Division algebras and supersymmetry IV

JOHN HUERTA

Recent work applying higher gauge theory to the superstring has indicated the presence of 'higher symmetry', and the same methods work for the super-2-brane. In the previous paper in this series, we used a geometric technique to construct a 'Lie 2-supergroup' extending the Poincaré supergroup in precisely those spacetime dimensions where the classical Green–Schwarz superstring makes sense: 3, 4, 6 and 10. In this paper, we use the same technique to construct a 'Lie 3-supergroup' extending the Poincaré supergroup in precisely those spacetime dimensions where the classical Green–Schwarz super-2-brane makes sense: 4, 5, 7 and 11. Because the geometric tools are identical, our focus here is on the precise definition of a Lie 3-supergroup.

1. Introduction

There is a deep connection between supersymmetry and the normed division algebras: the real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$. This is visible in super-Yang–Mills theory and in the classical supersymmetric string and 2-brane. Most simply, it is seen in the dimensions for which these theories make sense. The normed division algebras have dimension $n = 1$, 2, 4 and 8, while the classical supersymmetric string and super-Yang–Mills make sense in spacetimes of dimension two higher than these: $n + 2 = 3$, 4, 6 and 10. Similarly, the classical supersymmetric 2-brane makes sense in dimensions three higher: $n + 3 = 4$, 5, 7 and 11. At a deeper level, we find this connection leads to 'higher gauge theory', a kind of gauge theory suitable for describing the parallel transport not merely of particles but of extended objects, such as strings and membranes.

This is the fourth in a series of papers exploring the relationship between supersymmetry and division algebras [6, 7, 28], the first two of which were coauthored with John Baez. In the first paper [6], we reviewed the known story of how the division algebras give rise to the supersymmetry of super-Yang–Mills theory. In the second [7], we made contact with higher gauge theory: we showed how the division algebras can be used to construct 'Lie 2-superalgebras' $\mathfrak{superstring}(n + 1, 1)$, which extend the Poincaré superalgebra in the superstring dimensions $n + 2 = 3$, 4, 6 and 10, and 'Lie 3-superalgebras' $2\text{-}\mathfrak{brane}(n + 2, 1)$ in the super-2-brane dimensions $n + 3 = 4$, 5, 7 and 11. In our previous paper [28], we described a geometric method to integrate these Lie 2-superalgebras to 'Lie 2-supergroups'. Here we employ the same method to integrate the Lie 3-superalgebras to 'Lie 3-supergroups'. Indeed, since our method of integration is the same, our focus here is on giving the precise definition of a Lie 3-supergroup.
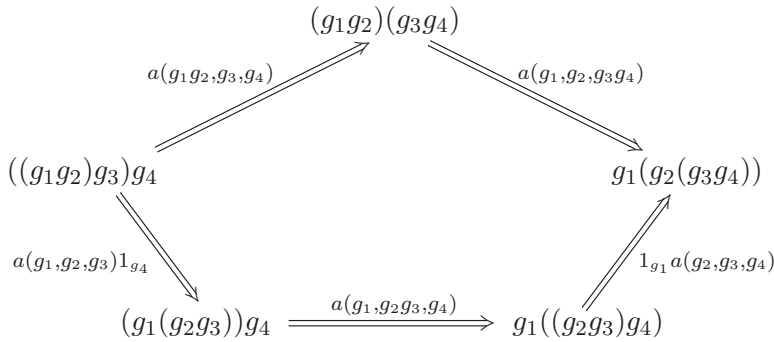
As a prelude, let us recall a bit about '2-groups'. Roughly, 2-group is a mathematical gadget like a group [8], but where the group axioms, such as the associative law:

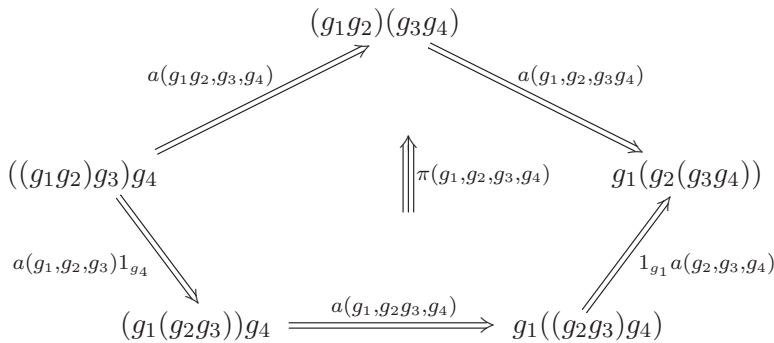$$(g_1 g_2) g_3 = g_1 (g_2 g_3)$$

no longer hold. Instead, they are replaced by isomorphisms, such as the 'associator':

$$a(g_1, g_2, g_3) \colon (g_1 g_2) g_3 \Rightarrow g_1 (g_2 g_3)$$

which must satisfy axioms of their own. For instance, the associator satisfies the **pentagon identity**, which says the following pentagon commutes:

$$(g_1g_2)(g_3g_4)$$

$$a(g_1g_2,g_3,g_4) \qquad\qquad a(g_1,g_2,g_3g_4)$$

$$((g_1g_2)g_3)g_4 \qquad\qquad g_1(g_2(g_3g_4))$$

$$a(g_1,g_2,g_3)1_{g_4} \qquad\qquad 1_{g_1}a(g_2,g_3,g_4)$$

$$(g_1(g_2g_3))g_4 \xrightarrow{a(g_1,g_2g_3,g_4)} g_1((g_2g_3)g_4)$$

Moving up in dimension, a '3-group' is a mathematical gadget like a 2-group, but where the 2-group axioms *no longer hold*. Instead, they are replaced by isomorphisms, such as the 'pentagonator':

$$(g_1g_2)(g_3g_4)$$

$$a(g_1g_2,g_3,g_4) \qquad\qquad a(g_1,g_2,g_3g_4)$$

$$((g_1g_2)g_3)g_4 \qquad \pi(g_1,g_2,g_3,g_4) \qquad g_1(g_2(g_3g_4))$$

$$a(g_1,g_2,g_3)1_{g_4} \qquad\qquad 1_{g_1}a(g_2,g_3,g_4)$$

$$(g_1(g_2g_3))g_4 \xrightarrow{a(g_1,g_2g_3,g_4)} g_1((g_2g_3)g_4)$$

which must satisfy axioms of their own. For us, the most significant of the 3-group axioms will be the **pentagonator identity**, which says the following polyhedron commutes:
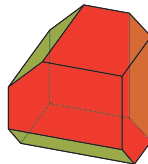
This shape is Stasheff's **associahedron**. This is the polyhedron where:

- vertices are parenthesized lists of five things, e.g. $(((fg)h)k)p$;

- edges connect any two vertices related by an application of the associator, e.g.

$$(((fg)h)k)p \Rightarrow ((fg)(hk))p.$$

The pentagon faces of the associahedron are filled with pentagonators, and the square faces are filled with a naturality condition for the associator, which will be trivial in the examples we consider.

What is the meaning of the pentagonator identity? For a certain class of 3-groups that we call 'slim 3-groups', it is a *cocycle condition*. When $\mathcal{G}$ is a slim 3-group, it is entirely determined by the following data:

- a group G;

- an abelian group H on which G acts;

- a 4-cocycle

$$\pi \colon G^4 \to H$$

   in group cohomology.

The condition that the map π be a 4-cocycle is an equation equivalent to the pentagonator identity on $\mathcal{G}$. Thus, we can build slim 3-groups merely by giving a 4-cocycle in group cohomology. This will be our strategy for constructing 3-groups.

Just as we can view a Lie group as the global version of a Lie algebra, the 3-groups we will construct will be the global versions of 'Lie 3-superalgebras'. In a general, a 'Lie n-algebra' is a gadget like a Lie algebra, but defined on an n-term chain complex, where the Lie algebra axioms hold only up to coherent chain homotopy. In turn, this is a special case of Lada and Stasheff's L_∞-algebras. All of these algebras have 'super' variants, with a $\mathbb{Z}_2$-grading throughout.

The Lie 3-superalgebra in question comes to us from physics. As we show in our previous paper [7], the existence of the super-2-brane in dimensions $n + 3 = 4$, 5, 7 and 11 secretly gives rise to a way to extend the Poincaré superalgebra, $\mathfrak{siso}(n + 2, 1)$, to a Lie 3-superalgebra we like to call 2-$\mathfrak{brane}(n + 2, 1)$. Here, the Poincaré superalgebra is a Lie superalgebra whose even part consists of the infinitesimal rotations $\mathfrak{so}(n + 2, 1)$ and translations V on Minkowski spacetime, and whose odd part consists of 'supertranslations' S, or spinors:

$$\mathfrak{siso}(n + 2, 1) = \mathfrak{so}(n + 2, 1) \ltimes (V \oplus S).$$

Naturally, we can identify the translations V with the vectors in Minkowski spacetime, so V carries a Minkowski inner product h of signature $(n+2,1)$. We extend $\mathfrak{siso}(n+2,1)$ to a Lie 3-superalgebra $2\text{-}\mathfrak{brane}(n+2,1)$ defined on a 3-term chain complex

$$\mathfrak{siso}(n+2,1) \xleftarrow{d} 0 \xleftarrow{d} \mathbb{R}$$

equipped with some extra structure.

The most interesting Lie 3-algebra of this type, $2\text{-}\mathfrak{brane}(10,1)$, plays an important role in 11-dimensional supergravity. This idea goes back to the work of Castellani, D'Auria and Fré [12, 14]. These authors derived the field content of 11-dimensional supergravity starting from a differential graded-commutative algebra, a mathematical gadget 'dual' to a Lie n-algebra. Later, Sati, Schreiber and Stasheff [35] explained that these fields can be reinterpreted as a kind of generalized 'connection' valued in a Lie 3-superalgebra which they called '$\mathfrak{sugra}(10,1)$'. This is the Lie 3-superalgebra we are calling $2\text{-}\mathfrak{brane}(10,1)$. If we follow these authors and consider a 3-connection valued in $2\text{-}\mathfrak{brane}(10,1)$, we find it can be described locally by these fields:

$2\text{-}\mathfrak{brane}(n+2,1)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 3-form
$\downarrow$	
0	
$\downarrow$	
$\mathfrak{siso}(n+2,1)$	$\mathfrak{siso}(n+2,1)$-valued 1-form

Here, a $\mathfrak{siso}(n+2,1)$-valued 1-form contains familiar fields: the Levi-Civita connection, the vielbein, and the gravitino. But now we also see a 3-form, called the C field. This is something we might expect on physical grounds, at least in dimension 11, because for the quantum theory of a 2-brane to be consistent, it must propagate in a background obeying the equations of 11-dimensional supergravity, in which the C field naturally shows up [40].

What sort of mathematical object is the C field? In electromagnetism, the potential is locally described by 1-form A on spacetime, but globally it has a beautiful geometric meaning: it is a connection on a U(1) bundle. Similarly, in string theory, the B field is locally described by a 2-form, but globally is a connection on a 'U(1) gerbe' — this is a mathematical gadget like a bundle, but with fibers that are categories rather than sets [9]. Likewise, the C field is locally described by a 3-form, but globally is a connection on a 'U(1) 2-gerbe' — a mathematical gadget like a gerbe, but

with fibers that are 2-categories rather than categories. This is shown in the work of Diaconescu, Freed, and Moore [15] using the language of differential characters. The work of Aschieri and Jurco [2] is also relevant here.

Meanwhile, the mathematical theory of 3-connections is in its infancy. Their holonomy has been studied by Martins and Picken [32], and their gauge transformations by Saemann and Wolf [34] and in a different formulation by Wang [44]. Saemann and Wolf develop these ideas as an attack on six-dimensional superconformal field theories, which physicists will recognize as part of the program to develop M5-brane models in M-theory. In a related direction, Sati, Schreiber and Fiorenza [20] have discovered that 2-$\mathfrak{brane}(n+2,1)$ fits into a much larger picture they are calling 'the brane bouquet'. In particular, Sati, Schreiber and Fiorenza find that the 3-group we construct in this paper may be the correct background for the M5-brane, a much more mysterious object than the 2-brane that is our focus here.

So far, we have focused on the super-2-brane Lie 3-superalgebras and generalized connections valued in them. This connection data is infinitesimal: it tells us how to parallel transport 2-branes a little bit. Ultimately, we would like to understand this parallel transport globally, as we do with particles in ordinary gauge theory.

To achieve this global description, we will need 'Lie n-groups' rather than Lie n-algebras. Naively, one expects that there is a Lie 3-supergroup 2-Brane$(n+2,1)$ for which the Lie 3-superalgebra 2-$\mathfrak{brane}(n+2,1)$ is an infinitesimal approximation. In fact, this is precisely what we will construct.

This paper is organized as follows. In Section 2, we recall the definition of a Lie n-superalgebra, and describe how particularly simple examples of these arise from $(n+1)$-cocycles in Lie superalgebra cohomology. We then describe our key example: the Lie 3-superalgebra we want to integrate, 2-$\mathfrak{brane}(n+2,1)$, in Section 2.1.

Section 3 is the heart of the paper. There we sketch how an n-group may be defined using an $(n+1)$-cocycle in group cohomology, and recall just enough tricategory theory to handle the case of interest to us: 3-groups coming from 4-cocycles. We give this definition in a way that makes it easy to generalize to Lie 3-groups and Lie 3-supergroups, using the supermanifold theory we outline next.

In Section 4, we provide a brief overview of supergeometry, focusing on the supermanifolds of greatest interest to us: supermanifolds diffeomorphic to super vector spaces, and smooth maps between them, though we outline the general definition. In Section 5, we use this supergeometry to define a Lie 3-supergroup, and show that 4-cocycles in Lie supergroup cohomology give rise to simple examples thereof.

Having defined Lie n-superalgebras from $(n + 1)$-cocycles on Lie super-algebras and Lie n-supergroups from $(n + 1)$-cocycles on Lie supergroups, we have a plausible strategy to integrate a Lie n-superalgebra to a Lie n-supergroup. In Section 6, we recall how to do just that, for the special case where the Lie superalgebra is nilpotent. Finally, in Section 7, we use this method to integrate the Lie 3-superalgebra 2-$\mathfrak{brane}(n + 2, 1)$ to a Lie 3-supergroup.

2. Lie n-superalgebras from Lie superalgebra cohomology

Unlike the Lie n-groups we shall meet in later sections, 'Lie n-algebras' are much simpler than their group-like counterparts. This is as one might expect from experience with ordinary Lie groups and Lie algebras. It is straightforward to define a Lie n-algebra for all n. It also straightforward to incorporate the 'super' case immediately.

Better still, the 'slim Lie n-superalgebras' we define in this section are a prelude to the more difficult 'slim Lie n-groups' and 'slim Lie n-supergroups' we will describe later: they are constructed simply from an $(n + 1)$-cocycle in Lie superalgebra cohomology, much as the group versions will be constructed from cocycles in the corresponding cohomology theories for Lie groups and Lie supergroups. Later, in Section 2.1, we introduce our principal example of a Lie n-superalgebra: the Lie 3-superalgebra 2-$\mathfrak{brane}(n + 2, 1)$, which arises from the theory of the supersymmetric 2-brane in spacetimes of dimension 4, 5, 7 and 11.

As we touched on in the Introduction, a Lie n-superalgebra is a certain kind of L_∞-superalgebra, which is the super version of an L_∞-algebra. This last is a chain complex, V:

$$V_0 \xleftarrow{\ d\ } V_1 \xleftarrow{\ d\ } V_2 \xleftarrow{\ d\ } \cdots$$

equipped with a structure like that of a Lie algebra, but where the Jacobi identity only holds *up to chain homotopy*, and this chain homotopy satifies its own identity up to chain homotopy, and so on. For an L_∞-*super*algebra, each term in the chain complex has a $\mathbb{Z}_2$-grading, and we introduce extra signs. A Lie n-superalgebra is an L_∞-superalgebra in which only the first n terms are nonzero, starting with V_0.

Now, we shall describe how a Lie superalgebra $(n + 1)$-cocycle:

$$\omega \colon \Lambda^n \mathfrak{g} \to \mathfrak{h}$$

can be used to construct an especially simple Lie n-superalgebra, defined on a chain complex with $\mathfrak{g}$ in grade 0, $\mathfrak{h}$ in grade $n - 1$, and all terms in between trivial. To make this precise, we had better start with some definitions.

To begin at the beginning, a **super vector space** is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ where V_0 is called the **even** part, and V_1 is called the **odd** part. There is a symmetric monoidal category SuperVect which has:

- $\mathbb{Z}_2$-graded vector spaces as objects;

- Grade-preserving linear maps as morphisms;

- A tensor product $\otimes$ that has the following grading: if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, then $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$;

- A braiding

$$B_{V,W} \colon V \otimes W \to W \otimes V$$

  defined as follows: if $v \in V$ and $w \in W$ are of grade $|v|$ and $|w|$, then

$$B_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

The braiding encodes the 'the rule of signs': in any calculation, when two odd elements are interchanged, we introduce a minus sign. We can see this in the axioms of a Lie superalgebra, which resemble those of a Lie algebra with some extra signs.

Briefly, a **Lie superalgebra** $\mathfrak{g}$ is a Lie algebra in the category of super vector spaces. More concretely, it is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, equipped with a graded-antisymmetric bracket:

$$[-, -] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g},$$

which satisfies the Jacobi identity up to signs:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|} [Y, [X, Z]].$$

for all homogeneous $X, Y, Z \in \mathfrak{g}$. Note how we have introduced an extra minus sign upon interchanging odd X and Y, exactly as the rule of signs says we should.

It is straightforward to generalize the cohomology of Lie algebras, as defined by Chevalley–Eilenberg [1, 13], to Lie superalgebras [31]. Suppose $\mathfrak{g}$ is a Lie superalgebra and $\mathfrak{h}$ is a representation of $\mathfrak{g}$. That is, $\mathfrak{h}$ is a super vector space equipped with a Lie superalgebra homomorphism $\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$.

The **cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$** is computed using the **Lie superalgebra cochain complex**, which consists of graded-antisymmetric p-linear maps at level p:

$$C^p(\mathfrak{g}, \mathfrak{h}) = \{\omega \colon \Lambda^p \mathfrak{g} \to \mathfrak{h}\}.$$

We call elements of this set **$\mathfrak{h}$-valued p-cochains on $\mathfrak{g}$**. Note that the $C^p(\mathfrak{g}, \mathfrak{h})$ is a super vector space, in which grade-preserving elements are even, while grade-reversing elements are odd. When $\mathfrak{h} = \mathbb{R}$, the trivial representation, we typically omit it from the cochain complex and all associated groups, such as the cohomology groups. Thus, we write $C^\bullet(\mathfrak{g})$ for $C^\bullet(\mathfrak{g}, \mathbb{R})$.

Next, we define the coboundary operator $d \colon C^p(\mathfrak{g}, \mathfrak{h}) \to C^{p+1}(\mathfrak{g}, \mathfrak{h})$. Let ω be a homogeneous p-cochain and let $X_1, \ldots, X_{p+1}$ be homogeneous elements of $\mathfrak{g}$. Now define:

$$
\begin{aligned}
&d\omega(X_1, \ldots, X_{p+1}) \\
&= \sum_{i=1}^{p+1} (-1)^{i+1}(-1)^{|X_i||\omega|} \epsilon_1^{i-1}(i) \rho(X_i) \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\
&\quad + \sum_{i<j} (-1)^{i+j}(-1)^{|X_i||X_j|} \epsilon_1^{i-1}(i) \epsilon_1^{j-1}(j) \\
&\qquad\qquad \times \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1}).
\end{aligned}
$$

Here, we denote the action of $\mathfrak{g}$ on $\mathfrak{h}$ by juxtaposition, and $\epsilon_i^j(k)$ is shorthand for the sign one obtains by moving X_k through $X_i, X_{i+1}, \ldots, X_j$. In other words,

$$\epsilon_i^j(k) = (-1)^{|X_k|(|X_i|+|X_{i+1}|+\cdots+|X_j|)}.$$

Following the usual argument for Lie algebras, one can check that:

Proposition 1. *The Lie superalgebra coboundary operator d satisfies $d^2 = 0$.*

We thus say an $\mathfrak{h}$-valued p-cochain ω on $\mathfrak{g}$ is a **p-cocycle** or **closed** when $d\omega = 0$, and a **p-coboundary** or **exact** if there exists a $(p-1)$-cochain θ such that $\omega = d\theta$. Every p-coboundary is a p-cocycle, and we say a p-cocycle is **trivial** if it is a coboundary. We denote the super vector spaces of p-cocycles and p-coboundaries by $Z^p(\mathfrak{g}, \mathfrak{h})$ and $B^p(\mathfrak{g}, \mathfrak{h})$ respectively. The pth **Lie superalgebra cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$**, denoted

$H^p(\mathfrak{g}, \mathfrak{h})$, is defined by

$$H^p(\mathfrak{g}, \mathfrak{h}) = Z^p(\mathfrak{g}, \mathfrak{h})/B^p(\mathfrak{g}, \mathfrak{h}).$$

This super vector space is nonzero if and only if there is a nontrivial p-cocycle. In what follows, we shall be especially concerned with the even part of this super vector space, which is nonzero if and only if there is a nontrivial even p-cocycle. Our motivation for looking for even cocycles is simple: these parity-preserving maps can regarded as morphisms in the category of super vector spaces, which is crucial for the construction in Theorem 3 and everything following it.

Suppose $\mathfrak{g}$ is a Lie superalgebra with a representation on a super vector space $\mathfrak{h}$. Then we shall prove that an even $\mathfrak{h}$-valued $(n + 1)$-cocycle ω on $\mathfrak{g}$ lets us construct an Lie n-superalgebra, called $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$, of the following form:

$$\mathfrak{g} \xleftarrow{\;d\;} 0 \xleftarrow{\;d\;} \cdots \xleftarrow{\;d\;} 0 \xleftarrow{\;d\;} \mathfrak{h}.$$

Now let us make all of these ideas precise. In what follows, we shall use **super chain complexes**, which are chain complexes in the category SuperVect of $\mathbb{Z}_2$-graded vector spaces:

$$V_0 \xleftarrow{\;d\;} V_1 \xleftarrow{\;d\;} V_2 \xleftarrow{\;d\;} \cdots$$

Thus each V_p is $\mathbb{Z}_2$-graded and d preserves this grading.

There are thus two gradings in play: the $\mathbb{Z}$-grading by **degree**, and the $\mathbb{Z}_2$-grading on each vector space, which we call the **parity**. We shall require a sign convention to establish how these gradings interact. If we consider an object of odd parity and odd degree, is it in fact even overall? By convention, we assume that it is. That is, whenever we interchange something of parity p and degree q with something of parity p' and degree q', we introduce the sign $(-1)^{(p+q)(p'+q')}$. We shall call the sum $p + q$ of parity and degree the **overall grade**, or when it will not cause confusion, simply the grade. We denote the overall grade of X by $|X|$.

We require a compressed notation for signs. If $x_1, \ldots, x_n$ are graded, $\sigma \in S_n$ a permutation, we define the **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ by

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \cdots x_{\sigma(n)},$$

the sign we would introduce in the free graded-commutative algebra generated by $x_1, \ldots, x_n$. Thus, $\epsilon(\sigma)$ encodes all the sign changes that arise from

permuting graded elements. Now define:

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) := \operatorname{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

Thus, $\chi(\sigma)$ is the sign we would introduce in the free graded-anticommutative algebra generated by $x_1, \ldots, x_n$.

Yet we shall only be concerned with particular permutations. If n is a natural number and $1 \leq j \leq n - 1$ we say that $\sigma \in S_n$ is an $(\boldsymbol{j}, \boldsymbol{n} - \boldsymbol{j})$-**unshuffle** if

$$\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j) \quad \text{and} \quad \sigma(j+1) \leq \sigma(j+2) \leq \cdots \leq \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses. A *shuffle* of two ordered sets (such as a deck of cards) is a permutation of the ordered union preserving the order of each of the given subsets. An *unshuffle* reverses this process. We denote the collection of all $(j, n - j)$ unshuffles by $S_{(j,n-j)}$.

The following definition of an L_∞-algebra was formulated by Schlessinger and Stasheff in 1985 [36]:

Definition 2. An $\boldsymbol{L_\infty}$**-superalgebra** is a graded vector space V equipped with a system $\{l_k | 1 \leq k < \infty\}$ of linear maps $l_k \colon V^{\otimes k} \to V$ with $\deg(l_k) = k - 2$ which are totally antisymmetric in the sense that

(1)
$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n)$$

for all $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$, and, moreover, the following generalized form of the Jacobi identity holds for $0 \leq n < \infty$:

(2)
$$\sum_{i+j=n+1} \sum_{\sigma \in S_{(i,n-i)}} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0,$$

where the inner summation is taken over all $(i, n - i)$-unshuffles with $i \geq 1$.

A **Lie $\boldsymbol{n}$-superalgebra** is an L_∞-superalgebra where only the first n terms of the chain complex are nonzero. A **slim Lie $\boldsymbol{n}$-superalgebra** is a Lie n-superalgebra V with only two nonzero terms, V_0 and V_{n-1}, and $d = 0$. Given an $\mathfrak{h}$-valued $(n + 1)$-cocycle ω on a Lie superalgebra $\mathfrak{g}$, we can construct a slim Lie n-superalgebra $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$ with:

- $\mathfrak{g}$ in degree 0, $\mathfrak{h}$ in degree $n - 1$, and trivial super vector spaces in between,

- $d = 0$,
- $l_2 \colon (\mathfrak{g} \oplus \mathfrak{h})^{\otimes 2} \to \mathfrak{g} \oplus \mathfrak{h}$ given by:
  - the Lie bracket on $\mathfrak{g} \otimes \mathfrak{g}$,
  - the action on $\mathfrak{g} \otimes \mathfrak{h}$,
  - zero on $\mathfrak{h} \otimes \mathfrak{h}$, as required by degree.
- $l_{n+1} \colon (\mathfrak{g} \oplus \mathfrak{h})^{\otimes(n+1)} \to \mathfrak{g} \oplus \mathfrak{h}$ given by the cocycle ω on $\mathfrak{g}^{\otimes(n+1)}$, and zero otherwise, as required by degree,
- all other maps l_k zero, as required by degree.

It remains to prove that this is, in fact, a Lie n-superalgebra. Indeed, more is true: every slim Lie n-superalgebra is precisely of this form.

Theorem 3. $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$ *is a Lie n-superalgebra. Conversely, every slim Lie n-superalgebra is of the form* $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$ *for some Lie superalgebra* $\mathfrak{g}$, *representation* $\mathfrak{h}$, *and* $\mathfrak{h}$-*valued* $(n+1)$-*cocycle* ω *on* $\mathfrak{g}$.

Proof. See [7, Thm. 17], which is a straightforward generalization of the theorem found in Baez and Crans [3] to the super case. $\qquad\square$

The key to above theorem is recognizing that ω is a Lie superalgebra cocycle if and only if the generalized Jacobi identity, Equation 2, holds. When $\mathfrak{h}$ is the trivial representation $\mathbb{R}$, we omit it, and write $\mathfrak{brane}_\omega(\mathfrak{g})$ for $\mathfrak{brane}_\omega(\mathfrak{g}, \mathbb{R})$.

2.1. The super-2-brane Lie 3-superalgebra

The supersymmetric 2-brane exists only in spacetimes of dimension 4, 5, 7 and 11. Secretly, this is because the infinitesimal symmetries of spacetimes in these dimensions can be extended to a Lie 3-algebra, as we now describe. For full details, see our previous paper with Baez [7].

One of the principal themes of theoretical physics over the last century has been the search for the underlying symmetries of nature. This began with special relativity, which could be summarized as the discovery that the laws of physics are invariant under the action of the Poincaré group:

$$\mathrm{ISO}(V) = \mathrm{Spin}(V) \ltimes V.$$

Here, V is the set of vectors in Minkowski spacetime and acts on Minkowski spacetime by translation, while $\mathrm{Spin}(V)$ is the **Lorentz group**: the double cover of $\mathrm{SO}_0(V)$, the connected component of the group of symmetries of

the Minkowski norm. Much of the progress in physics since special relativity has been associated with the discovery of additional symmetries, like the $U(1) \times SU(2) \times SU(3)$ symmetries of the Standard Model of particle physics [5].

Today, 'supersymmetry' could be summarized as the hypothesis that the laws of physics are invariant under the 'Poincaré supergroup', which is larger than the Poincaré group:

$$\mathrm{SISO}(V) = \mathrm{Spin}(V) \ltimes T.$$

Here, V is again the set of vectors in Minkowski spacetime and $\mathrm{Spin}(V)$ is the Lorentz group, but T is the supergroup of translations on Minkowski 'superspacetime'. Though we have not yet discussed enough supergeometry to talk about T precisely, at the moment we only need its infinitesimal approximation: the **supertranslation algebra**, $\mathcal{T}$. This is the super vector space with

$$\mathcal{T}_0 = V, \quad \mathcal{T}_1 = S,$$

where V, as before, is the set of vectors in Minkowski spacetime, and S is a spinor representation of $\mathrm{Spin}(V)$. We think of the spinor representation S as giving extra, supersymmetric translations, or 'supersymmetries'.

For a suitable choice of spinor representation, there is a symmetric map, equivariant with respect to the action of $\mathrm{Spin}(V)$:

$$[-, -] \colon \mathrm{Sym}^2 S \to V.$$

This makes $\mathcal{T}$ into a Lie superalgebra where the bracket of two elements of S is given by the map above, and all other brackets are trivial. It is clear that this satisfies the Jacobi identity, in a trivial way: bracketing with a bracket yields zero.

Everything we have said so far makes sense in spacetimes of any dimension. Yet the dimensions where we can write down the Green–Schwarz super 2-brane action [17], $n + 3 = 4, 5, 7$ and 11 are special, thanks to the normed division algebras [7, 21]. Recall that a **normed division algebra** is a nonassociative real algebra $\mathbb{K}$ equipped with a norm $|-|$ such that $|xy| = |x||y|$ for any $x, y \in \mathbb{K}$. There are precisely four such algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, with dimensions $n = 1, 2, 4$ and 8.

In the super 2-brane dimensions $n + 3 = 4, 5, 7$ and 11, we can use the normed division algebras to construct the supertranslation algebra $\mathcal{T}$ and a nontrivial 4-cocycle. In our previous paper [7], we showed how the vectors

V are a certain subspace of the 4×4 matrices over $\mathbb{K}$, the spinors $S = \mathbb{K}^4$. This yields an obvious map

$$V \otimes S \to S$$

given by matrix multiplication. We can show this map is $\mathrm{Spin}(V)$-equivariant. Moreover, we used this description to construct a $\mathrm{Spin}(V)$-invariant pairing

$$\langle -, - \rangle \colon S \otimes S \to \mathbb{R}$$

and a $\mathrm{Spin}(V)$-equivariant bracket

$$[-, -] \colon \mathrm{Sym}^2 S \to V.$$

The latter gives a supertranslation algebra $\mathcal{T} = V \oplus S$, while the former allows us to construct a 4-cocycle on $\mathcal{T}$. We can decompose the space of n-cochains on $\mathcal{T}$ into summands by counting how many of the arguments are vectors and how many are spinors:

$$C^n(\mathcal{T}) \cong \bigoplus_{p+q=n} (\Lambda^p(V) \otimes \mathrm{Sym}^q(S))^*.$$

We call an element of $(\Lambda^p(V) \otimes \mathrm{Sym}^q(S))^*$ a **(p, q)-form**.

Theorem 4. *In dimensions 4, 5, 7 and 11, the supertranslation algebra $\mathcal{T}$ has a nontrivial, Lorentz-invariant even 4-cocycle, namely the unique $(2,2)$-form with*

$$\beta(\Psi, \Phi, \mathcal{A}, \mathcal{B}) = \langle \Psi, \mathcal{A}(\mathcal{B}\Phi) - \mathcal{B}(\mathcal{A}\Phi) \rangle$$

for spinors $\Psi, \Phi \in S$ and vectors $\mathcal{A}, \mathcal{B} \in V$. Here the vectors $\mathcal{A}$ and $\mathcal{B}$ can act on the spinor Φ because they are 4×4 matrices with entries in $\mathbb{K}$, while Φ is an element of $\mathbb{K}^4$.

Proof. See [7, Thm. 15].   □

In spacetime dimensions 4, 5, 7 and 11, Theorem 4 states that there is a 4-cocycle β, which is nonzero only when its arguments consist of two spinors and two vectors:

$$\begin{array}{ccc} \beta \colon & \Lambda^4(\mathcal{T}) & \to & \mathbb{R} \\ & \mathcal{A} \wedge \mathcal{B} \wedge \Psi \wedge \Phi & \mapsto & \langle \Psi, \mathcal{A}(\mathcal{B}\Phi) - \mathcal{B}(\mathcal{A}\Phi) \rangle. \end{array}$$

There is thus a Lie 3-superalgebra, the **supertranslation Lie 3-superalgebra**, $\mathfrak{brane}_\beta(\mathcal{T})$. There is much more that one can do with the

cocycle β, however. We can use it to extend not just the supertranslations $\mathcal{T}$ to a Lie 3-superalgebra, but the full Poincaré superalgebra, $\mathfrak{so}(V) \ltimes \mathcal{T}$. This is because β is invariant under the action of $\mathrm{Spin}(V)$, by construction: it is made of $\mathrm{Spin}(V)$-equivariant maps.

Proposition 5. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie superalgebras such that $\mathfrak{g}$ acts on $\mathfrak{h}$, and let R be a representation of $\mathfrak{g} \ltimes \mathfrak{h}$. Given any R-valued n-cochain ω on $\mathfrak{h}$, we can uniquely extend it to an n-cochain $\tilde{\omega}$ on $\mathfrak{g} \ltimes \mathfrak{h}$ that takes the value of ω on $\mathfrak{h}$ and vanishes on $\mathfrak{g}$. When ω is even, we have:*

1) *$\tilde{\omega}$ is closed if and only if ω is closed and $\mathfrak{g}$-equivariant.*

2) *$\tilde{\omega}$ is exact if and only if $\omega = d\theta$, for θ a $\mathfrak{g}$-equivariant $(n-1)$-cochain on $\mathfrak{h}$.*

Proof. See [7, Prop. 20]. □

Theorem 6. *In dimensions 4, 5, 7 and 11, there exists a Lie 3-superalgebra formed by extending the Poincaré superalgebra $\mathfrak{siso}(n+2,1)$ by the 4-cocycle β, which we call the **2-brane Lie 3-superalgebra, 2-brane$(n+2,1)$**.*

3. Lie 3-groups from group cohomology

Roughly speaking, an 'n-group' is a weak n-groupoid with one object — an n-category with one object in which all morphisms are weakly invertible, up to higher-dimensional morphisms. This definition is a rough one because there are many possible definitions to use for 'weak n-category', but despite this ambiguity, it can still serve to motivate us.

The richness of weak n-categories, no matter what definition we apply, makes n-groups a complicated subject. In the midst of this complexity, we seek to define a class of n-groups that have a simple description, and which are straightforward to internalize, so that we may easily construct Lie n-groups and Lie n-supergroups, as we shall do later in this paper. The motivating example for this is what Baez and Lauda [8] call a 'special 2-group', which has a concrete description using group cohomology. Since Baez and Lauda prove that all 2-groups are equivalent to special ones, group cohomology also serves to classify 2-groups. We seek a similar class of Lie 3-groups.

So, we will define 'slim Lie 3-groups'. This is an Lie 3-group which is trivial in the middle: all 2-morphisms are the identity. In more generality, 'slim Lie n-groups' should be a Lie n-group where all k-morphisms are the identity for $1 < k < n$, though we will make this precise only for $n = 3$. The

concept is useful because such Lie n-groups can be completely classified by
Lie group cohomology. They are also easy to 'superize', and their super
versions can be completely classified using Lie supergroup cohomology, as
we shall see in Section 5. Finally, we note that we could equally well define
'slim n-groups', working in the category of sets rather than the category of
smooth manifolds. Indeed, when $n = 2$, this is what Baez and Lauda call a
'special 2-group', though we prefer the adjective 'slim'.

   We should stress that the definition of Lie 3-group we give here, while
it is good enough for our needs, is known to be too naive in some important
respects. For instance, it does not seem possible to integrate every Lie 3-
algebra to a Lie 3-group of this type, while Henriques's definition of Lie
n-group does make this possible [26].

   First we need to review the cohomology of Lie groups, as originally
defined by van Est [43], who was working in parallel with the definition of
group cohomology given by Eilenberg and Mac Lane. Fix a Lie group G,
an abelian Lie group H, and a smooth action of G on H which respects
addition in H. That is, for any $g \in G$ and $h, h' \in H$, we have:

$$g(h + h') = gh + gh'.$$

Then **the cohomology of G with coefficients in H** is given by the **Lie
group cochain complex**, $C^\bullet(G, H)$. At level p, this consists of the smooth
functions from G^p to H:

$$C^p(G, H) = \{f \colon G^p \to H\}.$$

We call elements of this set **H-valued p-cochains on G**. The boundary
operator is the same as the one defined by Eilenberg–Mac Lane. On a p-
cochain f, it is given by the formula:

$$
\begin{aligned}
df(g_1, \ldots, g_{p+1}) = {} & g_1 f(g_2, \ldots, g_{p+1}) \\
& + \sum_{i=1}^{p} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{p+1}) \\
& + (-1)^{p+1} f(g_1, \ldots, g_p).
\end{aligned}
$$

The proof that $d^2 = 0$ is routine. All the usual terminology applies: a p-
cochain f for which $df = 0$ is called **closed**, or a **cocycle**, a p-cochain $f = dg$
for some $(p-1)$-cochain g is called **exact**, or a **coboundary**. A p-cochain
is said to be **normalized** if it vanishes when any of its entries is 1. Every
cohomology class can be represented by a normalized cocycle. Finally, when

$H = \mathbb{R}$ with trivial G action, we omit it when writing the complex $C^\bullet(G)$, and we call real-valued cochains, cocycles, or coboundaries, simply cochains, cocycles or coboundaries, respectively.

This last choice, that $\mathbb{R}$ will be our default coefficient group, may seem innocuous, but there is another one-dimensional abelian Lie group we might have chosen: U(1), the group of phases. This would have been an equally valid choice, and perhaps better for some physical applications, but we have chosen $\mathbb{R}$ because it simplifies our formulas slightly.

We now sketch how to build a slim Lie 3-group from an 4-cocycle. In essence, given a normalized H-valued 4-cocycle π on a Lie group G, we want to construct a Lie 3-group $\mathrm{Brane}_\pi(G, H)$, which is the smooth, weak 3-groupoid with:

- One object. We can depict this with a dot, or '0-cell': •

- For each element $g \in G$, a 1-automorphism of the one object, which we depict as an arrow, or '1-cell':

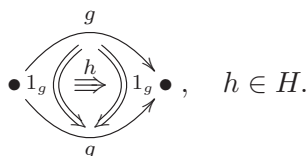$$\bullet \xrightarrow{\ g\ } \bullet\ , \quad g \in G.$$

  Composition corresponds to multiplication in the group:

$$\bullet \xrightarrow{\ g\ } \bullet \xrightarrow{\ g'\ } \bullet \ = \ \bullet \xrightarrow{\ gg'\ } \bullet \ .$$

- Trivial 2-morphisms. If we depict 2-morphisms with 2-cells, we are saying there is just one of these (the identity) for each 1-morphism:
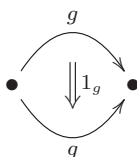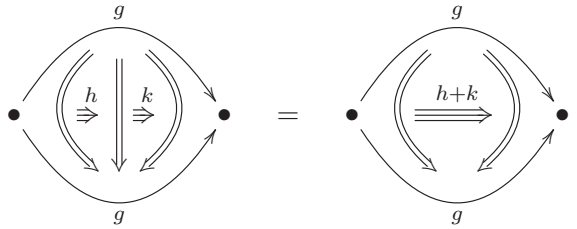

- For each element $h \in H$, an 3-automorphism on the identity of the 1-morphism g, and no 3-morphisms which are not 3-automorphisms:
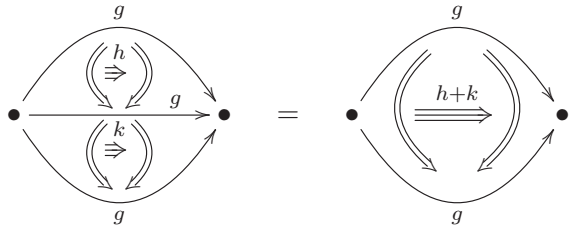

  $h \in H.$

- There are three ways of composing 3-morphisms, given by different ways of sticking 3-cells together — we can glue two 3-cells along a

2-cell, which should just correspond to addition in H:

We also can glue two 3-cells along a 1-cell, which should again just be addition in H:

And finally, we can glue two 3-cells at the 0-cell, the object $\bullet$. This is the only composition of 3-morphisms where the attached 1-morphisms can be distinct, which distinguishes it from the first two cases. It should be addition *twisted by the action of G*:
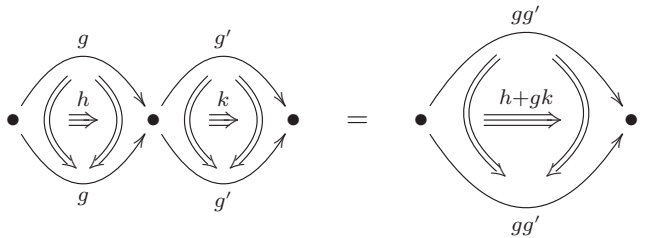
- For any 4-tuple of 1-morphisms, a 3-automorphism $\pi(g_1, g_2, g_3, g_4)$ on the identity of the 1-morphism $g_1 g_2 g_3 g_4$. We call π the **pentagonator**.

- π satisfies an equation corresponding to the 3-dimensional associahe-dron, which is equivalent to the cocycle condition.

The rest of this section is concerned with making this definition precise. We proceed as follows. In Section 3.1, we recall the definition of smooth categories and bicategories from our earlier paper. In Section 3.2, we outline the definition of a smooth tricategory. We do not need the full details, because

our slim Lie 3-groups will only use some of the structure. In Section 3.3, we discuss inverses and weak inverses in a tricategory. Finally, in Section 3.4 we give the definition of Lie 3-groups, and we show how 4-cocycles in Lie group cohomology give rise to slim Lie 3-groups.

3.1. Smooth categories and bicategories

In order to define Lie 3-groups, we now sketch the definition of a 'smooth weak n-category', where every set is a smooth manifold and every structure map is smooth, at least for $n \leq 3$. We focus especially on the case of smooth tricategories, as this part is new. Our previous paper [28] includes the full definition of smooth categories and smooth bicategories, so here we only recall the main ideas.

The key idea at play in this section is internalization. Due to Ehresmann [19] in the 1960s, internalization has become part of the toolkit of the working category theorist. The essence of the idea should be familiar to all mathematicians: any mathematical structure that can be defined using sets, functions, and equations between functions can be defined in categories other than Set. For instance, a group in the category of smooth manifolds is a Lie group. To perform internalization, we apply this idea to the definition of category itself. We recall the essentials here to define 'smooth categories'. More generally, one can define a 'category in K' for many categories K, though here we will work exclusively with the example where K is the category of smooth manifolds. For a readable treatment of internalization, see Borceux's handbook [11]. A general treatment of internal bicategories appears in the work of Douglas and Henriques [16].

A word of caution is needed here before we proceed: *in this section only*, we are bucking standard mathematical practice by writing the result of doing first α and then β as $\alpha \circ \beta$ rather than $\beta \circ \alpha$, as one would do in most contexts where $\circ$ denotes composition of *functions*. This has the effect of changing how we read commutative diagrams. For instance, the commutative triangle:

$$f \xrightarrow{\ \alpha\ } g$$
$$\gamma \searrow \quad \downarrow \beta$$
$$h$$

reads $\gamma = \alpha \circ \beta$ rather than $\gamma = \beta \circ \alpha$.

As the reader knows, a category has a collection of objects

$$x \bullet,$$

and a collection of morphisms

$$x \bullet \xrightarrow{\ f\ } \bullet\, y$$

such that suitable pairs of morphisms can be composed:

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \;=\; x \xrightarrow{\ fg\ } z \;.$$

Composition is associative:

$$(fg)h = f(gh)$$

and has left and right units:

$$1_x f = f = f 1_y$$

for $f\colon x \to y$, and units $1_x\colon x \to x$ and $1_y\colon y \to y$.

A **smooth category** C is just like a category, but now the collection of objects C_0 and morphisms C_1 are smooth manifolds and composition, along with some less obvious operations, are given by smooth maps. Specifically,

- the **source** and **target** maps $s, t\colon C_1 \to C_0$, sending a morphism $f\colon x \to y$ to x and y, respectively, are smooth.

- the **identity-assigning** map $i\colon C_0 \to C_1$, sending an object x to its identity map $1_x\colon x \to x$, is smooth.

- and, as already mentioned, **composition** $\circ\colon C_1 \times_{C_0} C_1 \to C_1$ is smooth, where $C_1 \times_{C_0} C_1$ is the pullback of the source and target maps:

$$C_1 \times_{C_0} C_1 = \{(f,g) \in C_1 \times C_1 : t(f) = s(g)\}\,,$$

  and is assumed to be a smooth manifold.

These maps satisfy equations making C into a category. Between smooth categories C and D, we have smooth functors $F\colon C \to D$, and between smooth functors we have smooth natural transformations

$$C \underset{G}{\overset{F}{\Longrightarrow_{\theta}}} C' \;.$$

Both of these are defined in the obvious way.

With this bit of smooth category theory, we are now ready to move on to smooth bicategory theory. Before we give this definition, let us review the idea of a 'bicategory', so that its basic simplicity is not obscured in technicalities. A bicategory has objects:

$$x \bullet,$$

morphisms going between objects,

$$x \bullet \xrightarrow{\ f\ } \bullet y\ ,$$

and 2-morphisms going between morphisms:

Morphisms in a bicategory can be composed just as morphisms in a category:

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \quad = \quad x \xrightarrow{\ f \cdot g\ } z\ .$$

But there are two ways to compose 2-morphisms — vertically:

and horizontally:

Unlike a category, composition of morphisms need not be associative or have left and right units. The presence of 2-morphisms allows us to *weaken the axioms*. Rather than demanding $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, for composable

morphisms f, g and h, the presence of 2-morphisms allow for the weaker condition that these two expressions are merely isomorphic:

$$a(f, g, h): (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h),$$

where $a(f, g, h)$ is an 2-isomorphism called the **associator**. In the same vein, rather than demanding that:

$$1_x \cdot f = f = f \cdot 1_y,$$

for $f: x \to y$, and identities $1_x: x \to x$ and $1_y: y \to y$, the presence of 2-morphisms allows us to weaken these equations to isomorphisms:

$$l(f): 1_x \cdot f \Rightarrow f, \quad r(f): f \cdot 1_y \Rightarrow f.$$

Here, $l(f)$ and $r(f)$ are 2-isomorphisms called the **left and right unitors**.

Of course, these 2-isomorphisms obey rules of their own. The associator satisfies its own axiom, called the **pentagon identity**, which says that this pentagon commutes:
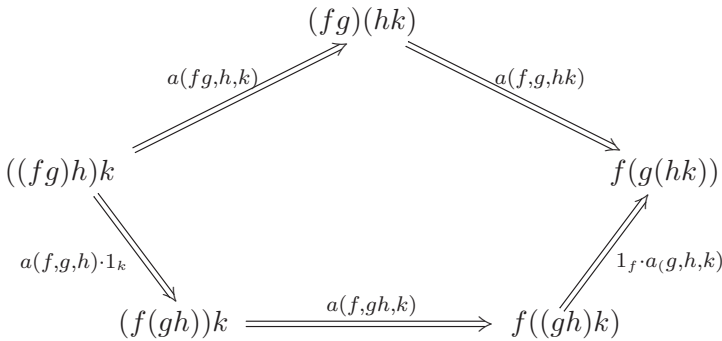
Finally, the associator and left and right unitors satisfy the **triangle identity**, which says the following triangle commutes:
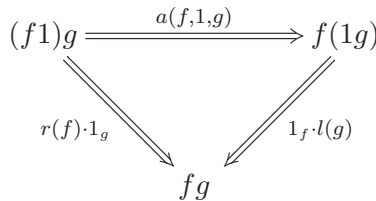
For a **smooth bicategory** B, the collection of objects, B_0, collection of morphisms B_1, and collection of 2-morphisms B_2, are all smooth manifolds, and all structure is given by smooth maps. In particular, B is equipped with:

- a smooth category structure on $\underline{\mathrm{Mor}}B$, with
  - B_1 as the smooth manifold of objects;
  - B_2 as the smooth manifold of morphisms;

  The composition in $\underline{\mathrm{Mor}}B$ is called **vertical composition** and denoted $\circ$.

- smooth **source** and **target maps**:

$$s, t \colon B_1 \to B_0.$$

- a smooth **identity-assigning map**:

$$i \colon B_0 \to B_1.$$

- a smooth **horizontal composition** functor:

$$\cdot \colon \underline{\mathrm{Mor}}B \times_{B_0} \underline{\mathrm{Mor}}B \to \underline{\mathrm{Mor}}B.$$

- a smooth natural isomorphism, the **associator**:

$$a(f, g, h) \colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h).$$

- smooth natural isomorphisms, the **left** and **right unitors**:

$$l(f) \colon 1 \cdot f \Rightarrow f, \quad r(f) \colon f \cdot 1 \Rightarrow f.$$

3.2. Smooth tricategories

We now sketch the definition of tricategories, originally defined by Gordon, Power and Street [23], but extensively studied by Gurski. We use the definition from his thesis [24].
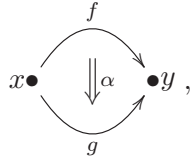
First, we approach the definition informally. A 'tricategory' has objects:
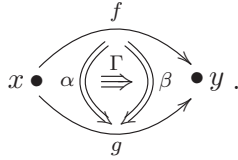
$$x \bullet,$$

morphisms going between objects,

$$x \bullet \xrightarrow{\ f\ } \bullet y \,,$$
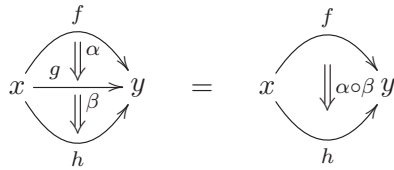
2-morphisms going between morphisms,

$$x\bullet \quad \Downarrow\alpha \quad \bullet y \ ,$$

and 3-morphisms going between 2-morphisms:

$$x \bullet \quad \alpha \left(\Downarrow^{\Gamma}\right) \beta \quad \bullet y \ .$$

Morphisms in a bicategory can be composed just as morphisms in a bicategory:

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \quad = \quad x \xrightarrow{\ f\cdot g\ } z \ .$$

And 2-morphisms can be composed in two different ways, again just as in a bicategory. They can be composed at a morphism:

$$x \xrightarrow{g} \begin{array}{c} f \\ \Downarrow\alpha \\ \Downarrow\beta \\ h \end{array} y \quad = \quad x \begin{array}{c} f \\ \Downarrow\alpha\circ\beta \\ h \end{array} y$$

or at an object:

$$x \begin{array}{c} f \\ \Downarrow\alpha \\ g \end{array} y \begin{array}{c} f' \\ \Downarrow\beta \\ g' \end{array} z \quad = \quad x \begin{array}{c} f\cdot f' \\ \Downarrow\alpha\cdot\beta \\ g\cdot g' \end{array} z \ .$$
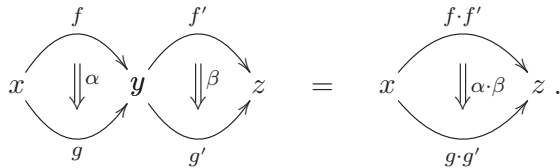
But now 3-morphisms can be composed in three different ways; we can compose two 3-morphisms at a 2-morphism, which we call **composition at**

a 2-cell:
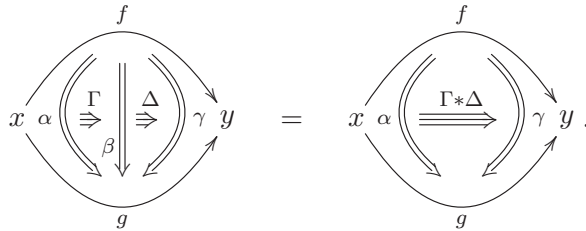
We can also compose two 3-morphisms at a morphism, which we call **composition at a 1-cell**:
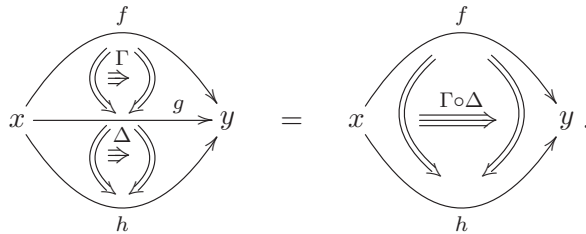
And finally, we can compose two 3-morphisms at an object, which we call **composition at a 0-cell**:
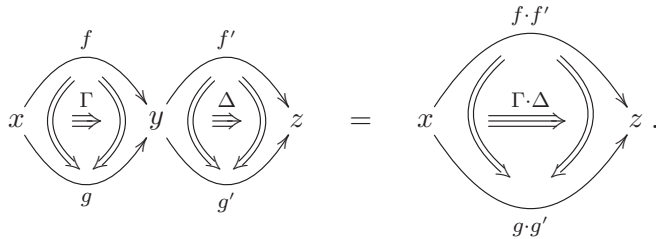
As in a bicategory, a tricategory has an associator and left and right unitors:

$$a(f,g,h)\colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h), \quad l(f)\colon 1_x \cdot f \Rightarrow f, \quad r(f)\colon f \cdot 1_y \Rightarrow f.$$

But the presence of 3-morphisms allows us to *weaken the axioms*; the associator only satisfies the pentagon identity up to a 3-isomorphism we call the

pentagonator:

$$
\begin{array}{c}
(fg)(hk) \\
\end{array}
$$

Diagram: Pentagon with vertices $(fg)(hk)$ (top), $((fg)h)k$ (upper left), $f(g(hk))$ (upper right), $(f(gh))k$ (lower left), $f((gh)k)$ (lower right).

- $((fg)h)k \xrightarrow{a(fg,h,k)} (fg)(hk)$
- $(fg)(hk) \xrightarrow{a(f,g,hk)} f(g(hk))$
- $((fg)h)k \xrightarrow{a(f,g,h)\cdot 1_k} (f(gh))k$
- $(f(gh))k \xrightarrow{a(f,gh,k)} f((gh)k)$
- $f((gh)k) \xrightarrow{1_f\cdot a(g,h,k)} f(g(hk))$
- $\pi(f,g,h,k)$ in the center

and the triangle axiom is replaced by three distinct 3-isomorphisms, called the 'triangulators'. These 3-isomorphisms must in turn satisfy axioms of their own.

Of course, we actually aim to define a 'smooth tricategory', T. This has a smooth manifold of objects, T_0, a smooth manifold of morphisms, T_1, a smooth manifold of 2-morphisms, T_2, and a smooth manifold of 3-morphisms, T_3, such that:

- T_1, T_2 and T_3 fit together to form a smooth bicategory;

- composition at a 0-cell is a smooth 'pseudofunctor';

- satisfying associativity and left and right unit laws up to smooth 'adjoint equivalences', the associator and left and right unitors;

- satisfying the pentagon and triangle identities up to invertible smooth 'modifications';

- satisfying some identities of their own.

Each of the above quoted terms — pseudofunctor, adjoint equivalences, modification — would usually need to be defined completely in order to understand tricategories. For us, the key new concept is that of modification, because our functors will not be pseudo, and our adjoint equivalences will be identities. Nonetheless, let us describe each of these concepts in turn.

- 'Pseudofunctor' is to 'bicategory' as 'functor' is to 'category': it is a map $F\colon B \to B'$ between bicategories B and B', preserving all structure in sight *except* horizontal composition and identities, which are

only preserved up to specified 2-isomorphisms:

$$F(f \cdot g) \Rightarrow F(f) \cdot F(g), \quad F(1_x) \Rightarrow 1_{F(x)}.$$

For the tricategories we construct, all pseudofunctors will be strict: the above 2-isomorphisms are identities.

- 'Adjoint equivalences' involve both 'pseudonatural transformations' and 'modifications':
  - 'Pseudonatural transformation' is to 'pseudofunctor' as 'natural transformation' is to 'functor': given two pseudofunctors

$$
\begin{array}{c}
\overset{F}{\frown} \\
B \qquad B' \\
\underset{G}{\smile}
\end{array}
$$
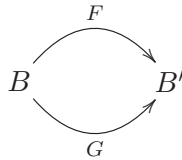
  a pseudonatural transformation is a map:

$$
\begin{array}{c}
\overset{F}{\frown} \\
B \quad \Downarrow \theta \quad B' \; . \\
\underset{G}{\smile}
\end{array}
$$
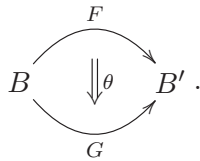
  Like a natural transformation, this consists of a morphism for each object x in B:

$$\theta(x)\colon F(x) \to G(x).$$

  Unlike a natural transformation, it is only natural up to a specified 2-isomorphism. That is, the naturality square:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle\theta(x)}\downarrow & & \downarrow{\scriptstyle\theta(y)} \\
G(x) & \xrightarrow{G(f)} & G(y)
\end{array}
$$

does *not* commute. It is replaced with a 2-isomorphism:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{\;F(f)\;} & F(y) \\
{\scriptstyle\theta(x)}\downarrow & {\scriptstyle\theta(f)} & \downarrow{\scriptstyle\theta(y)} \\
G(x) & \xrightarrow[\;G(f)\;]{} & G(y)
\end{array}
$$

that satisfies some equations of its own. For the tricategories we construct, all pseudonatural transformations will be strict: the 2-isomorphism above is the identity.

– A 'modification' is something new: it is a map between pseudonatural transformations. Given two pseudonatural transformations:

$$
B \;\; {\scriptstyle\theta} \overset{F}{\underset{G}{\Longrightarrow\!\Longleftarrow}} {\scriptstyle\eta} \;\; B'
$$

a modification Γ is a map:

$$
B \;\; {\scriptstyle\theta} \left(\overset{\Gamma}{\Longrightarrow}\right) {\scriptstyle\eta} \;\; B' \;.
$$

Just as a pseudonatural transformation consists of a morphism for each object x in B, a modification consists of a 2-morphism for each object x in B:

$$
F(x) \quad \overset{\theta(x)}{\underset{\eta(x)}{\Bigg\Downarrow{\scriptstyle\Gamma(x)}}} \quad G(x) \;.
$$

These 2-morphisms satisfy an equation that will hold trivially in the tricategories we consider, so we omit it. See Leinster [30] for more details.
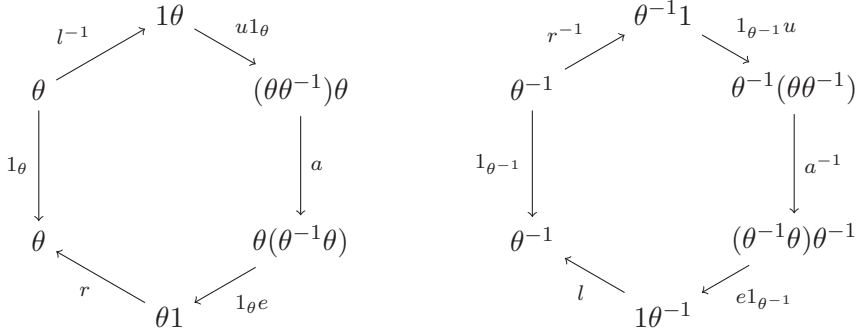
Finally, an **adjoint equivalence** is a quadruple $(\theta, \theta^{-1}, e, u)$ consisting of two pseudonatural transformations, θ and θ^{-1}, going in opposite directions:
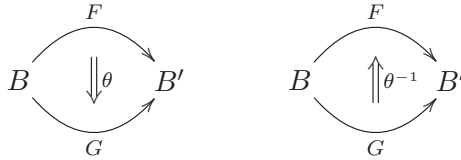
along with invertible modifications, e and u, that exhibit θ and θ^{-1} as weak inverses to each other:

$$e \colon \theta^{-1}\theta \Rightarrow 1_F, \quad u \colon 1_G \Rightarrow \theta\theta^{-1}.$$

We further demand that e and u also satisfy the **zig-zag identities**:

In these diagrams, we have drawn the triple arrows for the modifications as single arrows to avoid clutter. Juxtaposition of modifications comes from horizontal composition of 2-morphisms in the target bicategory, along with the associator and unitors in that bicategory. The word "adjoint" appears because these identities are analogous to those satisfied by the unit and counit of an adjoint pair of functors. While an adjoint equivalence refers to the entire quadruple $(\theta, \theta^{-1}, e, u)$, we will often abuse terminology and mention only θ.

With these preliminaries in mind, we can now sketch the definition of a smooth tricategory.

Definition 7. A **smooth tricategory** T consists of:

- a **manifold of objects**, T_0;
- a **manifold of morphisms**, T_1;

- a **manifold of 2-morphisms**, T_2;

- a **manifold of 3-morphisms**, T_3;

equipped with:

- a smooth bicategory structure on $\underline{\mathrm{Mor}}\,T$, with
  - T_1 as the smooth manifold of objects;
  - T_2 as the smooth manifold of morphisms;
  - T_3 as the smooth manifold of 2-morphisms;

  We call the vertical composition in $\underline{\mathrm{Mor}}\,T$ **composition at a 2-cell**, and the horizontal composition in $\underline{\mathrm{Mor}}\,T$ **composition at a 1-cell**.

- smooth **source** and **target maps**:
  $$s, t \colon T_1 \to T_0.$$

- a smooth **identity-assigning map**:
  $$i \colon T_0 \to T_1.$$

- a smooth **composition** pseudofunctor, called **composition at a 0-cell**, which is strict in the tricategories we consider:
  $$\cdot \colon \underline{\mathrm{Mor}}\,T \times_{T_0} \underline{\mathrm{Mor}}\,T \to \underline{\mathrm{Mor}}\,T.$$

  That is, three smooth maps:
  $$\cdot \colon T_1 \times_{T_0} T_1 \to T_1$$
  $$\cdot \colon T_2 \times_{T_0} T_2 \to T_2$$
  $$\cdot \colon T_3 \times_{T_0} T_3 \to T_3$$

  satisfying the axioms of a strict functor. Here, the pullbacks give us the spaces of 1-, 2- and 3-morphisms which are composable at a 0-cell, and we assume they are smooth manifolds.
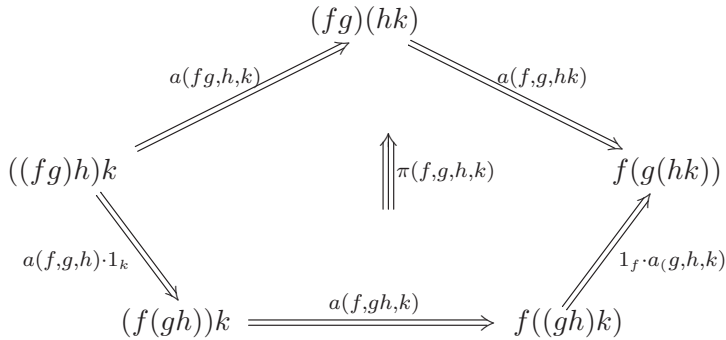
- a smooth adjoint equivalence, the **associator**, trivial in the tricategories we consider:
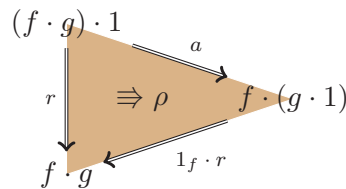 $$a(f, g, h) \colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h).$$

- smooth adjoint equivalences, the **left** and **right unitors**, both trivial in the tricategories we consider:
  $$l(f) \colon 1 \cdot f \Rightarrow f, \quad r(f) \colon f \cdot 1 \Rightarrow f.$$

- a smooth invertible modification called the **pentagonator**:

$$(fg)(hk)$$

$$a(fg,h,k) \qquad a(f,g,hk)$$

$$((fg)h)k \qquad \pi(f,g,h,k) \qquad f(g(hk))$$

$$a(f,g,h) \cdot 1_k \qquad 1_f \cdot a_{(g,h,k)}$$

$$(f(gh))k \xrightarrow{\quad a(f,gh,k) \quad} f((gh)k)$$

- smooth invertible modifications called the **middle, left** and **right triangulators**, all trivial in the tricategories we consider:

$$(1 \cdot f) \cdot g$$

$$l \cdot 1_g \qquad \Rightarrow \lambda \qquad 1 \cdot (f \cdot g)$$

$$f \cdot g \qquad \xleftarrow{\ l\ }$$

$$(f \cdot 1) \cdot g$$

$$r \cdot 1_g \qquad \Rightarrow \mu \qquad f \cdot (1 \cdot g)$$

$$f \cdot g \qquad \xleftarrow{\ 1_f \cdot l\ }$$

$$(f \cdot g) \cdot 1$$

$$r \qquad \Rightarrow \rho \qquad f \cdot (g \cdot 1)$$

$$f \cdot g \qquad \xleftarrow{\ 1_f \cdot r\ }$$
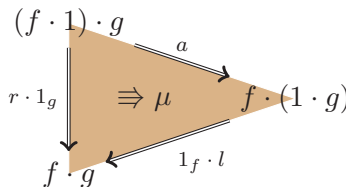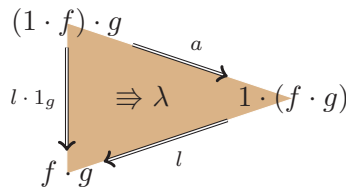
The pentagonator and triangulators satisfy some axioms. When λ, ρ and μ are trivial, as they are for us, the axioms involving the triangulators just say π is trivial whenever one of its arguments is 1, so we omit these. This leaves one axiom, involving only the pentagonator. This key axiom is the **pentagonator identity**:

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

$$(f(g(hk)))p$$

$$a \qquad 1_f(1_g a)$$

$$f(g(h(kp)))$$

$$(1_f a)1_p \qquad a1_p \qquad \Downarrow \pi \qquad a \qquad \cong \qquad a$$

$$(f((gh)k))p \qquad \Rrightarrow \pi \cdot 1_{1_p}$$

$$((fg)(hk))p \xrightarrow{a} (fg((hk)p))$$

$$a1_p \qquad a1_p \qquad (1_f 1_g)a$$

$$((f(gh))k)p \qquad \Downarrow \pi \qquad (fg)(h(kp))$$

$$(a1_k)1_p \qquad$$

$$(((fg)h)k)p \xrightarrow{a} ((fg)h)(kp) \qquad a$$

$$=$$

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

$$a \qquad 1_f(1_g a)$$

$$(f(g(hk)))p \qquad 1_f(a1_p) \qquad f(g(h(kp)))$$

$$\cong \qquad \Downarrow 1_{1_f} \cdot \pi \qquad 1_f a$$

$$(1_f a)1_p \qquad f(((gh)k)p) \xrightarrow{1_f a} f((gh)(kp))$$

$$a$$

$$(f((gh)k))p$$

$$a1_p \qquad \Downarrow \pi \qquad a \qquad \Downarrow \pi \qquad a$$

$$((f(gh))k)p$$

$$(a1_k)1_p \qquad a$$

$$(((fg)h)k)p \qquad \cong \qquad (f(gh))(kp) \qquad (fg)(h(kp))$$

$$a \qquad a(1_k 1_p) \qquad a$$

$$((fg)h)(kp)$$
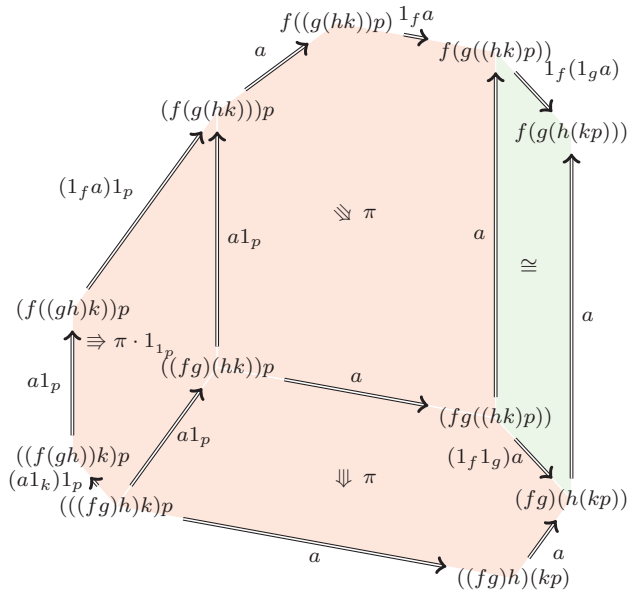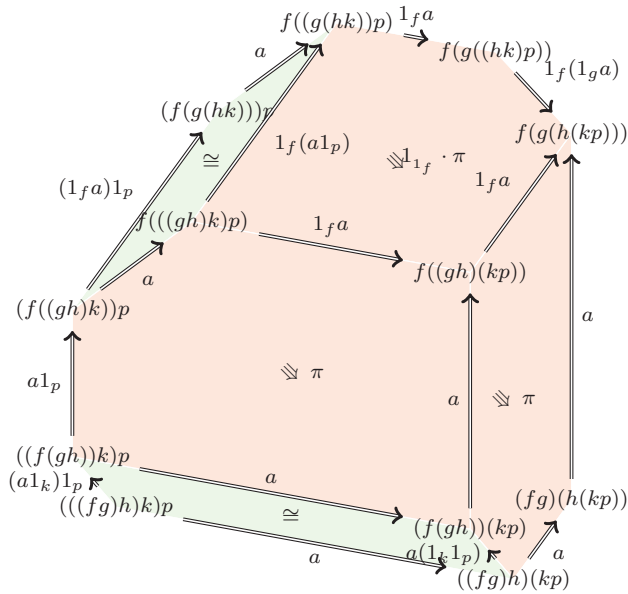
In the pentagonator identity, we have omitted · everywhere except the faces to save space, and the unlabeled squares are naturality squares for the associator, which will be trivial in the examples we consider. This identity comes from a 3-dimensional solid called the **associahedron**. This is the polyhedron where:

- vertices are parenthesized lists of five morphisms, e.g. $(((fg)h)k)p$;

- edges connect any two vertices related by an application of the associator, e.g.

$$(((fg)h)k)p \Rightarrow ((fg)(hk))p.$$

In fact, the pentagonator identity gives us a picture of the associahedron. Regarding the left-hand side of the equation as the back and the right-hand side as the front, we assemble the following polyhedron:
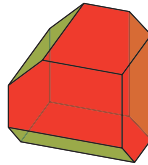
Identifying the vertices, edges and faces of this polyhedron with the corresponding morphisms, 2-morphisms and 3-morphisms from the pentagonator identity, we see the identity just says that the associahedron commutes.

3.3. Inverses

To talk about Lie 3-groups, we need to talk about inverses of morphisms, 2-morphisms and 3-morphisms. This is complicated by the fact that inverses in a 3-group should be weak inverses, satisfying the left and right inverse laws

$$f^{-1}f = 1, \quad ff^{-1} = 1$$

only up to higher morphisms. Fortunately, for the examples we care about, the inverses are strict, and these laws hold, but nonetheless we will describe the general situation.

For 3-morphisms, there are no higher morphisms, so the inverse of a 3-morphism Γ is the usual, strict notion: a 3-morphism Γ^{-1} satisfying the left and right inverse laws:

$$\Gamma^{-1}\Gamma = 1, \quad \Gamma\Gamma^{-1} = 1.$$

Here, juxtaposition denotes composition at a 2-cell, and 1 is the identity of the appropriate 2-cell. Provided such an inverse exists, it is automatically unique, so we are justified in saying *the* inverse.

For 2-morphisms, we can weaken the notion of inverse. A 2-morphism α has a **weak inverse** α^{-1} if there are invertible 3-morphisms e and u which weaken the left and right inverse laws:

$$e\colon \alpha^{-1}\alpha \Rightarrow 1, \quad u\colon 1 \Rightarrow \alpha\alpha^{-1}.$$

Now, juxtaposition denotes composition at a 1-cell, and the 1 denotes the identity of the appropriate 1-cell. Unlike strict inverses, weak inverses are no longer unique, but any two of them must be isomorphic. A 2-morphism α with a weak inverse is called an **equivalence**. Yet, to maximize the utility of a weak inverse, it is best if we impose some axioms on u and e. We have already seen the desired axioms in the last section, when we introduced the concept of an adjoint equivalence. They are the **zig-zag identities**:
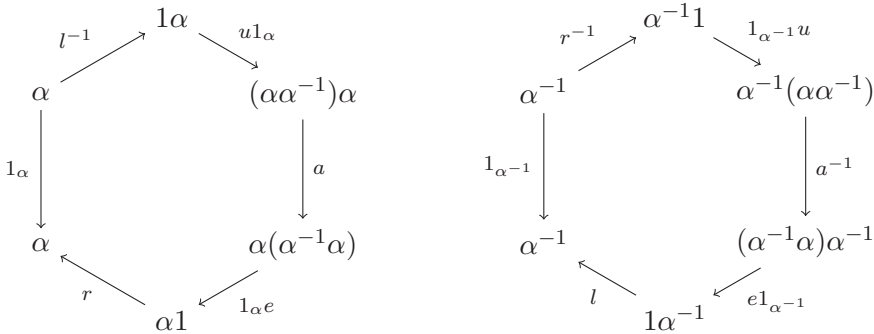

When e and u satisfy these identities, the quadruple $(\alpha, \alpha^{-1}, e, u)$ is called an **adjoint equivalence**, though we generally abuse terminology and say α itself is an adjoint equivalence. By adjusting e and u, every equivalence can be made into an adjoint equivalence. Unlike the choice of α^{-1}, which is only unique up to isomorphism, the choice of (α^{-1}, e, u) making α an adjoint equivalence is unique up to canonical isomorphism.

Moving down to morphisms, we can weaken the notion of invertibility even further. Whereas a 2-morphism had a weak inverse up to isomorphism, a morphism has a weak inverse up to equivalence. That is, a morphism f has a weak inverse f^{-1} if there are equivalences ϵ and η weakening the left and right inverse laws:

$$\epsilon\colon f^{-1}f \Rightarrow 1, \quad \eta\colon 1 \Rightarrow ff^{-1}$$

Of course, the weak inverse of a morphism is only unique up to equivalence. A morphism that has a weak inverse is called a **biequivalence**. Just as every equivalence can be improved by making it part of an adjoint equivalence, every biequivalence can be improved by making it a part of a 'biadjoint biequivalence' [25]. A **biadjoint biequivalence** is a biequivalence where ϵ and η are both adjoint equivalences, satisfying the zig-zag identities up to invertible 3-morphisms:
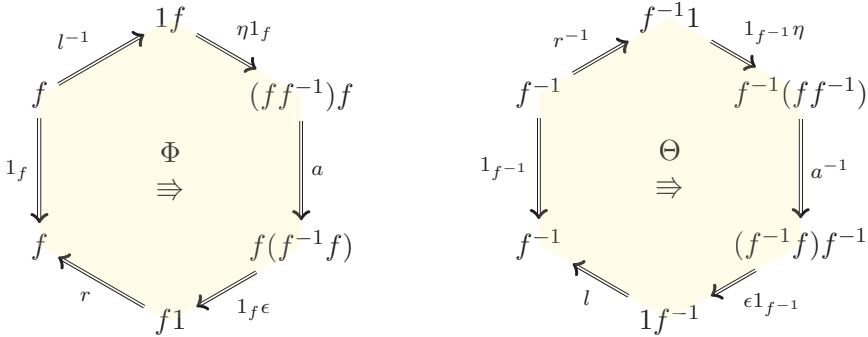
We require these 3-morphisms to satisfy a new equation, called the 'swallowtail identity'. The name comes from the 'swallowtail castastrophe' in castastrophe theory, and is motivated by a conjectured relationship between higher categories and tangles known as the tangle hypothesis [4]. To fully appreciate this conjecture, it is necessary to see the axioms in this section drawn as string diagrams, which can be found in the paper of Stay [39]. Here is the **swallowtail identity**:
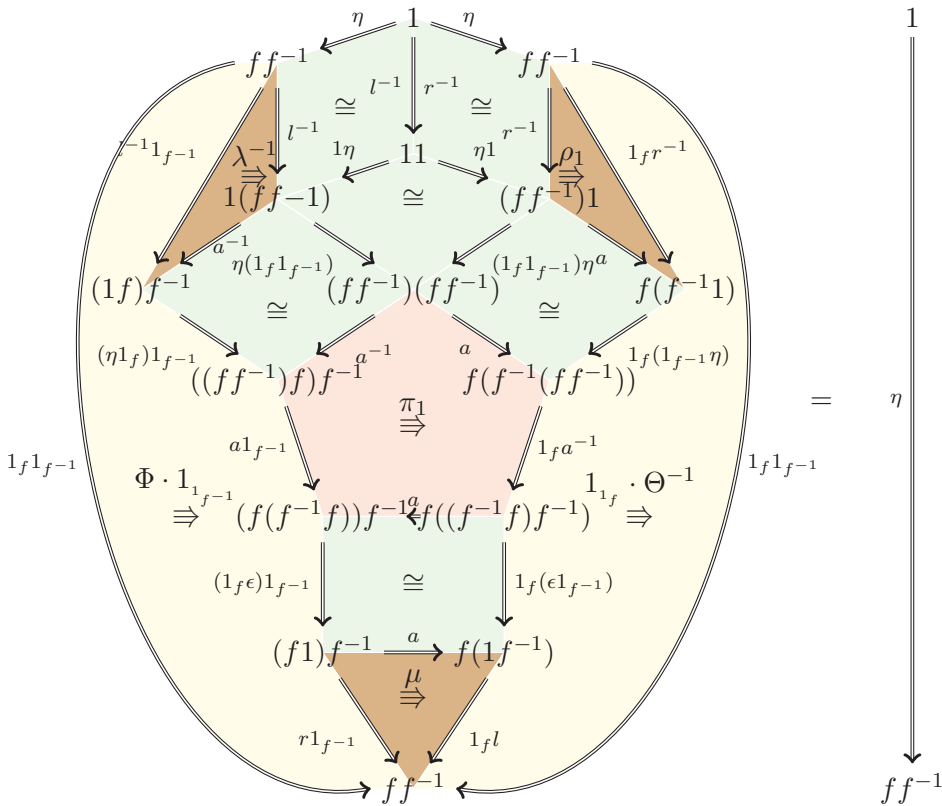
In this identity, we have omitted · everywhere except the faces to save space, and the unlabeled squares are naturality squares, which will be trivial in the examples we consider. The 3-morphisms π_1 and ρ_1 are obtained from π and ρ by using the fact that the associator and unitors are adjoint equivalences to reverse some arrows. For us, these adjoint equivalences will be trivial.

Ultimately, we are interested in having inverses in a smooth tricategory. For this, we need smooth choices of all of these data. In particular, a smooth tricategory T is said to have **invertible 3-morphisms** if there is a smooth map

$$\mathrm{inv}_3 \colon T_3 \to T_3$$

assigning to any 3-morphism Γ its inverse $\Gamma^{-1} = \mathrm{inv}_3(\Gamma)$. We say T has **weakly invertible 2-morphisms** if there are smooth maps:

$$\mathrm{inv}_2 \colon T_2 \to T_2, \quad e \colon T_2 \to T_3, \quad u \colon T_2 \to T_3$$

assigning to any 2-morphism α a choice of weak inverse $\alpha^{-1} = \mathrm{inv}_2(\alpha)$ and invertible 3-morphisms $e = e(\alpha)$ and $u = u(\alpha)$ such that $(\alpha, \alpha^{-1}, e, u)$ is an adjoint equivalence. Finally, we say T has **weakly invertible morphisms** if there are smooth maps:

$$\mathrm{inv}_1 \colon T_1 \to T_1, \quad \epsilon \colon T_1 \to T_2^2 \times T_3^2, \quad \eta \colon T_1 \to T_2^2 \times T_3^2,$$
$$\Phi \colon T_1 \to T_3, \quad \Theta \colon T_1 \to T_3$$

assigning to any morphism f a choice of weak inverse $f^{-1} = \mathrm{inv}_1(f)$ such that $(f, f^{-1}, \epsilon, \eta, \Phi, \Theta)$ is a biadjoint biequivalence. Note that targets of ϵ and η are what we need to specify adjoint equivalences. Here, we have suppressed the dependence of the last four entries on f for brevity.

3.4. Lie 3-groups

A **Lie 3-group** is a smooth tricategory with one object where morphisms, 2-morphisms and 3-morphisms are weakly invertible. Though it looks quite complex, we can construct an example using only a 4-cocycle in group cohomology, because the pentagonator identity is secretly a cocycle condition. Given a normalized H-valued 4-cocycle π on a Lie group G, we can construct a Lie 3-group $\mathrm{Brane}_\pi(G, H)$ with:

- One object, $\bullet$, regarded as a manifold in the trivial way.

- For each element $g \in G$, an automorphism of the one object:

$$\bullet \xrightarrow{g} \bullet$$

  Composition at a 0-cell given by multiplication in the group:

$$\cdot \colon G \times G \to G.$$

  The source and target maps are trivial, and identity-assigning map takes the one object to $1 \in G$.

- Only the identity 2-morphism on any 1-morphism, and no 2-morphisms between distinct 1-morphisms:
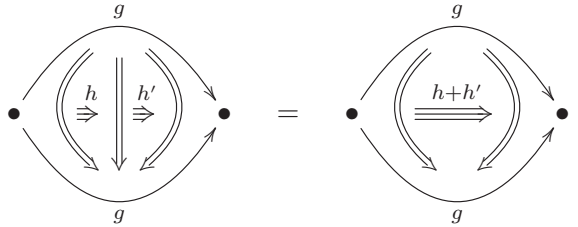

$$g \in G.$$

So the space of 2-morphisms is also G. The source, target and identity-assigning maps are all the identity on G. Composition at a 1-cell is trivial, while composition at a 0-cell is again multiplication in G.

- For each $h \in H$, a 3-automorphism of the 2-morphism 1_g, and no 3-morphisms between distinct 2-morphisms:

$$\bullet\,1_g \left(\!\!\left(\underset{h}{\Rightarrow}\right)\!\!\right) 1_g\,\bullet\,, \quad h \in H.$$
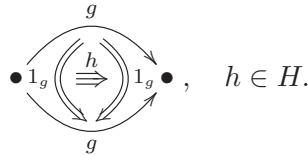
Thus the space of 3-morphisms is $G \times H$. The source and target maps are projection onto G, and the identity assigning map takes 1_g to $0 \in H$, for all $g \in G$.

- Three kinds of composition of 3-morphisms: given a pair of 3-morphisms on the same 2-morphism, we can compose them at a 2-cell, which we take to be addition in H:

$$\bullet \left(\!\!\left(\underset{\Rightarrow}{h}\,\bigg\|\,\underset{\Rightarrow}{h'}\right)\!\!\right) \bullet \quad = \quad \bullet \left(\!\!\left(\underset{\Longrightarrow}{h+h'}\right)\!\!\right) \bullet\,.$$

We can also compose two 3-morphisms at a 1-cell, which we again take to be addition in H:

$$\bullet \left(\!\!\left(\underset{\Rightarrow}{h}\right)\!\!\right)\,g\,\left(\!\!\left(\underset{\Rightarrow}{h'}\right)\!\!\right) \bullet \quad = \quad \bullet \left(\!\!\left(\underset{\Longrightarrow}{h+h'}\right)\!\!\right) \bullet\,.$$
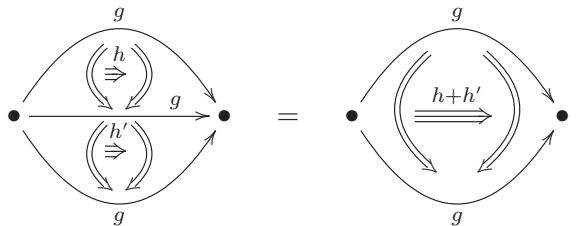
In terms of maps, both of these compositions are just:

$$1 \times + \colon G \times H \times H \to G \times H.$$

And finally, we can glue two 3-cells at the 0-cell, the object. We call this **composition at a 0-cell**, and define it to be addition *twisted by the action of G*:
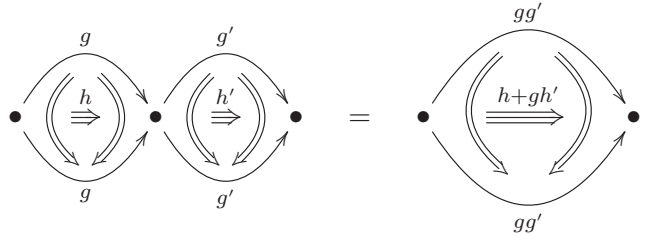
In terms of a map, $\cdot$ is just given by multiplication on the semidirect product:

$$\cdot \colon (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- The associator and left and right unitors are trivial.

- For each quadruple of 1-morphisms, a specified 3-isomorphism, the **2-associator** or **pentagonator**:

$$\pi(g_1, g_2, g_3, g_4) \colon 1_{g_1 g_2 g_3 g_4} \to 1_{g_1 g_2 g_3 g_4}.$$

given by the 4-cocycle $\pi \colon G^4 \to H$, which we think of as an element of H because the source (and target) are understood to be $1_{g_1 g_2 g_3 g_4}$.

- The three other specified 3-isomorphisms, λ, ρ and μ, are trivial.

- The inverse of 3-morphisms is just given by negation in H:
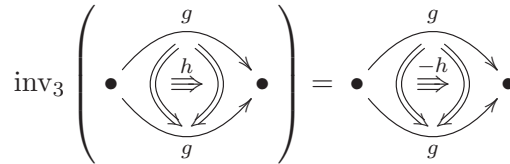
Or, as a map:

$$\mathrm{inv}_3 \colon G \times H \to G \times H$$

sending (g, h) to $(g, -h)$.

The inverses for 2-morphisms are trivial, because 2-morphisms are trivial.

Inverses for 1-morphisms are just inverses in G:

$$\text{inv}_1 \left(\bullet \xrightarrow{\ g\ } \bullet \right) = \bullet \xrightarrow{\ g^{-1}\ } \bullet$$

This is made into a biadjoint biequivalence with ϵ and η trivial, and Φ and Θ chosen to satisfy the swallowtail identity. There are many possible choices; here is a convenient one:

$$\Phi(g) = -\pi(g, g^{-1}, g, g^{-1}), \quad \Theta(g) = 0.$$

Again, we regard these 3-morphisms as elements of H, because their sources and targets are understood:

$$\Phi(g) \colon 1_g \to 1_g, \quad \Theta(g) \colon 1_{g^{-1}} \to 1_{g^{-1}}$$

A **slim Lie 3-group** is one of this form. It remains to check that it is, in fact, a Lie 3-group. We claim:

Proposition 8. $\text{Brane}_\pi(G, H)$ *is a Lie 3-group: a smooth tricategory with one object and all morphisms, 2-morphisms and 3-morphisms weakly invertible.*

Proof. As noted above, the triangulators λ, ρ and μ are trivial. The axioms they satisfy are automatic because π is normalized.

So to check that $\text{Brane}_\pi(G, H)$ is a tricategory, it remains to check that π satisfies the pentagonator identity. Since the 3-cells of the pentagonator identity commute (they represent elements of H), and all the faces not involving a π are trivial, the first half reads:

$$\pi(g_1, g_2, g_3, g_4) \cdot 0_{g_5} + \pi(g_1 g_2, g_3, g_4, g_5) + \pi(g_1, g_2, g_3 g_4, g_5).$$

Here, we write 0_{g_5} to denote the 3-morphism which is the identity on the identity of the 1-morphism g_5. We do not need to be worried about order of terms, since composition of 3-morphisms is addition in H. The second half of the pentagonator identity reads:

$$0_{g_1} \cdot \pi(g_2, g_3, g_4, g_5) + \pi(g_1, g_2 g_3, g_4, g_5) + \pi(g_1, g_2, g_3, g_4 g_5).$$

Here, we write 0_{g_1} for to denote the 3-morphism which is the identity on the identity of the 1-morphism g_1. Applying the definition of $\cdot$ as the semidirect

product, we see the equality of the first half with the second half is just the cocycle condition on π:

$$\pi(g_1, g_2, g_3, g_4) + \pi(g_1 g_2, g_3, g_4, g_5) + \pi(g_1, g_2, g_3 g_4, g_5)$$
$$= g_1 \pi(g_2, g_3, g_4, g_5) + \pi(g_1, g_2 g_3, g_4, g_5) + \pi(g_1, g_2, g_3, g_4 g_5).$$

So, $\mathrm{Brane}_\pi(G, H)$ is a tricategory. It is smooth because everything in sight is smooth: G, H, and the map $\pi \colon G^4 \to H$. And it is a Lie 3-group: the 1-morphisms in G, the trivial 2-morphisms, and the 3-morphisms in H are all strictly invertible, and thus weakly invertible. The swallowtail identity reduces to:

$$\Phi(g) \cdot 0_{g^{-1}} + \pi(g, g^{-1}, g, g^{-1}) - 0_g \cdot \Theta(g) = 0$$

Upon substituting the values of Φ and Θ we chose above, this becomes:

$$-\pi(g, g^{-1}, g, g^{-1}) + \pi(g, g^{-1}, g, g^{-1}) + 0 = 0$$

which is indeed true. $\square$

We can say something a bit stronger about $\mathrm{Brane}_\pi(G, H)$, if we let π be any normalized H-valued 4-cochain, rather than requiring it to be a cocycle. In this case, $\mathrm{Brane}_\pi(G, H)$ is a Lie 3-group if and only if π is a 4-cocycle, because π satisfies the pentagon identity if and only if it is a cocycle.

4. Supergeometry

We would now like to generalize our work from Lie algebras and Lie groups to Lie superalgebras and Lie supergroups. Of course, this means that we need a way to talk about Lie supergroups, their underlying supermanifolds, and the maps between supermanifolds. This task is made easier because we do not need the full machinery of supermanifold theory. Our key examples of supergroups will be exponential, meaning that the exponential map

$$\exp \colon \mathfrak{g} \to G$$

is a diffeomorphism, when G is a supergroup and $\mathfrak{g}$ is its Lie superalgebra. So, we only need to work with supermanifolds that are diffeomorphic to super vector spaces.

Nonetheless, we will give a lightning review of supermanifold theory from the perspective that suits us best, which could loosely be called the 'functor of points' approach. A more leisurely and carefully motivated account is in

our previous paper [28]. That account and this one are based on the work
of Sachse [33] and Balduzzi, Carmeli and Fioresi [10] developing the functor
of points for supermanifolds. Their work goes back to ideas of Schwarz [38]
and Voronov [42].

To begin, a **Grassmann algebra** is a finite-dimensional exterior algebra

$$A = \Lambda \mathbb{R}^n,$$

equipped with the grading:

$$A_0 = \Lambda^0 \mathbb{R}^n \oplus \Lambda^2 \mathbb{R}^n \oplus \cdots, \quad A_1 = \Lambda^1 \mathbb{R}^n \oplus \Lambda^3 \mathbb{R}^n \oplus \cdots.$$

Let us write GrAlg for the category with Grassmann algebras as objects
and grade-preserving homomorphisms as morphisms. We will define a su-
permanifold to be a functor from the category of Grassmann algebras to
smooth manifolds

$$M \colon \mathrm{GrAlg} \to \mathrm{Man}$$

equipped with some extra structure. Let us write M_A for the value of M at
the Grassmann algebra A. We call this manifold the **A-points** of M. For
$f \colon A \to B$ a homomorphism of Grassmann algebras A and B, we write the
induced smooth map as $M_f \colon M_A \to M_B$.

As we already remarked, our most important example of a supermani-
fold is a super vector space. Indeed, given a finite-dimensional super vector
space V, define the **supermanifold associated to V**, or just the **super-
manifold V** to be the functor:

$$V \colon \mathrm{GrAlg} \to \mathrm{Man}$$

which takes:

- each Grassmann algebra A to the vector space:

$$V_A = (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$$

  regarded as a manifold in the usual way;

- each homomorphism $f \colon A \to B$ of Grassmann algebras to the linear
  map $V_f \colon V_A \to V_B$ that is the identity on V and f on A:

$$V_f = (f \otimes 1)_0 \colon (A \otimes V)_0 \to (B \otimes V)_0.$$

  This map, being linear, is also smooth.

We also need to know how to smoothly map one super vector space to another. First note that V_A is more than a mere vector space; it is an A_0-module. Given two finite-dimensional super vector spaces V and W, we define a **smooth map between super vector spaces**:

$$\varphi\colon V \to W,$$

to be a natural transformation between the supermanifolds V and W such that the derivative

$$(\varphi_A)_*\colon T_x V_A \to T_{\varphi_A(x)} W_A$$

is A_0-linear at each A-point $x \in V_A$, where the A_0-module structure on each tangent space comes from the canonical identification of a vector space with its tangent space:

$$T_x V_A \cong V_A, \quad T_{\varphi(x)} W_A \cong W_A.$$

Note that each component $\varphi_A\colon V_A \to W_A$ is smooth in the ordinary sense, by virtue of living in the category of smooth manifolds. We say that a smooth map $\varphi_A\colon V_A \to W_A$ whose derivative is A_0-linear at each point is **A_0-smooth** for short.

These definitions, of the supermanifold associated to a super vector space and of smooth maps between super vector spaces, are the most important for us. Nonetheless, we now sketch how to define a general supermanifold, M. Since M will be locally isomorphic to a super vector space V, it helps to have local pieces of V to play the same role that open subsets of $\mathbb{R}^n$ play for ordinary manifolds. So, fix a super vector space V, and let $U \subseteq V_0$ be open. The **superdomain** over U is the functor:

$$\mathcal{U}\colon \mathrm{GrAlg} \to \mathrm{Man}$$

that takes each Grassmann algebra A to the following open subset of V_A:

$$\mathcal{U}_A = V_{\epsilon_A}^{-1}(U)$$

where $\epsilon_A\colon A \to \mathbb{R}$ is the projection of the Grassmann algebra A that kills all nilpotent elements. We say that $\mathcal{U}$ is a **superdomain in V**, and write $\mathcal{U} \subseteq V$.

If $\mathcal{U} \subseteq V$ and $\mathcal{U}' \subseteq W$ are two superdomains in super vector spaces V and W, a **smooth map of superdomains** is a natural transformation:

$$\varphi \colon \mathcal{U} \to \mathcal{U}'$$

such that for each Grassmann algebra A, the component on A-points is smooth:

$$\varphi_A \colon \mathcal{U}_A \to \mathcal{U}'_A.$$

and the derivative:

$$(\varphi_A)_* \colon T_x \mathcal{U}_A \to T_{\varphi_A(x)} \mathcal{U}'_A$$

is A_0-linear at each A-point $x \in \mathcal{U}_A$, where the A_0-module structure on each tangent space comes from the canonical identification with the ambient vector spaces:

$$T_x \mathcal{U}_A \cong V_A, \quad T_{\varphi(x)} \mathcal{U}'_A \cong W_A.$$

Again, we say that a smooth map $\varphi_A \colon \mathcal{U}_A \to \mathcal{U}'_A$ whose derivative is A_0-linear at each point is $\boldsymbol{A_0}$**-smooth** for short.

At long last, a **supermanifold** is a functor

$$M \colon \mathrm{GrAlg} \to \mathrm{Man}$$

equipped with an atlas

$$(\mathcal{U}_\alpha, \varphi_\alpha \colon \mathcal{U} \to M),$$

where each $\mathcal{U}_\alpha$ is a superdomain, each φ_α is a natural transformation, and one can define transition functions that are smooth maps of superdomains.

Finally, a **smooth map of supermanifolds** is a natural transformation:

$$\psi \colon M \to N$$

which induces smooth maps between the superdomains in the atlases. Equivalently, each component

$$\varphi_A \colon M_A \to N_A$$

is $\boldsymbol{A_0}$**-smooth**: it is smooth and its derivative

$$(\varphi_A)_* \colon T_x M_A \to T_{\varphi_A(x)} N_A$$

is A_0-linear at each A-point $x \in M_A$, where the A_0-module structure on each tangent space comes from the superdomains in the atlases. Thus, there is a category SuperMan of supermanifolds. See Sachse [33] for more details.

Finally, note that there is a supermanifold:

$$1 \colon \mathrm{GrAlg} \to \mathrm{Man},$$

which takes each Grassmann algebra to the one-point manifold. We call this the **one-point supermanifold**, and note that it is the supermanifold associated to the super vector space $\mathbb{R}^{0|0}$. The one-point supermanifold is the terminal object in the category of supermanifolds.

4.1. Supergroups from nilpotent Lie superalgebras

We now describe a procedure to integrate a nilpotent Lie superalgebra to a Lie supergroup. This is a partial generalization of Lie's Third Theorem, which describes how any Lie algebra can be integrated to a Lie group. In fact, the full theorem generalizes to Lie supergroups [41], but we do not need it here.

Recall from Section 2 that a **Lie superalgebra** $\mathfrak{g}$ is a Lie algebra in the category of super vector spaces. More concretely, it is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, equipped with a graded-antisymmetric bracket:

$$[-,-] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g},$$

which satisfies the Jacobi identity up to signs:

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]].$$

for all homogeneous $X, Y, Z \in \mathfrak{g}$. A Lie superalgebra $\mathfrak{n}$ is called **k-step nilpotent** if any k nested brackets vanish, and it is called **nilpotent** if it is k-step nilpotent for some k. Nilpotent Lie superalgebras can be integrated to a unique supergroup N defined on the same underlying super vector space $\mathfrak{n}$.

A **Lie supergroup**, or **supergroup**, is a group object in the category of supermanifolds. That is, it is a supermanifold G equipped with the following maps of supermanifolds:

- **multiplication**, $m \colon G \times G \to G$;

- **inverse**, $\mathrm{inv} \colon G \to G$;

- **identity**, $\mathrm{id} \colon 1 \to G$, where 1 is the one-point supermanifold;

such that the following diagrams commute, encoding the usual group axioms:

- the associative law:

$$
\begin{array}{ccc}
& G \times G \times G & \\
{}^{m\times 1}\swarrow & & \searrow^{1\times m} \\
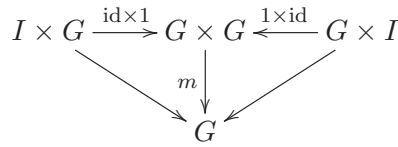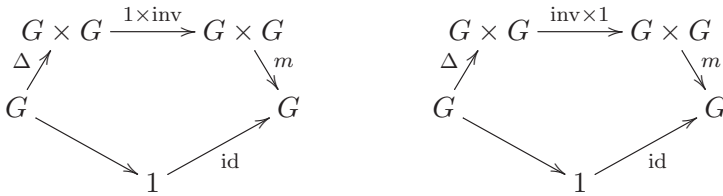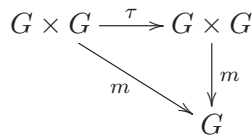G \times G & & G \times G \\
{}_{m}\searrow & & \swarrow_{m} \\
& G &
\end{array}
$$
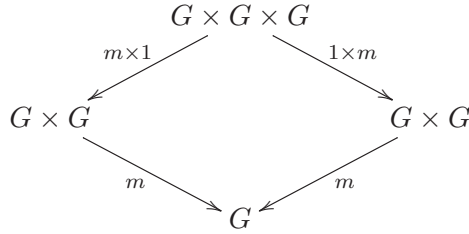
- the right and left unit laws:

$$
I \times G \xrightarrow{\mathrm{id}\times 1} G \times G \xleftarrow{1\times\mathrm{id}} G \times I
$$
$$
\searrow \quad \downarrow^{m} \quad \swarrow
$$
$$
G
$$

- the right and left inverse laws:

$$
\begin{array}{ccc}
G \times G & \xrightarrow{1\times\mathrm{inv}} & G \times G \\
\Delta\uparrow & & \downarrow m \\
G & & G \\
& \searrow \quad \nearrow_{\mathrm{id}} & \\
& 1 &
\end{array}
\qquad
\begin{array}{ccc}
G \times G & \xrightarrow{\mathrm{inv}\times 1} & G \times G \\
\Delta\uparrow & & \downarrow m \\
G & & G \\
& \searrow \quad \nearrow_{\mathrm{id}} & \\
& 1 &
\end{array}
$$

where $\Delta \colon G \to G \times G$ is the diagonal map. In addition, a supergroup is **abelian** if the following diagram commutes:

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\ \tau\ } & G \times G \\
{}_{m}\searrow & & \downarrow^{m} \\
& & G
\end{array}
$$

where $\tau \colon G \times G \to G \times G$ is the **twist map**. Using A-points, it is defined to be:

$$
\tau_A(x,y) = (y,x),
$$

for $(x,y) \in G_A \times G_A$.

Examples of supergroups arise easily from Lie groups. We can regard any ordinary manifold as a supermanifold, and so any Lie group G is also a supergroup. In this way, any classical Lie group, such as $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, becomes a supergroup.

To obtain more interesting examples, we will integrate a nilpotent Lie superalgebra, $\mathfrak{n}$ to a supergroup N. For any Grassmann algebra A, the bracket

$$[-,-]\colon \Lambda^2\mathfrak{n} \to \mathfrak{n}$$

induces an A_0-linear map between the A-points:

$$[-,-]_A\colon \Lambda^2\mathfrak{n}_A \to \mathfrak{n}_A,$$

where $\Lambda^2\mathfrak{n}_A$ denotes the exterior square of the A_0-module $\mathfrak{n}_A$. Thus $[-,-]_A$ is antisymmetric, and it easy to check that it makes $\mathfrak{n}_A$ into a Lie algebra which is also nilpotent.

On each such A_0-module $\mathfrak{n}_A$, we can thus define a Lie group N_A where the multplication is given by the Baker–Campbell–Hausdorff formula, inversion by negation, and the identity is 0. Because we want to write the group N_A multiplicatively, we write $\exp_A\colon n_A \to N_A$ for the identity map, and then define the multiplication, inverse and identity:

$$m_A\colon N_A \times N_A \to N_A, \quad \mathrm{inv}_A\colon N_A \to N_A, \quad \mathrm{id}_A\colon 1_A \to N_A,$$

as follows:

$$m_A(\exp_A(X), \exp_A(Y)) = \exp_A(X)\exp_A(Y)$$
$$= \exp_A\left(X + Y + \frac{1}{2}[X,Y]_A + \cdots\right)$$
$$\mathrm{inv}_A(\exp_A(X)) = \exp_A(X)^{-1} = \exp_A(-X),$$
$$\mathrm{id}_A(1) = 1 = \exp_A(0),$$

for any A-points $X, Y \in \mathfrak{n}_A$, where the first 1 in the last equation refers to the single element of 1_A. But it is clear that all of these maps are natural in A. Furthermore, they are all A_0-smooth, because as polynomials with coefficients in A_0, they are smooth with derivatives that are A_0-linear. They thus define smooth maps of supermanifolds:

$$m\colon N \times N \to N, \quad \mathrm{inv}\colon N \to N, \quad \mathrm{id}\colon 1 \to N,$$

where N is the supermanifold $\mathfrak{n}$. And because each of the N_A is a group, N is a supergroup. We have thus proved:

Proposition 9. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra. Then there is a supergroup N defined on the supermanifold $\mathfrak{n}$, obtained by integrating the nilpotent*

Lie algebra $\mathfrak{n}_A$ with the Baker–Campbell–Hausdorff formula for all Grassmann algebras A. More precisely, we define the maps:

$$m \colon N \times N \to N, \quad \mathrm{inv} \colon N \to N, \quad \mathrm{id} \colon 1 \to N,$$

by defining them on A-points as follows:

$$m_A(\exp_A(X), \exp_A(Y)) = \exp_A(Z(X, Y)),$$
$$\mathrm{inv}_A(\exp_A(X)) = \exp_A(-X),$$
$$\mathrm{id}_A(1) = \exp_A(0),$$

where

$$\exp \colon \mathfrak{n} \to N$$

is the identity map of supermanifolds, and:

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y]_A + \cdots$$

denotes the Baker–Campbell–Hausdorff series on $\mathfrak{n}_A$, which terminates because $\mathfrak{n}_A$ is nilpotent.

Experience with ordinary Lie theory suggests that, in general, there will be more than one supergroup which has Lie superalgebra $\mathfrak{n}$. To distinguish the one above, we call N the **exponential supergroup** of $\mathfrak{n}$.

5. 3-supergroups from supergroup cohomology

We saw in Section 3 that Lie 3-groups can be defined from a 4-cocycle in Lie group cohomology. We now generalize this to supergroups. The most significant barrier is that we now work internally to the category of supermanifolds instead of the much more familiar category of smooth manifolds. Our task is to show that this change of categories does not present a problem. The main obstacle is that the category of supermanifolds is not a concrete category: morphisms are determined not by their value on the underlying set of a supermanifold, but by their value on A-points for all Grassmann algebras A.

The most common approach is to define morphisms without reference to elements, and to define equations between morphisms using commutative diagrams. As an alternative to commutative diagrams, for supermanifolds, one can use A-points to define morphisms and specify equations between them. This tends to make equations look friendlier, because they look like equations between functions. We shall use this approach.

First, let us define the cohomology of a supergroup G with coefficients in an abelian supergroup H, on which G **acts by automorphism**. This means that we have a morphism of supermanifolds:

$$\alpha \colon G \times H \to H,$$

which, for any Grassmann algebra A, induces an action of the group G_A on the abelian group H_A:

$$\alpha_A \colon G_A \times H_A \to H_A.$$

For this action to be by automorphism, we require:

$$\alpha_A(g)(h + h') = \alpha_A(g)(h) + \alpha_A(g)(h'),$$

for all A-points $g \in G_A$ and $h, h' \in H_A$.

We define supergroup cohomology using **the supergroup cochain complex**, $C^\bullet(G, H)$, which at level p just consists of the set of maps from G^p to H as supermanifolds:

$$C^p(G, H) = \{f \colon G^p \to H\}.$$

Addition on H makes $C^p(G, H)$ into an abelian group for all p. The differential is given by the usual formula, but using A-points:

$$
\begin{aligned}
df_A(g_1, \ldots, g_{p+1}) = {} & g_1 f_A(g_2, \ldots, g_{p+1}) \\
& + \sum_{i=1}^{p} (-1)^i f_A(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\
& + (-1)^{p+1} f_A(g_1, \ldots, g_p),
\end{aligned}
$$

where $g_1, \ldots, g_{p+1} \in G_A$ and the action of g_1 is given by α_A. Noting that f_A, α_A, multiplication and $+$ are all:

- natural in A;
- A_0-smooth: smooth with derivatives which are A_0-linear;

we see that df_A is:

- natural in A;
- A_0-smooth: smooth with a derivative which is A_0-linear;

so it indeed defines a map of supermanifolds:

$$df \colon G^{p+1} \to H.$$

Furthermore, it is immediate that:

$$d^2 f_A = 0$$

for all A, and thus

$$d^2 f = 0.$$

So $C^\bullet(G, H)$ is truly a cochain complex. Its cohomology $H^\bullet(G, H)$ is the **supergroup cohomology of G with coefficients in H**. Of course, if $df = 0$, f is called a **cocycle**, and f is **normalized** if
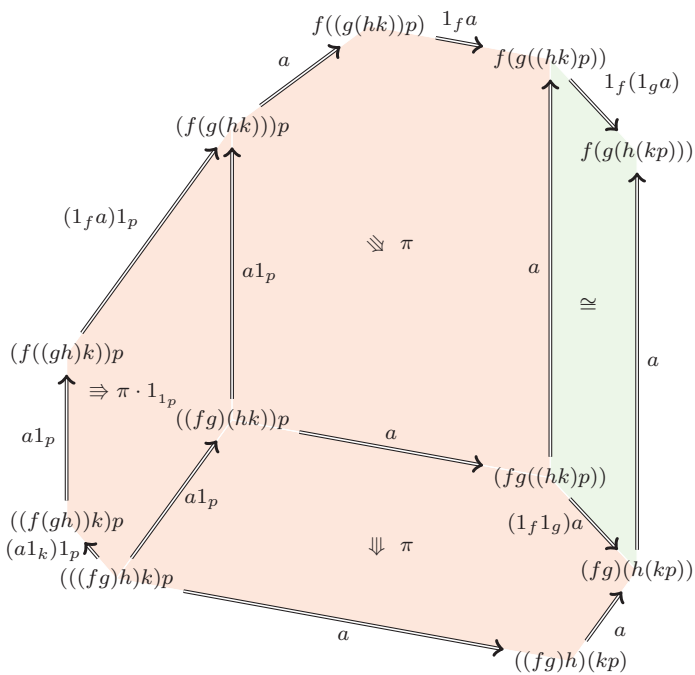
$$f_A(g_1, \ldots, g_p) = 0$$

for any Grassmann algebra A, whenever one of the A-points $g_1, \ldots, g_p$ is 1. When $H = \mathbb{R}$, we omit reference to it, and write $C^\bullet(G, \mathbb{R})$ as $C^\bullet(G)$.

We can generalize our construction of Lie 3-groups to 'Lie 3-supergroups'. A **super tricategory** T has

- a **supermanifold of objects** T_0;

- a **supermanifold of morphisms** T_1;

- a **supermanifold of 2-morphisms** T_2;

- a **supermanifold of 3-morphisms** T_3;

equipped with maps of supermanifolds as described in Definition 7: source, target, identity-assigning, composition at 0-cells, 1-cells and 2-cells, associator and left and right unitors, pentagonator and triangulators all maps of supermanifolds, and satisfying the same axioms as a smooth tricategory. We express the pentagonator identity in terms of A-points: the following equation holds:

$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$

a

$1_f(1_g a)$

$(f(g(hk)))p$

$f(g(h(kp)))$

$(1_f a)1_p$

$a1_p$

$\Downarrow \pi$

a

$\cong$

$(f((gh)k))p$

a

$\Rrightarrow \pi \cdot 1_{1_p}$

$((fg)(hk))p \xrightarrow{a} (fg((hk)p))$

$a1_p$

$((f(gh))k)p$

$(a1_k)1_p$

$a1_p$

$\Downarrow \pi$

$(1_f 1_g)a$

$(((fg)h)k)p$

$(fg)(h(kp))$

a

a

$((fg)h)(kp)$

$=$

$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$

a

$1_f(1_g a)$

$(f(g(hk)))p$

$f(g(h(kp)))$

$\cong$

$1_f(a1_p)$

$\Downarrow 1_{1_f} \cdot \pi$

$(1_f a)1_p$

$f(((gh)k)p) \xrightarrow{1_f a} f((gh)(kp))$

$1_f a$

a

$(f((gh)k))p$

$a1_p$

$\Downarrow \pi$

a

$\Downarrow \pi$

$((f(gh))k)p$

$(a1_k)1_p$

a

$(fg)(h(kp))$

$(((fg)h)k)p$

$\cong$

$(f(gh))(kp)$

a

$a(1_k 1_p)$

a
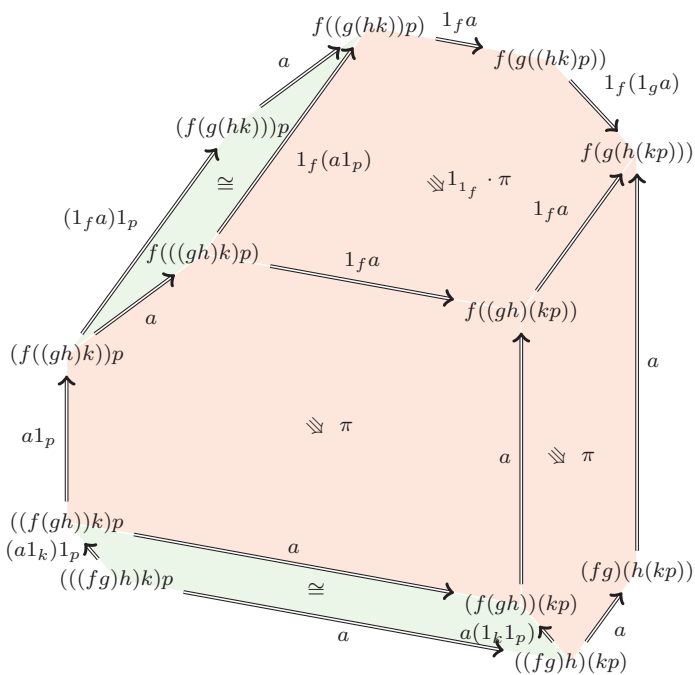
$((fg)h)(kp)$

for any 'composable quintet of morphisms':

$$(f, g, h, k, p) \in (T_1 \times_{T_0} T_1 \times_{T_0} T_1 \times_{T_0} T_1 \times_{T_0} T_1)_A$$

A **Lie 3-supergroup**, or **3-supergroup** is a super tricategory with one object (more precisely, the one-point supermanifold) and all morphisms, 2-morphisms and 3-morphisms weakly invertible. Given a normalized H-valued 4-cocycle π on G, we can construct a 3-supergroup $\mathrm{Brane}_\pi(G, H)$ in the same way we constructed the Lie 3-group $\mathrm{Brane}_\pi(G, H)$ when G and H were Lie groups, but deleting every reference to elements of G or H:

- The supermanifold of objects is the one-point supermanifold, 1.

- The supermanifold of morphisms is the supergroup G. Composition at a 0-cell is given by multiplication in the group:

$$\cdot : G \times G \to G.$$

  The source and target maps are the unique maps to 1. The identity-assigning map is the identity-assigning map for G:

$$\mathrm{id} : 1 \to G.$$

- The supermanifold of 2-morphisms is again G. The source, target and identity-assigning maps are all the identity on G. Composition at a 1-cell is the identity on G, while composition at a 0-cell is again multiplication in G. This encodes the idea that all 2-morphisms are trivial.

- The supermanifold of 3-morphisms is $G \times H$. The source and target maps are projection onto G. The identity-assigning map is the inclusion:

$$G \to G \times H$$

  that takes A-points $g \in G_A$ to $(g, 0) \in G_A \times H_A$, for all A.

- There are three kinds of composition of 3-morphisms: composition at a 2-cell and at a 3-cell are both given by addition on H:

$$1 \times + : G \times H \times H \to G \times H.$$

  While composition at a 0-cell is just given by multiplication on the semidirect product:

$$\cdot : (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- The associator and left and right unitors are trivial.

- The triangulators are trivial.

- The **2-associator** or **pentagonator** is given by the 4-cocycle $\pi\colon G^4 \to H$, where the source (and target) is understood to come from multiplication on G.

- The inverse of 3-morphisms is just given by negation in H. The map

$$\mathrm{inv}_3\colon G \times H \to G \times H$$

sends the A-point (g, h) to $(g, -h)$.

The inverses for 2-morphisms are trivial, because 2-morphisms are trivial.

Inverses for 1-morphisms are just inverses in G. The map

$$\mathrm{inv}_1\colon G \to G$$

is just the usual inverse map on G. This is made into a biadjoint biequivalence with ϵ and η trivial, and Φ and Θ chosen to satisfy the swallowtail identity. There are many possible choices; here is a convenient one in terms of A-points:

$$\Phi(g) = -\pi(g, g^{-1}, g, g^{-1}), \quad \Theta(g) = 0.$$

A **slim 3-supergroup** is one of this form. It remains to check that it is, indeed, a 3-supergroup.

Proposition 10. $\mathrm{Brane}_\pi(G, H)$ *is a 3-supergroup: a super tricategory with one object and all morphisms, 2-morphisms and 3-morphisms weakly invertible.*

Proof. This proof is a duplicate of Proposition 8, but with A-points instead of elements. $\square$

6. Integrating nilpotent Lie n-superalgebras

Any mathematician worth her salt knows that we can easily construct Lie algebras as the infinitesimal versions of Lie groups, and that a more challenging inverse construction exists: we can 'integrate' Lie algebras to get Lie groups. In fact, the same is true of Lie supergroups and Lie superalgebras, and indeed for Lie n-supergroups and Lie n-superalgebras for all n!

In the following, we recall our solution to this integration problem for slim, nilpotent Lie n-superalgebras, which appeared in our previous paper [28]. As we saw in Section 2, slim Lie n-superalgebras are built from $(n+1)$-cocycles in Lie superalgebra cohomology. Remember, p-cochains on the Lie superalgebra $\mathfrak{g}$ are linear maps:

$$C^p(\mathfrak{g}, \mathfrak{h}) = \{\omega \colon \Lambda^p \mathfrak{g} \to \mathfrak{h}\},$$

where $\mathfrak{h}$ is a representation of $\mathfrak{g}$, though we shall restrict ourselves to the trivial representation $\mathfrak{h} = \mathbb{R}$ in this section.

On the other hand, in Section 5, we saw that slim Lie 3-supergroups are built from 4-cocycles in Lie supergroup cohomology. Remember, p-cochains on G are smooth maps:

$$C^p(G, H) = \{f \colon G^p \to H\},$$

where H is an abelian supergroup on which G acts by automorphism, though we shall restrict ourselves to $H = \mathbb{R}$ with trivial action in this section.

This parallel suggests a naive scheme to integrate Lie 3-superalgebras. Given a slim Lie 3-superalgebra $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$, we seek a slim Lie 3-supergroup $\mathrm{Brane}_\pi(G, H)$ where:

- G is a Lie supergroup with Lie superalgebra $\mathfrak{g}$; i.e. it is a Lie supergroup integrating $\mathfrak{g}$,

- H is a Lie supergroup with Lie superalgebra $\mathfrak{h}$; i.e. it is a Lie supergroup integrating $\mathfrak{h}$,

- π is a Lie supergroup 4-cocycle on G that, in some suitable sense, integrates the Lie superalgebra 4-cocycle ω on $\mathfrak{g}$.

In this section, we describe an elegant, geometric procedure to integrate Lie superalgebra cocycles to obtain supergroup cocycles, which works when the Lie superalgebra in question is nilpotent.

Of course, this falls far short of integrating a general Lie n-superalgebra to an Lie n-supergroup, which has been done by others. Building on the earlier work of Getzler [22] on integrating nilpotent Lie n-algebras, Henriques [26] has shown that any Lie n-algebra can be integrated to a 'Lie n-group', which Henriques defines as a sort of smooth Kan complex in the category of Banach manifolds. More recently, Schreiber [37] has generalized this integration procedure to a setting much more general than that of Banach manifolds, including both supermanifolds and manifolds with infinitesimals. For both Henriques and Schreiber, the definition of Lie n-group is weaker

than the one we sketched in Section 3 — it weakens the notion of multi-plication so that the product of two group 'elements' is only defined up to equivalence. This level of generality seems essential for the construction to work for *every* Lie n-algebra.

However, for *some* Lie n-algebras, we can integrate them using the more naive idea of Lie n-group we prefer in this paper: a smooth n-category with one object in which every k-morphism is weakly invertible, for all $1 \le k \le n$. We shall see that, for some slim Lie n-algebras, we can integrate the defining Lie algebra $(n+1)$-cocycle to obtain a Lie group $(n+1)$-cocycle. In other words, for certain Lie groups G with Lie algebra $\mathfrak{g}$, there is a cochain map:

$$\int \colon C^{\bullet}(\mathfrak{g}) \to C^{\bullet}(G),$$

which is a chain homotopy inverse to differentiation.

When is this possible? We can always differentiate Lie group cochains to obtain Lie algebra cochains, but if we can also integrate Lie algebra cochains to obtain Lie group cochains, the cohomology of the Lie group and its Lie algebra will coincide:

$$H^{\bullet}(\mathfrak{g}) \cong H^{\bullet}(G).$$

By a theorem of van Est [43], this happens when all the homology groups of G, as a topological space, vanish.

Thus, we should look to Lie groups with vanishing homology for our examples. How bad can things be when the Lie group is not homologically trivial? To get a sense for this, recall that any semisimple Lie group G is diffeomorphic to the product of its maximal compact subgroup K and a contractible space C:

$$G \approx K \times C.$$

When K is a point, G is contractible, and certainly has vanishing homology. At the other extreme, when C is a point, G is compact. And indeed, in this case there is no hope of obtaining a nontrivial cochain map from Lie algebra cochains to Lie group cochains:

$$\int \colon C^{\bullet}(\mathfrak{g}) \to C^{\bullet}(G)$$

because *every smooth cochain on a compact group is trivial*.

Nonetheless, there is a large class of Lie n-algebras for which our Lie n-groups *are* general enough. In particular, when G is an 'exponential' Lie group, the story is completely different. A Lie group or Lie algebra is called

exponential if the exponential map

$$\exp\colon \mathfrak{g} \to G$$

is a diffeomorphism. For instance, all simply-connected nilpotent Lie groups are exponential, though the reverse is not true. Certainly, all exponential Lie groups have vanishing homology, because $\mathfrak{g}$ is contractible. We caution the reader that some authors use the term 'exponential' merely to indicate that exp is surjective.

When G is an exponential Lie group with Lie algebra $\mathfrak{g}$, we can use a geometric technique developed by Houard [27] to construct a cochain map:

$$\smallint\colon C^\bullet(\mathfrak{g}) \to C^\bullet(G).$$

The basic idea behind this construction is simple, a natural outgrowth of a familiar concept from the cohomology of Lie algebras. Because a Lie algebra p-cochain is a linear map:

$$\omega\colon \Lambda^p \mathfrak{g} \to \mathbb{R},$$

using left translation, we can view ω as defining a p-form on the Lie group G. So, we can integrate this p-form over p-simplices in G. Thus we can define a smooth function:

$$\smallint \omega\colon G^p \to \mathbb{R},$$

by viewing the integral of ω as a function of the vertices of a p-simplex:

$$\smallint \omega(g_1, g_2, \ldots, g_p) = \int_{[1,g_1,g_1g_2,\ldots,g_1g_2\cdots g_p]} \omega.$$

For the right-hand side to truly be a function of the p-tuple $(g_1, g_2, \ldots, g_p)$, we will need a standard way to 'fill out' the p-simplex $[1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_p]$, based only on its vertices. It is here that the fact that G is exponential is key: in an exponential group, we can use the exponential map to define a unique path from the identity 1 to any group element. We think of this path as giving a 1-simplex, $[1, g]$, and we can extend this idea to higher dimensional p-simplices.

Therefore, when G is exponential, we can construct $\smallint$. Using this cochain map, it is possible to integrate the slim Lie n-algebra $\mathfrak{brane}_\omega(\mathfrak{g})$ to the slim Lie n-group $\mathrm{Brane}_{\smallint \omega}(G)$.

Definition 11. Let Δ^p denote $\{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} : \sum x_i = 1, x_i \geq 0\}$, the standard p-simplex in $\mathbb{R}^{p+1}$. Given a collection of smooth maps

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G$$

for each $p \geq 0$, we say this collection defines a **left-invariant notion of simplices** in G if it satisfies:

1) **The vertex property.** For any $(p+1)$-tuple, the restriction

$$\varphi_p \colon \Delta^p \times \{(g_0, \ldots, g_p)\} \to G$$

   sends the vertices of Δ^p to $g_0, \ldots, g_p$, in that order. We denote this restriction by

$$[g_0, \ldots, g_p].$$

   We call this map a **p-simplex**, and regard it as a map from Δ^p to G.

2) **Left-invariance.** For any p-simplex $[g_0, \ldots, g_p]$ and any $g \in G$, we have:

$$g[g_0, \ldots, g_p] = [gg_0, \ldots, gg_p].$$

3) **The face property.** For any p-simplex

$$[g_0, \ldots, g_p] \colon \Delta^p \to G$$

   the restriction to a face of Δ^p is a $(p-1)$-simplex.

As we noted above, every exponential Lie group can be equipped with a left-invariant notion of simplices [28]. On any such group, we have the following result:

Proposition 12. *Let G be a Lie group equipped with a left-invariant notion of simplices, and let $\mathfrak{g}$ be its Lie algebra. Then there is a cochain map from the Lie algebra cochain complex to the Lie group cochain complex*

$$\textstyle\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

given by integration — that is, if ω is a left-invariant p-form on G, then define:

$$(\textstyle\int \omega)(g_1, g_2, \cdots, g_p) = \int_{[1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_p]} \omega.$$

Proof. See [28, Prop. 13].   □

Proposition 13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there is a cochain map from the Lie group cochain complex to the Lie algebra cochain complex:*

$$D\colon C^\bullet(G) \to C^\bullet(\mathfrak{g})$$

given by differentiation — that is, if F is a homogeneous p-cochain on G, and $X_1, \ldots, X_p \in \mathfrak{g}$, then we can define:

$$DF(X_1, \ldots, X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) X^1_{\sigma(1)} \ldots X^p_{\sigma(p)} F(1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_p),$$

where by X^j_i we indicate that the operator X_i differentiates only the jth variable, g_j.

Proof. See Houard [27], p. 224, Lemma 1.   □

Having now defined cochain maps

$$\textstyle\int\colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

and

$$D\colon C^\bullet(G) \to C^\bullet(\mathfrak{g}),$$

the obvious next question is whether or not this defines a homotopy equivalence of cochain complexes. Indeed, as proved by Houard, they do:

Theorem 14. *Let G be a Lie group equipped with a left-invariant notion of simplices, and $\mathfrak{g}$ its Lie algebra. The cochain map*

$$\textstyle D\int\colon C^\bullet(\mathfrak{g}) \to C^\bullet(\mathfrak{g}),$$

is the identity, whereas the cochain map

$$\textstyle\int D\colon C^\bullet(G) \to C^\bullet(G)$$

is cochain-homotopic to the identity. Therefore the Lie algebra cochain complex $C^\bullet(\mathfrak{g})$ and the Lie group cochain complex $C^\bullet(G)$ are homotopy equivalent and thus have isomorphic cohomology.

Proof. See Houard [27], p. 234, Proposition 2.   □

We now generalize the above results from Lie groups to Lie supergroups. As above, our concern will be for **exponetial supergroups**, where the exponential map

$$\exp\colon \mathfrak{g} \to G$$

from Lie superalgebra $\mathfrak{g}$ to supergroup G is a diffeomorphism. Our first concern, however, is to translate Lie superalgebra cocycles into Lie algebra cocycles. Note that for any Lie superalgebra $\mathfrak{g}$, we have Lie algebra on the A_0-module $\mathfrak{g}_A$. We call this an A_0-Lie algebra, because it is a Lie algebra over A_0.

Proposition 15. *Let $\mathfrak{g}$ be a Lie superalgebra, and let $\mathfrak{g}_A$ be the A_0-Lie algebra of its A-points. Then there is a cochain map:*

$$C_0^\bullet(\mathfrak{g}) \to C^\bullet(\mathfrak{g}_A)$$

given by taking the even p-cochain ω

$$\omega\colon \Lambda^p \mathfrak{g} \to \mathbb{R}$$

to the induced A_0-linear map ω_A:

$$\omega_A\colon \Lambda^p \mathfrak{g}_A \to A_0,$$

where $\Lambda^p \mathfrak{g}_A$ denotes the pth exterior power of $\mathfrak{g}_A$ as an A_0-module.

Proof. See [28, Prop. 23]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This proposition says that from any Lie superalgebra cocycle on $\mathfrak{n}$ we obtain a Lie algebra cocycle on $\mathfrak{n}_A$, albeit now valued in A_0. Since N_A is an exponential Lie group with Lie algebra $\mathfrak{n}_A$, we can integrate ω_A to a group cocycle, $\int \omega_A$, on N_A.

As before, we need a notion of simplices in N. Since N is a supermanifold, the vertices of a simplex should not be points of N, but rather A-points for arbitrary Grassmann algebras A. This means that for any $(p+1)$-tuple of A-points, we want to get a p-simplex:

$$[n_0, n_1, \ldots, n_p]\colon \Delta^p \to N_A,$$

where, once again, Δ^p is the standard p-simplex in $\mathbb{R}^{p+1}$, and this map is required to be smooth. But this only defines a p-simplex in N_A. To really get our hands on a p-simplex in N, we need it to depend functorially on

the choice of Grassmann algebra A we use to probe N. So if $f\colon A \to B$ is a homomorphism between Grassmann algebras and $N_f\colon N_A \to N_B$ is the induced map between A-points and B-points, we require:

$$N_f \circ [n_0, n_1, \ldots, n_p] = [N_f(n_0), N_f(n_1), \ldots, N_f(n_p)]$$

Thus given a collection of maps:

$$(\varphi_p)_A \colon \Delta^p \times (N_A)^{p+1} \to N_A$$

for all A and $p \geq 0$, we say this collection defines a **left-invariant notion of simplices** in N if

- each $(\varphi_p)_A$ is smooth, and for each $x \in \Delta^p$, the restriction:

$$(\varphi_p)_A \colon \{x\} \times N_A^{p+1} \to N_A$$

  is A_0-smooth;

- it defines a left-invariant notion of simplices in N_A for each A, as in Definition 11;

- the following diagram commutes for all homomorphisms $f\colon A \to B$:

$$
\begin{array}{ccc}
\Delta^p \times N_A^{p+1} & \xrightarrow{\ (\varphi_p)_A\ } & N_A \\
{\scriptstyle 1 \times N_f^{p+1}}\Big\downarrow & & \Big\downarrow{\scriptstyle N_f} \\
\Delta^p \times N_B^{p+1} & \xrightarrow[\ (\varphi_p)_B\ ]{} & N_B
\end{array}
$$

In fact, every exponential supergroup can be equipped with a left-invariant notion of simplices [28]. We can use this left-invariant notion of simplices to define a cochain map $\smallint\colon C^\bullet(\mathfrak{n}) \to C^\bullet(N)$:

Proposition 16. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra, and let N be the exponential supergroup which integrates $\mathfrak{n}$. There is a cochain map:*

$$\smallint\colon C_0^\bullet(\mathfrak{n}) \to C^\bullet(N)$$

which sends the even Lie superalgebra p-cochain ω to the supergroup p-cochain $\smallint \omega$, given on A-points by:

$$(\smallint \omega)_A(n_1, \ldots, n_p) = \int_{[1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} \omega_A$$

for $n_1, \ldots, n_p \in N_A$.

Proof. See [28, Prop. 24]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

7. The super-2-brane Lie 3-supergroup

We are now ready to unveil the Lie 3-supergroup which integrates our favorite Lie 3-superalgebra, 2-$\mathfrak{brane}(n + 2, 1)$. Remember, this is the Lie 3-superalgebra which occurs only in the dimensions for which the classical 2-brane makes sense. It is *not* nilpotent, simply because the Poincaré superalgebra $\mathfrak{siso}(n + 2, 1)$ that forms degree 0 of 2-$\mathfrak{brane}(n + 2, 1)$ is not nilpotent. Nonetheless, we are equipped to integrate this Lie 3-superalgebra using only the tools we have built to perform this task for nilpotent Lie n-superalgebras.

The road to this result has been a long one, and there is yet some ground to cover before we are finished. So, let us take stock of our progress before we move ahead:

- In spacetime dimensions $n + 3 = 4, 5, 7$ and 11, we used division algebras to construct a 4-cocycle β on the supertranslation algebra:

$$\mathcal{T} = V \oplus S$$

  which is nonzero only when it eats two vectors and two spinors:

$$\beta(\mathcal{A}, \mathcal{B}, \Psi, \Phi) = \langle \Psi, \mathcal{A}(\mathcal{B}\Phi) - \mathcal{B}(\mathcal{A}\Phi) \rangle.$$

- Because β is invariant under the action of $\mathfrak{so}(n + 2, 1)$, it can be extended to a 3-cocycle on the Poincaré superalgebra:

$$\mathfrak{siso}(n + 2, 1) = \mathfrak{so}(n + 2, 1) \ltimes \mathcal{T}.$$

  The extension is just defined to vanish outside of $\mathcal{T}$, and we call it β as well.

- Therefore, in spacetime dimensions $n + 3$, we get a Lie 3-superalgebra 2-$\mathfrak{brane}(n + 2, 1)$ by extending $\mathfrak{siso}(n + 2, 1)$ by the 4-cocycle β.

In the last section, we built the technology necessary to integrate Lie superalgebra cocycles to supergroup cocycles, *provided* the Lie superalgebra in question is nilpotent. This allows us to integrate nilpotent Lie n-superalgebras to n-supergroups. But 2-$\mathfrak{brane}(n + 2, 1)$ is not nilpotent, so we cannot use this directly here.

However, the cocycle β is supported on a nilpotent subalgebra: the supertranslation algebra, $\mathcal{T}$, for the appropriate dimension. This saves the day: we can integrate β as a cocycle on $\mathcal{T}$. This gives us a cocycle $\int \beta$ on the supertranslation supergroup, T, for the appropriate dimension. We will then be able to extend this cocycle to the Poincaré supergroup, thanks to its invariance under Lorentz transformations.

The following proposition helps us to accomplish this, but takes its most beautiful form when we work with 'homogeneous supergroup cochains', which we have not yet defined. Rest assured — they are not difficult. If G is a supergroup that acts on the abelian supergroup M by automorphism, a **homogeneous M-valued p-cochain** on G is a smooth map:

$$F \colon G^{p+1} \to M$$

that is equivariant with respect to the action of G. This means, for any Grassmann algebra A and A-points $g, g_0, \ldots, g_p \in G_A$:

$$F_A(gg_0, gg_1, \ldots, gg_p) = gF_A(g_1, \ldots, g_p).$$

In contrast, we call the p-cochains we have used until this point, namely a smooth, not necessarily equivariant map:

$$f \colon G^p \to M$$

an **inhomogeneous M-valued p-cochain**. We can define the supergroup cohomology of G using homogeneous cochains or inhomogeneous cochains. For the special case of discrete groups, this idea goes back to the founding paper on group cohomology by Eilenberg and Mac Lane [18].

Proposition 17. *Let G and H be Lie supergroups such that G acts on H, and let M be an abelian supergroup on which $G \ltimes H$ acts by automorphism. Given a homogeneous M-valued p-cochain F on H:*

$$F \colon H^{p+1} \to M,$$

we can extend it to a map of supermanifolds:

$$\tilde{F} \colon (G \ltimes H)^{p+1} \to M$$

by pulling back along the projection $(G \ltimes H)^{p+1} \to H^{p+1}$. In terms of A-points

$$(g_0, h_0), \ldots, (g_p, h_p) \in G_A \ltimes H_A,$$

this means $\tilde{F}$ is defined by:

$$\tilde{F}_A((g_0, h_0), \dots, (g_p, h_p)) = F_A(h_0, \dots, h_p),$$

Then $\tilde{F}$ is a homogeneous p-cochain on $G \ltimes H$ if and only if F is G-equivariant, and in this case $d\tilde{F} = \widetilde{dF}$.

Proof. See [28, Prop. 26]. □

Now, at long last, we are ready to integrate β. In the following proposition, T denotes the **supertranslation group**, the exponential supergroup of the supertranslation algebra $\mathcal{T}$.

Proposition 18. *In dimensions 4, 5, 7 and 11, the Lie supergroup 4-cocycle $\int \beta$ on the supertranslation group T is invariant under the action of $\mathrm{Spin}(n+2, 1)$.*

This is an immediate consequence of the following:

Proposition 19. *Let H be a nilpotent Lie supergroup with Lie superalgebra $\mathfrak{h}$. Assume H is equipped with its standard left-invariant notion of simplices, and let G be a Lie supergroup that acts on H by automorphism. If $\omega \in C^p(\mathfrak{h})$ is an even Lie superalgebra p-cochain which is invariant under the induced action of G on $\mathfrak{h}$, then $\int \omega \in C^p(H)$ is a Lie supergroup p-cochain which is invariant under the action of G on H.*

Proof. See [28, Prop. 28]. □

It thus follows that in dimensions 4, 5, 7 and 11, the cocycle $\int \beta$ can be extended to the Poincaré supergroup:

$$\mathrm{SISO}(n+2, 1) = \mathrm{Spin}(n+2, 1) \ltimes T.$$

By a slight abuse of notation, we continue to denote this extension by $\int \beta$. As an immediate consequence, we have:

Theorem 20. *In dimensions 4, 5, 7 and 11, there exists a slim Lie 3-supergroup formed by extending the Poincaré supergroup $\mathrm{SISO}(n+2, 1)$ by the 4-cocycle $\int \beta$, which we call the **2-brane Lie 3-supergroup**, **2-Brane($n+2, 1$)**.*

Acknowledgements

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CAMGSD, Instituto Superior Técnico, U Lisboa
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail address: `jhuerta@math.ist.utl.pt`