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Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras

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This paper studies two types of 3-Lie bialgebras whose compatibility conditions between the multiplication and comultiplication are given by local cocycles and double constructions respectively, and are thus called the local cocycle 3-Lie bialgebra and double construction 3-Lie bialgebra. They are two extensions of the well-known Lie bialgebra in the context of 3-Lie algebras. The local cocycle 3-Lie bialgebra extends the connection between Lie bialgebras and the classical Yang-Baxter equation. Its relationship with a ternary variation of the classical Yang-Baxter equation, called the 3-Lie classical Yang-Baxter equation, a ternary $\mathcal{O}$-operator and a 3-pre-Lie algebra is established. In particular, solutions of the 3-Lie classical Yang-Baxter equation give (coboundary) local cocycle 3-Lie bialgebras, whereas, 3-pre-Lie algebras give rise to solutions of the 3-Lie classical Yang-Baxter equation. The double construction 3-Lie bialgebra is also introduced to extend to the 3-Lie algebra context the connection between Lie bialgebras and double constructions of Lie algebras. Their related Manin triples give a natural construction of pseudo-metric 3-Lie algebras with neutral signature. The double construction 3-Lie bialgebra can be regarded as a special class of the local cocycle 3-Lie bialgebra. Explicit examples of double construction 3-Lie bialgebras are provided.

1. Introduction

1.1. Bialgebras

For a given algebraic structure determined by a set of multiplications of various arities and a set of relations among the operations (which can be made precise in the context of either universal algebra or operads [24, 32]), a bialgebra structure on this algebra is obtained by a corresponding set of comultiplications together with a set of compatibility conditions between the multiplications and comultiplications. For a finite dimensional vector space V with the given algebraic structure, this can be achieved by equipping the dual space V^* with the same algebraic structure and a set of compatibility conditions between the structures on V and those on V^*.

The associative bialgebra and infinitesimal bialgebra [1, 28] are well-known bialgebra structures. Note that these two structures have the same associative multiplications on V and V^*. They are distinguished only by the compatibility conditions, with the comultiplication acting as a homomorphism (resp. a derivation) on the multiplication for the associative bialgebra (resp. the infinitesimal bialgebra). In general, it is quite common to have multiple bialgebra structures that differ only by their compatibility conditions.

A good compatibility condition is prescribed on one hand by a strong motivation and potential applications, and on the other hand by a rich structure theory and effective constructions.

1.2. Lie bialgebras

In the Lie algebra context, the most common bialgebra structure is the Lie bialgebra, consisting of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ where $[\cdot,\cdot] : \otimes^2\mathfrak{g} \to \mathfrak{g}$ is a Lie bracket, a Lie coalgebra $(\mathfrak{g}, \Delta)$ where $\Delta : \mathfrak{g} \to \otimes^2\mathfrak{g}$ is a Lie comultiplication, and a suitable compatibility condition between the Lie bracket $[\cdot,\cdot]$ and the Lie comultiplication Δ. The Lie bialgebra is the algebraic structure corresponding to a Poisson-Lie group and the classical structure of a quantized universal enveloping algebra [12, 17]. Such great importance of the Lie bialgebra serves as the main motivation for our interest in a suitable bialgebra theory for the 3-Lie algebra in this paper.

There are several equivalent statements for the compatibility condition of a Lie bialgebra. It is these multiple manifestations of the Lie bialgebra that determine its importance in both theory and applications. However the equivalence of similar conditions in the 3-Lie algebra context no longer holds. Thus, for our purpose of developing a suitable bialgebra theory for the 3-Lie algebra, we first differentiate the roles played by these equivalent conditions.

For a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, there is a graded Lie algebra structure on the exterior algebra $\Lambda^\bullet\mathfrak{g}$. The compatibility condition can be concisely stated as the condition that the Lie comultiplication Δ is a derivation with respect to the graded Lie algebra structure on $\Lambda^\bullet\mathfrak{g}$:

$$\Delta[x,y] = [\Delta x, y] + [x, \Delta y], \quad \forall x, y \in \mathfrak{g}. \tag{1.1}$$

Note that, in fact, $[x, \Delta y] = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)\Delta y$, where $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation.

If one goes beyond the simplicity of this definition of the compatibility condition, one sees that the importance of Lie bialgebra mainly comes from two other equivalent statements of the compatibility condition.

One reason for the usefulness of the Lie bialgebra is that it has a coboundary theory, which leads to the construction of Lie bialgebras from solutions of the classical Yang-Baxter equation. This coboundary theory comes from the following equivalent statement of the compatibility condition (1.1) as cocycles: the Lie algebra $\mathfrak{g}$ acts on $\otimes^2\mathfrak{g}$ via the map $\mathrm{ad} \otimes 1 + 1 \otimes \mathrm{ad}$ (i.e. the tensor representation of two adjoint representations), and $\Delta : \mathfrak{g} \to \otimes^2\mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with coefficients in the representation $\mathrm{ad} \otimes 1 + 1 \otimes \mathrm{ad}$.

On the other hand, some important applications of Lie bialgebras to the related fields [12] have relied on the Lie algebras with symmetric nondegenerate invariant bilinear forms (the so-called "pseudo-metric" or "self-dual" or "quadratic" Lie algebras) coming from Lie bialgebras. These Lie algebras are obtained from the following Manin triple characterization of the Lie bialgebra: the compatibility condition (1.1) of a Lie bialgebra is given by the condition that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ whose underlying vector space is the dual space of $\mathfrak{g}$ are subalgebras of a third Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ such that the bilinear form

$$(x + u^*, y + v^*)_+ = \langle x, v^*\rangle + \langle u^*, y\rangle, \ \ \forall x, y \in \mathfrak{g}, u^*, v^* \in \mathfrak{g}^*, \tag{1.2}$$

is invariant on $\mathfrak{g} \oplus \mathfrak{g}^*$. Here $\langle \cdot,\cdot\rangle$ is the usual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

To recap, we have the following three compatibility conditions of Lie bialgebras. They are all equivalent, yet each has its own advantages.

Condition 1.1. (a) the comultiplication Δ satisfies the derivation condition in Eq. (1.1);

(b) the comultiplication $\Delta : \mathfrak{g} \to \otimes^2 \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$;

(c) there is a Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$.

1.3. 3-Lie algebras

Generalizations of Lie algebras to higher arities, including 3-Lie algebras and more generally, n-Lie algebras [22, 29, 30], have attracted attention from several fields of mathematics and physics. It is the algebraic structure corresponding to Nambu mechanics [2, 34, 38]. In particular, the study of 3-Lie algebras plays an important role in string theory [5, 14, 20, 25–27, 33]. For example, the structure of 3-Lie algebras is applied to the study of supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the generalized identity for a 3-Lie algebra is essential to define the action with $N = 8$ supersymmetry and the Jacobi identity can be regarded as a generalized Plücker relation in the physics literature.

Metric 3-Lie algebras are of particular interest in physics. More precisely, to obtain the correct equations of motion for the Begger-Lambert theory from a Lagrangian that is invariant under all aforementioned symmetries seems to require the 3-Lie algebra to admit an invariant inner product. The signature of this metric determines the relative signs of the kinetic terms for scalar and fermion fields in the Bagger-Lambert Lagrangian [5, 6, 25]. In ordinary gauge theory, a positive-definite metric is required in order to ensure that the theory has positive-definite kinetic terms and to prevent violations of unitarity due to propagating ghost-like degrees of freedom. However, there are few 3-Lie algebras which admit positive-definite metrics. In fact, it has been shown [23, 35] that all finite-dimensional real 3-Lie algebras with positive-definite metrics are the direct sums of a special 4-dimensional real simple 3-Lie algebra and a trivial 3-Lie algebra. On the other hand, in order to find new interesting Bagger-Lambert Lagrangians, one is led to contemplating 3-Lie algebras with pseudo-metrics having any signature (p, q) or with degenerate invariant symmetric bilinear forms, despite the possibility of negative-norm states, since in certain dynamical systems a zero-norm generator corresponds to a gauge symmetry while a negative norm generator corresponds to a ghost. Thus it seems worthwhile and interesting, from both physical and mathematical considerations, to find new 3-Lie algebras with symmetric invariant bilinear forms. In a related direction, it is shown in [15] that generalized metric 3-Lie algebras are metric Lie algebras together with a faithful orthogonal Lie algebra module. It would be interesting to see how this connection can be generalized to 3-Lie bialgebras. See also [16, 21].

1.4. 3-Lie bialgebras

Given the importance of Lie bialgebras and 3-Lie algebras, it is natural to develop a suitable bialgebra theory for 3-Lie algebras.

Motivated by the equivalent compatibility conditions of Lie bialgebras in Condition 1.1, one is naturally led to defining a 3-Lie bialgebra as a pair consisting of a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$, a 3-Lie coalgebra (A, Δ) such that (A^*, Δ^*) is also a 3-Lie algebra and one of the following compatibility conditions is satisfied.

Condition 1.2. (a) the comultiplication Δ satisfies certain "derivation" condition;

(b) the comultiplication $\Delta : A \to \otimes^3 A$ is a 1-cocycle on A;

(c) there is a Manin triple $(A \oplus A^*, A, A^*)$.

Contrary to the case of the Lie bialgebra, suitable extensions of these conditions to the context of 3-Lie algebras are not equivalent, leading to different extensions of the Lie bialgebra to the ternary case. Thus there might not be a unique "perfect" definition of a 3-Lie bialgebra, but three different versions serving different purposes. This is in reminiscent to the case of bialgebras. The following is an overview of the three approaches and also serves as an outline of this paper. The first approach was given in [8] based on Condition 1.2. (a). The other two are the subjects of study of this paper.

1.4.1. 3-Lie bialgebras with the derivation compatibility. An approach of bialgebra theory for 3-Lie algebras based on Condition 1.2.(a) was taken in [8], in which the authors generalized the compatibility condition (1.1) formally to the following equality:

$$\Delta[x,y,z] = [\Delta x, y, z] + [x, \Delta y, z] + [x, y, \Delta z], \tag{1.3}$$

where

$$\begin{aligned}[x,y,\Delta z] &= -[x,\Delta z, y] = [\Delta z, x, y]\\ &= (\mathrm{ad}_{x,y} \otimes 1 \otimes 1 + 1 \otimes \mathrm{ad}_{x,y} \otimes 1 + 1 \otimes 1 \otimes \mathrm{ad}_{x,y})\Delta(z),\end{aligned} \tag{1.4}$$

with $\mathrm{ad}_{x,y}(z) = [x,y,z]$ for all $x, y, z \in A$.

However, for this formal generalization, neither a coboundary theory nor the structure on the double space $A \oplus A^*$ is known. Unlike the case of Lie algebras, it is still unknown whether there is 3-Lie algebra structure on $\Lambda^\bullet A$, making it challenging for such a bialgebra structure to develop an expected structure theory and applications.

1.4.2. 3-Lie bialgebras with the cocycle compatibility. Since a large part of the Lie bialgebra theory is based on the cocycle characterization of its compatibility condition, it is natural to extend this approach to 3-Lie algebras via Condition 1.2.(b). We will carry out this approach in Section 3, after some preparation in Section 2. Unfortunately, unlike the case of the Lie bialgebra, such a cocycle description by itself does not make sense in the 3-Lie algebra context since there is no natural representation of a 3-Lie algebra A on $\otimes^3 A$. To get around this obstacle, we observe that, for a Lie bialgebra $(\mathfrak{g}, \Delta)$, there are two more 1-cocycles Δ_1 and Δ_2 with coefficients in the representations $(\otimes^2\mathfrak{g}, \mathrm{ad}\otimes 1)$ and $(\otimes^2\mathfrak{g}, 1\otimes \mathrm{ad})$ respectively and the cocycle condition for $\Delta : \mathfrak{g} \to \otimes^2\mathfrak{g}$ follows from the assumption that Δ is a sum $\Delta_1 + \Delta_2$ of 1-cocycles with certain additional constrains. Noting that $(\otimes^3 A, \mathrm{ad}\otimes 1\otimes 1)$, $(\otimes^3 A, 1\otimes \mathrm{ad}\otimes 1)$ and $(\otimes^3 A, 1\otimes 1\otimes \mathrm{ad})$ are representations of a 3-Lie algebra A, we are naturally led to the following modification of Condition 1.2.(b):

(b') The comultiplication $\Delta : A \to \otimes^3 A$ is a sum $\Delta_1 + \Delta_2 + \Delta_3$, where each Δ_i for $i = 1, 2, 3$, is a 1-cocycle with coefficients in $(\otimes^3 A, \mathrm{ad}\otimes 1\otimes 1)$, $(\otimes^3 A, 1\otimes \mathrm{ad}\otimes 1)$ and $(\otimes^3 A, 1\otimes 1\otimes \mathrm{ad})$ respectively.

With this definition, we obtain a bialgebra structure for the 3-Lie algebra that also has a coboundary theory. We call this structure a local cocycle 3-Lie bialgebra (Definition 3.1). Although there is no natural 3-Lie algebra structure on $A \oplus A^*$, there are still some interesting properties for local cocycle 3-Lie bialgebras. In particular, it also leads to an analogue of the classical Yang-Baxter equation defined on a 3-Lie algebra, called the 3-Lie classical Yang-Baxter equation (Definition 3.11). Solutions of this equation give natural constructions of local cocycle 3-Lie bialgebras (Theorem 3.10). There is also a one-to-one correspondence between invertible skew-symmetric solutions of this equation and symplectic 3-Lie algebras. On the other hand, such a bialgebra theory is also related to $\mathcal{O}$-operators and the so-called 3-pre-Lie-algebras which provide solutions of the 3-Lie classical Yang-Baxter equation and hence constructions of local cocycle 3-Lie bialgebras on certain double spaces. This is done in Section 3.3.

1.4.3. 3-Lie bialgebras with the Manin triple compatibility. Motivated by the double structure of a Manin triple characterizing a Lie bialgebra, it is natural to consider the following analogue of a Manin triple for 3-Lie algebras based on Condition 1.2.(c):

(c') $A \oplus A^*$ can be equipped with a 3-Lie algebra structure such that A and A^* are 3-Lie subalgebras of $A \oplus A^*$ and the bilinear form on $A \oplus A^*$ defined by Eq. (1.2) is invariant.

Note that if such a structure exists, then $(A \oplus A^*, (\cdot,\cdot)_+)$ is a 3-Lie algebra with a pseudo-metric having the signature (n, n), where $n := \dim A$. Thus, such an approach provides a natural construction of pseudo-metric 3-Lie algebras with signature (n, n) for the aforementioned study of Bagger-Lambert Lagrangians. Following the theory of Lie bialgebras, we utilize a Manin triple of 3-Lie algebras to define a bialgebra structure for 3-Lie algebras, called a double construction 3-Lie bialgebra. We will present this approach in Section 4. We also show that the double construction 3-Lie bialgebra can be regarded as a special case of the local cocycle 3-Lie bialgebra and provide explicit examples of double construction 3-Lie bialgebras.

This completes the overview of the paper. A more detailed summary of each section can be found at the beginning of the section. Moreover, throughout this paper, all the vector spaces and algebras are assumed to be of finite dimension, although many results still hold in the infinite dimensional cases.

2. Some preliminary results on 3-Lie algebras

In this section, we give some general results on 3-Lie algebras and their cohomology theory that will be used in later sections.

Definition 2.1. [22] A **3-Lie algebra** is a vector space A together with a skew-symmetric linear map (3-Lie bracket) $[\cdot,\cdot,\cdot] : \otimes^3 A \to A$ such that the following **Fundamental Identity (FI)** holds:

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]] \tag{2.1}$$

for $x_i \in A, 1 \le i \le 5$.

In other words, for $x_1, x_2 \in A$, the operator

$$\mathrm{ad}_{x_1,x_2} : A \to A, \quad \mathrm{ad}_{x_1,x_2} x := [x_1, x_2, x], \quad \forall x \in A, \tag{2.2}$$

is a derivation in the sense that

$$\begin{aligned}\mathrm{ad}_{x_1,x_2}[x_3,x_4,x_5] &= [\mathrm{ad}_{x_1,x_2}x_3,x_4,x_5]+[x_3,\mathrm{ad}_{x_1,x_2}x_4,x_5]\\ &\quad+[x_3,x_4,\mathrm{ad}_{x_1,x_2}x_5], \forall x_3,x_4,x_5\in A.\end{aligned}$$

A morphism between 3-Lie algebras is defined as usual, i.e. a linear map that preserves the 3-Lie brackets.

Proposition 2.2. *Let A be a vector space together with a skew-symmetric linear map $[\cdot,\cdot,\cdot]:\otimes^3 A\to A$. Then $(A,[\cdot,\cdot,\cdot])$ is a 3-Lie algebra if and only if the following identities hold:*

(a) $[[x_1,x_2,x_3],x_4,x_5]-[[x_1,x_2,x_4],x_3,x_5]+[[x_1,x_3,x_4],x_2,x_5]$ $-[[x_2,x_3,x_4],x_1,x_5]=0,$

(b) $[[x_1,x_2,x_5],x_3,x_4]+[[x_3,x_4,x_5],x_1,x_2]-[[x_1,x_3,x_5],x_2,x_4]$ $-[[x_2,x_4,x_5],x_1,x_3]+[[x_1,x_4,x_5],x_2,x_3]+[[x_2,x_3,x_5],x_1,x_4]=0,$

for $x_i\in A, 1\le i\le 5$.

Proof. If $(A,[\cdot,\cdot,\cdot])$ is a 3-Lie algebra, then applying Eq. (2.1) to the last term of Eq. (2.1), we have

$$\begin{aligned}[x_1,x_2,[x_3,x_4,x_5]] &= [[x_1,x_2,x_3],x_4,x_5]+[x_3,[x_1,x_2,x_4],x_5]\\ &\quad+[[x_3,x_4,x_1],x_2,x_5]+[x_1,[x_3,x_4,x_2],x_5]\\ &\quad+[x_1,x_2,[x_3,x_4,x_5]],\end{aligned}$$

proving Item (a).

Similarly, applying Eq. (2.1) to the first and second terms on the right hand side of Eq. (2.1), we obtain

$$\begin{aligned}[x_1,x_2,[x_3,x_4,x_5]] &= [[x_4,x_5,x_1],x_2,x_3]+[x_1,[x_4,x_5,x_2],x_3]\\ &\quad+[x_1,x_2,[x_4,x_5,x_3]]+[[x_5,x_3,x_1],x_2,x_4]\\ &\quad+[x_1,[x_5,x_3,x_2],x_4]+[x_1,x_2,[x_5,x_3,x_4]]\\ &\quad+[x_3,x_4,[x_1,x_2,x_5]],\end{aligned}$$

implying Item (b).

Conversely, suppose that Items (a) and (b) hold. First Item (a) gives

$$\begin{aligned}[[x_1,x_2,x_3],x_4,x_5] &= -[[x_1,x_2,x_5],x_3,x_4]+[[x_1,x_3,x_5],x_2,x_4]\\ &\quad-[[x_2,x_3,x_5],x_1,x_4]\end{aligned}$$

which equals to

$$[[x_3, x_4, x_5], x_1, x_2] - [[x_2, x_4, x_5], x_1, x_3] + [x_1, x_4, x_5], x_2, x_3]$$

by Item (b). Thus, A is a 3-Lie algebra. □

The notion of a representation of an n-Lie algebra was introduced in [29]. See also [19].

Definition 2.3. Let V be a vector space. A **representation of a 3-Lie algebra** A on V is a skew-symmetric linear map $\rho : \otimes^2 A \to \mathfrak{gl}(V)$ such that

(i) $\rho(x_1, x_2)\rho(x_3, x_4) - \rho(x_3, x_4)\rho(x_1, x_2) = \rho([x_1, x_2, x_3], x_4) - \rho([x_1, x_2, x_4], x_3)$;

(ii) $\rho([x_1, x_2, x_3], x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4)$,

for $x_i \in A, 1 \le i \le 4$.

It is straightforward to obtain

Lemma 2.4. *Let A be a 3-Lie algebra, V a vector space and $\rho : \otimes^2 A \to \mathfrak{gl}(V)$ a skew-symmetric linear map. Then (V, ρ) is a representation of A if and only if there is a 3-Lie algebra structure (called the semi-direct product) on the direct sum $A \oplus V$ of vector spaces, defined by*

$$[x_1 + v_1, x_2 + v_2, x_3 + v_3]_{A\oplus V} = [x_1, x_2, x_3] + \rho(x_1, x_2)v_3 + \rho(x_3, x_1)v_2 + \rho(x_2, x_3)v_1, \tag{2.3}$$

for $x_i \in A, v_i \in V, 1 \le i \le 3$. We denote this semi-direct product 3-Lie algebra by $A \ltimes_\rho V$.

From the proof of Proposition 2.2, we immediately obtain

Proposition 2.5. *Let (V, ρ) be a representation of a 3-Lie algebra A. Then the following identities hold:*

(a) $\rho([x_1, x_2, x_3], x_4) - \rho([x_1, x_2, x_4], x_3]) + \rho([x_1, x_3, x_4], x_2) - \rho([x_2, x_3, x_4], x_1) = 0$;

(b) $\rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4) + \rho(x_3, x_4)\rho(x_1, x_2) + \rho(x_1, x_4)\rho(x_2, x_3) + \rho(x_2, x_4)\rho(x_3, x_1) = 0$, *for* $x_i \in A, 1 \le i \le 4$.

Let (V,ρ) be a representation of a 3-Lie algebra A. Define $\rho^*: \otimes^2 A \longrightarrow \mathfrak{gl}(V^*)$ by

$$\langle \rho^*(x_1,x_2)\alpha, v\rangle = -\langle \alpha, \rho(x_1,x_2)v\rangle, \quad \forall \alpha \in V^*,\ x_1,x_2 \in A,\ v \in V. \tag{2.4}$$

A straightforward computation applying Definition 2.3 and Proposition 2.5 gives

Proposition 2.6. *With the above notations,* (V^*,ρ^*) *is a representation of* A*, called the* **dual representation** *of* (V,ρ).

Example 2.7. Let A be a 3-Lie algebra. The linear map $\mathrm{ad}: \otimes^2 A \to \mathfrak{gl}(A)$ sending $x_1 \otimes x_2$ to ad_{x_1,x_2} for any $x_1,x_2 \in A$ defines a representation (A,ad) which is called the **adjoint representation** of A, where ad_{x_1,x_2} is given by Eq. (2.2). The dual representation (A^*,ad^*) of the adjoint representation (A,ad) of a 3-Lie algebra A is called the **coadjoint representation**.

Given a representation (V,ρ), denote by $C^p(A;V)$ the set of p-cochains:

$$C^p(A;V) := \{\text{linear maps } f : \underbrace{\wedge^2 A \otimes \cdots \otimes \wedge^2 A}_{p-1 \text{ factors}} \wedge A \longrightarrow V\}.$$

The coboundary operators with coefficients in the trivial representation and the adjoint representation are given in [3, 4]. Similarly, using the notation $X_j := X_j^1 \wedge X_j^2 \in \wedge^2 A, 1 \le j \le p$, the coboundary operator $\delta : C^p(A;V) \longrightarrow C^{p+1}(A;V)$ is defined by

$$\begin{aligned}
&\delta f(X_1,\cdots,X_p,Z)\\
= &\sum_{1\le j\le k}^{p} (-1)^j f\big(X_1,\cdots,\widehat{X_j},\cdots,X_{k-1},\\
&\qquad\qquad [X_j^1,X_j^2,X_k^1]\wedge X_k^2 + X_k^1 \wedge [X_j^1,X_j^2,X_k^2],\cdots,X_p,Z\big)\\
&+\sum_{j=1}^{p}(-1)^j f(X_1,\cdots,\widehat{X_j},\cdots,X_p,[X_j^1,X_j^2,Z])\\
&+\sum_{j=1}^{p}(-1)^{j+1}\rho(X_j^1,X_j^2) f(X_1,\cdots,\widehat{X_j},\cdots,X_p,Z)\\
&+(-1)^{p+1}\rho(X_p^2,Z) f(X_1,\cdots,X_p,X_p^1)\\
&+(-1)^{p}\rho(X_p^1,Z) f(X_1,\cdots,X_p,X_p^2).
\end{aligned}$$

In particular, we obtain the following formula for a 1-cocycle.

Definition 2.8. Let A be a 3-Lie algebra and (V,ρ) a representation of A. A linear map $f : A \to V$ is called a **1-cocycle** on A with coefficients in (V,ρ) if it satisfies

$$f([x_1,x_2,x_3]) = \rho(x_1,x_2)f(x_3) + \rho(x_2,x_3)f(x_1) + \rho(x_3,x_1)f(x_2) \tag{2.5}$$

for $x_1,x_2,x_3 \in A$.

Example 2.9. Recall that a **derivation** D of a 3-Lie algebra A is defined to be a linear map $D : A \to A$ satisfying

$$D[x_1,x_2,x_3] = [D(x_1),x_2,x_3] + [x_1,D(x_2),x_3] + [x_1,x_2,D(x_3)]$$

for $x_1,x_2,x_3 \in A$.

So any derivation is a 1-cocycle of the 3-Lie algebra A with coefficients in the adjoint representation (A,ad) given by Eq. (2.2). In particular, there is a 1-cocycle (derivation) depending on two elements in the representation space A which can be regarded as a kind of "1-coboundary". Explicitly, for two fixed elements $u,v \in A$, the linear map $D : A \to A$ defined by

$$D(x) = \mathrm{ad}_{u,v}x = [u,v,x], \quad \forall x \in A,$$

is a 1-cocycle on A with coefficients in the adjoint representation (A,ad).

3. Local cocycle 3-Lie bialgebras

We study local cocycle 3-Lie bialgebras in this section. In Section 3.1, we introduce the notion of a local cocycle 3-Lie bialgebra. In Section 3.2, we study coboundary local cocycle 3-Lie bialgebras and introduce the notion of the 3-Lie classical Yang-Baxter equation. Given a solution of the 3-Lie classical Yang-Baxter equation, we can construct a local cocycle 3-Lie bialgebra (Theorem 3.10). In Section 3.3, we introduce the notion of an $\mathcal{O}$-operator on a 3-Lie algebra, which gives a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in some semi-direct product 3-Lie algebra (Theorem 3.21). Then we introduce the notion of a 3-pre-Lie algebra, and apply it to obtain a solution of the 3-Lie classical Yang-Baxter equation in some semi-direct product 3-Lie algebra associated to a 3-pre-Lie algebra.

3.1. Local cocycles and bialgebras

We first give the definition of a local cocycle 3-Lie bialgebra.

Definition 3.1. A **local cocycle** 3**-Lie bialgebra** is a pair (A, Δ), where A is a 3-Lie algebra and $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : A \to A \otimes A \otimes A$ is a linear map, such that $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a 3-Lie algebra structure on A^*, and the following conditions are satisfied:

- Δ_1 is a 1-cocycle with coefficients in the representation $(A \otimes A \otimes A, \mathrm{ad} \otimes 1 \otimes 1)$;
- Δ_2 is a 1-cocycle with coefficients in the representation $(A \otimes A \otimes A, 1 \otimes \mathrm{ad} \otimes 1)$;
- Δ_3 is a 1-cocycle with coefficients in the representation $(A \otimes A \otimes A, 1 \otimes 1 \otimes \mathrm{ad})$.

A local cocycle 3-Lie bialgebra is a natural generalization of a Lie bialgebra. Recall that a Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a linear map $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that Δ defines a Lie co-algebra and

$$\Delta([x, y]) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)\Delta(y) - (\mathrm{ad}_y \otimes 1 + 1 \otimes \mathrm{ad}_y)\Delta(x), \quad \forall\, x, y \in \mathfrak{g}. \tag{3.1}$$

The usual interpretation is that Δ is a 1-cocycle of $\mathfrak{g}$ with coefficients in the representation $\mathrm{ad} \otimes 1 + 1 \otimes \mathrm{ad}$ on the tensor space $\mathfrak{g} \otimes \mathfrak{g}$. Note that for a Lie algebra, due to the underlying enveloping algebra $U(\mathfrak{g})$ being a Hopf algebra, there is a natural representation on the tensor space. While this fact cannot be extended to 3-Lie algebras, the fact that both $(\mathfrak{g} \otimes \mathfrak{g}, \mathrm{ad} \otimes 1)$ and $(\mathfrak{g} \otimes \mathfrak{g}, 1 \otimes \mathrm{ad})$ are representations of $\mathfrak{g}$ can be extended, leading to the concept of a local cocycle 3-Lie bialgebra.

There is an alternative interpretation of a Lie bialgebra.

Lemma 3.2. *A pair $(\mathfrak{g}, \Delta)$ is a Lie bialgebra if $\Delta = \Delta_1 + \Delta_2$ such that for any $x, y \in \mathfrak{g}$,*

(a) *Δ_1 and Δ_2 are 1-cocycles of $\mathfrak{g}$ with coefficients in $\mathrm{ad} \otimes 1$ and $1 \otimes \mathrm{ad}$ respectively, i.e.*

$$\begin{aligned} \Delta_1[x, y] &= (\mathrm{ad}_x \otimes 1)\Delta_1(y) - (\mathrm{ad}_y \otimes 1)\Delta_1(x), \\ \Delta_2[x, y] &= (1 \otimes \mathrm{ad}_x)\Delta_2(y) - (1 \otimes \mathrm{ad}_y)\Delta_2(x); \end{aligned} \tag{3.2}$$

(b) *the following compatibility condition holds:*

$$(\mathrm{ad}_x \otimes 1)\Delta_2(y) + (1 \otimes \mathrm{ad}_x)\Delta_1(y) - (\mathrm{ad}_y \otimes 1)\Delta_2(x) - (1 \otimes \mathrm{ad}_y)\Delta_1(x) = 0. \tag{3.3}$$

Proof. It follows from the fact that Eqs. (3.2) and (3.3) imply Eq. (3.1). □

In the theory of Lie bialgebras, it is essential to consider the coboundary case, which is related to the theory of the classical Yang-Baxter equation. In the coboundary case, we have $\Delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)r$, for a fixed $r \in \mathfrak{g} \otimes \mathfrak{g}$ and any $x \in \mathfrak{g}$. In view of Lemma 3.2, it is natural to take

$$\Delta_1(x) = (\mathrm{ad}_x \otimes 1)(r), \quad \Delta_2(x) = (1 \otimes \mathrm{ad}_x)(r), \quad \forall x \in \mathfrak{g}. \tag{3.4}$$

Under this condition, Eq. (3.3) holds automatically by a straightforward computation. Thus, for the purpose of the classical Yang-Baxter equation, it is enough to only require Condition (a) in Lemma 3.2. The compatibility condition for local cocycle 3-Lie bialgebras is a natural extension of this condition. Note that Condition (a) alone cannot guarantee that $\mathfrak{g} \oplus \mathfrak{g}^*$ is a Lie algebra.

We end this subsection with an interpretation of the Condition (a) in Lemma 3.2 from an operadic point of view. See [31] for the background and notations. The compatibility condition of a Lie bialgebra can be expressed as

$$\Delta[x, y] = x_{(1)} \otimes [x_{(2)}, y] + [x_{(1)}, y] \otimes x_{(2)} + [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}] \tag{3.5}$$

in Sweedler's notation. Even though it does not fit in the frame of the generalized bialgebra of Loday in the sense that it gives a good triple in [31], it has a "unitarization" called **Lie^c-Lie-bialgebra** that does. Its compatibility condition is

$$\Delta[x, y] = 2(x \otimes y - y \otimes x) + \frac{1}{2}\big(x_{(1)} \otimes [x_{(2)}, y] + [x_{(1)}, y] \otimes x_{(2)} + [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}]\big).$$

In a similar way, a half of the compatibility condition in Eq. (3.5)

$$\Delta[x, y] = [x_{(1)}, y] \otimes x_{(2)} + [x, y_{(1)}] \otimes y_{(2)}$$

has the unitarization

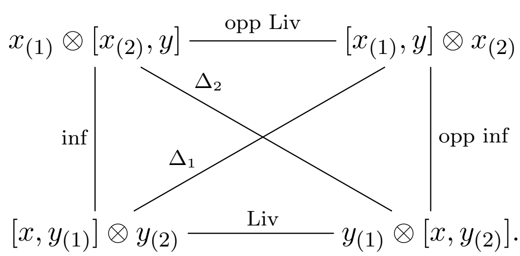
$$\Delta[x,y] = x \otimes y + [x_{(1)}, y] \otimes x_{(2)} + [x, y_{(1)}] \otimes y_{(2)}.$$

This is the compatibility condition of the co-variation of the **NAP^c-PreLie-**bialgebra of Livernet which is a good triple. Likewise, the other half of Eq. (3.5)

$$\Delta[x,y] = x_{(1)} \otimes [x_{(2)}, y] + y_{(1)} \otimes [x, y_{(2)}]$$

has its unitarization the compatibility condition of another good triple.

In fact, we can put the four terms of Eq. (3.5) in a diagram



Then the sum of the left (resp. right) two terms is the compatibility condition of the infinitesimal (resp. opposite infinitesimal) operads. The sum of the bottom (resp. top) two terms are compatibility of the Livernet (resp. opposite Livernet) operad. The sum of the diagonal (resp. opposite diagonal) two terms are for Δ_1 (resp. Δ_2) in our discussion in Eq. (3.3).

3.2. Coboundary local cocycle 3-Lie bialgebras and the 3-Lie CYBE

In this subsection, we study coboundary local cocycle 3-Lie bialgebras, i.e. construct a local cocycle 3-Lie bialgebra from an element $r \in A \otimes A$. First we give some preliminary notations.

Let A be a vector space. For any $T = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \otimes^n A$ and $1 \leq i < j \leq n$, define the (ij)-switching operator

$$\sigma_{ij}(T) = x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n.$$

For any $1 \leq p \neq q \leq n$, define an inclusion $\cdot_{pq} : \otimes^2 A \longrightarrow \otimes^n A$ by sending $r = \sum_i x_i \otimes y_i \in A \otimes A$ to

$$r_{pq} := \sum_i z_{i1} \otimes \cdots \otimes z_{in}, \qquad \text{where } z_{ij} = \begin{cases} x_i, & j = p, \\ y_i, & j = q, \\ 1, & j \neq p, q. \end{cases}$$

In other words, r_{pq} puts x_i at the p-th position, y_i at the q-th position and 1 elsewhere in an n-tensor, where 1 is the unit if A is a unital algebra, otherwise, 1 is a symbol playing a similar role of unit. For example, when $n = 4$, we have

$$r_{12} = \sum_i x_i \otimes y_i \otimes 1 \otimes 1 \in A^{\otimes 4}, \quad r_{21} = \sum_i y_i \otimes x_i \otimes 1 \otimes 1 \in A^{\otimes 4}.$$

When A is a 3-Lie algebra with the 3-Lie bracket $[\cdot,\cdot,\cdot]$, for any $r = \sum_i x_i \otimes y_i \in A \otimes A$, we define $[[r, r, r]] \in \otimes^4 A$ by

$$\begin{aligned} [[r,r,r]] &:= [r_{12}, r_{13}, r_{14}] + [r_{12}, r_{23}, r_{24}] + [r_{13}, r_{23}, r_{34}] + [r_{14}, r_{24}, r_{34}] \\ &= \sum_{i,j,k} \big([x_i, x_j, x_k] \otimes y_i \otimes y_j \otimes y_k + x_i \otimes [y_i, x_j, x_k] \otimes y_j \otimes y_k \\ &\quad + x_i \otimes x_j \otimes [y_i, y_j, x_k] \otimes y_k + x_i \otimes x_j \otimes x_k \otimes [y_i, y_j, y_k]\big). \end{aligned} \tag{3.6}$$

For any $r = \sum_i x_i \otimes y_i \in A \otimes A$, set

$$\begin{cases} \Delta_1(x) := \sum_{i,j} [x, x_i, x_j] \otimes y_j \otimes y_i; \\ \Delta_2(x) := \sum_{i,j} y_i \otimes [x, x_i, x_j] \otimes y_j; \\ \Delta_3(x) := \sum_{i,j} y_j \otimes y_i \otimes [x, x_i, x_j], \end{cases} \tag{3.7}$$

where $x \in A$.

Lemma 3.3. *With the above notations, we have*

(i) Δ_1 *is a* 1*-cocycle with coefficients in the representation* $(A \otimes A \otimes A, \mathrm{ad} \otimes 1 \otimes 1)$*;*

(ii) Δ_2 *is a* 1*-cocycle with coefficients in the representation* $(A \otimes A \otimes A, 1 \otimes \mathrm{ad} \otimes 1)$*;*

(iii) Δ_3 *is a* 1*-cocycle with coefficients in the representation* $(A \otimes A \otimes A, 1 \otimes 1 \otimes \mathrm{ad})$*.*

Proof. For all $x, y, z \in A$, we have

$$\begin{aligned}
&\Delta_1([x,y,z]) = \sum_{i,j}[[x,y,z],x_i,x_j] \otimes y_j \otimes y_i \\
&= \sum_{i,j}([[x,x_i,x_j],y,z] + [x,[y,x_i,x_j],z] + [x,y,[z,x_i,x_j]]) \otimes y_j \otimes y_i \\
&= (\mathrm{ad}_{y,z} \otimes 1 \otimes 1)\Delta_1(x) + (\mathrm{ad}_{z,x} \otimes 1 \otimes 1)\Delta_1(y) + (\mathrm{ad}_{x,y} \otimes 1 \otimes 1)\Delta_1(z).
\end{aligned}$$

Therefore, Δ_1 is 1-cocycle with coefficients in the representation $(A \otimes A \otimes A, \mathrm{ad} \otimes 1 \otimes 1)$. The other two statements can be proved similarly. □

Proposition 3.4. *Let A be a 3-Lie algebra and $r \in A \otimes A$. Let $\Delta = \Delta_1 + \Delta_2 + \Delta_3$, where $\Delta_1, \Delta_2, \Delta_3$ are induced by r as in Eq.* (3.7)*. Then $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a skew-symmetric operation.*

Proof. We only need to prove that for all $x \in A$,

$$\Delta(x) + \sigma_{12}\Delta(x) = 0, \ \Delta(x) + \sigma_{23}\Delta(x) = 0.$$

In fact, we have

$$\begin{aligned}
\sigma_{12}\Delta_1(x) &= \sum_{i,j} y_j \otimes [x,x_i,x_j] \otimes y_i = \sum_{i,j} y_i \otimes [x,x_j,x_i] \otimes y_j = -\Delta_2(x); \\
\sigma_{12}\Delta_2(x) &= \sum_{i,j}[x,x_i,x_j] \otimes y_i \otimes y_j = -\Delta_1(x); \\
\sigma_{12}\Delta_3(x) &= \sum_{i,j} y_i \otimes y_j \otimes [x,x_i,x_j] = -\Delta_3(x).
\end{aligned}$$

Hence $\sigma_{12}\Delta(x) = -\Delta(x)$. Similarly, we have $\sigma_{23}\Delta(x) = -\Delta(x)$. This completes the proof. □

Remark 3.5. In fact, the above Δ_i $(i = 1, 2, 3)$ can be regarded as a kind of "1-coboundaries" that generalizes the one in Example 2.9. There, for $x \in A$, the derivation

$$D(x) := \mathrm{ad}_{u,v}x = [u,v,x], \quad \forall u, v \in A,$$

defines a 1-cocycle on A. Analogously, for $u, v, a, b \in A$, the linear map

$$\Delta_1' : A \to \otimes^3 A, \quad x \mapsto \mathrm{ad}_{u,v}x \otimes a \otimes b = [u,v,x] \otimes a \otimes b, \quad \forall x \in A,$$

is a 1-cocycle with coefficients in $(A\otimes A\otimes A, \mathrm{ad}\otimes 1\otimes 1)$. More generally, for four families of elements $u_i, v_i, a_i, b_i \in A$, we define the 1-cocycle with coefficients in $(A\otimes A\otimes A, \mathrm{ad}\otimes 1\otimes 1)$:

$$\Delta_1'(x) = \sum_i \mathrm{ad}_{u_i,v_i}x \otimes a_i \otimes b_i = \sum_i [u_i, v_i, x]\otimes a_i\otimes b_i, \quad \forall x\in A.$$

Similarly, for $u_j', v_j', a_j', b_j', u_k'', v_k'', a_k'', b_k'' \in A$, the linear maps

$$\Delta_2'(x) = \sum_j a_j' \otimes \mathrm{ad}_{u_j',v_j'}x \otimes b_j', \;\; \Delta_3'(x) = \sum_k a_k''\otimes b_k''\otimes \mathrm{ad}_{u_k'',v_k''}x, \;\; \forall x\in A$$

are 1-cocycles with coefficients in $(A\otimes A\otimes A, 1\otimes \mathrm{ad}\otimes 1)$ and $(A\otimes A\otimes A, 1\otimes 1\otimes \mathrm{ad})$ respectively.

Moreover, set

$$\Delta = \Delta_1' + \Delta_2' + \Delta_3'.$$

From the proof of Proposition 3.4, in order for Δ^* to define a "natural" skew-symmetric operation, there should be some constraint conditions for the choice of the above elements $u_i, v_i, a_i, b_i, u_j', v_j', a_j', b_j', u_k'', v_k'', a_k'', b_k'' \in A$. Here, "natural" means that there should not be any additional condition for the skew-symmetry of Δ^*. In particular, by a straightforward observation, it seems reasonable to assume that the following conditions should be satisfied: (the following sets are multi-sets, in the sense that elements can repeat in each of them)

(a) The sets $\{u_j', v_j', a_j', b_j'\}$, $\{u_k'', v_k'', a_k'', b_k''\}$ and $\{u_i, v_i, a_i, b_i\}$ coincide.

(b) The sets $\{u_i\}$ and $\{v_i\}$ coincide; while the sets $\{a_i\}$ and $\{b_i\}$ coincide.

Note that the above two conditions force $\Delta_1', \Delta_2', \Delta_3'$ to depend on only two family of elements $\{u_i\}$ and $\{a_i\}$. With some more constraints on the indices involving $u_i = x_i, a_i = y_i$, the Δ_i $(i = 1, 2, 3)$ given in Eq. (3.7) are what we need in the above sense.

In the sequel, we will apply the notation $r = \sum_i x_i \otimes y_i$ to represent the two families of elements $x_i, y_i \in A$ when they are used to define the cocycles Δ_i $(i = 1, 2, 3)$. One of the advantages of using the notation r here is that Δ_i can be expressed more concisely and conventionally as

$$\begin{aligned} \Delta_1(x) &= \sum_{i,j}[x_i, x_j, x]\otimes y_j\otimes y_i =: x.[r_{13}, r_{12}], \\ \Delta_2(x) &= x.[r_{21}, r_{23}], \quad \Delta_3(x) = x.[r_{32}, r_{31}], \quad \forall x\in A. \end{aligned}$$

The following result is straightforward to verify.

Lemma 3.6. *Let V be a vector space and $\Delta : V \to V \otimes V \otimes V$ a linear map. Then $\Delta^* : V^* \otimes V^* \otimes V^* \to V^*$ defines a 3-Lie algebra structure on V^* if and only if Δ^* is a skew-symmetric operation and Δ satisfies*

$$(\Delta \otimes 1 \otimes 1)\Delta(x) + \sigma_{23}\sigma_{12}((1 \otimes \Delta \otimes 1)\Delta(x)) + \sigma_{13}\sigma_{24}((1 \otimes 1 \otimes \Delta)\Delta(x)) - (1 \otimes 1 \otimes \Delta)\Delta(x) = 0, \tag{3.8}$$

for $x \in A$.

With this preparation, we can begin our discussion on the 3-Lie classical Yang-Baxter equation. We introduce a notation that is needed in the next theorem. For $a \in A$ and $1 \le i \le 5$, define the linear map $\otimes_i a : \otimes^4 A \longrightarrow \otimes^5 A$ by inserting a at the i-th position. For example, for any $t = t_1 \otimes t_2 \otimes t_3 \otimes t_4$, we have $t \otimes_2 a = t_1 \otimes a \otimes t_2 \otimes t_3 \otimes t_4$.

Theorem 3.7. *Let A be a 3-Lie algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$. Define the linear map $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : A \to A \otimes A \otimes A$, where $\Delta_1, \Delta_2, \Delta_3$ are given by Eq. (3.7). Then $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a 3-Lie algebra structure if and only if for any $x \in A$, the following equation holds:*

$$\begin{aligned}
&\sum_i (\mathrm{ad}_{x_i,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)([[r,r,r]]_1 \otimes_2 y_i)\\
&+ \sum_i (1 \otimes \mathrm{ad}_{x,x_i} \otimes 1 \otimes 1 \otimes 1)([[r,r,r]]_1 \otimes_1 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes \mathrm{ad}_{x,x_i} \otimes 1 \otimes 1)([[r,r,r]]_2 \otimes_5 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes \mathrm{ad}_{x_i,x} \otimes 1 \otimes 1)([[r,r,r]]_2 \otimes_4 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x,x_i} \otimes 1)([[r,r,r]]_2 \otimes_3 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x_i,x} \otimes 1)([[r,r,r]]_3 \otimes_5 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x,x_i})([[r,r,r]]_3 \otimes_4 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x_i,x})([[r,r,r]]_3 \otimes_3 y_i)\\
&= 0,
\end{aligned}$$

where

$$
\begin{aligned}
[[r,r,r]]_1 &:= [r_{12},r_{13},r_{14}]+[r_{12},r_{23},r_{24}]-[r_{13},r_{32},r_{34}]+[r_{14},r_{42},r_{43}];\\
[[r,r,r]]_2 &:= [r_{12},r_{31},r_{14}]-[r_{21},r_{32},r_{24}]-[r_{31},r_{32},r_{34}]-[r_{41},r_{42},r_{34}];\\
[[r,r,r]]_3 &:= -[r_{12},r_{13},r_{41}]+[r_{21},r_{23},r_{42}]-[r_{31},r_{32},r_{43}]-[r_{41},r_{42},r_{43}].
\end{aligned}
$$

Proof. By Proposition 3.4, Δ^* is skew-symmetric. Thus we only need to give the condition for which Eq. (3.8) holds. Since each Δ contains three terms, there are 36 terms in Eq. (3.8). Let $G_i, 1 \le i \le 5$, denote the sum of these terms where x is at the i-th position in the 5-tensors. We then obtain

$$G_1+G_2+G_3+G_4+G_5=0.$$

There are 6 terms in G_1:

$$G_1 = G_{11}+G_{12}+G_{13}+G_{14}+G_{15}+G_{16},$$

where

$$
\begin{aligned}
G_{11} &= \sum_{ijkl}[[x,x_i,x_j],x_k,x_l]\otimes y_l\otimes y_k\otimes y_j\otimes y_i,\\
G_{12} &= \sum_{ijkl}[[x,x_i,x_j],x_k,x_l]\otimes y_l\otimes y_i\otimes y_k\otimes y_j,\\
G_{13} &= \sum_{ijkl}[[x,x_i,x_j],x_k,x_l]\otimes y_l\otimes y_j\otimes y_i\otimes y_k,\\
G_{14} &= -\sum_{ijkl}[x,x_i,x_j]\otimes y_j\otimes [y_i,x_k,x_l]\otimes y_l\otimes y_k,\\
G_{15} &= -\sum_{ijkl}[x,x_i,x_j]\otimes y_j\otimes y_k\otimes [y_i,x_k,x_l]\otimes y_l,\\
G_{16} &= -\sum_{ijkl}[x,x_i,x_j]\otimes y_j\otimes y_l\otimes y_k\otimes [y_i,x_k,x_l].
\end{aligned}
$$

By Condition (a) in Proposition 2.2, we have

$$
\begin{aligned}
G_{11}+G_{12}+G_{13} &= \sum_{ijkl}[[x_i,x_j,x_k],x,x_l]\otimes y_l\otimes y_k\otimes y_j\otimes y_i\\
&= \sum_{ijkl}(\mathrm{ad}_{x_l,x}\otimes 1\otimes 1\otimes 1\otimes 1)[x_k,x_j,x_i]\otimes y_l\otimes y_k\otimes y_j\otimes y_i\\
&= \sum_{l}(\mathrm{ad}_{x_l,x}\otimes 1\otimes 1\otimes 1\otimes 1)[r_{12},r_{13},r_{14}]\otimes_2 y_l.
\end{aligned}
$$

Furthermore, we have

$$\begin{aligned} G_{14} &= \sum_{ijkl} (\mathrm{ad}_{x_j,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1) x_i \otimes y_j \otimes [y_i, x_l, x_k] \otimes y_l \otimes y_k \\ &= \sum_{j} (\mathrm{ad}_{x_j,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)[r_{12}, r_{23}, r_{24}] \otimes_2 y_j, \end{aligned}$$

and similarly,

$$\begin{aligned} G_{15} &= -\sum_{j} (\mathrm{ad}_{x_j,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)[r_{13}, r_{32}, r_{34}] \otimes_2 y_j, \\ G_{16} &= \sum_{j} (\mathrm{ad}_{x_j,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)[r_{14}, r_{42}, r_{43}] \otimes_2 y_j. \end{aligned}$$

Therefore, we obtain

$$G_1 = \sum_{i} (\mathrm{ad}_{x_i,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)([[r, r, r]]_1 \otimes_2 y_i).$$

In a similar manner, we have

$$G_2 = \sum_{i} (1 \otimes \mathrm{ad}_{x,x_i} \otimes 1 \otimes 1 \otimes 1)([[r, r, r]]_1 \otimes_1 y_i).$$

There are 8 terms in G_3:

$$G_3 = G_{31} + G_{32} + G_{33} + G_{34} + G_{35} + G_{36} + G_{37} + G_{38},$$

where

$$\begin{aligned} G_{31} &= \sum_{ijkl} y_l \otimes y_k \otimes [[x, x_i, x_j], x_k, x_l] \otimes y_j \otimes y_i, \\ G_{32} &= -\sum_{ijkl} y_j \otimes y_i \otimes [[x, x_i, x_j], x_k, x_l] \otimes y_l \otimes y_k, \end{aligned}$$

$$
\begin{aligned}
G_{33} &= \sum_{ijkl} [y_j, x_k, x_l] \otimes y_l \otimes [x, x_i, x_j] \otimes y_k \otimes y_i, \\
G_{34} &= \sum_{ijkl} y_k \otimes [y_j, x_k, x_l] \otimes [x, x_i, x_j] \otimes y_l \otimes y_i, \\
G_{35} &= \sum_{ijkl} y_l \otimes y_k \otimes [x, x_i, x_j] \otimes [y_j, x_k, x_l] \otimes y_i, \\
G_{36} &= \sum_{ijkl} [y_i, x_k, x_l] \otimes y_l \otimes [x, x_i, x_j] \otimes y_j \otimes y_k, \\
G_{37} &= \sum_{ijkl} y_k \otimes [y_i, x_k, x_l] \otimes [x, x_i, x_j] \otimes y_j \otimes y_l, \\
G_{38} &= \sum_{ijkl} y_l \otimes y_k \otimes [x, x_i, x_j] \otimes y_j \otimes [y_i, x_k, x_l].
\end{aligned}
$$

We have

$$
\begin{aligned}
G_{31} + G_{32} &= \sum_{ijkl} y_l \otimes y_k \otimes ([[x, [x_i, x_k, x_l], x_j] + [x, x_i, [x_j, x_k, x_l]]) \otimes y_j \otimes y_i \\
&= -\sum_{ijkl} (1 \otimes 1 \otimes \mathrm{ad}_{x_j, x} \otimes 1 \otimes 1) y_l \otimes y_k \otimes [x_l, x_k, x_i] \otimes y_j \otimes y_i \\
&\quad - \sum_{ijkl} (1 \otimes 1 \otimes \mathrm{ad}_{x, x_i} \otimes 1 \otimes 1) y_l \otimes y_k \otimes [x_l, x_k, x_j] \otimes y_j \otimes y_i \\
&= -\sum_{j} (1 \otimes 1 \otimes \mathrm{ad}_{x_j, x} \otimes 1 \otimes 1) [r_{31}, r_{32}, r_{34}] \otimes_4 y_j \\
&\quad - \sum_{i} (1 \otimes 1 \otimes \mathrm{ad}_{x, x_i} \otimes 1 \otimes 1) [r_{31}, r_{32}, r_{34}] \otimes_5 y_i.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
G_{33} + G_{36} &= \sum_{ijkl} (1 \otimes 1 \otimes \mathrm{ad}_{x, x_i} \otimes 1 \otimes 1) [x_l, y_j, x_k] \otimes y_l \otimes x_j \otimes y_k \otimes y_i \\
&\quad + \sum_{ijkl} (1 \otimes 1 \otimes \mathrm{ad}_{x_j, x} \otimes 1 \otimes 1) [x_l, y_i, x_k] \otimes y_l \otimes x_i \otimes y_j \otimes y_k \\
&= \sum_{i} (1 \otimes 1 \otimes \mathrm{ad}_{x, x_i} \otimes 1 \otimes 1) [r_{12}, r_{31}, r_{14}] \otimes_5 y_i \\
&\quad + \sum_{j} (1 \otimes 1 \otimes \mathrm{ad}_{x_j, x} \otimes 1 \otimes 1) [r_{12}, r_{31}, r_{14}] \otimes_4 y_j,
\end{aligned}
$$

and similarly,

$$\begin{aligned} G_{34}+G_{37} &= -\sum_i (1\otimes 1\otimes \mathrm{ad}_{x,x_i}\otimes 1\otimes 1)[r_{21},r_{32},r_{24}]\otimes_5 y_i \\ &\quad -\sum_i (1\otimes 1\otimes \mathrm{ad}_{x_j,x}\otimes 1\otimes 1)[r_{21},r_{32},r_{24}]\otimes_4 y_j, \\ G_{35}+G_{38} &= -\sum_i (1\otimes 1\otimes \mathrm{ad}_{x,x_i}\otimes 1\otimes 1)[r_{41},r_{42},r_{34}]\otimes_5 y_i \\ &\quad -\sum_j (1\otimes 1\otimes \mathrm{ad}_{x_j,x}\otimes 1\otimes 1)[r_{41},r_{42},r_{34}]\otimes_4 y_j. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} G_3 &= \sum_i (1\otimes 1\otimes \mathrm{ad}_{x,x_i}\otimes 1\otimes 1)([[r,r,r]]_2\otimes_5 y_i) \\ &\quad +\sum_i (1\otimes 1\otimes \mathrm{ad}_{x_i,x}\otimes 1\otimes 1)([[r,r,r]]_2\otimes_4 y_i). \end{aligned}$$

We similarly obtain

$$\begin{aligned} G_4 &= \sum_i \Big(1\otimes 1\otimes 1\otimes \mathrm{ad}_{x,x_i}\otimes 1)([[r,r,r]]_2\otimes_3 y_i) \\ &\qquad + (1\otimes 1\otimes 1\otimes \mathrm{ad}_{x_i,x}\otimes 1)([[r,r,r]]_3\otimes_5 y_i)\Big), \\ G_5 &= \sum_i \Big((1\otimes 1\otimes 1\otimes 1\otimes \mathrm{ad}_{x,x_i})([[r,r,r]]_3\otimes_4 y_i) \\ &\qquad + (1\otimes 1\otimes 1\otimes 1\otimes \mathrm{ad}_{x_i,x})([[r,r,r]]_3\otimes_3 y_i)\Big). \end{aligned}$$

This completes the proof. □

A direct checking gives

Lemma 3.8. *With the notations above, if r is skew-symmetric, then*

$$[[r,r,r]]_1 = [[r,r,r]],\quad [[r,r,r]]_2 = -[[r,r,r]],\quad [[r,r,r]]_3 = [[r,r,r]].$$

We then have

Corollary 3.9. *Let A be a 3-Lie algebra and $r=\sum_i x_i\otimes y_i\in A\otimes A$ skew-symmetric. Define the linear map $\Delta=\Delta_1+\Delta_2+\Delta_3 : A\to A\otimes A\otimes A$, where $\Delta_1,\Delta_2,\Delta_3$ are induced by r as in Eq. (3.7). Then $\Delta^* : A^*\otimes A^*\otimes$*

$A^* \to A^*$ defines a 3-Lie algebra structure if and only if for any $x \in A$, the following equation holds:

$$\begin{aligned}
&\sum_i (\mathrm{ad}_{x_i,x} \otimes 1 \otimes 1 \otimes 1 \otimes 1)([[r,r,r]] \otimes_2 y_i)\\
&+ \sum_i (1 \otimes \mathrm{ad}_{x,x_i} \otimes 1 \otimes 1 \otimes 1)([[r,r,r]] \otimes_1 y_i)\\
&- \sum_i (1 \otimes 1 \otimes \mathrm{ad}_{x,x_i} \otimes 1 \otimes 1)([[r,r,r]] \otimes_5 y_i)\\
&- \sum_i (1 \otimes 1 \otimes \mathrm{ad}_{x_i,x} \otimes 1 \otimes 1)([[r,r,r]] \otimes_4 y_i)\\
&- \sum_i (1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x,x_i} \otimes 1)([[r,r,r]] \otimes_3 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x_i,x} \otimes 1)([[r,r,r]] \otimes_5 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x,x_i})([[r,r,r]] \otimes_4 y_i)\\
&+ \sum_i (1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathrm{ad}_{x_i,x})([[r,r,r]] \otimes_3 y_i)\\
&= 0.
\end{aligned}$$

In particular, if $[[r,r,r]] = 0$, then $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a 3-Lie algebra structure.

Summarizing the above discussions, we obtain

Theorem 3.10. *Let A be a 3-Lie algebra and $r \in A \otimes A$ skew-symmetric. If*

$$[[r,r,r]] = 0,$$

then Δ^* defines a 3-Lie algebra structure on A^*, where $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : A \to A \otimes A \otimes A$, in which $\Delta_1, \Delta_2, \Delta_3$ are induced by r as in Eq. (3.7). Furthermore, (A, Δ) is a local cocycle 3-Lie bialgebra.

Theorem 3.10 can be regarded as a 3-Lie algebra analogue of the fact that a skew-symmetric solution of the classical Yang-Baxter equation gives a Lie bialgebra, leading us to give the following definition.

Definition 3.11. Let A be a 3-Lie algebra and $r \in A \otimes A$. The equation

$$[[r, r, r]] = 0$$

is called the **3-Lie classical Yang-Baxter equation (3-Lie CYBE)**.

This can be regarded as a natural extension of the classical Yang-Baxter equation

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

to the context of 3-Lie algebras.

Example 3.12. Let A be the (unique) non-trivial 3-dimensional complex 3-Lie algebra whose non-zero product with respect to a basis $\{e_1, e_2, e_3\}$ is given by [10]

$$[e_1, e_2, e_3] = e_1.$$

If $r \in A \otimes A$ is skew-symmetric, then r is a solution of the 3-Lie CYBE in A. Moreover, for $r = \sum_{i<j}^{3} r_{ij}(e_i \otimes e_j - e_j \otimes e_i)$, with the notation introduced before Theorem 3.7, the corresponding local cocycle 3-Lie bialgebra is given by

$$\Delta_i(e_1) = (-1)^i r_{23} r \otimes_i e_1, \quad \Delta_i(e_2) = (-1)^{i+1} r_{13} r \otimes_i e_1,$$
$$\Delta_i(e_3) = (-1)^i r_{12} r \otimes_i e_1, \quad i = 1, 2, 3,$$

and the comultiplication $\Delta : A \longrightarrow \wedge^3 A$ is given by

$$\Delta(e_1) = -r_{23}^2 e_1 \wedge e_2 \wedge e_3, \quad \Delta(e_2) = r_{13} r_{23} e_1 \wedge e_2 \wedge e_3,$$
$$\Delta(e_3) = -r_{12} r_{23} e_1 \wedge e_2 \wedge e_3,$$

where $e_1 \wedge e_2 \wedge e_3 = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ and S_3 is the permutation group on $\{1, 2, 3\}$. In particular, when $r_{23} \neq 0$, we get a local cocycle 3-Lie bialgebra whose coproduct is not zero.

Let $r \in A \otimes A$. Then r induces a linear map $A^* \to A$ that we still denote by r:

$$\langle r(\xi), \eta \rangle = \langle r, \xi \otimes \eta \rangle, \quad \forall \xi, \eta \in A^*. \tag{3.9}$$

Furthermore, we denote the ternary operation $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ by $[\cdot, \cdot, \cdot]^*$.

Proposition 3.13. *Let A be a 3-Lie algebra and $r \in A \otimes A$. Suppose that r is skew-symmetric and $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : A \to A \otimes A \otimes A$, in which Δ_1, Δ_2, Δ_3 are induced by r as in Eq.* (3.7)*. Then we have*

$$[\xi, \eta, \gamma]^* = \mathrm{ad}^*_{r(\xi),r(\eta)}\gamma + \mathrm{ad}^*_{r(\eta),r(\gamma)}\xi + \mathrm{ad}^*_{r(\gamma),r(\xi)}\eta, \quad \forall \xi, \eta, \gamma \in A^*. \tag{3.10}$$

Furthermore, we have

$$[r(\xi), r(\eta), r(\gamma)] - r([\xi, \eta, \gamma]^*) = [[r, r, r]](\xi, \eta, \gamma), \quad \forall \xi, \eta, \gamma \in A^*. \tag{3.11}$$

Proof. Let $x \in A, \xi, \eta, \gamma \in A^*$. For the first conclusion, we only need to prove

$$\langle \Delta(x), \xi \otimes \eta \otimes \gamma \rangle = \langle x, [\xi, \eta, \gamma]^* \rangle.$$

Let $r = \sum_i x_i \otimes y_i$. Since r is skew-symmetric, we have

$$\begin{aligned}
\langle x, \mathrm{ad}^*_{r(\xi),r(\eta)}\gamma \rangle &= \langle -[r(\xi), r(\eta), x], \gamma \rangle = -\langle r, \eta \otimes \mathrm{ad}^*_{r(\xi),x}\gamma \rangle \\
&= \sum_i \langle y_i, \eta \rangle \langle x_i, \mathrm{ad}^*_{r(\xi),x}\gamma \rangle = \sum_i \langle y_i, \eta \rangle \langle r, \xi \otimes \mathrm{ad}^*_{x,x_i}\gamma \rangle \\
&= \sum_{i,j} \langle y_i, \eta \rangle \langle y_j, \xi \rangle \langle [x, x_i, x_j], \gamma \rangle \\
&= \Big\langle \sum_{ij} y_j \otimes y_i \otimes [x, x_i, x_j], \xi \otimes \eta \otimes \gamma \Big\rangle \\
&= \langle \Delta_3(x), \xi \otimes \eta \otimes \gamma \rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\langle x, \mathrm{ad}^*_{r(\eta),r(\gamma)}\xi \rangle &= \langle \Delta_1(x), \xi \otimes \eta \otimes \gamma \rangle, \\
\langle x, \mathrm{ad}^*_{r(\gamma),r(\xi)}\eta \rangle &= \langle \Delta_2(x), \xi \otimes \eta \otimes \gamma \rangle.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\langle \Delta(x), \xi \otimes \eta \otimes \gamma \rangle &= \langle \Delta_1(x) + \Delta_2(x) + \Delta_3(x), \xi \otimes \eta \otimes \gamma \rangle \\
&= \langle x, \mathrm{ad}^*_{r(\eta),r(\gamma)}\xi \rangle + \langle x, \mathrm{ad}^*_{r(\gamma),r(\xi)}\eta \rangle + \langle x, \mathrm{ad}^*_{r(\xi),r(\eta)}\gamma \rangle \\
&= \langle x, [\xi, \eta, \gamma]^* \rangle.
\end{aligned}$$

This finishes the proof of Eq. (3.10).

Applying the left hand side of Eq. (3.11) to $\kappa \in A^*$ gives

$$\begin{aligned}
&\langle [r(\xi), r(\eta), r(\gamma)] - r([\xi,\eta,\gamma]^*), \kappa\rangle \\
=\ &\langle \kappa, [r(\xi), r(\eta), r(\gamma)]\rangle - \langle \gamma, [r(\xi), r(\eta), r(\kappa)]\rangle \\
&- \langle \xi, [r(\eta), r(\gamma), r(\kappa)]\rangle - \langle \eta, [r(\gamma), r(\xi), r(\kappa)]\rangle.
\end{aligned}$$

Applying the right hand side of Eq. (3.11) to $\kappa \in A^*$ gives

$$\begin{aligned}
&[[r,r,r]](\xi,\eta,\gamma,\kappa) \\
=&\sum_{i,j,k}\Big([x_i,x_j,x_k]\otimes y_i\otimes y_j\otimes y_k(\xi,\eta,\gamma,\kappa) \\
&\qquad + x_i\otimes [y_i,x_j,x_k]\otimes y_j\otimes y_k(\xi,\eta,\gamma,\kappa) \\
&\qquad + x_i\otimes x_j\otimes [y_i,y_j,x_k]\otimes y_k(\xi,\eta,\gamma,\kappa) \\
&\qquad + x_i\otimes x_j\otimes x_k\otimes [y_i,y_j,y_k](\xi,\eta,\gamma,\kappa)\Big) \\
=&\sum_{i,j,k}\Big(\langle \xi, [x_i,x_j,x_k]\rangle\langle \eta, y_i\rangle\langle \gamma, y_j\rangle\langle \kappa, y_k\rangle \\
&\qquad + \langle \eta, [y_i,x_j,x_k]\rangle\langle \xi, x_i\rangle\langle \gamma, y_j\rangle\langle \kappa, y_k\rangle \\
&\qquad + \langle \gamma, [y_i,y_j,x_k]\rangle\langle \xi, x_i\rangle\langle \eta, x_j\rangle\langle \kappa, y_k\rangle \\
&\qquad + \langle \kappa, [y_i,y_j,y_k]\rangle\langle \xi, x_i\rangle\langle \eta, x_j\rangle\langle \gamma, x_k\rangle\Big) \\
=&-\langle \xi, [r(\eta), r(\gamma), r(\kappa)]\rangle - \langle \eta, [r(\gamma), r(\xi), r(\kappa)]\rangle \\
&- \langle \gamma, [r(\xi), r(\eta), r(\kappa)]\rangle + \langle \kappa, [r(\xi), r(\eta), r(\gamma)]\rangle.
\end{aligned}$$

Therefore, Eq. (3.11) holds. This completes the proof. □

As a direct consequence, we obtain

Corollary 3.14. *Suppose that r is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation. Then the linear map r defined in Eq. (3.9) is a 3-Lie algebra morphism from $(A^*, [\cdot,\cdot,\cdot]^*)$ to $(A, [\cdot,\cdot,\cdot])$.*

We further give the following interpretation of the invertible skew-symmetric solutions of the 3-Lie classical Yang-Baxter equation which is parallel to a similar result for the classical Yang-Baxter equation in a Lie algebra given by Drinfeld [17].

Proposition 3.15. *Let A be a 3-Lie algebra and $r \in A\otimes A$. Suppose that r is skew-symmetric and nondegenerate. Then r is a solution of the 3-Lie classical Yang-Baxter equation if and only if the nondegenerate skew-symmetric* *bilinear form B on A defined by $B(x,y) := \langle r^{-1}(x), y\rangle$ for any $x, y \in A$ satisfies*

$$B([x,y,z],w) - B([x,y,w],z) + B([x,z,w],y) - B([y,z,w],x) = 0, \quad \forall x,y,z,w \in A. \tag{3.12}$$

Proof. For any $x, y, z, w \in A$, since r is nondegenerate, there exist ξ, η, γ, $\kappa \in A^*$ such that $r(\xi) = x, r(\eta) = y, r(\gamma) = z, r(\kappa) = w$. By Eq. (3.11), if $[[r,r,r]] = 0$, we have

$$\begin{aligned} B([x,y,z],w) &= \langle r^{-1}[r(\xi), r(\eta), r(\gamma)], w\rangle \\ &= \langle \mathrm{ad}^*_{r(\xi),r(\eta)}\gamma + \mathrm{ad}^*_{r(\eta),r(\gamma)}\xi + \mathrm{ad}^*_{r(\gamma),r(\xi)}\eta, w\rangle \\ &= \langle -\gamma, [x,y,w]\rangle - \langle \xi, [y,z,w]\rangle - \langle \eta, [z,x,w]\rangle \\ &= -B(z,[x,y,w]) - B(x,[y,z,w]) - B(y,[z,x,w]). \end{aligned}$$

Hence the conclusion follows. □

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Recall that $\omega \in \wedge^2\mathfrak{g}^*$ is a 2-cocycle on $\mathfrak{g}$ if

$$\omega([x,y]_{\mathfrak{g}}, z) + \omega([z,x]_{\mathfrak{g}}, y) + \omega([y,z]_{\mathfrak{g}}, x) = 0, \quad \forall\, x,y,z \in \mathfrak{g}.$$

A symplectic Lie algebra is defined to be a Lie algebra equipped with a nondegenerate 2-cocycle. There is a one-to-one correspondence between invertible skew-symmetric solutions of the classical Yang-Baxter equation and symplectic Lie algebras. This motivates us to give the following definition.

Definition 3.16. A **symplectic 3-Lie algebra** is a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ together with a nondegenerate skew-symmetric bilinear form $B \in \wedge^2 A^*$ such that Eq. (3.12) holds. We denote a symplectic 3-Lie algebra by $(A, [\cdot,\cdot,\cdot], B)$.

The close relationship between symplectic Lie algebras and pre-Lie algebras will be generalized to the 3-Lie algebra context in Proposition 3.30.

Remark 3.17. In [13], the author introduced the notion of a symplectic 3-algebra in the study of superconformal Chern-Simons-matter (CSM) theory, as a 3-algebra A (it is not a 3-Lie algebra since the structure constants are symmetric in the first two indices) with a nondegenerate skew-symmetric bilinear form $B \in \wedge^2 A^*$ satisfying

$$B([x,y,z],w) - B([x,y,w],z) = 0, \quad \forall x,y,z,w \in A. \tag{3.13}$$

It is obvious that a skew-symmetric bilinear form satisfying Eq. (3.13) automatically satisfies Eq. (3.12).

3.3. $\mathcal{O}$-operators, 3-pre-Lie algebras and solutions of the 3-Lie CYBE

Pre-Lie algebras and $\mathcal{O}$-operators are useful tools for the construction of solutions of the classical Yang-Baxter equation. For further details, see [11] for pre-Lie algebras and [7] for the relationship between pre-Lie algebras, $\mathcal{O}$-operators and CYBE. In this subsection, we first introduce the notion of an $\mathcal{O}$-operator in the 3-Lie algebra context, which could give solutions of the 3-Lie classical Yang-Baxter equation. Then we introduce the notion of a 3-pre-Lie algebra which is closely related to $\mathcal{O}$-operators. We show that there is a natural 3-pre-Lie algebra from the underlying structure of a symplectic 3-Lie algebra. In particular, there is a construction of solutions of the 3-Lie classical Yang-Baxter equation in some special 3-Lie algebras obtained from 3-pre-Lie algebras.

Definition 3.18. Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and (V,ρ) a representation. A linear operator $T: V \to A$ is called an **$\mathcal{O}$-operator** associated to (V,ρ) if T satisfies

$$[Tu,Tv,Tw] = T\big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v\big), \quad \forall u,v,w \in V. \tag{3.14}$$

Example 3.19. Let A be a 3-Lie algebra and $r \in A \otimes A$. Suppose that r is skew-symmetric. Then r is a solution of the 3-Lie classical Yang-Baxter equation if and only if r is an $\mathcal{O}$-operator of A associated to the coadjoint representation (A^*, ad^*).

Example 3.20. Let A be a 3-Lie algebra. An $\mathcal{O}$-operator of A associated to the adjoint representation (A, ad) is called a **Rota-Baxter operator of weight zero**. See [9] for more details.

Let $T : V \to A$ be a linear map and $\overline{T} \in V^* \otimes A$ the corresponding tensor, i.e.

$$\overline{T}(v,\xi) = \langle \xi, Tv\rangle, \quad \forall\, v \in V,\ \xi \in A^*.$$

The following result is the 3-Lie algebra analogue of the relationship between $\mathcal{O}$-operators and the classical Yang-Baxter equation on Lie algebras [7].

Theorem 3.21. *With the above notations, T is an $\mathcal{O}$-operator if and only if*

$$r = \overline{T} - \sigma_{12}(\overline{T}) \tag{3.15}$$

is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A \ltimes_{\rho^*} V^*$.

Proof. Let $\{v_1, \cdots, v_n\}$ be a basis of V and $\{v_1^*, \cdots, v_n^*\}$ the dual basis. Then we have

$$\overline{T} = \sum_i v_i^* \otimes Tv_i \in V^* \otimes A.$$

Therefore, we derive

$$\begin{aligned}
[r_{12}, r_{13}, r_{14}] = \sum_{ijk} \Big(&- [Tv_i, Tv_j, Tv_k] \otimes v_i^* \otimes v_j^* \otimes v_k^* \\
&+ [Tv_i, v_j^*, Tv_k] \otimes v_i^* \otimes Tv_j \otimes v_k^* \\
&+ [Tv_i, Tv_j, v_k^*] \otimes v_i^* \otimes v_j^* \otimes Tv_k \\
&+ [v_i^*, Tv_j, Tv_k] \otimes Tv_i \otimes v_j^* \otimes v_k^* \Big); \\
[r_{12}, r_{23}, r_{24}] = \sum_{ijk} \Big(&v_i^* \otimes [Tv_i, Tv_j, Tv_k] \otimes v_j^* \otimes v_k^* \\
&- Tv_i \otimes [v_i^*, Tv_j, Tv_k] \otimes v_j^* \otimes v_k^* \\
&- v_i^* \otimes [Tv_i, Tv_j, v_k^*] \otimes v_j^* \otimes v_k^* \\
&- v_i^* \otimes [Tv_i, v_j^*, Tv_k] \otimes Tv_j \otimes v_k^* \Big); \\
[r_{13}, r_{23}, r_{34}] = \sum_{ijk} \Big(&- v_i^* \otimes v_j^* \otimes [Tv_i, Tv_j, Tv_k] \otimes v_k^* \\
&+ Tv_i \otimes v_j^* \otimes [v_i^*, Tv_j, Tv_k] \otimes v_k^* \\
&+ v_i^* \otimes Tv_j \otimes [Tv_i, v_j^*, Tv_k] \otimes v_k^* \\
&+ v_i^* \otimes v_j^* \otimes [Tv_i, Tv_j, v_k^*] \otimes Tv_k \Big);
\end{aligned}$$

$$
\begin{aligned}
[r_{14}, r_{24}, r_{34}] = \sum_{ijk} \Big(& - Tv_i \otimes v_j^* \otimes v_k^* \otimes [v_i^*, Tv_j, Tv_k] \\
& + v_i^* \otimes v_j^* \otimes v_k^* \otimes [Tv_i, Tv_j, Tv_k] \\
& - v_i^* \otimes Tv_j \otimes v_k^* \otimes [Tv_i, v_j^*, Tv_k] \\
& - v_i^* \otimes v_j^* \otimes Tv_k \otimes [Tv_i, Tv_j, v_k^*] \Big).
\end{aligned}
$$

Set

$$
\begin{aligned}
OT(u, v, w) = & [Tu, Tv, Tw] \\
& - T\Big(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v\Big), \quad \forall u, v, w \in A.
\end{aligned}
$$

Note

$$
\begin{aligned}
\sum_i Tv_i \otimes [v_i^*, Tv_j, Tv_k] &= \sum_i Tv_i \otimes \rho^*(Tv_j, Tv_k)v_i^* \\
&= \sum_i Tv_i \otimes \sum_m \langle \rho^*(Tv_j, Tv_k)v_i^*, v_m \rangle v_m^* \\
&= \sum_{i,m} Tv_i \otimes -\langle \rho(Tv_j, Tv_k)v_m, v_i^* \rangle v_m^* \\
&= \sum_m -T(\rho(Tv_j, Tv_k)v_m \otimes v_m^*.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& [[r, r, r]] = [r_{12}, r_{13}, r_{14}] + [r_{12}, r_{23}, r_{24}] + [r_{13}, r_{23}, r_{34}] + [r_{14}, r_{24}, r_{34}] \\
= & \sum_{i,j,k} \Big(- OT(v_i, v_j, v_k) \otimes v_i^* \otimes v_j^* \otimes v_k^* + v_i^* \otimes OT(v_i, v_j, v_k) \otimes v_j^* \otimes v_k^* \\
& - v_i^* \otimes v_j^* \otimes OT(v_i, v_j, v_k) \otimes v_k^* + v_i^* \otimes v_j^* \otimes v_k^* \otimes OT(v_i, v_j, v_k) \Big).
\end{aligned}
$$

Therefore r satisfies the 3-Lie classical Yang-Baxter equation, i.e. $[[r, r, r]] = 0$ if and only if $OT(v_i, v_j, v_k) = 0$ for all i, j, k, i.e. T is an $\mathcal{O}$-operator. □

Definition 3.22. Let A be a vector space with a linear map $\{\cdot, \cdot, \cdot\} : A \otimes A \otimes A \to A$. The pair $(A, \{\cdot, \cdot, \cdot\})$ is called a **3-pre-Lie algebra** if the following identities hold:

$$\{x,y,z\} = -\{y,x,z\}, \tag{3.16}$$

$$\begin{aligned}\{x_1,x_2,\{x_3,x_4,x_5\}\} &= \{[x_1,x_2,x_3]_C,x_4,x_5\} \\ &\quad + \{x_3,[x_1,x_2,x_4]_C,x_5\} \\ &\quad + \{x_3,x_4,\{x_1,x_2,x_5\}\},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\{[x_1,x_2,x_3]_C,x_4,x_5\} &= \{x_1,x_2,\{x_3,x_4,x_5\}\} \\ &\quad + \{x_2,x_3,\{x_1,x_4,x_5\}\} \\ &\quad + \{x_3,x_1,\{x_2,x_4,x_5\}\},\end{aligned} \tag{3.18}$$

where $x,y,z,x_i \in A, 1 \le i \le 5$ and $[\cdot,\cdot,\cdot]_C$ is defined by

$$[x,y,z]_C = \{x,y,z\} + \{y,z,x\} + \{z,x,y\}, \quad \forall x,y,z \in A. \tag{3.19}$$

This agrees with the general construction of splitting of operads applied to the operad of the 3-Lie algebra [36].

Proposition 3.23. *Let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Then the induced 3-commutator given by Eq. (3.19) defines a 3-Lie algebra.*

Proof. By Eq. (3.16), the induced 3-commutator $[\cdot,\cdot,\cdot]_C$ given by Eq. (3.19) is skew-symmetric. For $x_1,x_2,x_3,x_4,x_5 \in A$, we have

$$\begin{aligned}
&[x_1,x_2,[x_3,x_4,x_5]_C]_C - [[x_1,x_2,x_3]_C,x_4,x_5]_C - [x_3,[x_1,x_2,x_4]_C,x_5]_C \\
&\quad - [x_3,x_4,[x_1,x_2,x_5]_C]_C \\
&= \{x_1,x_2,\{x_3,x_4,x_5\}\} + \{x_1,x_2,\{x_4,x_5,x_3\}\} + \{x_1,x_2,\{x_5,x_3,x_4\}\} \\
&\quad + \{x_2,[x_3,x_4,x_5]_C,x_1\} + \{[x_3,x_4,x_5]_C,x_1,x_2\} \\
&\quad - \{x_4,x_5,\{x_1,x_2,x_3\}\} - \{x_4,x_5,\{x_2,x_3,x_1\}\} - \{x_4,x_5,\{x_3,x_1,x_2\}\} \\
&\quad - \{[x_1,x_2,x_3]_C,x_4,x_5\} - \{x_5,[x_1,x_2,x_3]_C,x_4\} \\
&\quad - \{x_5,x_3,\{x_1,x_2,x_4\}\} - \{x_5,x_3,\{x_2,x_4,x_1\}\} - \{x_5,x_3,\{x_4,x_1,x_2\}\} \\
&\quad - \{x_3,[x_1,x_2,x_4]_C,x_5\} - \{[x_1,x_2,x_4]_C,x_5,x_3\} \\
&\quad - \{x_3,x_4,\{x_1,x_2,x_5\}\} - \{x_3,x_4,\{x_2,x_5,x_1\}\} - \{x_3,x_4,\{x_5,x_1,x_2\}\} \\
&\quad - \{x_4,[x_1,x_2,x_5]_C,x_3\} - \{[x_1,x_2,x_5]_C,x_3,x_4\} \\
&= 0.
\end{aligned}$$

This is because

$$\begin{aligned}
\{x_1,x_2,\{x_3,x_4,x_5\}\} &= \{[x_1,x_2,x_3]_C,x_4,x_5\}+\{x_3,[x_1,x_2,x_4]_C,x_5\}\\
&\quad+\{x_3,x_4,\{x_1,x_2,x_5\}\},\\
\{x_1,x_2,\{x_4,x_5,x_3\}\} &= \{[x_1,x_2,x_4]_C,x_5,x_3\}+\{x_4,[x_1,x_2,x_5]_C,x_3\}\\
&\quad+\{x_4,x_5,\{x_1,x_2,x_3\}\},\\
\{x_1,x_2,\{x_5,x_3,x_4\}\} &= \{x_5,[x_1,x_2,x_3]_C,x_4\}+\{[x_1,x_2,x_5]_C,x_3,x_4\}\\
&\quad+\{x_5,x_3,\{x_1,x_2,x_4\}\},\\
\{x_2,[x_3,x_4,x_5]_C,x_1\} &= \{x_4,x_5,\{x_2,x_3,x_1\}\}+\{x_5,x_3,\{x_2,x_4,x_1\}\}\\
&\quad+\{x_3,x_4,\{x_2,x_5,x_1\}\},\\
\{[x_3,x_4,x_5]_C,x_1,x_2\} &= \{x_3,x_4,\{x_5,x_1,x_2\}\}+\{x_4,x_5,\{x_3,x_1,x_2\}\}\\
&\quad+\{x_5,x_3,\{x_4,x_1,x_2\}\}.
\end{aligned}$$

Thus the proof is completed. □

Definition 3.24. Let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. The 3-Lie algebra $(A,[\cdot,\cdot,\cdot]_C)$ is called the **sub-adjacent 3-Lie algebra** of $(A,\{\cdot,\cdot,\cdot\})$ and $(A,\{\cdot,\cdot,\cdot\})$ is called a **compatible 3-pre-Lie algebra** on the 3-Lie algebra $(A,[\cdot,\cdot,\cdot]_C)$.

Let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Define a skew-symmetric linear map $L:\otimes^2 A\to\mathfrak{gl}(A)$ by

$$L(x,y)z=\{x,y,z\},\quad \forall x,y,z\in A. \tag{3.20}$$

By the definitions of a 3-pre-Lie algebra and a representation of a 3-Lie algebra, we immediately obtain

Proposition 3.25. *With the above notations, (A,L) is a representation of the 3-Lie algebra $(A,[\cdot,\cdot,\cdot]_C)$. On the other hand, let A be a vector space with a linear map $\{\cdot,\cdot,\cdot\}:A\otimes A\otimes A\to A$ satisfying Eq.* (3.16)*. Then $(A,\{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra if $[\cdot,\cdot,\cdot]_C$ defined by Eq* (3.19) *is a 3-Lie algebra and the left multiplication L defined by Eq.* (3.20) *gives a representation of this 3-Lie algebra.*

New identities of 3-pre-Lie algebras can be derived from Proposition 2.2. For example,

Corollary 3.26. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Then the following identities hold:*

$$\{[x_1, x_2, x_3]_C, x_4, x_5\} - \{[x_1, x_2, x_4]_C, x_3, x_5\} + \{[x_1, x_3, x_4]_C, x_2, x_5\} - \{[x_2, x_3, x_4]_C, x_1, x_5\} = 0,$$

$$\{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_3, x_4, \{x_1, x_2, x_5\}\} + \{x_2, x_4, \{x_3, x_1, x_5\}\} + \{x_3, x_1, \{x_2, x_4, x_5\}\} + \{x_2, x_3, \{x_1, x_4, x_5\}\} + \{x_1, x_4, \{x_2, x_3, x_5\}\} = 0,$$

for $x_i \in A, 1 \leq i \leq 5$.

Proposition 3.27. *Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and (V, ρ) a representation. Suppose that the linear map $T : V \to A$ is an $\mathcal{O}$-operator associated to (V, ρ). Then there exists a 3-pre-Lie algebra structure on V given by*

$$\{u, v, w\} = \rho(Tu, Tv)w, \quad \forall\ u, v, w \in V. \tag{3.21}$$

Proof. Let $u, v, w \in V$. It is obvious that

$$\{u, v, w\} = \rho(Tu, Tv)w = -\rho(Tv, Tu)w = -\{v, u, w\}.$$

Furthermore, the following equation holds.

$$[u, v, w]_C = \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v.$$

Since T is an $\mathcal{O}$-operator, we have

$$T[u, v, w]_C = [Tu, Tv, Tw].$$

For $v_1, v_2, v_3, v_4, v_5 \in V$, we have

$$\begin{aligned}
\{v_1, v_2, \{v_3, v_4, v_5\}\} &= \rho(Tv_1, Tv_2)\rho(Tv_3, Tv_4)v_5;\\
\{[v_1, v_2, v_3]_C, v_4, v_5\} &= \rho(T[v_1, v_2, v_3]_C, Tv_4)v_5 = \rho([Tv_1, Tv_2, Tv_3], Tv_4)v_5;\\
\{v_3, [v_1, v_2, v_4]_C, v_5\} &= \rho(Tv_3, T[v_1, v_2, v_4]_C)v_5 = \rho(Tv_3, [Tv_1, Tv_2, Tv_4])v_5;\\
\{v_3, v_4, \{v_1, v_2, v_5\}\} &= \rho(Tv_3, Tv_4)\rho(Tv_1, Tv_2)v_5.
\end{aligned}$$

Since (V,ρ) is a representation of A, by Condition (i) in Definition 2.3, Eq. (3.17) holds. On the other hand, we have

$$\begin{aligned}
\{[v_1,v_2,v_3]_C,v_4,v_5\} &= \rho(T[v_1,v_2,v_3]_C,Tv_4)v_5 = \rho([Tv_1,Tv_2,Tv_3],Tv_4)v_5;\\
\{v_1,v_2,\{v_3,v_4,v_5\}\} &= \rho(Tv_1,Tv_2)\rho(Tv_3,Tv_4)v_5;\\
\{v_2,v_3,\{v_1,v_4,v_5\}\} &= \rho(Tv_2,Tv_3)\rho(Tv_1,Tv_4)v_5;\\
\{v_3,v_1,\{v_2,v_4,v_5\}\} &= \rho(Tv_3,Tv_1)\rho(Tv_2,Tv_4)v_5.
\end{aligned}$$

By Condition (ii) in Definition 2.3, Eq. (3.18) holds. This completes the proof. □

Corollary 3.28. *With the above conditions, $(V,[\cdot,\cdot,\cdot]_C)$ is a 3-Lie algebra as the sub-adjacent 3-Lie algebra of the 3-pre-Lie algebra given in Proposition 3.27, and T is a 3-Lie algebra morphism from $(V,[\cdot,\cdot,\cdot]_C)$ to $(A,[\cdot,\cdot,\cdot])$. Furthermore, $T(V)=\{Tv|v\in V\}\subset A$ is a 3-Lie subalgebra of A and there is an induced 3-pre-Lie algebra structure $\{\cdot,\cdot,\cdot\}_{T(V)}$ on $T(V)$ given by*

$$\{Tu,Tv,Tw\}_{T(V)} := T\{u,v,w\}, \quad \forall u,v,w\in V. \tag{3.22}$$

Proposition 3.29. *Let $(A,[\cdot,\cdot,\cdot])$ be a 3-Lie algebra. Then there exists a compatible 3-pre-Lie algebra if and only if there exists an invertible $\mathcal{O}$-operator on A.*

Proof. Let T be an invertible $\mathcal{O}$-operator of A associated to a representation (V,ρ). Then there exists a 3-pre-Lie algebra structure on V defined by

$$\{u,v,w\}=\rho(Tv,Tw)u, \quad \forall u,v,w\in V.$$

Moreover, there is an induced 3-pre-Lie algebra structure $\{\cdot,\cdot,\cdot\}_A$ on $A=T(V)$ given by

$$\{x,y,z\}_A = T\{T^{-1}x,T^{-1}y,T^{-1}z\} = T\rho(x,y)T^{-1}(z)$$

for all $x,y,z\in A$. Since T is an $\mathcal{O}$-operator, we have

$$\begin{aligned}
[x,y,z] &= T\left(\rho(y,z)T^{-1}(x)+\rho(z,x)T^{-1}(y)+\rho(x,y)T^{-1}z\right)\\
&= \{x,y,z\}_A+\{y,z,x\}_A+\{z,x,y\}_A.
\end{aligned}$$

Therefore $(A,\{\cdot,\cdot,\cdot\}_A)$ is a compatible 3-pre-Lie algebra. Conversely, the identity map id is an $\mathcal{O}$-operator of the sub-adjacent 3-Lie algebra of a 3-pre-Lie algebra associated to the representation (A,L). □

The following result shows that, in analogous to the fact that pre-Lie algebras naturally arise from the underlying structures of symplectic Lie algebras, 3-pre-Lie algebras naturally arise from the underlying structures of symplectic 3-Lie algebras.

Proposition 3.30. *Let $(A, [\cdot,\cdot,\cdot], B)$ be a symplectic 3-Lie algebra. Then there exists a compatible 3-pre-Lie algebra structure $\{\cdot,\cdot,\cdot\}$ on A given by*

$$B(\{x,y,z\},w) = -B(z,[x,y,w]), \quad \forall x,y,z,w \in A. \tag{3.23}$$

Proof. Define a linear map $T : A^* \longrightarrow A$ by $\langle T^{-1}x, y\rangle = B(x,y)$, or equivalently, $B(T\xi, y) = \langle \xi, y\rangle$ for all $x,y \in A$ and $\xi \in A^*$. By Eq. (3.12), we obtain that T is an invertible $\mathcal{O}$-operator associated to the coadjoint representation (A^*, ad^*). By Proposition 3.29, there exists a compatible 3-pre-Lie algebra on A given by $\{x,y,z\}_A = T(\mathrm{ad}^*_{x,y}T^{-1}z)$ for $x,y,z \in A$. Then we have

$$\begin{aligned} B(\{x,y,z\}_A, w) &= B(T(\mathrm{ad}^*_{x,y}T^{-1}z), w) = \langle \mathrm{ad}^*_{x,y}T^{-1}z, w\rangle \\ &= \langle T^{-1}(z), -[x,y,w]\rangle = -B(z,[x,y,w]), \end{aligned}$$

for $x,y,z,w \in A$. This completes the proof. □

We end this subsection by obtaining solutions of the 3-Lie classical Yang-Baxter equation from 3-pre-Lie algebras.

Theorem 3.31. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Let $\{e_i\}$ be a basis of A and $\{e_i^*\}$ the dual basis. Then*

$$r = \sum_i e_i \otimes e_i^* - e_i^* \otimes e_i \tag{3.24}$$

is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in $A \ltimes_{L^*} A^*$.

Proof. It follows from Theorem 3.21 and the fact that the identity map id is an $\mathcal{O}$-operator of the sub-adjacent 3-Lie algebra of a 3-pre-Lie algebra associated to the representation (A, L). □

4. Double construction 3-Lie bialgebras

We consider double construction 3-Lie bialgebras in this section. In Section 4.1, we introduce the notions of a Manin triple and a matched pair of 3-Lie algebras. In Section 4.2, we introduce the notion of a double construction

3-Lie bialgebra, and establish the correspondence between a Manin triple of 3-Lie algebras, a matched pair of 3-Lie algebras and a double construction 3-Lie bialgebra (Theorem 4.15). In Section 4.3, we establish the relationship between local cocycle 3-Lie bialgebras and double construction 3-Lie bialgebras, and provide explicit examples of the latter.

4.1. Manin triples and matched pairs of 3-Lie algebras

Definition 4.1. Let A be a 3-Lie algebra. A bilinear form $(\cdot,\cdot)_A$ on A is called **invariant** if it satisfies

$$([x_1,x_2,x_3],x_4)_A+([x_1,x_2,x_4],x_3)_A=0,\quad \forall x_1,x_2,x_3,x_4\in A. \tag{4.1}$$

A 3-Lie algebra A is called **pseudo-metric** if there is a nondegenerate symmetric invariant bilinear form on A.

Definition 4.2. A **Manin triple of 3-Lie algebras** consists of a pseudo-metric 3-Lie algebra $(\mathcal{A},(\cdot,\cdot)_{\mathcal{A}})$ and 3-Lie algebras A_1, A_2 such that

(a) A_1 and A_2 are 3-Lie subalgebras of $\mathcal{A}$ which are isotropic, that is, $(x_1,y_1)_{\mathcal{A}}=(x_2,y_2)_{\mathcal{A}}=0$ for $x_1,y_1\in A_1$ and $x_2,y_2\in A_2$;

(b) $\mathcal{A}=A_1\oplus A_2$ as the direct sum of vector spaces;

(c) For all $x_1,y_1\in A_1$ and $x_2,y_2\in A_2$, we have $\mathrm{pr}_1[x_1,y_1,x_2]=0$ and $\mathrm{pr}_2[x_2,y_2,x_1]=0$, where pr_1 and pr_2 are the projections from $A_1\oplus A_2$ to A_1 and A_2 respectively.

An **isomorphism** between two Manin triples $((\mathcal{A},(\cdot,\cdot)_{\mathcal{A}}),A_1,A_2)$ and $((\mathcal{A}',(\cdot,\cdot)_{\mathcal{A}'}),A_1',A_2')$ is an isomorphism $f:\mathcal{A}\to\mathcal{A}'$ of 3-Lie algebras satisfying

$$f(A_1)\subset A_1',\quad f(A_2)\subset A_2',\quad (x_1,x_2)_{\mathcal{A}}=(f(x_1),f(x_2))_{\mathcal{A}'},\quad \forall x_1,x_2\in\mathcal{A}.$$

Remark 4.3. In [37], the authors introduced the notion of a product structure on a 3-Lie algebra, which gives rise to a decomposition of the 3-Lie algebra. In particular, there are four special product structures, which are called strict product structure, abelian product structure, strong abelian structure and perfect product structure respectively, each of them gives rise to a special decomposition of the original 3-Lie algebra. The above condition (c) actually means that there exists a perfect product structure on the 3-Lie algebra $\mathcal{A}$. See [37] for more details.

Let $(A, [\cdot,\cdot,\cdot])$ and $(A^*, [\cdot,\cdot,\cdot]^*)$ be 3-Lie algebras. On $A \oplus A^*$, there is a natural nondegenerate symmetric bilinear form $(\cdot,\cdot)_+$ given by

$$(x + \xi, y + \eta)_+ = \langle x, \eta\rangle + \langle \xi, y\rangle, \quad \forall x, y \in A, \xi, \eta \in A^*. \tag{4.2}$$

There is also a bracket operation $[\cdot,\cdot,\cdot]_{A\oplus A^*}$ on $A \oplus A^*$ given by

$$\begin{aligned}[x + \xi, y + \eta, z + \gamma]_{A\oplus A^*} = {} & [x, y, z] + \mathrm{ad}^*_{x,y}\gamma + \mathrm{ad}^*_{y,z}\xi + \mathrm{ad}^*_{z,x}\eta \\ & + \mathfrak{ad}^*_{\xi,\eta} z + \mathfrak{ad}^*_{\eta,\gamma} x + \mathfrak{ad}^*_{\gamma,\xi} y + [\xi, \eta, \gamma]^*,\end{aligned} \tag{4.3}$$

where $x, y, z \in A, \xi, \eta, \gamma \in A^*$, ad^* and $\mathfrak{ad}^*$ are the coadjoint representations of A and A^* on A^* and A respectively. Note that the bracket operation $[\cdot,\cdot,\cdot]_{A\oplus A^*}$ is naturally invariant with respect to the symmetric bilinear form $(\cdot,\cdot)_+$, and satisfies Condition (c) in Definition 4.2. If $(A \oplus A^*, [\cdot,\cdot,\cdot]_{A\oplus A^*})$ is a 3-Lie algebra, then obviously A and A^* are isotropic subalgebras. Consequently, $((A \oplus A^*, (\cdot,\cdot)_+), A, A^*)$ is a Manin triple, which we call **the standard Manin triple of 3-Lie algebras.**

Proposition 4.4. *Any Manin triple of* 3*-Lie algebras is isomorphic to a standard one.*

Proof. For any Manin triple $((\mathcal{A}, (\cdot,\cdot)_{\mathcal{A}}), A_1, A_2)$, through the nondegenerate bilinear form $(\cdot,\cdot)_{\mathcal{A}}$, A_2 is isomorphic to A_1^* as a vector space. Moreover, A_1^* is equipped with a 3-Lie algebra structure from A_2 via this isomorphism. Then $((\mathcal{A}, (\cdot,\cdot)_{\mathcal{A}}), A_1, A_2)$ is isomorphic to the standard Manin triple $((A_1 \oplus A_1^*, (\cdot,\cdot)_+), A_1, A_1^*)$. □

Now we turn our attention to matched pairs of 3-Lie algebras.

Proposition 4.5. *Let $(A, [\cdot,\cdot,\cdot])$ and $(A', [\cdot,\cdot,\cdot]')$ be two* 3*-Lie algebras. Suppose that there are skew-symmetric linear maps $\rho : \otimes^2 A \to \mathfrak{gl}(A')$ and $\mu : \otimes^2 A' \to \mathfrak{gl}(A)$ satisfying the following conditions:*

(a) *(A', ρ) is a representation of A;*

(b) *(A, μ) is a representation of A';*

(c) *For all $x_i \in A$ and $a_i \in A', 1 \le i \le 5$, ρ, μ satisfy the following compatibility conditions:*

$$\mu(a_4,a_5)[x_1,x_2,x_3] = [\mu(a_4,a_5)x_1,x_2,x_3] + [x_1,\mu(a_4,a_5)x_2,x_3] + [x_1,x_2,\mu(a_4,a_5)x_3]; \tag{4.4}$$

$$-\mu(\rho(x_1,x_2)a_3,a_5)x_4 = -\mu(\rho(x_1,x_4)a_5,a_3)x_2 + \mu(\rho(x_2,x_4)a_5,a_3)x_1 - [x_1,x_2,\mu(a_3,a_5)x_4]; \tag{4.5}$$

$$[\mu(a_2,a_3)x_1,x_4,x_5] = \mu(a_2,a_3)[x_1,x_4,x_5] + \mu(\rho(x_4,x_5)a_2,a_3)x_1 + \mu(a_2,\rho(x_4,x_5)a_3)x_1; \tag{4.6}$$

$$\rho(x_4,x_5)[a_1,a_2,a_3]' = [\rho(x_4,x_5)a_1,a_2,a_3]' + [a_1,\rho(x_4,x_5)a_2,a_3]' + [a_1,a_2,\rho(x_4,x_5)a_3]'; \tag{4.7}$$

$$-\rho(\mu(a_1,a_2)x_3,x_5)a_4 = -\rho(\mu(a_1,a_4)x_5,x_3)a_2 + \rho(\mu(a_2,a_4)x_5,x_3)a_1 - [a_1,a_2,\rho(x_3,x_5)a_4]'; \tag{4.8}$$

$$[\rho(x_2,x_3)a_1,a_4,a_5]' = \rho(x_2,x_3)[a_1,a_4,a_5]' + \rho(\mu(a_4,a_5)x_2,x_3)a_1 + \rho(x_2,\rho(a_4,a_5)x_3)a_1, \tag{4.9}$$

Then there is a 3*-Lie algebra structure on* $A \oplus A'$ (*as the direct sum of vector spaces*) *defined by*

$$\begin{aligned} &[x_1+a_1,x_2+a_2,x_3+a_3]_{A\oplus A'} \\ :=\ &[x_1,x_2,x_3] + \rho(x_1,x_2)a_3 + \rho(x_3,x_1)a_2 + \rho(x_2,x_3)a_1 \\ &+ [a_1,a_2,a_3]' + \mu(a_1,a_2)x_3 + \mu(a_3,a_1)x_2 + \mu(a_2,a_3)x_1, \end{aligned}$$

for $x_i \in A, a_i \in A', 1 \le i \le 3$.

Proof. It follows from straightforward applications of the Fundamental Identity for the bracket operation $[\cdot,\cdot,\cdot]_{A\oplus A'}$. □

Remark 4.6. Eq. (4.4) means that $\mu(a_1,a_2)$ is a derivation on the 3-Lie algebra A for all $a_1,a_2 \in A'$. Eq. (4.7) means that $\rho(x_1,x_2)$ is a derivation on the 3-Lie algebra A' for all $x_1,x_2 \in A$.

Definition 4.7. Let $(A,[\cdot,\cdot,\cdot])$ and $(A^*,[\cdot,\cdot,\cdot]^*)$ be 3-Lie algebras. Suppose that there are skew-symmetric linear maps $\rho : \otimes^2 A \to \mathfrak{gl}(A')$ and $\mu : \otimes^2 A' \to \mathfrak{gl}(A)$ such that (A',ρ) is a representation of A, (A,μ) is a representation of A' and ρ,μ satisfy Eqs. (4.4)-(4.9). Then we call (A,A') (or more precisely (A,A',ρ,μ)) a **matched pair of 3-Lie algebras**.

We immediately get the following relation between Manin triples and matched pairs of 3-Lie algebras:

Proposition 4.8. *Let* $(A, [\cdot,\cdot,\cdot])$ *and* $(A^*, [\cdot,\cdot,\cdot]^*)$ *be* 3*-Lie algebras. Then* $((A \oplus A^*, (\cdot,\cdot)_+), A, A^*)$ *is a standard Manin triple if and only if* $(A, A^*, \mathrm{ad}^*, \mathfrak{ad}^*)$ *is a matched pair.*

4.2. Double construction 3-Lie bialgebras

There are several equivalent conditions for matched pairs of 3-Lie algebras.

Lemma 4.9. *Let* $(A, [\cdot,\cdot,\cdot])$ *and* $(A^*, [\cdot,\cdot,\cdot]^*)$ *be* 3*-Lie algebras. Then* $(A, A^*, \mathrm{ad}^*, \mathfrak{ad}^*)$ *is a matched pair if and only if Eqs.* (4.4), (4.5) *and* (4.6) *hold for* $\rho = \mathrm{ad}^*, \mu = \mathfrak{ad}^*$.

Proof. We only need to prove that for $\rho = \mathrm{ad}^*, \mu = \mathfrak{ad}^*$, we have

$$\text{Eq. (4.4)} \Longleftrightarrow \text{Eq. (4.7)},\ \ \text{Eq. (4.5)} \Longleftrightarrow \text{Eq. (4.8)},\ \ \text{Eq. (4.6)} \Longleftrightarrow \text{Eq. (4.9)}.$$

We only prove the second equivalence since the proofs of the other cases are similar. Let $x_1, x_2, x_4 \in A$ and $a_3, a_5, a_6 \in A^*$. We have

$$\begin{aligned}
&\langle -\mathfrak{ad}^*_{\mathrm{ad}^*_{x_1,x_2}a_3,a_5}x_4 + \mathfrak{ad}^*_{\mathrm{ad}^*_{x_1,x_4}a_5,a_3}x_2 - \mathfrak{ad}^*_{\mathrm{ad}^*_{x_2,x_4}a_5,a_3}x_1 \\
&\quad + [x_1, x_2, \mathfrak{ad}^*_{a_3,a_5}x_4], a_6\rangle \\
&= \langle [\mathrm{ad}^*_{x_1,x_2}a_3, a_5, a_6]^*, x_4\rangle - \langle [\mathrm{ad}^*_{x_1,x_4}a_5, a_3, a_6]^*, x_2\rangle \\
&\quad + \langle [\mathrm{ad}^*_{x_2,x_4}a_5, a_3, a_6]^* - \mathrm{ad}^*_{x_2,\mathfrak{ad}^*_{a_3,a_5}x_4}a_6, x_1\rangle \\
&= \langle x_1, -\mathrm{ad}^*_{x_2,\mathfrak{ad}^*_{a_5,a_6}x_4}a_3 + \mathrm{ad}^*_{x_4,\mathfrak{ad}^*_{a_3,a_6}x_2}a_5 + [\mathrm{ad}^*_{x_2,x_4}a_5, a_3, a_6]^* \\
&\quad - \mathrm{ad}^*_{x_2,\mathfrak{ad}^*_{a_3,a_5}x_4}a_6\rangle,
\end{aligned}$$

which implies the equivalence between Eq. (4.5) and Eq. (4.8). □

Theorem 4.10. *Let* $(A, [\cdot,\cdot,\cdot])$ *be a* 3*-Lie algebra and* $\Delta : A \to A \otimes A \otimes A$ *a linear map. Suppose that* $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ *defines a* 3*-Lie algebra structure* $[\cdot,\cdot,\cdot]^*$ *on* A^*. *Then* $(A, A^*, \mathrm{ad}^*, \mathfrak{ad}^*)$ *is a matched pair if and only if for all* $x, y, z \in A$, *the following equations are satisfied:*

$$\begin{aligned}
\Delta([x,y,z]) &= (1 \otimes 1 \otimes \mathrm{ad}_{y,z})\Delta(x) + (1 \otimes 1 \otimes \mathrm{ad}_{z,x})\Delta(y) \\
&\quad + (1 \otimes 1 \otimes \mathrm{ad}_{x,y})\Delta(z);
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\Delta([x,y,z]) &= (1 \otimes 1 \otimes \mathrm{ad}_{y,z})\Delta(x) + (1 \otimes \mathrm{ad}_{y,z} \otimes 1)\Delta(x) \\
&\quad + (\mathrm{ad}_{y,z} \otimes 1 \otimes 1)\Delta(x);
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&(\mathrm{ad}_{x,y} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \mathrm{ad}_{x,y})\Delta(z) \\
&\quad = (1 \otimes \mathrm{ad}_{z,x} \otimes 1)\Delta(y) + (1 \otimes \mathrm{ad}_{y,z} \otimes 1)\Delta(x),
\end{aligned} \tag{4.12}$$

for $x, y, z, \in A$.

Proof. By Lemma 4.9, we only need to prove that Eqs. (4.4), (4.5) and Eq. (4.6) are equivalent to Eqs. (4.10), (4.11) and (4.12) respectively.

Let $\{e_1, \cdots, e_n\}$ be a basis of A and $\{e_1^*, \cdots, e_n^*\}$ the dual basis. Suppose

$$[e_i, e_j, e_k] = \sum_{l=1}^{n} c_{ijk}^l e_l, \quad [e_i^*, e_j^*, e_k^*]^* = \sum_{l=1}^{n} d_{ijk}^l e_l^*.$$

Then we have

$$\begin{aligned} \mathrm{ad}^*_{e_i,e_j} e_k^* &= -\sum_{l=1}^{n} c_{ijl}^k e_l^*; \quad \mathfrak{ad}^*_{e_i^*,e_j^*} e_k = -\sum_{l=1}^{n} d_{ijl}^k e_l, \\ \Delta(e_k) &= \sum_{i,j,l=1}^{n} d_{ijl}^k e_i \otimes e_j \otimes e_l. \end{aligned}$$

By Eq. (4.4), we have

$$\begin{aligned} \mathfrak{ad}^*_{e_\alpha^*,e_\beta^*}[e_i, e_j, e_k] &- [\mathfrak{ad}^*_{e_\alpha^*,e_\beta^*} e_i, e_j, e_k] \\ &- [e_i, \mathfrak{ad}^*_{e_\alpha^*,e_\beta^*} e_j, e_k] - [e_i, e_j, \mathfrak{ad}^*_{e_\alpha^*,e_\beta^*} e_k] = 0, \end{aligned}$$

which gives

$$\sum_{l=1}^{n} (-d_{\alpha\beta m}^l c_{ijk}^l + d_{\alpha\beta l}^i c_{ljk}^m + d_{\alpha\beta l}^j c_{ilk}^m + d_{\alpha\beta l}^k c_{ijl}^m) = 0, \quad \forall \alpha, \beta, i, j, k, m, \tag{4.13}$$

as the coefficient of e_m. On the other hand, the left hand side of the above equation is also the coefficient of $e_\alpha \otimes e_\beta \otimes e_m$ in

$$\begin{aligned} (1 \otimes 1 \otimes \mathrm{ad}_{e_j,e_k})\Delta(e_i) &+ (1 \otimes 1 \otimes \mathrm{ad}_{e_k,e_i})\Delta(e_j) \\ &+ (1 \otimes 1 \otimes \mathrm{ad}_{e_i,e_j})\Delta(e_k) - \Delta([e_i, e_j, e_k]). \end{aligned}$$

Thus, we deduce that Eq. (4.4) is equivalent to Eq. (4.10).

We can similarly prove that Eq. (4.5) and Eq. (4.6) are equivalent to Eq. (4.12) and Eq. (4.11) respectively. □

Remark 4.11. In fact, Eq. (4.10) means that Δ is a 1-cocycle with coefficients in $(1 \otimes 1 \otimes \mathrm{ad}, A \otimes A \otimes A)$ and Eq. (4.11) means that (see Eq. (1.4) for the notations, also see [8])

$$\Delta([x, y, z]) = [\Delta(x), y, z], \quad \forall x, y, z \in A.$$

Remark 4.12. Note that there are certain symmetries in the above proof. Let $x, y, z \in A$.

(i) Eq. (4.10) holds if and only if one of the following equations holds:
- $\Delta([x,y,z]) = (1 \otimes \mathrm{ad}_{y,z} \otimes 1)\Delta(x) + (1 \otimes \mathrm{ad}_{z,x} \otimes 1)\Delta(y) + (1 \otimes \mathrm{ad}_{x,y} \otimes 1)\Delta(z)$;
- $\Delta([x,y,z]) = (\mathrm{ad}_{y,z} \otimes 1 \otimes 1)\Delta(x) + (\mathrm{ad}_{z,x} \otimes 1 \otimes 1)\Delta(y) + (\mathrm{ad}_{x,y} \otimes 1 \otimes 1)\Delta(z)$.

In fact, the former corresponds to the coefficient of $e_\alpha \otimes e_m \otimes e_\beta$ in Eq. (4.13), whereas the latter corresponds to the coefficient of $e_m \otimes e_\alpha \otimes e_\beta$ in Eq. (4.13).

(ii) Eq. (4.11) holds if and only if one of the following equations holds:
- $\Delta([x,y,z]) = (1 \otimes 1 \otimes \mathrm{ad}_{z,x})\Delta(y) + (1 \otimes \mathrm{ad}_{z,x} \otimes 1)\Delta(y) + (\mathrm{ad}_{z,x} \otimes 1 \otimes 1)\Delta(y)$;
- $\Delta([x,y,z]) = (1 \otimes 1 \otimes \mathrm{ad}_{x,y})\Delta(z) + (1 \otimes \mathrm{ad}_{x,y} \otimes 1)\Delta(z) + (\mathrm{ad}_{x,y} \otimes 1 \otimes 1)\Delta(z)$.

(iii) Eq. (4.12) holds if and only if one of the following equations holds:
- $(\mathrm{ad}_{x,y} \otimes 1 \otimes 1)\Delta(z) + (1 \otimes \mathrm{ad}_{x,y} \otimes 1)\Delta(z) = (1 \otimes 1 \otimes \mathrm{ad}_{z,x})\Delta(y) + (1 \otimes 1 \otimes \mathrm{ad}_{y,z})\Delta(x)$;
- $(1 \otimes \mathrm{ad}_{x,y} \otimes 1)\Delta(z) + (1 \otimes 1 \otimes \mathrm{ad}_{x,y})\Delta(z) = (\mathrm{ad}_{z,x} \otimes 1 \otimes 1)\Delta(y) + (\mathrm{ad}_{y,z} \otimes 1 \otimes 1)\Delta(x)$.

The three equations (4.10) – (4.12) are not independent.

Proposition 4.13. *Any two equations in Eqs.* (4.10), (4.11) *and* (4.12) *imply the third one.*

Proof. Substituting Eq. (4.11) into Eq. (4.10) yields the first equation in Item (iii) in the above remark, which is equivalent to Eq. (4.12).

Substituting Eq. (4.12) into Eq. (4.11) yields the first equation in Item (i) in the above remark, which is equivalent to Eq. (4.10).

Eq. (4.10) and the first equation in Item (iii) in the above remark imply Eq. (4.11). Hence by the same remark, Eqs. (4.10) and (4.12) imply Eq. (4.11). $\square$

Since a Manin triple of 3-Lie algebras $((A \oplus A^*, (\cdot,\cdot)_+), A, A^*)$ gives a natural pseudo-metric 3-Lie algebra structure on the double space $A \oplus A^*$, we give the following definition:

Definition 4.14. Let A be a 3-Lie algebra and $\Delta : A \to A \otimes A \otimes A$ a linear map. Suppose that $\Delta^* : A^* \otimes A^* \otimes A^* \to A$ defines a 3-Lie algebra structure on A^*. If Δ satisfies Eqs. (4.10) and (4.11), then we call (A, Δ) a **double construction 3-Lie bialgebra**.

Combining Proposition 4.8, Theorem 4.10 and Proposition 4.13, we obtain

Theorem 4.15. *Let A be a 3-Lie algebra and $\Delta : A \to A \otimes A \otimes A$ a linear map. Suppose that $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a 3-Lie algebra structure on A^*. Then the following statements are equivalent.*

(a) *(A, Δ) is a double construction 3-Lie bialgebra;*

(b) *$((A \oplus A^*, (\cdot,\cdot)_+), A, A^*)$ is a standard Manin triple, where the bilinear form $(\cdot,\cdot)_+$ and the 3-Lie bracket $[\cdot,\cdot,\cdot]_{A\oplus A^*}$ are given by Eqs. (4.2) and (4.3) respectively;*

(c) *$(A, A^*, \mathrm{ad}^*, \mathfrak{ad}^*)$ is a matched pair of 3-Lie algebras.*

4.3. The relationship between the two types of 3-Lie bialgebras and examples

We begin with a relationship between the local cocycle 3-Lie bialgebra and the double construction 3-Lie bialgebra. We have the following result slightly improving Remark 4.12.(i):

Proposition 4.16. *Let A be a 3-Lie algebra and $\Delta : A \to A \otimes A \otimes A$ a linear map. Suppose that $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a skew-symmetric operation on A^*. Then the following conditions are equivalent:*

(a) *Δ is a 1-cocycle with coefficients in $(A \otimes A \otimes A, \mathrm{ad} \otimes 1 \otimes 1)$;*

(b) *Δ is a 1-cocycle with coefficients in $(A \otimes A \otimes A, 1 \otimes \mathrm{ad} \otimes 1)$;*

(c) *Δ is a 1-cocycle with coefficients in $(A \otimes A \otimes A, 1 \otimes 1 \otimes \mathrm{ad})$.*

Proof. We only prove that (a) holds if and only if (b) holds. The proofs of the other cases are similar. By the proof in Theorem 4.10, we have

$$(a) \Leftrightarrow \sum_{\alpha,\beta,m} \sum_{l} [-d^l_{\alpha\beta m} c^l_{ijk} + d^i_{\alpha\beta l} c^m_{jkl} + d^j_{\alpha\beta l} c^m_{kil} + d^k_{\alpha\beta l} c^m_{ijl}] e_\alpha \otimes e_\beta \otimes e_m = 0;$$

$$(b) \Leftrightarrow \sum_{\alpha,\beta,m} \sum_{l} [-d^l_{\alpha m\beta} c^l_{ijk} + d^i_{\alpha l\beta} c^m_{jkl} + d^j_{\alpha l\beta} c^m_{kil} + d^k_{\alpha l\beta} c^m_{ijl}] e_\alpha \otimes e_m \otimes e_\beta = 0.$$

The skew-symmetry of Δ^* means $d_{\alpha\beta m} = -d_{\alpha m\beta}$, which implies that (a) holds if and only if (b) holds. □

As a direct consequence, we obtain

Corollary 4.17. *A double construction* 3*-Lie bialgebra gives a local cocycle* 3*-Lie bialgebra.*

Proof. Let (A, Δ) be a double construction 3-Lie bialgebra. Let k_1, k_2, k_3 be scalars such that $k_1 + k_2 + k_3 = 1$. Denote $\Delta_i = k_i\Delta, i = 1, 2, 3$. Then by definition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ defines a local cocycle 3-Lie bialgebra. □

Remark 4.18. From the corollary, it is natural to consider whether there is a "coboundary double construction 3-Lie bialgebra". As in Section 2, for any $r = \sum_i x_i \otimes y_i \in A \otimes A$, set

$$\Delta_1(x) = \sum_{i,j}[x, x_i, x_j] \otimes y_j \otimes y_i,$$
$$\Delta_2(x) = \sum_{i,j} y_i \otimes [x, x_i, x_j] \otimes y_j,$$
$$\Delta_3(x) = \sum_{i,j} y_j \otimes y_i \otimes [x, x_i, x_j],$$

for any $x \in A$. If all $\Delta_i^* : A^* \otimes A^* \otimes A^* \longrightarrow A^*$ are skew-symmetric, it is straightforward to see that $\Delta_1 = \Delta_2 = \Delta_3$. Unfortunately, there is no natural condition for r such that Δ^* defines a skew-symmetric operation on A^* (in fact, it involves the concrete structural constants of A). Furthermore, if such an r exists, then we have a "modified" version of Theorem 3.7. However, an algebraic equation for r remains to be discovered.

We end the paper with some examples to illustrate these points and various other phenomena of double construction 3-Lie bialgebras. As a first example, we have

Example 4.19. For any 3-Lie algebra A, taking $\Delta = 0$, then (A, Δ) is a double construction 3-Lie bialgebra. In this case, the corresponding Manin triple gives a pseudo-metric 3-Lie algebra $(A \ltimes_{\mathrm{ad}^*} A^*, (\cdot, \cdot)_+)$. Dually, for any trivial 3-Lie algebra A (namely whose product is zero), any 3-Lie algebra structure Δ^* on the dual space A^* makes (A, Δ) a double construction 3-Lie bialgebra. Such double construction 3-Lie bialgebras are called **trivial double construction 3-Lie bialgebras**.

In general, it is not easy to determine whether there exists a non-trivial double construction 3-Lie bialgebra on a given non-trivial 3-Lie algebra. One reason is that there is certain inconsistence between the 1-cocycle or Eq. (4.11) and skew-symmetry. Before giving such examples, we recall the following results on the classification of complex 3-Lie algebras of dimensions 3 and 4.

Proposition 4.20. [10]

(a) *There is a unique non-trivial* 3*-dimensional complex* 3*-Lie algebra. It has a basis* $\{e_1, e_2, e_3\}$ *with respect to which the non-zero product is given by* $[e_1, e_2, e_3] = e_1$.

(b) *Let* A *be a non-trivial* 4*-dimensional complex* 3*-Lie algebra. Then* A *has a basis* $\{e_1, e_2, e_3, e_4\}$ *with respect to which the nonzero part of the product of the* 3*-Lie algebra is given by one of the following.*
  (1) $[e_1, e_2, e_3] = e_4,\ [e_2, e_3, e_4] = e_1,\ [e_1, e_3, e_4] = e_2,\ [e_1, e_2, e_4] = e_3.$
  (2) $[e_1, e_2, e_3] = e_1.$
  (3) $[e_2, e_3, e_4] = e_1.$
  (4) $[e_2, e_3, e_4] = e_1,\ [e_1, e_3, e_4] = e_2.$
  (5) $[e_2, e_3, e_4] = e_2,\ [e_1, e_3, e_4] = e_1.$
  (6) $[e_2, e_3, e_4] = \alpha e_1 + e_2,\ \alpha \neq 0,\ [e_1, e_3, e_4] = e_2.$
  (7) $[e_2, e_3, e_4] = e_1,\ [e_1, e_3, e_4] = e_2,\ [e_1, e_2, e_4] = e_3.$

In the following two examples, the double construction 3-Lie bialgebras have only the zero coproduct, exhibiting the aforementioned inconsistence between 1-cocycle and skew-symmetry. One can compare them with Example 3.12 which gives the local cocycle 3-Lie bialgebras with non-zero coproduct on the non-trivial 3-dimensional complex 3-Lie algebra.

Example 4.21. Let A be the unique non-trivial 3-dimensional complex 3-Lie algebra in Proposition 4.20. If a linear map $\Delta : A \to A \otimes A \otimes A$ satisfies Eq. (4.10) with the property that $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a skew-symmetric operation on A^*, then $\Delta = 0$.

Example 4.22. For classes (2), (5) and (6) in Proposition 4.20, if a linear map $\Delta : A \to A \otimes A \otimes A$ satisfies Eq. (4.10) and Eq. (4.11) such that $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a skew-symmetric operation on A^*, then $\Delta = 0$.

The following is an example of non-trivial double construction 3-Lie bialgebras and an illustration of Remark 4.18.

Example 4.23. Consider the first class of the 4-dimensional non-trivial 3-Lie algebras in Proposition 4.20. Note that A is the (unique) simple complex 3-Lie algebra whose only ideals are the zero ideal and A itself. Define a linear map $\Delta : A \to A \otimes A \otimes A$ by (see Example 3.12 for notations)

$$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \quad \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,$$
$$\Delta(e_3) = e_1 \wedge e_2 \wedge e_4, \quad \Delta(e_4) = e_1 \wedge e_2 \wedge e_3.$$

Then Δ satisfies Eqs. (4.10) and (4.11). Moreover, $\Delta^* : A^* \otimes A^* \otimes A^* \to A^*$ defines a 3-Lie algebra structure on A^* which is isomorphic to A. So (A, Δ) is a double construction 3-Lie bialgebra.

Furthermore, let

$$r = e_1 \otimes e_1 - e_2 \otimes e_2 + e_3 \otimes e_3 - e_4 \otimes e_4 = \sum_i (-1)^{i+1} e_i \otimes e_i.$$

Note that r is symmetric. We have

$$\Delta(e_i) = \sum_{k,l} (-1)^{k+l} [e_i, e_k, e_l] \otimes e_l \otimes e_k = \Delta_1(e_i), \quad 1 \leq i \leq 4,$$

where Δ_1 is given by Eq. (3.7). However, for the eight items in Theorem 3.7, every item is not zero, but the sum is zero.

This double construction 3-Lie bialgebra structure can be induced by an $r \in A \otimes A$. But unfortunately, such an r does not satisfy the 3-Lie CYBE or even a well-constructed algebraic equation on A.

The other classes in Proposition 4.20 also give non-trivial double construction 3-Lie bialgebras. All these examples follow from straightforward computations. For details see [18] which is based on the general framework in this paper.

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