values. In the case of type II strings, giving those fields' vacuum expectation values involved formally adding terms to the lagrangian coupled to the superconformal ghosts, which is problematic and is the reason that Ramond–Ramond field vevs are problematic in basic formulations of type II strings. In a heterotic string, however, giving a vev to a gauge spinor does *not* involve coupling to superconformal ghosts, unlike the type II case, so there is no obstruction in principle. Thus, from this consideration, one is led to believe that E_8 bundles with connection that cannot be reduced to $\mathrm{Spin}(16)/\mathbb{Z}_2$ should nevertheless define well-behaved CFTs, even though they cannot be described within traditional heterotic worldsheet constructions.

In the remainder of this paper, we will describe alternative constructions of perturbative heterotic strings that can explicitly realize more general E_8 bundles with connection. First, in the next section we will describe how subgroups other than $\mathrm{Spin}(16)/\mathbb{Z}_2$ can be used to build E_8 in 10 dimensions and will check by comparing modular forms that corresponding current algebra constructions realize all of the degrees of freedom of the left-moving part of the standard constructions. To make such constructions practical in less than 10 dimensions, however, one needs suitable technology for fibering current algebras over a base, and so we introduce "fibered WZW models," which will enable us to fiber a current algebra for any group at any level over a base, using a principal bundle with connection to define the fibering.

6 Alternative constructions of 10-dimensional heterotic strings

The reader might ask whether the heterotic string could be formulated in some alternative fashion that might be more amenable to some of the constructions above. For example, might it be possible to formulate a worldsheet string with, for each E_8, two sets of five complex fermions, realizing the E_8 from $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$? Unfortunately, two sets of five complex fermions would have a $U(5) \times U(5)$ global symmetry, and if we try to gauge each $U(1)$ on the worldsheet, we would encounter a $U(1)^2$ anomaly which would force c_2 of each bundle to vanish separately.

Instead, we are going to take an alternative approach to this issue. We are going to develop a notion of fibered current algebras, realized by fibered WZW models, which will allow us to realize current algebras at any level and associated to any group G, fibered nontrivially over any compactification manifold. The standard $E_8 \times E_8$ heterotic string construction is, after all, one realization of a fibered $E_8 \times E_8$ current algebra at level 1; our technology will enable us to talk about fibering G-current algebras at level k.

Before doing that, however, we will check to what extent subgroups of E_8 other than $\mathrm{Spin}(16)/\mathbb{Z}_2$ can be used to build up the left-moving E_8 partition function in 10 dimensions. For example, one could take a pair of SU(5) current algebras, then perform a $\mathbb{Z}_5$ orbifold (replacing the "left-moving GSO" used to build $\mathrm{Spin}(16)/\mathbb{Z}_2$ from a Spin(16) current algebra in the usual construction) so as to get an $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ global symmetry on the worldsheet, or take an SU(9) global symmetry and perform a $\mathbb{Z}_3$ orbifold to get an $\mathrm{SU}(9)/\mathbb{Z}_3$ global symmetry. Both $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ and $\mathrm{SU}(9)/\mathbb{Z}_3$ are subgroups of E_8, and we will find that such alternative subgroups correctly reproduce the E_8 partition function and so give alternative constructions of the E_8 current algebra in 10 dimensions. At the level of characters and abstract affine algebras, the idea that E_8 can be built from other subgroups has appeared previously in [3]; we shall review some pertinent results and also describe how those character decompositions are realized physically in partition functions, via orbifold twisted sectors.

First, let us recall how E_8 is built from $\mathrm{Spin}(16)/\mathbb{Z}_2$ in 10 dimensions. The adjoint representation of E_8 decomposes as

$$\mathbf{248} = \mathbf{120} + \mathbf{128} \tag{6.1}$$

under $\mathrm{Spin}(16)/\mathbb{Z}_2$. At the level of ordinary Lie algebras, we get the elements of the E_8 Lie algebra from the adjoint plus a spinor representation of $\mathrm{Spin}(16)/\mathbb{Z}_2$ and assigning them suitable commutation relations. At the level of WZW conformal families, we could write

$$[\mathbf{1}] = [\mathbf{1}] + [\mathbf{128}]$$

which implicitly includes equation (6.1) as a special case, since the (adjoint-valued) currents are nonprimary descendants of the identity operator. That statement about conformal families implies a statement about characters of the corresponding affine Lie algebras, namely that

$$\chi_{E_8}(\mathbf{1}, q) = \chi_{\mathrm{Spin}(16)}(\mathbf{1}, q) + \chi_{\mathrm{Spin}(16)}(\mathbf{128}, q) \tag{6.2}$$

where [34, Section 6.4.8]

$$\chi_{E_8}(\mathbf{1}, q) = \frac{E_2(q)}{\eta(q)^8}$$

and where $E_2(q)$ is the degree four Eisenstein modular form

$$\begin{aligned}
E_2(q) &= 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \\
&= 1 + 240[q + (1^3 + 2^3)q^2 + (1^3 + 3^3)q^3 + \cdots] \\
&= 1 + 240q + 2160q^2 + 6720q^3 + 17{,}520q^4 \\
&\quad + 30{,}240q^5 + 60{,}480q^6 + \cdots
\end{aligned}$$

with

$$\sigma_3(m) = \sum_{d|m} d^3.$$

Identity (6.2) is discussed in, for example, [34, Section 6.4; 35, equation (3.4a)]. The $\mathbb{Z}_2$ orbifold plays a crucial role in the expression above. Without the $\mathbb{Z}_2$ orbifold, we would only consider the single conformal family [$\mathbf{1}$] and the single character $\chi_{\mathrm{Spin}(16)}(\mathbf{1}, q)$. The [$\mathbf{128}$] arises from the $\mathbb{Z}_2$ orbifold twisted sector. (The fact that the twisted sector states are still representatives of the same affine Lie algebra as the untwisted sector states, despite being in a twisted sector, is a consequence of the fact that the orbifold group action preserves the currents — it acts on the center of the group, preserving the algebra structure.)

Next, we shall check to what extent other subgroups of E_8 can be used to duplicate the same left-moving degrees of freedom.

6.1 Some maximal-rank subgroups

In this subsection, we shall argue that the left-moving E_8 degrees of freedom can be reproduced by using the maximal-rank $\mathrm{SU}(5)^2/\mathbb{Z}_5$ and $\mathrm{SU}(9)/\mathbb{Z}_3$ subgroups of E_8, in place of $\mathrm{Spin}(16)/\mathbb{Z}_2$. Just as for $\mathrm{Spin}(16)/\mathbb{Z}_2$, the finite group quotients will be realized by orbifolds and will play a crucial role. At the level of characters of affine algebras, the ideas have appeared previously in, e.g., [3], but we shall also explain how those character decompositions are realized physically in partition functions. For more information on determining such finite group quotients, see Appendix A.

First, let us check central charges. From [36, Section 15.2], the central charge of a bosonic WZW model at level k is

$$\frac{k \dim G}{k + C}$$

where C is the dual Coxeter number. For the case of $G = \mathrm{SU}(N)$, $\dim G = N^2 - 1$ and $C = N$ (see, e.g., [37, p. 502]); hence, the central charge of the bosonic $\mathrm{SU}(N)$ WZW is

$$\frac{k(N^2 - 1)}{k + N}.$$

For $k = 1$, this reduces to $N - 1$. Thus, the $\mathrm{SU}(5)$ current algebra at level 1 has central charge 4, and the $\mathrm{SU}(9)$ current algebra has central charge 8. In particular, this means that the $\mathrm{SU}(5) \times \mathrm{SU}(5)$ current algebra at level 1 has central charge $4 + 4 = 8$, just right to be used in critical heterotic strings to build an E_8. Similarly, the $\mathrm{SU}(9)$ current algebra at level 1 has central charge 8, also just right to be used in critical heterotic strings to build an E_8.

Similarly, for E_6, E_7, and E_8, the dual Coxeter numbers are 12, 18, and 30, respectively, and it is easy to check that at level 1, each current algebra has central charge equal to 6, 7, and 8, respectively. More generally, for ADE groups, level 1 current algebras have central charge equal to the rank of the group.

For $\mathrm{SU}(5)$, the integrable representations (defining WZW primaries) are $\mathbf{5}$, $\mathbf{10} = \Lambda^2 \mathbf{5}$, $\overline{\mathbf{10}} = \Lambda^3 \mathbf{5}$, and $\overline{\mathbf{5}} = \Lambda^4 \mathbf{5}$. The fusion rules obeyed by the WZW conformal families have the form

$$[\mathbf{5}] \times [\mathbf{5}] = [\mathbf{10}]$$
$$[\mathbf{5}] \times [\overline{\mathbf{5}}] = [\mathbf{1}]$$
$$[\overline{\mathbf{10}}] \times [\overline{\mathbf{5}}] = [\mathbf{10}]$$
$$[\mathbf{10}] \times [\overline{\mathbf{5}}] = [\mathbf{5}]$$
$$[\overline{\mathbf{10}}] \times [\overline{\mathbf{10}}] = [\mathbf{5}]$$
$$[\overline{\mathbf{10}}] \times [\mathbf{10}] = [\mathbf{1}].$$

The adjoint representation of E_8 decomposes under $\mathrm{SU}(5)^2/\mathbb{Z}_5$ as [38]

$$\mathbf{248} = (\mathbf{1}, \mathbf{24}) + (\mathbf{24}, \mathbf{1}) + (\mathbf{5}, \overline{\mathbf{10}}) + (\overline{\mathbf{5}}, \mathbf{10}) + (\mathbf{10}, \mathbf{5}) + (\overline{\mathbf{10}}, \overline{\mathbf{5}})$$

from which one would surmise that the corresponding statement about conformal families is

$$[\mathbf{1}] = [\mathbf{1}, \mathbf{1}] + [\mathbf{5}, \overline{\mathbf{10}}] + [\overline{\mathbf{5}}, \mathbf{10}] + [\mathbf{10}, \mathbf{5}] + [\overline{\mathbf{10}}, \overline{\mathbf{5}}] \tag{6.3}$$

which can be checked by noting that the right-hand side above squares into itself under the fusion rules.

Next, we shall check partition functions, which will provide the conclusive demonstration that the E_8 of a 10-dimensional heterotic string can be built from $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ instead of $\mathrm{Spin}(16)/\mathbb{Z}_2$.

The character of the identity representation of SU(5) is

$$\chi_{\mathrm{SU}(5)}(\mathbf{1}, q) = \frac{1}{\eta(\tau)^4} \sum_{\vec{m} \in \mathbb{Z}^4} q^{(\sum m_i^2 + (\sum m_i)^2)/2}.$$

Taking modular transformations, the characters of the other needed integrable representations are

$$\chi_{\mathrm{SU}(5)}(\mathbf{5}, q) = \frac{1}{\eta(\tau)^4} \sum_{\vec{m} \in \mathbb{Z}^4, \, \sum m_i = 1 \bmod 5} q^{(\sum m_i^2 - \frac{1}{5}(\sum m_i)^2)/2}$$

and

$$\chi_{\mathrm{SU}(5)}(\mathbf{10}, q) = \frac{1}{\eta(\tau)^4} \sum_{\vec{m} \in \mathbb{Z}^4, \, \sum m_i = 2 \bmod 5} q^{(\sum m_i^2 - \frac{1}{5}(\sum m_i)^2)/2}.$$

The remaining two characters (given by $\sum m_i = 3, 4 \bmod 5$) are equal to these, by taking $\vec{m} \to -\vec{m}$.

Now, we need to verify that

$$\chi_{E_8}(\mathbf{1}, q) = \chi_{\mathrm{SU}(5)}(\mathbf{1}, q)^2 + 4\chi_{\mathrm{SU}(5)}(\mathbf{5}, q)\, \chi_{\mathrm{SU}(5)}(\mathbf{10}, q) \tag{6.4}$$

which corresponds to equation (6.3) for the conformal families. This character decomposition, along with character decompositions for other subgroups, has appeared previously in [3], but since it plays a crucial role in our arguments, we shall explain in detail why it is true and then explain how it is realized physically in partition functions. The E_8 character is

given by [34, Section 6.4.8]

$$\chi_{E_8}(\mathbf{1}, q) = \frac{E_2(q)}{\eta(\tau)^8}$$

where $E_2(q)$ denotes the relevant Eisenstein series. The $\mathbb{Z}_5$ orbifold is implicit here — $\chi(\mathbf{1}, q)^2$ arises from the untwisted sector, and each of the four $\chi(\mathbf{5}, q)\chi(\mathbf{10}, q)$'s arises from a twisted sector. (As for $\mathrm{Spin}(16)/\mathbb{Z}_2$, since the orbifold action preserves the currents, the twisted sector states must form a well-defined module over the (unorbifolded) affine Lie algebra.)

Ample numerical evidence for equation (6.4) is straightforward to generate. For example,

$$\begin{aligned}
\eta(\tau)^4 \chi_{\mathrm{SU}(5)}(\mathbf{1}, q) &= 1 + 20q + 30q^2 + 60q^3 + 60q^4 + 120q^5 + 40q^6 + 180q^7 \\
&\quad + 150q^8 + 140q^9 + 130q^{10} + 240q^{11} \\
&\quad + 180q^{12} + 360q^{13} + \cdots \\
\eta(\tau)^4 \chi_{\mathrm{SU}(5)}(\mathbf{5}, q) &= 5q^{2/5} + 30q^{7/5} + 30q^{12/5} + 80q^{17/5} + 60q^{22/5} + 100q^{27/5} \\
&\quad + 104q^{32/5} + 168q^{37/5} + 54q^{42/5} + 206q^{47/5} + 168q^{52/5} \\
&\quad + 172q^{57/5} + 140q^{62/5} + 270q^{67/5} + 153q^{72/5} + \cdots \\
\eta(\tau)^4 \chi_{\mathrm{SU}(5)}(\mathbf{10}, q) &= 10q^{3/5} + 25q^{8/5} + 60q^{13/5} + 35q^{18/5} + 110q^{23/5} + 90q^{28/5} \\
&\quad + 120q^{33/5} + 96q^{38/5} + 198q^{43/5} + 98q^{48/5} + 244q^{53/5} \\
&\quad + 126q^{58/5} + 192q^{63/5} + 208q^{68/5} + 300q^{73/5} + \cdots
\end{aligned}$$

Putting this together, we find

$$\begin{aligned}
\eta(\tau)^8 &\left(\chi_{\mathrm{SU}(5)}(\mathbf{1}, q)^2 + 4\chi_{\mathrm{SU}(5)}(\mathbf{5}, q)\, \chi_{\mathrm{SU}(5)}(\mathbf{10}, q) \right) \\
&= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + \cdots
\end{aligned}$$

which are precisely the first few terms of the appropriate Eisenstein series $E_2(q)$, numerically verifying prediction (6.4).

More abstractly, the equivalence can be proven as follows.[10]   In the notation of [35], we need to relate the theta function of the E_8 lattice to a

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[10]This argument is due to E. Scheidegger, and we would like to thank him for allowing us to print it here.

product of theta functions for SU(5) lattices. Briefly, first one argues that

$$\Theta(E_8) = \Theta(\{A_4, A_4\}[1, 2]).$$

Using [35, equations (1.1), (1.5)], this can be written as

$$\Theta\left(\bigcup_{i=1}^{5}[ig]\{A_4, A_4\}\right) = \sum_{i=1}^{5}\Theta\left([ig]\{A_4, A_4\}\right)$$

where g denotes the generator of the $\mathbb{Z}_5$ action (shift by 1 on first A_4, shift by 2 on second). Using [35, equation (1.4)], this can be written as

$$\sum_{i=1}^{5}\Theta([ig]A_4)\Theta([ig]A_4) = \sum_{i=1}^{5}\Theta([i]A_4)\Theta([2i]A_4).$$

Using the symmetry

$$\Theta([5-i]A_4) = \Theta([i]A_4)$$

the result then follows after making the identifications

$$\eta(\tau)^4\chi(\mathbf{1}, q) = \Theta(A_4), \quad \eta(\tau)^4\chi(\mathbf{5}, q) = \Theta([1]A_4),$$
$$\eta(\tau)^4\chi(\mathbf{10}, q) = \Theta([2]A_4).$$

Merely verifying the existence of a character decomposition does not suffice to explain how this can be used in alternative constructions of heterotic strings — one must also explain how that character decomposition is realized physically. In the case of $\mathrm{Spin}(16)/\mathbb{Z}_2$, the two components of the character decomposition were realized physically as the untwisted and twisted sectors of a $\mathbb{Z}_2$ orbifold of a $\mathrm{Spin}(16)$ current algebra. That orbifold structure precisely correlates with the group-theoretic fact that the subgroup of E_8 is $\mathrm{Spin}(16)/\mathbb{Z}_2$ and not $\mathrm{Spin}(16)$ or $\mathrm{SO}(16)$ — the finite group factor that one gets from the group theory of E_8 appears physically as the orbifold of the current algebra that one needs in order to reproduce the correct character decomposition.

There is a closely analogous story here. Group-theoretically, the subgroup of E_8 is not $\mathrm{SU}(5) \times \mathrm{SU}(5)$ but rather $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$, and so one should expect that a $\mathbb{Z}_5$ orbifold of the $\mathrm{SU}(5) \times \mathrm{SU}(5)$ current algebra should appear. Indeed, that is precisely what happens. If we only considered an $\mathrm{SU}(5) \times \mathrm{SU}(5)$ current algebra without an orbifold, the only

contribution to the heterotic partition function would be from the characters $\chi_{\mathrm{SU}(5)}(\mathbf{1}, q)^2$, which would not reproduce the E_8 character. In order to realize the complete E_8 character decomposition, we need more, and the extra components of the character decomposition are realized in twisted sectors of a $\mathbb{Z}_5$ orbifold, the same $\mathbb{Z}_5$ arising in group-theoretic considerations. Each $\chi_{\mathrm{SU}(5)}(\mathbf{5}, q)\chi_{\mathrm{SU}(5)}(\mathbf{10}, q)$ arises in a twisted sector. The individual $\mathbf{5}$, $\mathbf{10}$, $\overline{\mathbf{5}}$, and $\overline{\mathbf{10}}$ are not invariant under the $\mathbb{Z}_5$, but the products $(\mathbf{5}, \overline{\mathbf{10}})$, $(\overline{\mathbf{5}}, \mathbf{10})$, $(\mathbf{10}, \mathbf{5})$, $(\overline{\mathbf{10}}, \overline{\mathbf{5}})$ *are* invariant under the $\mathbb{Z}_5$ orbifold, as discussed in Appendix A.

For $\mathrm{SU}(9)/\mathbb{Z}_3$, there is an analogous[11] story. The adjoint representation of E_8 decomposes as [38]

$$\mathbf{248} = \mathbf{80} + \mathbf{84} + \overline{\mathbf{84}}$$

and so proceeding as before the conformal families of E_8, $\mathrm{SU}(9)$ should be related by

$$[\mathbf{1}] = [\mathbf{1}] + [\mathbf{84}] + [\overline{\mathbf{84}}] \tag{6.5}$$

(which includes the decomposition above as a special case as the currents in the current algebra are descendants of the identity). The relevant $\mathrm{SU}(9)$, level 1, characters are given by

$$\chi_{\mathrm{SU}(9)}(\mathbf{1}, q) = \frac{1}{\eta(\tau)^8} \sum_{\vec{m} \in \mathbb{Z}^8} q^{(\sum m_i^2 + (\sum m_i)^2)/2}$$

and

$$\chi_{\mathrm{SU}(9)}(\mathbf{84}, q) = \frac{1}{\eta(\tau)^8} \sum_{\vec{m} \in \mathbb{Z}^8, \sum m_i = 3 \bmod 9} q^{(\sum m_i^2 - \frac{1}{9}(\sum m_i)^2)/2}.$$

(The character for $\overline{\mathbf{84}}$ is identical.) Then, from equation (6.5) it should be true that

$$\chi_{E_8}(\mathbf{1}, q) = \chi_{\mathrm{SU}(9)}(\mathbf{1}, q) + 2\chi_{\mathrm{SU}(9)}(\mathbf{84}, q).$$

This identity is proven in [35, table 1]. The same statement is also made for lattices in [39, Section A.3, p. 109; 40, equation (8.12)], and of course also appeared in [3].

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[11]At the level of character decompositions, this and other examples are discussed in, e.g., [3].

Again, it is important to check that this character decomposition really is realized physically in a partition function, and the story here closely mirrors the $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ and $\mathrm{Spin}(16)/\mathbb{Z}_2$ cases discussed previously. Group-theoretically, the subgroup of E_8 is $\mathrm{SU}(9)/\mathbb{Z}_3$ and not $\mathrm{SU}(9)$ or $\mathrm{SU}(9)/\mathbb{Z}_9$, so one would expect that we need to take a $\mathbb{Z}_3$ orbifold of the $\mathrm{SU}(9)$ current algebra. Indeed, if we did not take any orbifold at all, and only coupled the $\mathrm{SU}(9)$ current algebra by itself, then the only contribution to the heterotic partition function would be from the character $\chi_{\mathrm{SU}(9)}(\mathbf{1}, q)$, which does not suffice to reproduce the E_8 character. Instead, we take a $\mathbb{Z}_3$ orbifold, and each of the two characters $\chi_{\mathrm{SU}(9)}(\mathbf{84}, q)$ and $\chi_{\mathrm{SU}(9)}(\overline{\mathbf{84}}, q)$ appears in a $\mathbb{Z}_3$ orbifold twisted sector. Taking those orbifold twisted sectors into account correctly reproduces the E_8 character decomposition within the heterotic partition function.

6.2 A non-maximal-rank subgroup

So far we have discussed how E_8 can be built from maximal-rank subgroups.

Somewhat surprisingly, on the level of characters, it appears that one can build it from non-maximal-rank subgroups also. We will discuss the case of $G_2 \times F_4$. Although it satisfies many highly nontrivial checks, unfortunately we will eventually conclude that it cannot be used, unlike the maximal-rank subgroups discussed so far.

First, we should mention that the construction of the ordinary Lie group E_8 from $G_2 \times F_4$ is described in [12, Chapter 8]. Very roughly, the idea is that if one takes $\mathrm{Spin}(16)$ and splits it into $\mathrm{Spin}(7) \times \mathrm{Spin}(9)$, then $G_2 \subset \mathrm{Spin}(7)$ and $F_4 \subset \mathrm{Spin}(9)$. Under the $g_2 \times f_4$ subalgebra, the adjoint representation of e_8 decomposes as [38]

$$\mathbf{248} = (\mathbf{14}, \mathbf{1}) + (\mathbf{1}, \mathbf{52}) + (\mathbf{7}, \mathbf{26}). \tag{6.6}$$

The commutant of $G_2 \times F_4$ in E_8 has rank zero. One way to see this is from the construction outlined above, but a simpler way is from the decomposition of the adjoint representation of E_8: if the commutant had rank greater than zero, then the adjoint of the commutant would secretly appear in the decomposition of the adjoint of E_8, as a set of singlets, but there are no singlets in the E_8 adjoint decomposition, and so the commutant must have rank zero.

Thus, even though $G_2 \times F_4$ is not of maximal rank, its commutant in E_8 can be no more than a finite group. This may sound a little surprising to some readers but is in fact a relatively common occurrence in representation

theory. For example, a dimension n representation of SU(2) embeds SU(2) in SU(n), and has rank zero commutant inside SU(n), even though SU(2) is not a maximal-rank subgroup. This is a consequence of Schur's lemma.

We are going to discuss whether the E_8 degrees of freedom can be described by this non-maximal-rank subgroup, namely $G_2 \times F_4$. As one initial piece of evidence, the fact stated above that the commutant of $G_2 \times F_4$ in E_8 has rank zero is consistent. After all, if it is possible to describe all of the E_8 current algebra using $G_2 \times F_4$ on the internal space, then there will be no left-moving worldsheet degrees of freedom left over to describe any gauge symmetry in the low-energy compactified heterotic theory. That can only be consistent if the commutant has rank zero, i.e., if there is no low-energy gauge symmetry left over to describe.

Next, let us check that the central charges of the algebras work out correctly. The dual Coxeter number of G_2 is 4 and that of F_4 is 9, so the central charge of the G_2 algebra at level 1 is 14/5 and that of the F_4 algebra at level 1 is 52/10, which sum to 8, the same as the central charge of the E_8 algebra at level 1.

Both G_2 and F_4 affine algebras at level 1 have only two[12] integrable representations:

$$G_2 : [\mathbf{1}], [\mathbf{7}]$$
$$F_4 : [\mathbf{1}], [\mathbf{26}].$$

The conformal weights of the primary fields are, respectively, $h_7 = \frac{2}{5}$ and $h_{26} = \frac{3}{5}$. So, our proposed decomposition of E_8 level 1 (which has only one integrable representation)

$$[\mathbf{1}] = [\mathbf{1}, \mathbf{1}] + [\mathbf{7}, \mathbf{26}]$$

does, indeed, reproduce the correct central charge and the conformal weights and multiplicity of currents.

Under modular transformations,

$$\chi_{E_8}(\mathbf{1}, q) = \chi_{G_2}(\mathbf{1}, q)\chi_{F_4}(\mathbf{1}, q) + \chi_{G_2}(\mathbf{7}, q)\chi_{F_4}(\mathbf{26}, q) \qquad (6.7)$$

___

[12]This is a short exercise using [38]; let us briefly outline the details for G_2. The condition for a representation with highest weight λ to be integrable at level k is $2\psi \cdot \lambda/\psi^2 \leq k$, where ψ is the highest weight of the adjoint representation. Using [38, Tables 7 and 8], a representation of G_2 with Dynkin labels (a, b) has $2\psi \cdot \lambda/\psi^2 = 2a + b$, where a, b are nonnegative integers and so can only be ≤ 1 when $a = 0$ and b is either 0 or 1, which gives the $\mathbf{1}$ and $\mathbf{7}$ [38, Table 13] representations, respectively.

transform identically. To see this, note that the fusion rules of G_2 and F_4 at level 1 are, respectively,

$$G_2 : [\mathbf{7}] \times [\mathbf{7}] = [\mathbf{1}] + [\mathbf{7}]$$
$$F_4 : [\mathbf{26}] \times [\mathbf{26}] = [\mathbf{1}] + [\mathbf{26}]. \tag{6.8}$$

The modular S-matrix (for both G_2 and F_4) is

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 - 1/\sqrt{5}} & \sqrt{1 + 1/\sqrt{5}} \\ \sqrt{1 + 1/\sqrt{5}} & -\sqrt{1 - 1/\sqrt{5}} \end{pmatrix} \tag{6.9}$$

which, in both cases, satisfies $S^2 = (ST)^3 = 1$ and $N_{ijk} = \sum_m \frac{S_{im} S_{jm} S_{km}}{S_{1m}}$. Using this modular S-matrix, the particular combination of characters on the right-hand side of (6.7) is invariant, as it should be.

This, along with the transformation under T which we have already checked, proves (6.7).

However, it is clear from the fusion rules (6.8) that something is amiss. If we take the OPE of $[\mathbf{7}, \mathbf{26}]$ with itself, the fusion rules dictate that we should see, in addition to the desired $[\mathbf{1}, \mathbf{1}] + [\mathbf{7}, \mathbf{26}]$, terms involving $[\mathbf{7}, \mathbf{1}] + [\mathbf{1}, \mathbf{26}]$ as well.

While we have managed to reproduce the multiplicity of states correctly, it appears that we have failed to reproduce their interactions correctly. Moreover, Kac and Sanielevici [3] have found several other examples of non-maximal-rank embeddings of characters of affine algebras, of which this is, perhaps, the simplest example. As far as we can tell, the same criticism applies to their other examples: the multiplicity of states correctly reproduces that of the E_8 current algebra, but the interactions do not.

It is worth remarking that our previous examples were obtained as (asymmetric) orbifolds by some subgroup of the center. In the case at hand, G_2 and F_4 are center-less,[13] so there is no obvious orbifold construction that could give rise to (6.7).

---

[13] This fact is discussed in Appendix A. In addition, they also have no normal finite subgroup, as any discrete normal subgroup of a connected group is necessarily central, and there is no center in this case. The statement on discrete normal subgroups can be shown as follows. Let G be a connected group and N a discrete normal subgroup. Let G act on N by conjugation, which it does since N is normal. Then for any $n \in N$, every gng^{-1} is in N and connected to n within N, since G is connected. Since N is discrete, for gng^{-1} to be connected to n, they must be equal, hence N is central. We would like to thank A. Knutson for pointing this out to us.

7 Symmetric bosonic fibered WZW models

Now that we have seen alternative constructions of 10-dimensional heterotic strings using more general current algebras than $\mathrm{Spin}(16)/\mathbb{Z}_2$, we will next discuss how to fiber those current algebras over nontrivial spaces. As a warm-up, let us first describe a fibered WZW model in the symmetric case. This will not be useful for heterotic strings, but it will provide a good "stepping-stone" to the asymmetric fibered WZW models we will discuss in the next section.

Start with the total space of a G bundle in which across coordinate patches the fibers transform as $g \mapsto (g_{\alpha\beta})g(g_{\alpha\beta}^{-1})$. Let A_μ be a connection on this bundle.

First, recall from [41, equation (2.4)] that a WZW model in which the adjoint action has been gauged has the form

$$
\begin{aligned}
S = {} & -\frac{k}{4\pi} \int_\Sigma d^2z \mathrm{Tr}\, \left[g^{-1}\partial g g^{-1}\overline{\partial} g \right] \\
& -\frac{ik}{12\pi} \int_B d^3y \epsilon^{ijk}\mathrm{Tr}\, \left[g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g \right] \\
& +\frac{k}{2\pi} \int_\Sigma d^2z \mathrm{Tr}\, \left[A_{\overline{z}}g^{-1}\partial g - A_z \overline{\partial} g g^{-1} \right] \\
& +\frac{k}{2\pi} \int_\Sigma d^2z \mathrm{Tr}\, \left[A_{\overline{z}}g^{-1}A_z g - A_{\overline{z}}A_z \right]
\end{aligned}
$$

where A_z, $A_{\overline{z}}$ is a worldsheet gauge field.

To define a fibered WZW model, we will want to replace the worldsheet gauge fields with pullbacks of a gauge field on the target space (the connection on the G bundle). That way, gauge invariance across coordinate patches will be built in. Thus, consider a nonlinear sigma model on the total space of that bundle with action

$$
\begin{aligned}
S = {} & \frac{1}{\alpha'} \int_\Sigma d^2z \partial_\alpha \phi^\mu \partial^\alpha \phi^\nu g_{\mu\nu} \quad -\frac{k}{4\pi} \int_\Sigma d^2z\, \mathrm{Tr}\, \left[g^{-1}\partial g g^{-1}\overline{\partial} g \right] \\
& -\frac{ik}{12\pi} \int_B d^3y \epsilon^{ijk}\mathrm{Tr}\, \left[g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g \right] \\
& +\frac{k}{2\pi} \int_\Sigma d^2z \mathrm{Tr}\, \left[\overline{\partial}\phi^\mu A_\mu g^{-1}\partial g - \partial\phi^\mu A_\mu \overline{\partial} g g^{-1} \right] \\
& +\frac{k}{2\pi} \int_\Sigma d^2z \overline{\partial}\phi^\mu \partial\phi^\nu \mathrm{Tr}\, \left[A_\mu g^{-1}A_\nu g - A_\mu A_\nu \right]
\end{aligned}
$$

where the ϕ^μ are coordinates on the base and g is a coordinate on the fibers. On each coordinate patch on the base, the Wess–Zumino term is an ordinary Wess–Zumino term — the fields g are fields on the worldsheet, not functions of the ϕ — and so can be handled in the ordinary fashion.

Next, although we have deliberately engineered this action to be well defined across coordinate patches on the target space, let us explicitly check that the action is indeed gauge invariant. Under the following variation

$$g \mapsto hgh^{-1}$$
$$A_\mu \mapsto h\partial_\mu h^{-1} + hA_\mu h^{-1}$$

(where $h = h(\phi)$), the variation of all terms except the WZ term is given by

$$\delta = \frac{k}{4\pi} \int_\Sigma d^2z \mathrm{Tr} \ \left[-h^{-1}\overline{\partial}hg^{-1}\partial g + h^{-1}\partial h\overline{\partial}gg^{-1} - h^{-1}\partial hgh^{-1}\overline{\partial}hg^{-1} \right.$$
$$\left. + h^{-1}\partial hg^{-1}\overline{\partial}g - \partial gg^{-1}h^{-1}\overline{\partial}h + h^{-1}\partial hg^{-1}h^{-1}\overline{\partial}hg \right]$$

and where it is understood that, for example, $\partial h = \partial\phi^\mu \partial_\mu h$.

The variation of the WZ term is given by

$$-\frac{3ik}{12\pi} \int_B d^3y \epsilon^{ijk} \mathrm{Tr} \ [g^{-1}h^{-1}\partial_i hh^{-1}\partial_j h\partial_k g - g^{-1}h^{-1}\partial_i hh^{-1}\partial_j hgh^{-1}\partial_k h$$
$$+ h^{-1}\partial_i h\partial_j gg^{-1}\partial_k gg^{-1} - g^{-1}h^{-1}\partial_i h\partial_j gh^{-1}\partial_k h$$
$$- g^{-1}h^{-1}\partial_i hgh^{-1}\partial_j hg^{-1}\partial_k g + g^{-1}h^{-1}\partial_i hgh^{-1}\partial_j hh^{-1}\partial_k h$$
$$- g^{-1}\partial_i gg^{-1}\partial_j gh^{-1}\partial_k h + g^{-1}\partial_i gh^{-1}\partial_j hh^{-1}\partial_k h]$$
$$= -\frac{3ik}{12\pi} \int_B d\mathrm{Tr} \ [-h^{-1}dh \wedge dgg^{-1} - h^{-1}dh \wedge g^{-1}dg$$
$$+ g^{-1}h^{-1}(dh)g \wedge h^{-1}dh]$$
$$= -\frac{3ik}{12\pi} \int_\Sigma \mathrm{Tr} \ [-h^{-1}dh \wedge dgg^{-1} - h^{-1}dh \wedge g^{-1}dg$$
$$+ g^{-1}h^{-1}(dh)g \wedge h^{-1}dh].$$

If we write $z = x + iy$, then

$$dz \wedge d\overline{z} - d\overline{z} \wedge dz = 2i \left(dy \wedge dx - dx \wedge dy \right)$$

then we see that the terms generated by the variation of the WZ term are exactly what is needed to cancel the terms generated by everything else.

Note that the computation above, the check that the model is well defined across target-space coordinate patches, is identical to the computation needed to show that an ordinary gauged WZW model is invariant under gauge transformations.

The model we have described so far is bosonic, but one could imagine adding fermions along the base and demanding supersymmetry under transformations that leave the fibers invariant. A simpler version of this is obtained by taking a $(2,2)$ nonlinear sigma model and adding right- and left-moving fermions $\lambda_\pm$ coupling to a vector bundle over the $(2,2)$ base. Demanding that the resulting model be $(2,2)$ supersymmetric on-shell unfortunately forces the bundle to be flat: $F = 0$. Roughly, half of the constraints one obtains from supersymmetry force the curvature to be holomorphic, in the sense $F_{ij} = F_{\overline{ij}} = 0$, and the other half force the connection to be flat. We shall find in the next section that imposing merely $(0,2)$ supersymmetry is easier: one merely needs the curvature to be holomorphic, not necessarily flat.

8   Fibered (0,2) WZW models

8.1   Construction of the lagrangian

Begin with some principal G bundle with connection A_μ over some Calabi–Yau X. Consider a nonlinear sigma model on the total space of that bundle. We shall think of the fibers as defining, locally, WZW models, so we use the connection A_μ to define a chiral multiplication on the fibers of the bundle and have a WZ term to describe H flux in the fibers.

8.1.1   Gauge invariance and global well-definedness

We are going to write down a fibered WZW model in which each fiber is a gauged WZW model, gauging the action $g \mapsto hg$ across coordinate patches on the target space, the principal G bundle.

First, recall from [41, equation (2.9); 42], a gauged WZW model gauging the chiral multiplication $g \mapsto hg$ is given by

$$S' = -\frac{k}{4\pi} \int_\Sigma d^2z \, \mathrm{Tr} \left(g^{-1}\partial_z g g^{-1}\partial_{\bar z} g \right)$$

$$- \frac{\mathrm{i}k}{12\pi} \int_B d^3 y \epsilon^{ijk} \mathrm{Tr} \left(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g \right)$$

$$- \frac{k}{2\pi} \int_\Sigma d^2 z \mathrm{Tr} \left(A_z \partial_{\bar z} g g^{-1} + \frac{1}{2} A_z A_{\bar z} \right)$$

where A_z, $A_{\bar z}$ are worldsheet gauge fields.

With that in mind, to describe a fibered WZW model, one would replace the worldsheet gauge fields with pullbacks of a connection A_μ on the target space, the principal G bundle. In fact, one would initially suppose that the action should have the form

$$S = \frac{1}{\alpha'} \int_\Sigma d^2 z \left(\frac{1}{4} g_{i\bar j} \partial_\alpha \phi^i \partial^\alpha \phi^{\bar j} + \mathrm{i} g_{i\bar j} \psi_+^{\bar j} D_{\bar z} \psi_+^i \right)$$

$$- \frac{k}{4\pi} \int_\Sigma d^2 z \mathrm{Tr} \left(g^{-1} \partial_z g g^{-1} \partial_{\bar z} g \right)$$

$$- \frac{\mathrm{i}k}{12\pi} \int_B d^3 y \epsilon^{ijk} \mathrm{Tr} \left(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g \right)$$

$$- \frac{k}{2\pi} \int_\Sigma d^2 z \mathrm{Tr} \left((\partial_z \phi^\mu) A_\mu \partial_{\bar z} g g^{-1} + \frac{1}{2} (\partial_z \phi^\mu \partial_{\bar z} \phi^\nu) A_\mu A_\nu \right).$$

The field g defines a coordinate on the fibers of the bundle, and ϕ are coordinates on the base.

However, the full analysis is slightly more complicated. As described in [41,42,44] a WZW action is not invariant under chiral group multiplications, so the action above is not invariant across coordinate patches on the target space. Specifically, under the target space gauge transformation

$$g \mapsto hg$$
$$A_\mu \mapsto h A_\mu h^{-1} + h \partial_\mu h^{-1}$$

(where h is a group-valued function on the target space) the gauge transformation of the terms above excepting the Wess–Zumino term is given by

$$\frac{k}{4\pi} \int_\Sigma d^2 z \mathrm{Tr} \left(h^{-1} \partial h \bar\partial g g^{-1} - h^{-1} \bar\partial h \partial g g^{-1} + \bar\partial \phi^\mu A_\mu h^{-1} \partial h - \partial \phi^\mu A_\mu h^{-1} \bar\partial h \right)$$

where, for example, $\partial h = (\partial_z \phi^\mu)(\partial_\mu h)$, and the gauge transformation of the Wess–Zumino term is given by

$$- \frac{\mathrm{i}k}{12\pi} \int_B d^3 y \epsilon^{ijk} \mathrm{Tr} \left(h^{-1} \partial_i h h^{-1} \partial_j h h^{-1} \partial_k h \right) + \frac{\mathrm{i}k}{4\pi} \int_\Sigma \mathrm{Tr} \left(h^{-1} dh \wedge dg g^{-1} \right).$$

This lack of gauge invariance is exactly what one would expect of a bosonized description of the left-movers on a heterotic string worldsheet. There is a chiral gauge anomaly in the fermionic realization which after bosonization should be realized classically. On the other hand, a lack of gauge invariance across coordinate patches means that we have a problem with global well-definedness of the chiral fibered WZW model.

We can resolve this problem with gauge invariance in the standard way for heterotic strings: assign the B field nontrivial gauge transformation properties. So, we add a B field, coupling as

$$\frac{1}{\alpha'} \int_\Sigma d^2z B_{\mu\nu} \left(\partial\phi^\mu \overline{\partial}\phi^\nu - \overline{\partial}\phi^\mu \partial\phi^\nu \right)$$

and demand that under the gauge transformation above, the holonomy above pick up the terms

$$+ \frac{\mathrm{i}k}{12\pi} \int_B d^3y \epsilon^{ijk} \mathrm{Tr} \ \left(h^{-1}\partial_i h h^{-1}\partial_j h h^{-1}\partial_k h \right) + \frac{\mathrm{i}k}{4\pi} \int_\Sigma \mathrm{Tr} \ \left(h^{-1}dh \wedge d\phi^\mu A_\mu \right).$$

$$(8.1)$$

This transformation law manifestly restores gauge invariance.

Let us check for a minute that this transformation law is consistent. The second term is a two-form, and so it is completely consistent for the B field to pick up such a term. The first term, on the other hand, is a three-form, which in general will not even be closed on each overlap chart. As a result, the first term cannot be expressed even locally in terms of a two-form.

However, there is a fix. In addition to gauge invariance, we must also demand, as is standard in heterotic strings, that the B field transform under local-Lorentz transformations acting on the chiral right-moving fermions. These transformations are anomalous, and by demanding that the B field transform, we can restore the gauge invariance broken by the anomalies. Under such transformations, the B field will necessarily pick up two closely analogous terms, one of which will involve another problematic three-form. Thus, we need for the combination

$$k \, \mathrm{Tr} \ ((g_{\alpha\beta}^F)^{-1} dg_{\alpha\beta}^F)^3 - \mathrm{Tr} \ ((g_{\alpha\beta}^R)^{-1} dg_{\alpha\beta}^R)^3$$

to be exact on each overlap, where the $g_{\alpha\beta}$'s are transition functions for the gauge (F) and tangent (R) bundles. This turns out [43] to be implied by the statement that $k\mathrm{Tr} \ F^2$ and $\mathrm{Tr} \ R^2$ match in cohomology; writing Chern–Simons forms for both and interpreting in terms of Deligne cohomology, the condition that the difference across overlaps is exact is immediate. This is

the first appearance of the anomaly cancelation constraint that

$$k \left[\mathrm{Tr}\ F^2 \right] = \left[\mathrm{Tr}\ R^2 \right] \tag{8.2}$$

where k is the level of the fibered Kac–Moody algebra. We shall see this same constraint emerge several more times in different ways.

In any event, so long as condition (8.2) is obeyed, we see that the chiral fibered WZW model is well defined globally. Next we shall consider the fermion kinetic terms in this model.

In order to formulate a supersymmetric theory, we shall need to add a three-form flux $H_{\mu\nu\rho}$ to the connection appearing in the ψ kinetic terms. Ordinarily $H = dB$, but we need H to be gauge- and local-Lorentz neutral, whereas B transforms under both gauge- and local-Lorentz transformations. To fix this, we follow the standard procedure in heterotic strings of adding Chern–Simons terms. For example, the gauge terms (8.1) are the same as those arising in a gauge transformation of the Chern–Simons term

$$+ \frac{\mathrm{i}k}{4\pi} \int_B d^3 y \epsilon^{ijk} \partial_i \phi^\mu \partial_j \phi^\nu \partial_k \phi^\rho \mathrm{Tr}\ \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right)$$

and similarly one can cancel the terms picked up under local-Lorentz transformation by adding a term involving the Chern–Simons form coupling to the spin connection. Schematically, we have

$$H = dB + (\alpha') \mathrm{Tr}\ (k\,\mathrm{CS}(A) - \mathrm{CS}(\omega))$$

where k is the level of the fibered current algebra. H is now an ordinary gauge- and local-Lorentz-invariant three-form. This statement implies that $k\,\mathrm{Tr}\ F^2$ and $\mathrm{Tr}\ R^2$ must be in the same cohomology class. For a fibered current algebra defined by a principal $SU(n)$ bundle $\mathcal{E}$ over a space X, this is the statement that $k\,c_2(\mathcal{E}) = c_2(TX)$, which generalizes the ordinary anomaly cancelation condition of heterotic strings. This is the second appearance of this constraint; we shall see it again later.

As an aside, note that since this model has nonzero H flux, the metric cannot be Kähler [45]. More precisely, to zeroth order in α' a Kähler metric can be consistent, but to next leading order in α' the metric will be non-Kähler, with H measuring how far the metric is from being Kähler.

Also note that this analysis is analogous to, though slightly different from, that of $(0, 2)$ WZW models discussed in [41, 44]. There, WZW models with chiral group multiplications and chiral fermions were also considered. However, the fermions lived in the tangent bundle to the group manifold, so

the chiral group multiplication induced the right-moving fermion anomaly, and so that chiral fermion anomaly and the classical noninvariance of the action could be set to cancel each other out. Here, on the other hand, the chiral fermions live on the base, not the WZW fibers, and so do not see the chiral group multiplication (which only happens on the fibers). Thus, here we proceed in a more nearly traditional fashion, by adding a B field with nontrivial gauge- and local-Lorentz transformations, whose global well-definedness places constraints on the bundles involved.

Thus, the gauge-invariant action has the form

$$
\begin{aligned}
S = {}& \frac{1}{\alpha'} \int_\Sigma d^2z \left(\frac{1}{4} g_{i\bar{j}} \partial_\alpha \phi^i \partial^\alpha \phi^{\bar{j}} + \mathrm{i} g_{i\bar{j}} \psi_+^{\bar{j}} D_{\bar{z}} \psi_+^i \right) \\
& + \frac{1}{\alpha'} \int_\Sigma d^2z B_{\mu\nu} \left(\partial \phi^\mu \bar{\partial} \phi^\nu - \bar{\partial} \phi^\mu \partial \phi^\nu \right) \\
& - \frac{k}{4\pi} \int_\Sigma d^2z \mathrm{Tr} \left(g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g \right) \\
& - \frac{\mathrm{i}k}{12\pi} \int_B d^3y \epsilon^{ijk} \mathrm{Tr} \left(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g \right) \\
& - \frac{k}{2\pi} \int_\Sigma d^2z \mathrm{Tr} \left((\partial_z \phi^\mu) A_\mu \partial_{\bar{z}} g g^{-1} + \frac{1}{2} (\partial_z \phi^\mu \partial_{\bar{z}} \phi^\nu) A_\mu A_\nu \right).
\end{aligned}
$$

8.1.2 Worldsheet supersymmetry

Next, let us demand that the model possess $(0,2)$ supersymmetry, under the transformations

$$
\begin{aligned}
\delta \phi^i &= \mathrm{i} \alpha_- \psi_+^i \\
\delta \phi^{\bar{i}} &= \mathrm{i} \tilde{\alpha}_- \psi_+^{\bar{i}} \\
\delta \psi_+^i &= -\tilde{\alpha}_- \partial \phi^i \\
\delta \psi_+^{\bar{i}} &= -\alpha_- \partial \phi^{\bar{i}} \\
\delta g &= 0.
\end{aligned}
$$

Supersymmetry will require us to add the gauge-invariant term

$$
\frac{\mathrm{i}k}{4\pi} \int_\Sigma d^2z \mathrm{Tr} \left(F_{\mu\nu} \bar{\partial}_A g g^{-1} \right) \psi_+^\mu \psi_+^\nu
$$

where

$$
\begin{aligned}
\bar{\partial}_A g g^{-1} &= \left(\bar{\partial} g + \bar{\partial} \phi^\mu A_\mu g \right) g^{-1} \\
&= \bar{\partial} g g^{-1} + \bar{\partial} \phi^\mu A_\mu
\end{aligned}
$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The term above is an analog of the four-Fermi term appearing in standard heterotic string constructions. We shall also add an H flux field to the base. One finds that for the supersymmetry transformations to close, one needs $F_{ij} = F_{\overline{ij}} = 0$.

Let us outline how the α_- supersymmetry transformations work.

The α_- terms in the supersymmetry transformation of the base terms

$$\frac{1}{\alpha'} \int_\Sigma d^2 z \left(\frac{1}{4} g_{i\overline{j}} \partial_\alpha \phi^i \partial^\alpha \phi^{\overline{j}} + \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\overline{z}} \psi_+^\nu + B_{\mu\nu} \left(\partial \phi^\mu \overline{\partial} \phi^\nu - \overline{\partial} \phi^\mu \partial \phi^\nu \right) \right)$$

where

$$D_{\overline{z}} \psi_+^\nu = \overline{\partial} \psi_+^\nu + \overline{\partial} \phi^\mu \left(\Gamma^\nu{}_{\sigma\mu} - H^\nu{}_{\sigma\mu} \right) \psi_+^\sigma$$

are given by

$$\frac{1}{\alpha'} \int d^2 z \left[(i\alpha_- \psi_+^i)(\overline{\partial} \phi^\mu)(\partial \phi^\nu)(H_{i\mu\nu}) \right]$$

$$+ \frac{1}{\alpha'} \int_\Sigma d^2 z \left[\frac{i}{2} (i\alpha_- \psi_+^i)(\overline{\partial} \phi^\mu) \psi_+^j \psi_+^{\overline{k}} (H_{\overline{k}ij,\mu} - H_{\overline{k}i\mu,j} - H_{j\overline{k}\mu,i} + H_{ji\mu,\overline{k}}) \right]$$

$$+ \frac{1}{\alpha'} \int_\Sigma d^2 z (i\alpha_- \psi_+^i)(B_{\mu\nu,i} - B_{i\nu,\mu} - B_{\mu i,\nu})(\partial \phi^\mu \overline{\partial} \phi^\nu - \overline{\partial} \phi^\mu \partial \phi^\nu)$$

and where we needed to assume

$$H_{ijk} = H_{\overline{ij}\overline{k}} = 0$$

$$H_{ij\overline{k}} = \frac{1}{2} \left(g_{i\overline{k},j} - g_{j\overline{k},i} \right) = \Gamma_{ij\overline{k}}.$$

(This was derived off-shell, without using any equations of motion.)

The α_- terms in the supersymmetry transformation of the fiber terms

$$-\frac{k}{4\pi} \int_\Sigma d^2 z \, \mathrm{Tr} \, (g^{-1} \partial_z g g^{-1} \partial_{\overline{z}} g) - \frac{ik}{12\pi} \int_B d^3 y \epsilon^{ijk} \mathrm{Tr} \, (g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g)$$

$$-\frac{k}{2\pi} \int_\Sigma d^2 z \, \mathrm{Tr} \, \left((\partial_z \phi^\mu) A_\mu \partial_{\overline{z}} g g^{-1} + \frac{1}{2} (\partial_z \phi^\mu \partial_{\overline{z}} \phi^\nu) A_\mu A_\nu \right)$$

$$+\frac{ik}{4\pi} \int_\Sigma d^2 z \, \mathrm{Tr} \, \left(F_{\mu\nu} \overline{\partial}_A g g^{-1} \right) \psi_+^\mu \psi_+^\nu$$

are given by

$$\frac{\mathrm{i}k}{4\pi}\int_\Sigma d^2z \left(\mathrm{i}\alpha_-\psi_+^i\right)\mathrm{Tr}\ \left(F_{\mu\nu}F_{i\lambda}\right)\psi_+^\mu\psi_+^\nu(\overline{\partial}\phi^\lambda)$$

$$-\frac{k}{4\pi}\int_\Sigma d^2z(\mathrm{i}\alpha_-\psi_+^i)\overline{\partial}\phi^\mu\partial\phi^\nu\mathrm{Tr}\ \left(\left(A_i\partial_\mu A_\nu + \frac{2}{3}A_iA_\mu A_\nu\right)\right.$$

$$\left.\pm \text{ permutations }\right).$$

The supersymmetry transformations only close on-shell;[14] to get the result above requires using the classical equations of motion for g, namely

$$\partial_A(\overline{\partial}_Agg^{-1}) = \partial\phi^\mu\overline{\partial}\phi^\nu F_{\mu\nu} + \frac{\mathrm{i}}{2}[F_{\mu\nu}, \overline{\partial}_Agg^{-1}]\psi_+^\mu\psi_+^\nu + \frac{\mathrm{i}}{2}\overline{\partial}_A(F_{\mu\nu}\psi_+^\mu\psi_+^\nu) \tag{8.3}$$

where

$$\partial_A(\overline{\partial}_Agg^{-1}) = \partial(\overline{\partial}_Agg^{-1}) + [\partial\phi^\lambda A_\lambda, \overline{\partial}_Agg^{-1}].$$

Note equation (8.3) generalizes the chirality condition $\partial(\overline{\partial}gg^{-1}) = 0$ that appears in ordinary (nonfibered) WZW models.

We will also use equation (8.3) to define a second-class constraint — we are describing chiral nonabelian bosons, after all.

Also note equation (8.3) is the supersymmetrization of the anomaly in the chiral gauge current: defining $j = \overline{\partial}_Agg^{-1}$, and omitting fermions, this says $Dj \propto F$. If the WZW current were realized by fermions, this would be the chiral anomaly; here, we have bosonized, and so the anomaly is realized classically. In such a fermionic realization, the second term is a classical contribution to the divergence of the current from the four-Fermi term in the action, and the third term is a nonuniversal contribution to the anomaly from a one-loop diagram also involving the four-Fermi interaction.

In a fermionic realization of the left-movers, the terms in the supersymmetry transformations above would not appear at zeroth order in α'.

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[14]Alternatively, the supersymmetry transformations will close off-shell if instead of $\delta g = 0$ we take

$$\delta g = -(\mathrm{i}\alpha_-\psi_+^i)A_ig - (\mathrm{i}\tilde{\alpha}_-\psi_+^{\bar{\imath}})A_{\bar{\imath}}g.$$

(This is true for both α_- transformations considered here and $\tilde{\alpha}_-$ transformations.) In this form supersymmetry transformations explicitly commute with gauge transformations; on the other hand, the on-shell formulation $\delta g = 0$ makes it explicit that supersymmetry is only meaningfully acting on the base.

Classically, supersymmetry transformations of the action result in one-Fermi terms proportional to $H - dB$ and three-Fermi terms proportional to dH, both of which are proportional to α'. However, at next-to-leading-order in α' on the worldsheet, one has more interesting effects. Specifically, "supersymmetry anomalies" arise [46, 47]. These are phase factors picked up by the path integral measure. Unlike true anomalies, these are canceled by counterterms. In particular, the Chern–Simons terms added to make H gauge- and local-Lorentz invariant cancel out the effect of these "anomalies."

In more detail, if we realize the left-moving gauge degrees of freedom by chiral fermions λ_-, we can realize worldsheet supersymmetry off-shell[15] with supersymmetry transformations of the form

$$\delta\lambda_- = -(\mathrm{i}\alpha_-\psi_+^i)A_i\lambda_- - (\mathrm{i}\tilde{\alpha}_-\psi_+^{\bar{\imath}})A_{\bar{\imath}}\lambda_-$$

where A_μ is the target space gauge field. However, these supersymmetry transformations are equivalent to (anomalous chiral) gauge transformations with parameter

$$-(i\alpha_-\psi_+^i)A_i - (i\tilde{\alpha}_-\psi_+^{\bar{\imath}})A_{\bar{\imath}}. \tag{8.4}$$

Thus, the supersymmetry transformation implies an anomalous gauge transformation, and so the path integral measure picks up a phase factor. From the (universal) bosonic term in the divergence of the gauge current proportional to the curvature F, we will get a one-Fermi term in the anomalous transformation proportional to the Chern–Simons form. In our case, as we have bosonized the left-movers, we get such a one-Fermi term in supersymmetry transformations classically. In addition to the universal piece, there is a regularization-dependent multi-Fermi contribution as well. If we calculate the anomalous divergence of the gauge current in a fermionic realization, then because of the four-Fermi term $F\lambda\lambda\psi\psi$ there will be a two-Fermi contribution to the divergence of the gauge current proportional to $\bar{\partial}(F\psi\psi)$. Plugging into the gauge parameter (8.4) yields a three-Fermi term in the supersymmetry transformations proportional to $\mathrm{Tr}\, F \wedge F$, exactly as we have discovered in the classical supersymmetry transformations of our bosonized formulation.

There is a closely analogous phenomenon of supersymmetry anomalies in the right-moving fermions as well. Since we have not bosonized them, the analysis here is identical to that for ordinary heterotic string constructions discussed, for example, in [46, 47]. In terms of supersymmetry transformations of the right-moving fermions written with general-covariant indices,

---

[15]If we take $\delta\lambda_- = 0$, the worldsheet supersymmetry transformations close only if one uses the λ_- equations of motion.

e.g., $\delta\psi_+^i = -\tilde{\alpha}_-\partial\phi^i$, the source of the anomaly is not obvious. To make it more manifest, we must switch to local-Lorentz indices and define

$$\psi_+^a = e_\mu^a \psi_+^\mu.$$

Then, the supersymmetry transformations have the form

$$\begin{aligned}
\delta\psi_+^a = {} & (e_i^a(-\tilde{\alpha}_-\partial\phi^i) + e_{\bar{i}}^a(-\alpha_-\partial\phi^{\bar{i}})) \\
& + (e_{\mu,i}^a(\mathrm{i}\alpha_-\psi_+^i)\psi_+^\mu + e_{\mu,\bar{i}}^a(\mathrm{i}\tilde{\alpha}_-\psi_+^{\bar{i}})\psi_+^\mu).
\end{aligned}$$

The second set of terms above can be written as (anomalous, chiral) local-Lorentz transformations, and so the supersymmetry transformations induce anomalous local-Lorentz transformations. In particular, under a supersymmetry transformation the path integral measure will pick up a phase factor including a one-Fermi term proportional to the Chern–Simons form for the target space spin connection, whose origin is the (universal, bosonic) curvature term in the divergence of the local-Lorentz current. The path integral phase factor will also include a multifermi contribution. Here, the same analysis of four-Fermi terms as before would appear to imply that the multi-Fermi contribution will be proportional to FR, where F is the gauge curvature and R is the metric curvature. However, these multi-Fermi terms are sensitive to the choice of regulator, and to maintain (0,2) worldsheet supersymmetry we must be very careful about the choice of regulator here. For the correct choice of regularization, the multi-fermi contribution is a three-Fermi term proportional to Tr $R \wedge R$, where R is the curvature of the connection $\Gamma - H$, as discussed in, e.g., [46, 47].

As a check on this method, note that if we replace the right-moving chiral fermions with nonabelian bosons, then following the same analysis as for the gauge degrees of freedom the supersymmetry transformations will automatically generate one- and three-Fermi terms of the desired form.

For more information on supersymmetric anomalies in such two-dimensional theories, see also [48, 49]. See also [50, 51] for an interesting approach to the interaction of second-class constraints and worldsheet supersymmetry.

To summarize, under (anomalous) worldsheet supersymmetry transformations we have found one-Fermi terms proportional to

$$H - dB - (\alpha')(k\,\mathrm{CS}(A) - \mathrm{CS}(\omega - H))$$

and three-Fermi terms proportional to

$$dH - (\alpha')(k \operatorname{Tr} F \wedge F - \operatorname{Tr} R \wedge R)$$

where the terms involving the spin connection ω arise from quantum corrections, and the terms involving the gauge field A arise classically in our bosonic construction but from quantum corrections in fermionic realizations of left-movers. Closure of supersymmetry is guaranteed by our definition of H. Put another way, we see that worldsheet supersymmetry is deeply intertwined with the Green–Schwarz mechanism.

The $\tilde{\alpha}_-$ terms in the supersymmetry transformations are almost identical. The $\tilde{\alpha}_-$ terms in the supersymmetry transformation of the base terms are given by

$$\frac{1}{\alpha'} \int_\Sigma d^2 z (\mathrm{i}\tilde{\alpha}_- \psi_+^{\overline{k}})(\overline{\partial}\phi^\mu)(\partial\phi^\nu) H_{\overline{k}\mu\nu}$$

$$+ \frac{1}{\alpha'} \int_\Sigma d^2 z \frac{\mathrm{i}}{2} (i\tilde{\alpha}_- \psi_+^{\overline{k}})(\overline{\partial}\phi^\mu)\psi_+^i \psi_+^{\overline{j}} (H_{\overline{j}\overline{k}i,\mu} - H_{\overline{j}\overline{k}\mu,i} - H_{i\overline{j}\mu,\overline{k}} + H_{i\overline{k}\mu,\overline{j}})$$

$$+ \frac{1}{\alpha'} \int_\Sigma d^2 z (\mathrm{i}\tilde{\alpha}_- \psi_+^{\overline{\imath}})(B_{\mu\nu,\overline{\imath}} - B_{\overline{\imath}\nu,\mu} - B_{\mu\overline{\imath},\nu})(\partial\phi^\mu \overline{\partial}\phi^\nu - \overline{\partial}\phi^\mu \partial\phi^\nu)$$

which are virtually identical to the corresponding α_- terms.

The $\tilde{\alpha}_-$ terms in the supersymmetry transformation of the fiber terms are given by

$$\frac{\mathrm{i}k}{4\pi} \int_\Sigma d^2 z \operatorname{Tr} \, (F_{\mu\nu} F_{\overline{\imath}\rho}) (i\tilde{\alpha}_- \psi_+^{\overline{\imath}}) \psi_+^\mu \psi_+^\nu \overline{\partial}\phi^\rho$$

$$- \frac{k}{4\pi} \int_\Sigma d^2 z (\mathrm{i}\tilde{\alpha}_- \psi_+^{\overline{\imath}})(\overline{\partial}\phi^\mu)(\partial\phi^\nu) \operatorname{Tr}$$

$$\times \left(A_{\overline{\imath}} \partial_\mu A_\nu + \frac{2}{3} A_{\overline{\imath}} A_\mu A_\nu \pm \text{permutations} \right)$$

which are virtually identical to the corresponding α_- terms. (As before, to get the result above requires using the equations of motion for g.)

The supersymmetry anomaly story works here in the same way as for the α_- terms, and, just as for the α_- terms, one can show that the worldsheet theory is supersymmetric through first order in α'.

8.1.3 Full gauge-invariant supersymmetric lagrangian

Let us summarize the results of the last two subsections. The full lagrangian is given by

$$
S = \frac{1}{\alpha'} \int_\Sigma d^2 z \left(\frac{1}{4} g_{i\bar{\jmath}} \partial_\alpha \phi^i \partial^\alpha \phi^{\bar{\jmath}} + \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\bar{z}} \psi_+^\nu \right)
$$
$$
+ \frac{1}{\alpha'} \int_\Sigma d^2 z B_{\mu\nu} \left(\partial\phi^\mu \overline{\partial}\phi^\nu - \overline{\partial}\phi^\mu \partial\phi^\nu \right)
$$
$$
- \frac{k}{4\pi} \int_\Sigma d^2 z \mathrm{Tr} \left(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}} g \right)
$$
$$
- \frac{ik}{12\pi} \int_B d^3 y \epsilon^{ijk} \mathrm{Tr} \left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g \right)
$$
$$
- \frac{k}{2\pi} \int_\Sigma d^2 z \mathrm{Tr} \left((\partial_z\phi^\mu) A_\mu \partial_{\bar{z}} g g^{-1} + \frac{1}{2}(\partial_z\phi^\mu \partial_{\bar{z}}\phi^\nu) A_\mu A_\nu \right)
$$
$$
+ \frac{ik}{4\pi} \int_\Sigma d^2 z \mathrm{Tr} \left(F_{\mu\nu} \overline{\partial}_A g g^{-1} \right) \psi_+^\mu \psi_+^\nu
$$

where

$$
D_{\bar{z}} \psi_+^\nu = \overline{\partial}\psi_+^\nu + \overline{\partial}\phi^\mu \left(\Gamma^\nu{}_{\sigma\mu} - H^\nu{}_{\sigma\mu} \right) \psi_+^\sigma
$$

sand the metric $g_{\mu\nu}$ on the base will not be Kähler (except optionally at zeroth order in α').

The action is well defined under the gauge transformations

$$
g \mapsto hg
$$
$$
A_\mu \mapsto h A_\mu h^{-1} + h \partial_\mu h^{-1}
$$

across coordinate-charge changes on the base, where h is a group-valued function on the overlap patch on the target space, and the B field transforms to absorb both the gauge anomaly above and the local-Lorentz anomaly on the right-moving chiral fermions.

The action is also invariant under the (0,2) worldsheet supersymmetry transformations

$$
\delta\phi^i = i\alpha_- \psi_+^i
$$
$$
\delta\phi^{\bar{\imath}} = i\tilde{\alpha}_- \psi_+^{\bar{\imath}}
$$
$$
\delta\psi_+^i = -\tilde{\alpha}_- \partial\phi^i
$$
$$
\delta\psi_+^{\bar{\imath}} = -\alpha_- \partial\phi^{\bar{\imath}}
$$
$$
\delta g = 0
$$

where we assume $F_{ij} = F_{\bar{i}\bar{j}} = 0$, and that H has only (1,2) or (2,1) components, no (0,3) or (3,0), related to the metric by

$$H_{i\bar{j}k} = -\frac{1}{2}\left(g_{i\bar{j},k} - g_{k\bar{j},i}\right)$$

and where H is also given by the difference of Chern–Simons forms, in the form

$$H = dB + (\alpha')(k\,\mathrm{CS}(A) - \mathrm{CS}(\omega - H)).$$

The classical equations of motion for g are

$$\partial_A\left(\bar{\partial}_A g g^{-1}\right) = \partial\phi^\mu \bar{\partial}\phi^\nu F_{\mu\nu} + \frac{i}{2}[F_{\mu\nu}, \bar{\partial}_A g g^{-1}]\psi_+^\mu \psi_+^\nu + \frac{i}{2}\bar{\partial}_A\left(F_{\mu\nu}\psi_+^\mu \psi_+^\nu\right)$$

where

$$\partial_A\left(\bar{\partial}_A g g^{-1}\right) = \partial\left(\bar{\partial}_A g g^{-1}\right) + [\partial\phi^\lambda A_\lambda, \bar{\partial}_A g g^{-1}].$$

Note this equation generalizes the chirality condition $\partial(\bar{\partial}g g^{-1}) = 0$ that appears in ordinary (nonfibered) WZW models. Here, it also plays the role of a second-class constraint. Also note this is the supersymmetrization of the chiral anomaly in the current: defining $j = \bar{\partial}_A g g^{-1}$, and omitting fermions, this says $Dj \propto F$. Since we have bosonized, the anomaly is realized classically. In a fermionic description of the left-movers, the current $\bar{\partial}_A g g^{-1}$ would be given by $\lambda_- \bar{\lambda}_-$, the $[F, \bar{\partial}_A g g^{-1}]\psi\psi$ term would be a classical contribution to the divergence of the current, and the F and $\bar{\partial}(F\psi\psi)$ terms would arise as quantum corrections, from one-loop diagrams involving the interactions $A\lambda\lambda$ and the four-Fermi term $F\psi\psi\lambda\lambda$, respectively. The former (bosonic) contribution to the divergence is universal, the latter is in principle regularization dependent.

8.2 Anomaly cancelation

In order to make the action well defined, recall that we needed to demand that $k\int \mathrm{Tr}\,F^2$ and $\int \mathrm{Tr}\,R^2$ be in the same de Rham cohomology class. From that fact we can immediately read off the form of the anomaly cancelation condition for general levels of the fibered current algebra: if the condition at level 1 is that

$$c_2(\mathcal{E}) = c_2(TX)$$

then the condition at level k is

$$kc_2(\mathcal{E}) = c_2(TX).$$

We have already seen several independent derivations of the anomaly cancelation condition — it plays several roles in making the fibered WZW model self-consistent and supersymmetric, analogs of the same roles in heterotic worldsheets. Here is another quick test of this claim. Take the heterotic $E_8 \times E_8$ string on S^1 and orbifold by the action which translates halfway around the S^1 while simultaneously exchanging the two E_8's. The result is a theory, again on S^1, but with a single E_8 current algebra at level 2. We can understand anomaly cancelation in this theory by working on the covering space, before the orbifold action. Embed bundles $\mathcal{E}_1$, $\mathcal{E}_2$ ($\mathcal{E}_1 \cong \mathcal{E}_2 \cong \mathcal{E}$) in each of the E_8's, then for anomaly cancelation to hold we must have

$$c_2(\mathcal{E}_1) + c_2(\mathcal{E}_2) = c_2(TX)$$

but this is just the statement

$$2c_2(\mathcal{E}) = c_2(TX)$$

which is precisely the prediction above for anomaly cancelation in a level 2 fibered current algebra. (Attentive readers will note that the central charge of a single E_8 at level 2 is 15.5, not 16, and so this does not suffice for a critical heterotic string. However, the orbifold has massive structure in the twisted sector that is not captured purely by the description above, and so the central charge of level 2 E_8 current algebra does not suffice; put another way, in the flat 10-dimensional space limit, the S^1 unravels, the orbifold is undone, and some of the massive twisted sector states become massless, curing the naive problem with the central charge.)

We can outline another derivation of the anomaly cancelation constraint in the language of chiral de Rham complexes [52–55]. In those papers, the idea was to describe the perturbative physics of a nonlinear sigma model on a space in terms of a set of free field theories on patches on a good cover of the target space. Conditions such as the anomaly cancelation condition arise as consistency conditions on triple overlaps. (Technically, the local free field descriptions need not patch together nicely, so one need get nothing more than a stack over the target, in fact a special stack known as a gerbe. The anomaly cancelation condition arises as the condition for that stack/gerbe to be trivial.)

Here, we can follow a similar program, except that instead of associating free theories to patches, we associate solvable theories to patches, which

is the next best thing. So, consider the left-moving degrees of freedom, described by a current algebra at level k:

$$J_F^a(z) \cdot J_F^b(z') \sim \frac{k\delta^{ab}}{(z-z')^2} + i\sum_c f_{abc}\frac{J^c(z')}{z-z'} + \cdots.$$

Let T^a denote the generators of the Lie algebra, and suppose that they are functions of the base space, $T^a = T^a(\gamma(z))$ in the notation of [52,55]. Define

$$J_F(\gamma) = \sum_a J_F^a(z)T^a(\gamma(z)).$$

Using the expansion

$$T^a(\gamma(z')) = T^a(\gamma(z)) + (z'-z)\left(\partial_{z'}\gamma^j\right)\partial_j T^a + \cdots$$

it is trivial to derive that the following OPE includes the terms

$$J_F(\gamma(z)) \cdot J_F(\gamma(z')) \sim \cdots + i\sum_c f_{abc}\frac{J^c(z')}{z-z'}T^a(\gamma)T^b(\gamma)$$

$$+ k\frac{\left(\partial_{z'}\gamma^j\right)T^a(\gamma)\partial_j T^a(\gamma)}{z-z'} + \cdots.$$

The equation above should be compared to [55, equation (5.30)], for example. The essential difference between the two is that the second term above (which corresponds to the fourth term on the right-hand side of [55, equation (5.30)]) has an extra factor of k, the level. That k-dependence in the second term on the right-hand side is ultimately responsible for modifying the anomaly cancelation condition from $[\text{Tr }F^2] = [\text{Tr }R^2]$ to $k[\text{Tr }F^2] = [\text{Tr }R^2]$.

8.3   Massless spectra

Letting the currents of a Kac–Moody algebra be denoted $J^a(z)$, for a an index of the ordinary Lie algebra, the WZW primaries $\varphi_{(r)}(w)$ are fields whose OPEs with the currents have only simple poles [58, Section 9.1]:

$$J^a(z) \cdot \varphi_{(r)}(w) \sim \frac{t_{(r)}^a}{z-w}\varphi_{(r)}(w) + \cdots$$

where (r) denotes some representation of the ordinary Lie algebra. In other words, the WZW primaries transform under the currents just like ordinary representations of the ordinary Lie algebra.

When we fiber WZW models, each WZW primary will define a smooth vector bundle associated to the principal G bundle defining how the WZW models are fibered, since across coordinate patches the primaries will map just as sections of such a bundle. (In the language of chiral de Rham complexes and soluble field theories on coordinate patches, the WZW primaries transform just like sections of associated vector bundles when we cross from one coordinate patch to another.) If the theory has $(0,2)$ supersymmetry, then that C^∞ vector bundle is a holomorphic vector bundle (otherwise, the transition functions break the BRST symmetry in the twisted theory).

More generally, a primary together with its descendants form a "positive-energy representation" of a Kac–Moody algebra. Since $[J_0^a, L_0] = 0$, the states at any given mass level will break into irreducible representations of G (as described by the zero-mode components J_0^a of the currents). (In addition, their OPEs with the full currents will have higher-order poles, but this is not important here.) When fibering WZW models, each such representation will then define a vector bundle associated to the underlying principal bundle, and so for WZW models fibered over a base manifold X the states in the positive-energy representation can be thought of as sections of $K(X)[[q]]$, a fact that will be important to the analysis of elliptic genera.

Following the usual yoga, a chiral primary in the $(0,2)$ fibered WZW model is then of the form

$$f_{\bar{\imath}_1 \cdots \bar{\imath}_n} \psi^{\bar{\imath}_1} \cdots \psi^{\bar{\imath}_n}$$

where the ψ's are right-moving worldsheet fermions, coupling to the tangent bundle of the base manifold X, and f is a section of $V \otimes \Lambda^n TX$, where V is a vector bundle defined by an irreducible representation of G corresponding to some component of a positive-energy representation of the Kac–Moody algebra as above. In cases[16] in which the base space is a Calabi–Yau to zeroth order in α', for the state to the BRST closed, f will be a holomorphic section, and in fact following the usual procedure this will realize a sheaf cohomology group valued in V, i.e., $H^*(X, V)$.

Morally, the integrable (or "unitary") representations (which define WZW primaries) correspond to massless states, as they have the lowest-lying L_0 eigenvalues (though of course that need not literally be true in all cases).

---

[16]Our fibered WZW model construction also applies to cases in which the base space is non-Kähler to zeroth order in α'. However, that complicates the BRST condition, and so for present purposes we restrict to Calabi–Yau spaces.

Let us briefly consider an example. For SU(n) at level 1, the integrable representations (WZW primaries) correspond to antisymmetric powers of the fundamental **n**. The construction above predicts "massless states" counted by $H^*(X, \Lambda^* \mathcal{E})$ where $\mathcal{E}$ is a rank n vector bundle associated to a principal SU(n) bundle. These are precisely the left-Ramond-sector states described in [56], for ordinary heterotic worldsheets built with left-moving fermions, and this is a standard result. (Because [56] are concerned with heterotic compactifications, their SU(n) is embedded in Spin(16) and then a left $\mathbb{Z}_2$ orbifold is performed, so there are additional states, in $\mathbb{Z}_2$ twisted sectors.) At higher levels there are additional integrable representations. (In fact, the integrable representations of SU(n) at any level are classified by Young diagrams of width bounded by the level. Thus, at level 2, the adjoint representation becomes integrable, and so in addition to the WZW current there is a WZW primary which transforms as the adjoint.)

In ordinary heterotic compactifications, Serre duality has the effect of exchanging particles and antiparticles. Let us check that the same is true here. For any complex reductive algebraic group G and any representation ρ, let $\mathcal{E}_\rho$ denote the holomorphic vector bundle associated to ρ. Then on an n-dimensional complex manifold X, Serre duality is the statement

$$H^i(X, \mathcal{E}_\rho) \cong H^{n-i}(X, \mathcal{E}_{\rho^*} \otimes K_X)^*$$

where ρ^* denotes the representation dual to ρ. We have implicitly used the fact that $\mathcal{E}_{\rho^*} \cong \mathcal{E}_\rho^\vee$, an immediate consequence of the definition of dual representation (see, e.g., [57, Section 8.1]). For example, for the group SU(n), the dual of the representation $\Lambda^i V$ is $\Lambda^i V^* \cong \Lambda^{n-i} V$, exactly as needed to reproduce the usual form. Thus, for Serre duality on Calabi–Yau's to respect the spectrum, properties of fields associated to representations ρ must be symmetric with respect to the dual representations ρ^*. Suppose the original representation ρ is integrable, then it can be shown that[17] the dual representation ρ^* is also integrable. Furthermore, the conformal weights of

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[17] The unitarity bound is [58, equation (9.30)]

$$2 \frac{\psi \cdot \lambda}{\psi^2} \leq k$$

where λ is the highest weight of the representation in question and ψ is the highest weight of the adjoint representation. The highest weight of the dual representation is $-w_0 \lambda$, where w_0 is the longest Weyl group element [36, equation (13.117)]. (The weight $-\lambda$ is the lowest weight of the dual representation.) Since the Killing form is invariant under w_0, i.e., $A \cdot B = (w_0 A) \cdot (w_0 B)$, and $w_0 \psi = -\psi$, we see that the left-hand side of the inequality is invariant under $\lambda \mapsto -w_0 \lambda$, and so a representation is unitary if and only if its dual is also unitary. We would like to thank A. Knutson for a discussion of this matter.

the states are also invariant[18] under this dualization. Thus, Serre duality symmetrically closes states into other states, just as one would expect.

8.4   Physical applications

Some interesting examples of six-dimensional gauged supergravities exist in the literature [59–62], for which a string-theoretic interpretation does not seem to be clear at present. The technology of this paper may give some insight into this question. (The relevance of higher-level currents has been observed previously, see e.g., [10], but is worth repeating here.)

One of the six-dimensional theories in question [59] has a gauge group $E_6 \times E_7 \times U(1)$ with massless matter in the **912** representation of E_7. One basic problem with realizing this in ordinary string worldsheet constructions is that it is not clear how to build a massless **912**. If we apply a standard construction, then the e_7 algebra is built from an $so(12) \times su(2)$ subalgebra. Under that subalgebra the **912** decomposes as

$$\mathbf{912} = (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32}, \mathbf{3}) \oplus (\mathbf{352}, \mathbf{1}) \oplus (\mathbf{220}, \mathbf{2}).$$

However, the standard construction can only recreate adjoints (**66**) and spinors (**32**) of $\mathrm{Spin}(12)$ in massless states from left-moving fermions, not a **352** or **220**, and so it is far from clear how a **912** could arise.

By working with current algebras at higher levels, however, more representations become unitary. In particular, an E_7 current algebra at level greater than 1 could have a massless state given by a **912**, which is part of what one would need to reproduce the six-dimensional supergravity in [59].

---

[18]For a given WZW primary (which are also Virasoro primaries), the L_0 eigenvalue is [36, equation (15.87)]

$$h = \frac{(\lambda, \lambda + 2\rho)}{2(k + g)}$$

where k is the level, g is the dual Coxeter number, and ρ is the Weyl vector (half-sum of positive roots). Recall that for a highest weight λ, the highest weight of the dual representation is $-w_0\lambda$, where w_0 is the longest Weyl group element. Now, $w_0\rho = -\rho$, it takes all the positive roots to negatives. Thus, using the fact that the Killing metric is Weyl invariant

$$(\lambda, \lambda + 2\rho) = (-w_0\lambda, -w_0\lambda - 2w_0\rho) = (-w_0\lambda, -w_0\lambda + 2\rho)$$

and so we see that a representation and its dual define primaries with the same conformal weight.

This by itself does not suffice to give a string-theoretic interpretation of any of the six-dimensional theories described in [59–62], but at least is a bit of progress towards such a goal.

8.5 Elliptic genera

Elliptic genera are often described as one-loop partition functions of half-twisted heterotic theories. Since we are describing new heterotic worldsheet constructions, we are implicitly realizing some elliptic genera not previously considered by physicists.

However, although the elliptic genera implied by our work have not been realized previously by physics constructions, they have been studied formally in the mathematics community, in the recent[19] works [63, 64]. Those papers describe elliptic genera in which the left-moving degrees of freedom couple to some G-current algebra at some level k, fibered over the base in a fashion determined by a fixed principal G bundle, just as done in this paper.

In a little more detail, each positive energy representation, call it E, of the G-current algebra decomposes at each mass level into a sum of irreducible representations of G, and so fibering them over the base in a fashion determined by an underlying principal G bundle P yields an element $\psi(E, P) \in K(X)[[q]]$, where the coefficient of each power of q is the sum of vector bundle associated to P via the irreducible representations appearing in E at the corresponding mass level. Each such positive energy representation consists of the descendants of some WZW primary. The corresponding characters in an ordinary WZW model can be interpreted as sections of line bundles over the moduli space of flat G connections on an elliptic curve [66]. Replacing the coordinates on the moduli space with Chern roots of P gives the Chern character of $\psi(E, P)$. (For example, compare the χ_S in [63, p. 353] to the P_{++} in [65, equation (4.15)].)

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[19] We should briefly speak to a potential language confusion. Many mathematics papers on elliptic genera speak of genera "at level k." This does not usually refer to the level of the current algebra to which left-moving degrees of freedom couple but rather refers to the modular properties of the genus. Specifically, it means the form is modular with respect to the "level-k-principal congruence subgroup" $\Gamma_0(k) \subset \mathrm{SL}(2, \mathbb{Z})$ defined by matrices congruent mod k to the identity. Thus, Witten's elliptic genera are often called level 1 elliptic genera, not because the left-movers couple to a level 1 current algebra but rather because they have good modular properties with respect to all of $\mathrm{SL}(2, \mathbb{Z})$. The elliptic genera discussed in [63, 64], by contrast, have left-moving degrees of freedom coupling to level k current algebras, just as in our heterotic fibered WZW model construction.

The elliptic genera described by Witten [67, 68] are described and derived in this language in [63]. Ordinarily we think of the left-movers' contribution to Witten's elliptic genera in terms of boundary conditions on fermions; the precise relationship between those boundary conditions and positive energy representations of the left-moving current algebra is spelled out in [69, equation (11.102)].

For the elliptic genera of [67, 68], demanding that the genera have good modular properties implies the standard anomaly cancelation constraint $c_2(P) = c_2(TX)$, see, for example, [65, 70–72]. For fibered level k current algebras, it is shown in detail in [63, 64] that demanding the genera have good modular properties implies $kc_2(P) = c_2(TX)$, the same anomaly cancelation constraint we have already derived multiple times from the physics of fibered WZW models.

8.6   Relevance of principal LG bundles

We have described how to fiber WZW models, but we (as well as [63, 64]) have only discussed how to fiber in a fashion controlled by a principal G bundle with connection. Since the WZW models describe Kac–Moody algebras, since we are fibering current algebras, one might expect that one could more generally fiber according to the dictates of a principal LG bundle.

Any principal G bundle induces a principal LG bundle, as there is a map $BG \to BLG$. Indeed, we have implicitly used that fact — the Kac–Moody algebra determined by a WZW model fits into a principal LG bundle that is such an image of a principal G bundle. If G is simply connected then a principal LG bundle over X can be thought of as a principal G bundle on $X \times S^1$ [73–75]. Given a principal LG bundle so described, we can get a principal G bundle just by evaluating at a point on the S^1, but these maps are not terribly invertible. Thus, principal LG bundles are not the same as principal G bundles.

In fact, there is a physical difficulty with fibering Kac–Moody algebras using general principal LG bundles that do not arise from principal G bundles. Put briefly, a physical state condition would not be satisfied in that more general case, and so one cannot expect to find physical theories in which left-moving current algebras have been fibered with more general principal LG bundles.

Let us work through this in more detail. As discussed earlier, a positive energy representation of LG decomposes into irreducible representations of G at each mass level, essentially because $[J_0^a, L_0] = 0$. Thus, so long as we

are fibering with a principal G bundle, instead of a principal LG bundle, the L_0 eigenvalues of states should be well defined across coordinate patches. (This is also the reason why the descendants can all be understood in terms of $K(X)[[q]]$, as used in the discussion of elliptic genera.)

If we had a principal LG bundle that was not the image of a principal G bundle, then the transition functions would necessarily mix up states of different conformal weights, more or less by definition of LG bundle.

Now, the physical states need to satisfy a condition of the form $m_{\mathrm{L}}^2 = m_R^2$, which defines a matching between conformal weights of left- and right-moving parts.

In a large-radius limit, we can choose a basis of right-moving states with well-defined L_0 eigenvalues. For the left-movers, if the WZW model is fibered with a principal G bundle, then we can choose a basis of left-moving states that also have well-defined L_0 eigenvalues, and so we can hope to satisfy the physical state condition above. On the other hand, if the WZW model were to be fibered with a principal LG bundle, then we would not be able to choose a basis of left-moving states with well-defined L_0 eigenvalues and would not be able to satisfy the physical state condition.

Thus, in a heterotic context, the only way to get states that satisfy the physical state condition above is if the left-moving current algebra couples to a principal G bundle, and not a more general principal LG bundle.

Note, however, that in a symmetrically fibered WZW model, of the form discussed in Section 7, this argument would not apply.

8.7 T-duality

One natural question to ask is how heterotic T-duality works when one has fibered a current algebra of level greater than 1.

We have seen how the fibering structure of a fibered current algebra is determined by a principal G bundle and a connection on that bundle. In the special case of tori, when the flat connection over the torus can be rotated into a maximal torus of G, it is easy to speculate that heterotic T-duality should act on the connection in a fashion independent of k. After all, once one rotates the connection into a maximal torus, the connection only sees a product of $U(1)$'s, and for $U(1)$'s the level of the Kac–Moody algebra is essentially irrelevant. Thus, if this conjecture is correct, in such cases heterotic T-duality would proceed as usual.

However, even if this conjecture is correct, we have no conjectures regarding how heterotic T-duality at higher levels should act when the connection cannot be diagonalized into a maximal torus (as can happen for flat connections on tori), or if the base space is not a torus so that one only has a fiberwise notion of heterotic T-duality.
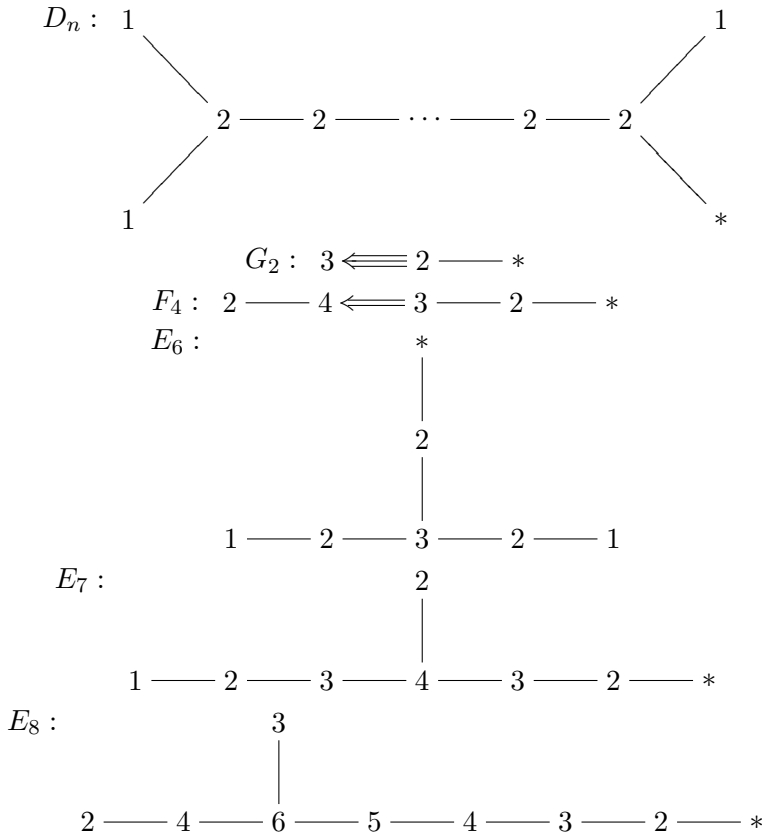
9   Conclusions

In this paper, we have done three things:

- We argued that conventional heterotic worldsheet theories do not suffice to describe arbitrary E_8 gauge fields in compactifications. The basic issue is that the conventional construction builds each E_8 using a $\mathrm{Spin}(16)/\mathbb{Z}_2$ subgroup, and only data reducible to $\mathrm{Spin}(16)/\mathbb{Z}_2$ can be described, but not all E_8 gauge fields are so reducible.
- We reviewed alternative constructions of the 10-dimensional E_8 algebra, using other subgroups than $\mathrm{Spin}(16)/\mathbb{Z}_2$. In examples we recalled the character decomposition of the affine algebras (see, e.g., [3] for earlier work) and also described how that character decomposition is realized physically in a heterotic partition function via orbifold twisted sectors that correlate to E_8 group theory. In addition to discussing maximal-rank subgroups, we also discussed whether it may be possible to use non-maximal-rank subgroups such as $G_2 \times F_4$.
- We developed[20] fibered WZW models to describe these more general E_8 constructions on arbitrary manifolds. In fact, this allows us to describe conformal field theories in which the left-movers couple to general G-current algebras at arbitrary levels, a considerable generalization of ordinary heterotic worldsheet constructions. This also enables us to give a physical realization of some new elliptic genera recently studied in the mathematics literature [63, 64].

It would be interesting if the elliptic genera discussed here appeared in any black hole entropy computations.

It would also be interesting to understand heterotic worldsheet instanton corrections in these theories, along the lines of [76–80]. Unfortunately, to produce the (0,2) analogs of the A and B models described in those papers

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[20]After the original publication of this paper it was pointed out to us that chiral fibered WZW models with $(0, 1)$ supersymmetry have been previously considered, under the name "lefton, righton Thirring models," see, for example, [4–8]. We believe we have pushed the notion somewhat further, by studying anomaly cancelation, spectra, elliptic genera, and so forth in chiral fibered WZW models with $(0, 2)$ supersymmetry.

required a left-moving topological twist involving a global $U(1)$ symmetry present because the left-moving fermions were realizing a $U(n)$ current algebra at level 1. In more general cases, there will not be such a global $U(1)$ symmetry, unless one adds it in by hand.

Acknowledgments

Appendix A   Group theory

In this appendix we will derive some results on subgroups of the Lie group E_8 that are used in the text. We would like to thank A. Knutson for explanations of the material below.

First, let us collect in the following table affine Dynkin diagrams for the simple Lie groups, labeled by the weights of the highest-weight state for the adjoint representation, which shall prove useful when determining subgroup structures:

D_n :

$$D_n: \quad \overset{1}{} \quad \overset{1}{}$$

$$2 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} \cdots \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} 2$$

1   *

$$G_2: \quad 3 \Longleftarrow 2 \rule[0.5ex]{1em}{0.4pt} *$$

$$F_4: \quad 2 \rule[0.5ex]{1em}{0.4pt} 4 \Longleftarrow 3 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} *$$

E_6 :

$$*$$
$$|$$
$$2$$
$$|$$
$$1 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} 3 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} 1$$

E_7 :

$$2$$
$$|$$
$$1 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} 3 \rule[0.5ex]{1em}{0.4pt} 4 \rule[0.5ex]{1em}{0.4pt} 3 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} *$$

E_8 :

$$3$$
$$|$$
$$2 \rule[0.5ex]{1em}{0.4pt} 4 \rule[0.5ex]{1em}{0.4pt} 6 \rule[0.5ex]{1em}{0.4pt} 5 \rule[0.5ex]{1em}{0.4pt} 4 \rule[0.5ex]{1em}{0.4pt} 3 \rule[0.5ex]{1em}{0.4pt} 2 \rule[0.5ex]{1em}{0.4pt} *$$

(Arrows point from long to short roots.)

Next, we need to compute the centers of the universal covers of each of the groups above. We can read this off very simply from the diagrams above: the order of the center is the sum of the number of copies of 1 appearing on each affine Dynkin diagram, counting the extra point $*$ as 1. Thus, for example, SU(n) (A_{n-1}) has center of order n; G_2, F_4, and E_8 have center of order 1, so no center at all; and E_7 has center of order 2, hence $\mathbb{Z}_2$. The technical reason for this is as follows. The vertices of the affine Dynkin diagram correspond to the corners of the Weyl alcove, corresponding to conjugacy classes of elements whose centralizer is semisimple. (The points in the Weyl alcove correspond to conjugacy classes in the simply connected compact group.) The label on a vertex is the order of the corresponding conjugacy class in the adjoint group. Any central element is its own conjugacy class and has semisimple centralizer — namely the whole group. Its order in the adjoint group is 1. The result follows.

Next, to read off a maximal-rank subalgebra from one of the affine Dynkin diagrams, first omit one of the nodes; what remains is the Dynkin diagram

for a subalgebra, generated by all the positive roots except the one omitted. (This will not produce all maximal-rank subalgebras in general: to do that, one will have to repeat a process of first omitting nodes then affinizing, possibly several times. However, only a single step will be required for the examples in which we are primarily interested.)

To read off a maximal-rank sub*group* takes a little more work. If the node we omit is labeled above with n, say, then the weight lattice for the ambient Lie algebra and the weight lattice for the subalgebra have relative index n. This means that the subgroup will have center whose order is n times larger than the center of the ambient Lie group.

For example, consider the group E_7. The Lie algebra e_7 contains (maximal-rank) su(8), obtained by omitting the 2 node sticking out at the top of the Dynkin diagram. The center of the maximal-rank subgroup of E_7 should then be two times larger than that of E_7. We computed above that E_7 has center $\mathbb{Z}_2$; hence, the center of the subgroup should have order $(2)(2) = 4$. Now, the group SU(8) has center $\mathbb{Z}_8$, so to get a center of order 4, we must quotient by $\mathbb{Z}_2$. Thus, a maximal-rank subgroup of E_7 is SU(8)/$\mathbb{Z}_2$.

Similarly, we can show that E_8 contains the subgroup $(E_7 \times \mathrm{SU}(2))/\mathbb{Z}_2$. The subalgebra $e_7 \times \mathrm{su}(2)$ is obtained from the affine Dynkin diagram for e_8 by omitting the 2 vertex next to the $*$. Thus, the center of the subgroup needs to be twice as large as the center of E_8, but E_8 has no center, so the center of the subgroup must be $\mathbb{Z}_2$. We computed that E_7 has center $\mathbb{Z}_2$, and it is a standard fact that SU(2) has center $\mathbb{Z}_2$, from which we deduce that the subgroup of E_8 is $(E_7 \times \mathrm{SU}(2))/\mathbb{Z}_2$.

In exactly the same fashion, one can show that E_8 has the subgroup $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$. Here we omit the 3 node next to 2 and $*$ on the affine Dynkin diagram for E_8, which means that the center of the subgroup must be three times larger than the center of E_8, hence $\mathbb{Z}_3$. One can compute that the center of E_6 is order 3, hence $\mathbb{Z}_3$, and the center of SU(3) is well known to be $\mathbb{Z}_3$, so for the subgroup to have center $\mathbb{Z}_3$ it must be $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$.

It can also be shown that E_8 has the subgroup $(\mathrm{Spin}(10) \times \mathrm{SU}(4))/\mathbb{Z}_4$, though here we have to work a little more. On the affine Dynkin diagram for E_8, we omit the 4 node, and since E_8 has no center, we see the subgroup should have center of order 4. Both $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are abelian of order 4, so we have to work slightly harder to determine whether the subgroup is $(\mathrm{Spin}(10) \times \mathrm{SU}(4))/\mathbb{Z}_4$ or $/\mathbb{Z}_2^2$. In this case, since E_8 has no center, the simply connected group and the adjoint group are the same, and so the labels

on the Dynkin diagram contain the element orders in the simply connected group, not just the adjoint group as would ordinarily be the case. The fact that we omitted a node marked 4 means that the subgroup should contain a central element of order 4, not just that the subgroup's center should be of order 4, from which we can deduce that the subgroup in question is $(\mathrm{Spin}(10) \times \mathrm{SU}(4))/\mathbb{Z}_4$.

A result that is more important for this paper is the fact that E_8 contains an $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ subgroup. We get this result by removing the 5 node on the labeled E_8 Dynkin diagram. The index of the two weight lattices is then 5, or put another way, the subgroup sits inside E_8 as the centralizer of a certain element of (adjoint) order 5, which is then the remaining center. Since $\mathrm{SU}(5) \times \mathrm{SU}(5)$ has center $\mathbb{Z}_5 \times \mathbb{Z}_5$, we see that the subgroup of E_8 must be $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$.

Analogous reasoning tells us that the $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ subgroup of E_8 above cannot be a subgroup of $\mathrm{Spin}(16)/\mathbb{Z}_2$, or vice versa. The $\mathrm{Spin}(16)/\mathbb{Z}_2$ subgroup is obtained by removing the leftmost 2 node above. The centralizer of that subgroup is then order 2, and because 2 and 5 are relatively prime, no element of $\mathbb{Z}_5$ contains an element of order 2 or vice versa; hence, neither is a subgroup of the other. This result is even true at the level of algebras. If $su(5)$ were a subalgebra of $so(8)$, then $su(5) \times su(5)$ would be a subalgebra of $so(8) \times so(8)$, itself a subalgebra of $so(16)$, and then there might be a way for $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ to be a subgroup of $\mathrm{Spin}(16)/\mathbb{Z}_2$. However, $so(8)$ does not contain the algebra $su(5)$ – the largest subalgebra it contains is $su(4) \times u(1)$.

Under the $su(5) \times su(5)$ subalgebra of e_8, the **248** (adjoint) representation decomposes as

$$(\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \overline{\mathbf{5}}) \oplus (\mathbf{5}, \overline{\mathbf{10}}) \oplus (\overline{\mathbf{5}}, \mathbf{10}).$$

How does the $\mathbb{Z}_5$ act on the representations above? In principle, since E_8 contains an $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ subgroup, the representations above must be representations of $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$ and so must be invariant under $\mathbb{Z}_5$. Suppose the first $\mathrm{SU}(5)$ acts on the five-dimensional vector space V in the fundamental representation, and the second acts on W in the fundamental representation. The **10**'s above can be understood as the second exterior power of V or W. In order for each of the representations to remain invariant under the $\mathbb{Z}_5$, the $\mathbb{Z}_5$ might act on basis elements of V by fifth roots of unity, and on basis elements of W by inverses of squares of fifth roots of unity. In other words, if g denotes the generator of the $\mathbb{Z}_5$,

then take

$$g : v \mapsto \zeta v$$
$$w \mapsto \zeta^{-2} w$$

for $v \in V$, $w \in W$, $\zeta = \exp(2\pi i/5)$. Then, with this choice of g action, we see that the four nonadjoint representations of $su(5) \times su(5)$ appearing in the decomposition of the adjoint representation of e_8, namely $(\Lambda^2 V) \otimes W$, $(\Lambda^2 V^*) \otimes W^*$, $V \otimes (\Lambda^2 W^*)$, and $V^* \otimes (\Lambda^2 W)$, are all invariant under $\mathbb{Z}_5$.

Similarly, one can show that E_8 has the subgroup $SU(9)/\mathbb{Z}_3$. To get the $su(9)$ subalgebra, we omit the top 3 node on the affine Dynkin diagram for E_8, so the center of the subgroup must be three times as large as the center of E_8, but since E_8 has no center, we see that the center of the subgroup must be $\mathbb{Z}_3$. Since $SU(9)$ has center $\mathbb{Z}_9$, we see that the subgroup of E_8 must be $SU(9)/\mathbb{Z}_3$.

Under the $su(9)$ subalgebra of e_8, the **248** (adjoint) representation decomposes as

$$\mathbf{80} \oplus \mathbf{84} \oplus \overline{\mathbf{84}}.$$

(The **84** is $\Lambda^3 V$ for V a nine-dimensional vector space, and the **80** is the adjoint representation of $SU(9)$.) To build E_8 from $SU(9)$, we first quotient $SU(9)$ by $\mathbb{Z}_3$. If V is a nine-dimensional vector space upon which $SU(9)$ acts in the fundamental representation, then notice that it is consistent for the $\mathbb{Z}_3$ to act as third roots of unity on each element of a basis for V (consistent in the sense that the representations of $su(9)$ forming the adjoint representation of e_8 are invariant under such a $\mathbb{Z}_3$ — in other words, the representations appearing above are representations of $SU(9)/\mathbb{Z}_3$ not just $SU(9)$).

Two cases that involve more work are the $su(2) \times su(8)$ and $su(2) \times su(3) \times su(6)$ subalgebras of E_8. From the analysis above, it is straightforward to determine that in the first case, the center of the subgroup should have order 4, so the subgroup should have the form $SU(2) \times SU(8)/G$ for some G of order $16/4 = 4$, which is ambiguous. In the second case, the center of the subgroup should be of order 6, so the subgroup should have the form $SU(2) \times SU(3) \times SU(6)/G$ for some G of order $(2)(3)(6)/6 = 6$, which is again ambiguous. We can resolve the ambiguity [81] by looking at the decomposition of the adjoint representation of e_8 under each subalgebra. In particular, in that decomposition one gets the tensor product of fundamental representations of the factors, corresponding to where the missing vertex is attached.

For example, for $\mathrm{su}(2) \times \mathrm{su}(8)$, the decomposition of the adjoint representation of e_8 includes the $(\mathbf{2}, \Lambda^2 \mathbf{8})$ of $\mathrm{su}(2) \times \mathrm{su}(8)$, corresponding to the diagrams
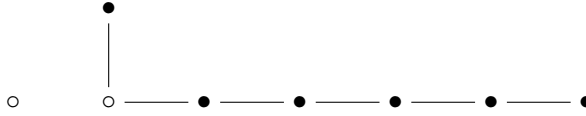
where $\circ$ indicates the position of the omitted e_8 diagram vertex. Thus, inside the

$$\mathbb{Z}_2 \times \mathbb{Z}_8 = <-1> \times <t>$$

center of $\mathrm{SU}(2) \times \mathrm{SU}(8)$, the kernel is generated by $(-1, t^2)$. (On the standard representation of $\mathrm{SU}(8)$, t acts as an eighth root of unity, but on $\Lambda^2 \mathbf{8}$ it acts as a fourth root, so t^2 acts as -1 and $(-1, t^2)$ acts as $+1$.) Thus, the subgroup of E_8 with algebra $\mathrm{su}(2) \times \mathrm{su}(8)$ is given by

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(8)}{\mathbb{Z}_4}$$

where the $\mathbb{Z}_4$ acts diagonally.

For $\mathrm{su}(2) \times \mathrm{su}(3) \times \mathrm{su}(6)$, the adjoint representation of e_8 decomposes to include the $(\mathbf{2}, \mathbf{3}, \mathbf{6})$ of $\mathrm{su}(2) \times \mathrm{su}(3) \times \mathrm{su}(6)$, judging by the diagrams
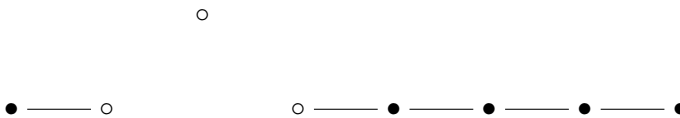
Inside the $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_6 = <1> \times <r> \times <s>$ center of $\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(6)$, the kernel is generated by $(-1, 1, s^3)$ and $(1, r, s^4)$; hence, the subgroup of E_8 with algebra $\mathrm{su}(2) \times \mathrm{su}(3) \times \mathrm{su}(6)$ is given by

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(6)}{\mathbb{Z}_2 \times \mathbb{Z}_3}$$

with action on the factors as indicated above. As a consistency check, let us apply this same reasoning to the $\mathrm{su}(5) \times \mathrm{su}(5)$ subalgebra of e_8. Here, from omitting a vertex from the extended Dynkin diagram, we get the diagrams
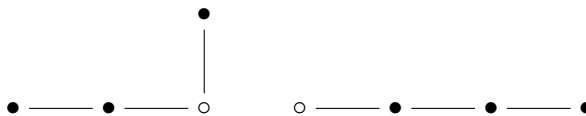
From the right diagram, we get a **5** of su(5), and from the left diagram, we get a $\Lambda^2 \mathbf{5} = \mathbf{10}$. Writing the center of $SU(5) \times SU(5)$ as $< r > \times < s >$, the kernel is (r^{-2}, s), and so the subgroup of E_8 with algebra su(5) $\times$ su(5) is

$$\frac{SU(5) \times SU(5)}{\mathbb{Z}_5}$$

as we worked out previously.

Since this appendix is rather lengthy, and many readers will be most interested in simply picking off results for maximal-rank subgroups of E_8, we have included a summary table below.

Maximal-rank subgroups of E_8
$(E_7 \times SU(2))/\mathbb{Z}_2$
$(E_6 \times SU(3))/\mathbb{Z}_3$
$(\mathrm{Spin}(10) \times SU(4))/\mathbb{Z}_4$
$(SU(5) \times SU(5))/\mathbb{Z}_5$
$SU(9)/\mathbb{Z}_3$
$(SU(2) \times SU(8))/\mathbb{Z}_4$
$(SU(2) \times SU(3) \times SU(6))/\mathbb{Z}_2 \times \mathbb{Z}_3$

Some references on these matters are [82, 83].

Appendix B   Notes on $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundles

Given the role that $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundles play in the analysis, we thought that a short section reviewing properties of such bundles would be useful.

First, any $SU(5) \times SU(5)$ bundle with connection defines an $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundle with connection. To get the bundle, one simply takes the image of the transition functions of the original bundle in the coset, and similarly, to get the connection, one takes the image of the holonomies of the original connection in the coset to get the holonomies of the connection on the $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundle.

However, the reverse need not be true — not every $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundle defines an $SU(5) \times SU(5)$ bundle.

In addition to ordinary Chern-like invariants, an $(SU(5) \times SU(5))/\mathbb{Z}_5$ bundle on a space X has a characteristic class in $H^2(X, \mathbb{Z}_5)$, which characterizes

the obstruction to lifting to an $SU(5)^2$ bundle. This class is defined as follows. The short exact sequence

$$\mathbb{Z}_5 \longrightarrow SU(5) \times SU(5) \longrightarrow \frac{SU(5) \times SU(5)}{\mathbb{Z}_5}$$

extends to the right as

$$\mathbb{Z}_5 \to SU(5)^2 \to (SU(5)^2)/\mathbb{Z}_5 \to B\mathbb{Z}_5 \to BSU(5)^2$$
$$\to B(SU(5)^2)/\mathbb{Z}_5 \to K(\mathbb{Z}_5, 2)$$

The characteristic class in $H^2(X, \mathbb{Z}_5)$ comes from composing the classifying map $X \to B(SU(5)^2)/\mathbb{Z}_5$ defining the bundle, with the map $B(SU(5)^2)/\mathbb{Z}_5 \to K(\mathbb{Z}_5, 2)$.

Appendix C $SU(N)_1$ characters

The character for the $\mathbf{1}$ representation of $SU(N)_1$ is

$$\chi_{SU(N)}(\mathbf{1}, \tau) = \frac{1}{\eta(\tau)^{N-1}} \sum_{\vec{m} \in \mathbb{Z}^{N-1}} q^{(\sum m_i^2 + (\sum m_i)^2)/2}$$

$$= \frac{1}{\eta(\tau)^{N-1}} \sum_{\vec{m} \in \mathbb{Z}^{N-1}} q^{(\vec{m} \cdot M \cdot \vec{m})/2} \qquad \text{(C.1)}$$

where

$$M = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix} \quad \text{and}$$

$$M^{-1} = \frac{1}{N} \begin{pmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \cdots & -1 \\ -1 & -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & N-1 \end{pmatrix}.$$

Under a modular transformation, $\chi_{\mathrm{SU}(N)}(\mathbf{1}, -1/\tau)$ is a linear combination of $\chi_{\mathrm{SU}(N)}(\wedge^k \mathbf{N}, \tau)$. Poisson-resumming (C.1), we obtain

$$\chi_{\mathrm{SU}(N)}(\wedge^k \mathbf{N}, \tau) = \frac{1}{\eta(\tau)^{N-1}} \sum_{\substack{\vec{m} \in \mathbb{Z}^{N-1} \\ \sum m_i = k \mod N}} q^{(\vec{m} \cdot M^{-1} \cdot \vec{m})/2} \qquad (\mathrm{C.2})$$

$$= \frac{1}{\eta(\tau)^{N-1}} \sum_{\substack{\vec{m} \in \mathbb{Z}^{N-1} \\ \sum m_i = k \mod N}} q^{(\sum m_i^2 - \frac{1}{N}(\sum m_i)^2)/2}. \qquad (\mathrm{C.3})$$

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