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Entropy formula and conserved charges of spin-3 Chern-Simons-like theories of gravity

M. R. Setare and H. Adami

In this paper we present the generalization of Chern-Simons-like theories of gravity (CSLTG) to spin-3. We propose a Lagrangian describing the spin-3 fields coupled to Chern-Simons-like theories of gravity. Then we obtain conserved charges of these theories by using a quasi-local formalism. We find a general formula for entropy of black holes solutions of Spin-3 CSLTG. As an example, we apply our formalism to the spin-3 Generalized minimal massive gravity (GMMG) model. We analysis this model at linearized level and show that this model propagates two massive spin-2 modes and two massive spin-3 modes. We find no-ghost and no-tachyon conditions, which can be satisfied in the parameter space of the model. Then we find energy, angular momentum and entropy of a special black hole solution of this model.

1. Introduction

We know that 3-dimensional gravity is the simplest model for studying gravitational dynamics. However since it has rich physics in both classical [1] and quantum versions [2, 3]. So there are many motivations for studying gravity in 3 dimensions. By this study we can address conceptual issues of quantum gravity, investigate black hole evaporation, information loss, and black hole microstate counting. Also we can understand the black hole holography deeper. Gauge/gravity duality can be extended to the beyond standard AdS/CFT [4], such as warped AdS, asymptotic Schrödinger/Lifshitz, nonrelativistic CFTs, flat space holography, logarithmic CFTs, and higher spin gravity, which last topic is the subject of this paper.

It is well known that Einstein-Hilbert action in the presence of negative cosmological constant in 3-dimensions can be reformulated as a Chern-Simons theory with gauge group $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ [2, 5]. Similarly, a $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory with the following action describes a three dimensional spin-3 gravity theory [6, 7] ,

$$S_{EH} = S_{CS}[A^{+}] - S_{CS}[A^{-}], \tag{1}$$

where

$$S_{CS}[A] = \frac{l}{8\pi G} \int tr \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}, \tag{2}$$

where $l^2 > 0$ corresponds to a negative cosmological constant, and G is Newton's constant. In the other hand, there is a class of gravitational theories in $(2+1)$-dimensions (e.g. Topological massive gravity (TMG) [8], New massive gravity (NMG)[9], Minimal massive gravity (MMG) [10], Generalized minimal massive gravity (GMMG) [11], etc), called the Chern-Simons-like theories of gravity (CSLTG) [12][1]. The authors of [14] have done the generalization of TMG to higher spins, specifically spin-3 (see also [15]). In this paper we propose a Lagrangian describing spin-3 fields coupled to CSLTG,

[1]Recently, further clarifications on models in 3D are also collected in the book [13].

then we will obtain conserved charges of black hole solutions of these theories by using a quasi-local formalism. In previous paper we have obtained conserved charges of spin-3 TMG by this method [16]. In Ref. [17] the energy of the higher spin black hole solutions of ordinary higher spin gravity has been obtained by canonical formalism (see also [18]).

The authors of [19] have obtained the quasi-local conserved charges for black holes in any diffeomorphically invariant theory of gravity. By considering an appropriate variation of the metric, they have established a one-to-one correspondence between the ADT approach and the linear Noether expressions. They have extended that work to a theory of gravity containing a gravitational Chern-Simons term in [20], and have computed the off-shell potential and quasi-local conserved charges of some black holes in TMG.

Our paper is organized as follows. In Section 2 we summarize some relevant aspects of spin-3 gravity in three dimensions in the first order formalism. In Section 3 we introduce our Lagrangian for spin-3 fields coupled to CSLTG, which is an extension of the ordinary CSLTG. One can obtain the spin-3 TMG as a special case of our generic Lagrangian. In Section 4 we find the conserved charges of the spin-3 CSLTG by quasi-local formalism. By using that formalism, one can obtain conserve charges of solutions which are not asymptotically (A)dS. Then in Section 5, we consider a black hole solution of the Spin-3 CSLTG. After that, using obtained general formula for conserved charges, we find a general formula for entropy of black holes solutions of Spin-3 CSLTG. In Section 6 we consider spin-3 generalized minimal massive gravity (Spin-3 GMMG) as an example of the Spin-3 CSLTG. In Subsection 6.1 we obtain a class of solutions for that model. In Subsection 6.2, we do a linear analysis of spin-3 GMMG. We show that model has two massive spin-2 modes and two massive spin-3 modes. We will obtain the quadratic Lagrangian for the fluctuations about AdS_3 vacuum with vanishing spin-3 field. Then from quadratic Lagrangian $L^{(2)}$, we find the no-ghost conditions for spin-2 and spin-3 modes. Also, we will show that no-tachyon condition is $|lm_I| \geq 1$. Then in Section 7 we find energy, angular momentum and entropy of a special black hole solution of Spin-3 GMMG. In this case, if we set ${e_\mu}^{ab} = {\omega_\mu}^{ab} = 0$, our entropy formula reduces to the entropy formula in the ordinary GMMG which is obtained in the paper [21]. Also in the limiting case $\alpha = 0$, $m^2 \to \infty$, where Spin-3 GMMG model reduces to the spin-3 TMG model, our results for energy, angular momentum and entropy reduce to the corresponding results in the spin-3 TMG [16]. Section 8 is devoted to conclusions and discussions.

2. Spin-3 gravity in three dimensions

In this section, we summarize some relevant aspects of spin-3 gravity in three dimensions in the first order formalism. In this frame work, spin-3 gravity can be described by generalized dreibein and generalized spin-connection which take values in the Lie algebra $sl(3,\mathbb{R})$ [22, 23]:

$$e = e_\mu{}^A J_A dx^\mu = (e_\mu{}^a J_a + e_\mu{}^{ab} T_{ab}) dx^\mu, \tag{3}$$

$$\omega = \omega_\mu{}^A J_A dx^\mu = (\omega_\mu{}^a J_a + \omega_\mu{}^{ab} T_{ab}) dx^\mu_{ab}, \tag{4}$$

respectively, where J_A $(A = 1, \dots, 8)$ stand for the generators of $sl(3,\mathbb{R})$ algebra, J_a $(a = 1, 2, 3)$ denote generators of $sl(2,\mathbb{R})$ algebra, and T_{ab} are symmetric and trace-less in the Lorentz indices (appendix contains more technical details about eight generators of $sl(3,\mathbb{R})$). Here, we use the lower case Greek letters for the spacetime indices, and the internal Lorentz indices are denoted by the lower case Latin letters. The frame-like formalism can be translated to the metric-like one by [24]

$$g_{\mu\nu} = \frac{1}{2!} tr(e_{(\mu} e_{\nu)}), \tag{5}$$

$$\varphi_{\mu\nu\lambda} = \frac{1}{3!} tr(e_{(\mu} e_\nu e_{\lambda)}). \tag{6}$$

The spacetime metric $g_{\mu\nu}$ and the spin-3 field $\varphi_{\mu\nu\lambda}$ both are invariant under Lorentz-like gauge transformation

$$\tilde{e}_\mu = L e_\mu L^{-1}, \tag{7}$$

where $L \in SL(3,\mathbb{R})$ and we can write $L = \exp(\lambda)$, where λ is the generator of Lorentz-like transformation and it is a $sl(3,\mathbb{R})$ Lie algebra valued quantity,

$$\lambda = \lambda^A J_A = \lambda^a J_a + \lambda^{ab} T_{ab}. \tag{8}$$

The exterior covariant derivative can be defined as

$$D_\mu e_\nu = \partial_\mu e_\nu + [\omega_\mu, e_\nu], \tag{9}$$

which is covariant under Lorentz-like transformations as well. Generalized spin-connection transforms as

$$\tilde{\omega} = L \omega L^{-1} + L dL^{-1}, \tag{10}$$

under Lorentz-like transformations, where d denotes the ordinary exterior derivative. One can define the total derivative

$$D^{(T)}_{\mu}e_{\nu}=\partial_{\mu}e_{\nu}+[\omega_{\mu},e_{\nu}]-\Gamma^{\lambda}_{\mu\nu}e_{\lambda}, \tag{11}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is affine connection. The metric-connection compatibility condition $\nabla_{\lambda}g_{\mu\nu}=0$ leads to $D^{(T)}_{\mu}e_{\nu}=0$. Similar to ordinary gravity case, one can define generalized torsion and generalized curvature 2-forms respectively as

$$\mathcal{T}=e_{\lambda}\Gamma^{\lambda}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}=De, \tag{12}$$

$$\mathcal{R}=d\omega+\omega\wedge\omega, \tag{13}$$

Also, the ordinary Lie derivative of dreibein along a curve generated by the vector filed ξ, $\pounds_{\xi}e=i_{\xi}de+di_{\xi}e$, can be generalized so that it becomes covariant under Lorentz-like transformations. Here i_{ξ} denotes interior product in ξ. That generalization can occur by adding variation of e_{μ} with respect to an infinitesimal Lorentz-like gauge transformation:

$$\mathfrak{L}_{\xi}e=\pounds_{\xi}e+[\lambda_{\xi},e]. \tag{14}$$

In order that Eq.(14) to be covariant under Lorentz-like gauge transformations, generator of Lorentz-like gauge transformation λ_{ξ} must transform as

$$\tilde{\lambda}_{\xi}=L\lambda_{\xi}L^{-1}+L\pounds_{\xi}L^{-1}, \tag{15}$$

In this way, under generalized local translations we have

$$\delta_{\xi}e=\mathfrak{L}_{\xi}e, \tag{16}$$

$$\delta_{\xi}\omega=\mathfrak{L}_{\xi}\omega-d\lambda_{\xi}. \tag{17}$$

These two equations are covariant under the Lorentz-like gauge transformations as well as diffeomorphisms.

λ is introduced in Eq.(8) as generator of Lorentz-like gauge transformation where it is just an arbitrary function of coordinates. As we mentioned earlier, ordinary Lie derivative of a Lorentz-like invariant quantity, say e, is not Lorentz-like covariant. On the other hand, change in e under Lorentz-like gauge transformation is given by Eq.(7). As discussed, by combining the change due to infinitesimal Lorentz-like gauge transformation and ordinary Lie derivative one can define generalized Lie derivative (14) which is Lorentz-like covariant provided that λ transforms as (15) under Lorentz-like gauge transformation. Simply one can see form Eq.(15) that change in λ , under infinitesimal Lorentz-like gauge transformation, is given by $\delta\lambda = -\pounds_\xi \lambda$. So, we expect that λ can be regarded as a function of ξ as well, i.e. $\lambda = \lambda_\xi(x)$. Another reason for this comes from the fact that we set total variation induced by ξ equal to generalized Lie derivative with respect to ξ and λ_ξ is obliged to generate variation induced by ξ (See Eq.(16)). Thus, in the formalism presented in this paper, λ should be a function of coordinates and of ξ.

The following action describes a three dimensional spin-3 gravity theory

$$S_{EH} = \frac{1}{16\pi} \int tr \left\{ e \wedge \mathcal{R} + \frac{1}{3l^2} e \wedge e \wedge e \right\}. \tag{18}$$

Equations of motion arising from above action are:

$$\mathcal{T}(\Omega) = 0, \qquad \mathcal{R}(\Omega) + \frac{1}{l^2} e \wedge e = 0, \tag{19}$$

where $\omega = \Omega$ is torsion-free spin-connection[2].

3. Spin-3 Chern-Simons-like theories of gravity

The ordinary Chern-Simons-like theories of gravity are investigated in some papers, for instance see [9–12]. These type of theories are an extension of general relativity in 3D which is a gauge theory (to know more about gauge-theoretic approach to gravity see [27]). The ordinary Chern-Simons-like theories of gravity can be generalized to spin-3 one (Spin-3 CSLTG) by introducing Lagrangian

$$L = tr \left\{ \frac{1}{2} \tilde{g}_{rs} a^r \wedge da^s + \frac{1}{3} \tilde{f}_{rst} a^r \wedge a^s \wedge a^t \right\}, \tag{20}$$

where $a^r = a^{rA}{}_\mu J_A dx^\mu$ are $sl(3,\mathbb{R})$ Lie algebra valued one-forms and $r = 1, \ldots, N$ refers to flavour index. Also, $\tilde{g}_{rs}$ is a symmetric constant metric on the flavour space and $\tilde{f}_{rst}$ is a totally symmetric "flavour tensor"so that its components can be interpreted as the coupling constants. We take $a^r = \{e, \omega, h, \ldots\}$, where h is an auxiliary field and so on. For all of our interesting spin-3 CSLTG, we have $\tilde{f}_{\omega rs} = \tilde{g}_{rs}$. If one sets $a^{rab}_\mu = 0$ and uses Eq.(A.2) the Lagrangian (20) will be reduced to the Lagrangian of ordinary CSLTG.

[2] Torsional extension of Einstein's GR in 3D have been considered in [25], also rotating black hole solutions in a generalized topological 3D gravity with torsion, have analyzed in [26].

The arbitrary variation of Lagrangian (20) is

$$\delta L = tr\{\delta a^r \wedge E_r\} + d\Theta(a, \delta a), \tag{21}$$

where

$$E_r = \tilde{g}_{rs} da^s + \tilde{f}_{rst} a^s \wedge a^t, \tag{22}$$

and

$$\Theta(a, \delta a) = tr\left\{\frac{1}{2}\tilde{g}_{rs}\delta a^r \wedge a^s\right\}. \tag{23}$$

The equations of motion of these theories are $E_r = 0$ and $\Theta(a, \delta a)$ is surface term. By considering equations (16) and (17), under generalized local translations, a^r transforms as

$$\delta_\xi a^r = \mathfrak{L}_\xi a^r - \delta^r_\omega d\lambda_\xi, \tag{24}$$

where δ^r_ω is Kronecker delta. If we take the non-zero components of the flavour metric and the flavour tensor as

$$\begin{aligned} &\tilde{g}_{e\omega} = -\sigma, \quad \tilde{g}_{eh} = 1, \quad \tilde{g}_{\omega\omega} = \frac{1}{\mu}, \\ &\tilde{f}_{e\omega\omega} = -\sigma, \quad \tilde{f}_{eh\omega} = 1, \quad \tilde{f}_{\omega\omega\omega} = \frac{1}{\mu}, \quad \tilde{f}_{eee} = \Lambda, \end{aligned} \tag{25}$$

the Spin-3 CSLTG reduce to spin-3 topologically massive gravity (Spin-3 TMG) which is investigated at the linearized level in the papers [14, 15]. In paper [16] we have obtained conserved charges of this model.

4. Quasi-local conserved charges

In this section we want to find the conserved charges of the spin-3 CSLTG. Since e, ω and auxiliary fields appeared in CSLTG are invariant under general coordinates transformation it is clear that the Lagrangian of spin-3 CSLTG (20) is invariant under such a transformation. Now we recall that under general Lorentz-like gauge transformation, e and ω transform as (7) and (10), respectively. Also it is easy to check that under general Lorentz-like gauge transformation generalized curvature 2-form (13) and generalized torsion 2-form (12) transform as $\tilde{\mathcal{R}} = L\mathcal{R}L^{-1}$ and $\tilde{\mathcal{T}} = L\mathcal{T}L^{-1}$. Therefore it is inferred that three dimensional spin-3 gravity theory described by action (18) is a covariant theory under general Lorentz-like gauge transformation. Also, it is clear that equations of motion of spin-3 gravity theory, Eq.(19), are covariant under general Lorentz-like gauge transformation. Nevertheless, there may be exist theories which are not covariant under general Lorentz-like gauge transformation. In other words, theories described by the Lagrangian (20) may contain terms that can break covariance under general Lorentz-like gauge transformation. Spin-3 topologically massive gravity term, $\frac{1}{2\mu}tr\left(\omega\wedge d\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega\right)$, is an example of such terms. This term is appeared in the spin-3 TMG [14–16] and in spin-3 generalized minimal massive gravity which will be introduced in Section 6. Under generalized local translations, the Lagrangian (20) transforms as

$$\delta_\xi L=\mathfrak{L}_\xi L+d\psi_\xi, \tag{26}$$

where

$$\psi_\xi=tr\left\{\frac{1}{2}\tilde{g}_{\omega r}d\lambda_\xi\wedge a^r\right\}. \tag{27}$$

which is equivalent to the statement that a symmetry is a transformation which leaves the Lagrangian form invariant, up to a total derivative. Despite the fact that a Lagrangian is not invariant under general Lorentz-like gauge transformation, if a Lagrangian behaves like (26) under generalized local translations, then ξ could be a symmetry generator. Although the Lagrangian (20) is not invariant under general Lorentz-like gauge transformation, but by virtue of equations (26) and (27), it is invariant under the infinitesimal Lorentz-like gauge transformation. Also, it is enough in obtaining generally covariant equations of motion that Lagrangian behaves like (26) under generalized local translations. Now, we consider the variation of Lagrangian induced by generalized local translations

$$\delta_\xi L=tr\{\delta_\xi a^r\wedge E_r\}+d\Theta(a,\delta_\xi a), \tag{28}$$

Comparing Eq.(26) and Eq.(28) leads to the following relation

$$d\left(\Theta(a,\delta_\xi a)-i_\xi L-\psi_\xi\right)=-tr\{\delta_\xi a^r\wedge E_r\}. \tag{29}$$

We can rewrite Eq.(24) as

$$\begin{aligned}\delta_\xi a^{r'}&=Di_\xi a^{r'}+i_\xi Da^{r'}+[(\lambda_\xi-i_\xi\omega),a^{r'}],\\ \delta_\xi\omega&=i_\xi\mathcal{R}-D(\lambda_\xi-i_\xi\omega),\end{aligned} \tag{30}$$

where the prime on r indicates that the sum runs over all the flavour indices except ω. Now, we substitute equations (30) into Eq.(25) and we find that

$$d\mathfrak{J}_\xi = -tr\{(\lambda_\xi - i_\xi\omega)(DE_\omega + a^{r'} \wedge E_{r'} - E_{r'} \wedge a^{r'})\} + tr\{i_\xi a^{r'} DE_{r'} - i_\xi Da^{r'} \wedge E_{r'} - i_\xi \mathcal{R} \wedge E_\omega\}, \tag{31}$$

where $\mathfrak{J}_\xi$ is given by

$$\mathfrak{J}_\xi = \Theta(a, \delta_\xi a) - i_\xi L - \psi_\xi + tr\{i_\xi a^r E_r - \lambda_\xi E_\omega\}. \tag{32}$$

We expect that the last line in Eq.(31) can be rewritten as

$$tr\{i_\xi a^{r'} DE_{r'} - i_\xi Da^{r'} \wedge E_{r'} - i_\xi \mathcal{R} \wedge E_\omega\} = tr\{i_\xi a^{r'} X_{r'}(a)\}, \tag{33}$$

see [16] for the Spin-3 TMG case. In order to have an off-shell conserved current $\mathfrak{J}_\xi$ we must have

$$DE_\omega + a^{r'} \wedge E_{r'} - E_{r'} \wedge a^{r'} = 0, \qquad X_{r'}(a) = 0. \tag{34}$$

These equations give us the Bianchi identities and they reduce to the following one for the Spin-3 TMG model [16]

$$\begin{aligned} D\mathcal{R} &= 0, \\ D\mathcal{T} + e \wedge \mathcal{R} - \mathcal{R} \wedge e &= 0. \end{aligned} \tag{35}$$

In this way, through the Bianchi identities, $\mathfrak{J}_\xi$ is conserved off-shell, i.e. $d\mathfrak{J}_\xi = 0$. Accordingly, by virtue of Poincaré lemma, we find that

$$\mathfrak{J}_\xi = dK_\xi, \tag{36}$$

where K_ξ is given by

$$K_\xi = tr\left\{\frac{1}{2}\tilde{g}_{rs} i_\xi a^r a^s - \tilde{g}_{r\omega}\lambda_\xi a^r\right\}. \tag{37}$$

It is straightforward to show that under generalized local translations, $\Theta(a, \delta a)$ transforms as

$$\delta_\xi \Theta(a, \delta a) = \mathfrak{L}_\xi \Theta(a, \delta a) + \Pi_\xi, \tag{38}$$

where

$$\Pi_\xi = tr\left\{\frac{1}{2}\tilde{g}_{r\omega} d\lambda_\xi \wedge \delta a^r\right\}. \tag{39}$$

By taking arbitrary variation of the equation (36) and using Eq.(32) we have

$$\begin{aligned} d\left(\delta K_\xi - i_\xi \Theta(a,\delta a)\right) = tr\{\delta a^r \wedge i_\xi E_r + i_\xi a^r \delta E_r - \lambda_\xi \delta E_\omega\} \\ + \delta\Theta(a,\delta_\xi a) - \delta_\xi \Theta(a,\delta a), \end{aligned} \tag{40}$$

where we have used the relation $\Pi_\xi = \delta\psi_\xi$ in the last step of calculations. On the other hand, if we demand that ξ be a Killing vector field admitted by spacetime everywhere, we have the following configuration space result given in [28]

$$\delta\Theta(a,\delta_\xi a) - \delta_\xi \Theta(a,\delta a) = 0, \tag{41}$$

then the right hand side of the equation (40) is the off-shell ADT current,

$$\mathcal{J}_{ADT} = tr\{\delta a^r \wedge i_\xi E_r + i_\xi a^r \delta E_r - \lambda_\xi \delta E_\omega\}. \tag{42}$$

By substituting the components of flavour metric and flavour tensor from Eq.(25) into above equation, current $\mathcal{J}_{ADT}$ reduces to the off-shell ADT current appearing in the Spin-3 TMG[16]. By demanding that field equations, $E_r = 0$, and linearized field equations, $\delta E_r = 0$, be held the right hand side of the equation (40) is simply the symplectic current

$$\Omega_{symplectic}(a,\delta a,\delta_\xi a) = \delta\Theta(a,\delta_\xi a) - \delta_\xi \Theta(a,\delta a). \tag{43}$$

Hence, it seems reasonable to generalize the off-shell ADT current so that it becomes conserved when spacetime admits ξ as an asymptotically Killing vector field rather than a Killing vector field admitted by spacetime everywhere. So, we can define generalized off-shell ADT current as

$$\tilde{\mathcal{J}}_{ADT} = \mathcal{J}_{ADT} + \Omega_{symplectic}. \tag{44}$$

As mentioned above, this generalized off-shell ADT current will reduce to the ordinary one when ξ is a Killing vector field admitted by spacetime everywhere and to the symplectic current when $E_r = \delta E_r = 0$, and ξ is an asymptotically Killing vector field. By substituting Eq.(44) into (40) we have

$$\tilde{\mathcal{J}}_{ADT} = d\left(\delta K_{\xi} - i_{\xi}\Theta(a, \delta a)\right), \tag{45}$$

so, it is obvious that the generalized off-shell ADT current is conserved for Killing vectors which are admitted by spacetime everywhere as well as asymptotically Killing vectors. Hence we can define generalized off-shell ADT conserved charge as

$$\tilde{\mathcal{Q}}_{ADT}(a, \delta a; \xi) = \delta K_{\xi} - i_{\xi}\Theta(a, \delta a), \tag{46}$$

for which $\tilde{\mathcal{J}}_{ADT} = d\tilde{\mathcal{Q}}_{ADT}$. Now, in the manner of papers [19, 20], we define quasi-local conserved charge conjugate to (asymptotically) Killing vector field ξ as

$$Q(\xi) = \frac{1}{8\pi}\int_0^1 ds \int_{\Sigma} \tilde{\mathcal{Q}}_{ADT}(a|s), \tag{47}$$

where Σ denotes an arbitrary codimension two space-like surface and integration with respect to s runs over one-parameter path in the solution space, where $s = 0$ and $s = 1$ are corresponded to the background solution and the interested solution, respectively.

By substituting the explicit forms of K_{ξ} and $\Theta(a, \delta a)$ into Eq.(47) we find that

$$Q(\xi) = \frac{1}{8\pi}\int_0^1 ds \int_{\Sigma} tr\{(\tilde{g}_{rs} i_{\xi} a^{s} - \tilde{g}_{r\omega}\lambda_{\xi})\delta a^{r}\}. \tag{48}$$

The advantage of this formalism is that it is applicable to solutions that are not asymptotically (A)dS. Assuming that $(\delta_1\delta_2 - \delta_2\delta_1)a^r = 0$, the conserved charge (48) satisfy the integrability condition $(\delta_1\delta_2 - \delta_2\delta_1)Q(\xi) = 0$ [28]. Hence, the result of Eq.(48) does not depend on the given path on the solution space.

5. A general formula for the entropy of black holes

Let us consider a black hole solution of the Spin-3 CSLTG. We take the codimension two surface Σ to be the bifurcation surface $\mathcal{B}$. Suppose that ξ is the horizon-generating Killing vector field, so we must set $\xi = 0$ on

$\mathcal{B}$.[3] Since Eq.(48) is conserved for Killing vectors which are admitted by spacetime everywhere and for asymptotically Killing vectors, so we can use Eq.(48) to find the entropy of black holes in the context of considered theory as a conserved charge corresponds to the horizon-generating Killing vector field. Thus, the conserved charge expression (48) reduces to

$$Q(\xi) = -\frac{1}{8\pi}\int_0^1 ds \int_{\mathcal{B}} tr\{\tilde{g}_{r\omega}\lambda_\xi \delta a^r\}. \tag{49}$$

Now, we take $s=0$ and $s=1$ correspond to the interested black hole spacetime and the perturbed one respectively. Thus, the equation (48) becomes

$$\delta Q(\xi) = -\frac{1}{8\pi}\tilde{g}_{r\omega}\int_{\mathcal{B}} tr\{\lambda_\xi \delta a^r\}. \tag{50}$$

Since λ_ξ does not depend on dynamical fields at all, we obtain

$$Q(\xi) = -\frac{1}{8\pi}\tilde{g}_{r\omega}\int_{\mathcal{B}} tr\{\lambda_\xi a^r\}. \tag{51}$$

By demanding that the generalized Lie derivative of generalized dreibein vanish explicitly when ξ is a Killing vector field, one can find an expression for λ_ξ [16]

$$\lambda_\xi = i_\xi \omega + \frac{1}{2}[e^\nu, D_\nu(i_\xi e) + (i_\xi \mathcal{T})_\nu]. \tag{52}$$

One can use Eq.(7) and Eq.(10) to show that Eq.(52) satisfy the transformation property (15). It has been shown that Eq.(52) reduces to [16]

$$\lambda_\xi = \frac{\kappa}{\sqrt{g_{\phi\phi}}} e_\phi, \tag{53}$$

on bifurcation surface $\mathcal{B}$, where κ is surface gravity. By substituting Eq.(53) into Eq.(51), we can define the entropy of a given black hole solution as

$$S = \frac{8\pi}{\kappa} Q(\xi) = -\tilde{g}_{r\omega}\int_{\mathcal{B}} \frac{d\phi}{\sqrt{g_{\phi\phi}}} tr\{e_\phi a^r_\phi\}. \tag{54}$$

[3] Here, we consider stationary black hole solutions and we assume that the event horizon of considered black hole is a non-degenerate Killing horizon. As we know, a cross-section of non-degenerate Killing horizon is bifurcation surface where $\xi = 0$.

6. Spin-3 generalized minimal massive gravity

In this section, we introduce spin-3 generalized minimal massive gravity (spin-3 GMMG) and analysis it at linearized level.

6.1. The model

We consider spin-3 GMMG as an example of the Spin-3 CSLTG. The ordinary GMMG have studied originally in [11]. The GMMG model is realized by adding the CS deformation term, the higher derivative deformation term, and an extra term to pure Einstein gravity with a negative cosmological constant. In Ref.[11], it has been discussed that this model is free of negative-energy bulk modes, and also avoids the aforementioned “bulk-boundary unitarity clash”. By a Hamiltonian analysis one can show that the GMMG model has no Boulware-Deser ghosts and propagates only two physical modes. In this model, there are four flavours of Lie algebra valued one-form $a^r = \{e, \omega, h, f\}$ and the non-zero components of flavour metric and flavour tensor are

$$\begin{aligned}&\tilde{g}_{e\omega} = -\sigma, \quad \tilde{g}_{eh} = 1, \quad \tilde{g}_{f\omega} = -\frac{1}{m^2}, \quad \tilde{g}_{\omega\omega} = \frac{1}{\mu},\\ &\tilde{f}_{e\omega\omega} = -\sigma, \quad \tilde{f}_{eh\omega} = 1, \quad \tilde{f}_{f\omega\omega} = -\frac{1}{m^2}, \quad \tilde{f}_{\omega\omega\omega} = \frac{1}{\mu},\\ &\tilde{f}_{eee} = \Lambda_0, \quad \tilde{f}_{ehh} = \alpha, \quad \tilde{f}_{eff} = -\frac{1}{m^2}.\end{aligned} \tag{55}$$

Thus, equations of motion (22) reduce to

$$\begin{aligned}-\sigma\mathcal{R}(\omega) + \Lambda_0 e\wedge e + D(\omega)h + \alpha h\wedge h - \frac{1}{m^2} f\wedge f = 0,\\ -\sigma\mathcal{T}(\omega) + \frac{1}{\mu}\mathcal{R}(\omega) + e\wedge h + h\wedge e - \frac{1}{m^2} D(\omega)f = 0,\\ \mathcal{R}(\omega) + e\wedge f + f\wedge e = 0,\\ \mathcal{T}(\omega) + \alpha(e\wedge h + h\wedge e) = 0.\end{aligned} \tag{56}$$

It is clear that the Spin-3 GMMG is not a torsion-free theory, but by redefinition of generalized spin-connection, as $\omega = \Omega - \alpha h$, we make it torsion-free. In this case we have

$$\begin{aligned}\mathcal{R}(\omega) = \mathcal{R}(\Omega) - \alpha D(\Omega)h + \alpha^2 h\wedge h,\\ D(\omega)f = D(\Omega)f - \alpha(h\wedge f + f\wedge h),\end{aligned} \tag{57}$$

then equations of motion (56) can be rewritten as

$$-\sigma\mathcal{R}(\Omega)+\Lambda_0 e\wedge e+(1+\alpha\sigma)(D(\Omega)h-\alpha h\wedge h)-\frac{1}{m^2}f\wedge f=0, \tag{58}$$

$$\mathcal{R}(\Omega)-\alpha D(\Omega)h+\alpha^2 h\wedge h+\mu(1+\alpha\sigma)(e\wedge h+h\wedge e) \\ -\frac{\mu}{m^2}D(\Omega)f+\frac{\mu\alpha}{m^2}(f\wedge h+h\wedge f)=0, \tag{59}$$

$$\mathcal{R}(\Omega)-\alpha D(\Omega)h+\alpha^2 h\wedge h+e\wedge f+f\wedge e=0, \tag{60}$$

$$\mathcal{T}(\Omega)=0. \tag{61}$$

Now, we want to find some solutions of the considered model. For this purpose, we consider the following ansatz for h and f

$$h=\beta e, \qquad f=\gamma e, \tag{62}$$

where e is generalized dreibein and β and γ are constants. By substituting Eq.(62) into equations (58)–(60) we have

$$\begin{aligned} -\sigma\mathcal{R}(\Omega)+\left(\Lambda_0-\alpha(1+\alpha\sigma)\beta^2-\frac{\gamma^2}{m^2}\right)\ e\wedge e=0,\\ \mathcal{R}(\Omega)+\left(\alpha^2\beta^2+2\mu(1+\alpha\sigma)\beta+\frac{2\mu\alpha}{m^2}\beta\gamma\right)e\wedge e=0,\\ \mathcal{R}(\Omega)+\left(\alpha^2\beta^2+2\gamma\right)e\wedge e=0. \end{aligned} \tag{63}$$

By comparing these equations with Eq.(19) we find that

$$\Lambda_0-\alpha(1+\alpha\sigma)\beta^2-\frac{\gamma^2}{m^2}=-\frac{\sigma}{l^2}, \tag{64}$$

$$\alpha^2\beta^2+2\mu(1+\alpha\sigma)\beta+\frac{2\mu\alpha}{m^2}\beta\gamma=\frac{1}{l^2}, \tag{65}$$

$$\alpha^2\beta^2+2\gamma=\frac{1}{l^2}. \tag{66}$$

In this way, all solutions of the spin-3 gravity (for instance, see [29–31]) are solutions of the Spin-3 GMMG when β, γ and parameters of the considered model satisfy equations (64)–(66). From equations (64)–(66), one finds that

$$l_{\pm}^2=\left[\alpha^2\beta^2+2\sigma m^2\pm2\sqrt{m^2\left(\sigma^2m^2-\alpha\beta^2+\Lambda_0\right)}\right]^{-1}, \tag{67}$$

$$\gamma_{\pm}=\sigma m^2\pm\sqrt{m^2\left(\sigma^2m^2-\alpha\beta^2+\Lambda_0\right)}, \tag{68}$$

where β should satisfy following quartic equation

$$\begin{aligned}(69)\qquad & -\Lambda_0 m^4 + 2\mu m^2\left[\alpha\Lambda_0 - \sigma m^2(1+\alpha\sigma)\right]\beta \\ & + \left[\mu^2 m^2(1+\alpha\sigma)(1+3\alpha\sigma) + \alpha\left(m^4 - \alpha\mu^2\Lambda_0\right)\right]\beta^2 \\ & - 2\mu\alpha^2 m^2\beta^3 + \mu\alpha^3\beta^4 = 0.\end{aligned}$$

We have following conditions on roots of quartic equation (69)

$$(70)\qquad \begin{aligned} \beta^2 &\leq \alpha^{-1}\left(\sigma^2 m^2 + \Lambda_0\right) \quad \text{for} \quad \alpha > 0, \\ \beta^2 &\geq \alpha^{-1}\left(\sigma^2 m^2 + \Lambda_0\right) \quad \text{for} \quad \alpha < 0,\end{aligned}$$

which ensure that l^2 and γ are real. It should be noted that β and γ could not be complex numbers because they appear in energy, angular momentum and entropy experssions (see equations (113), (116) and (119)).

6.2. Linearized Analysis

The Lagrangian of spin-3 GMMG up to a surface term can be written as

$$\begin{aligned}(71)\qquad L_{\text{spin-3 GMMG}} = tr\bigg\{ & -\sigma e\wedge\mathcal{R}(\omega) + \frac{\Lambda_0}{3}e\wedge e\wedge e + h\wedge\mathcal{T}(\omega) \\ & + \alpha e\wedge h\wedge h + \frac{1}{2\mu}\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right) \\ & - \frac{1}{m^2}\left(f\wedge\mathcal{R}(\omega) + e\wedge f\wedge f\right)\bigg\}.\end{aligned}$$

We assume that the background generalized dreibein and torsion-free generalized spin-connection are given by $\bar{e}$ and $\bar{\Omega}$ respectively, where $\bar{e}_\mu{}^{ab} = 0$ and $\bar{\Omega}_\mu{}^{ab} = 0$.[4] Also, $\bar{e}_\mu{}^a$ and $\bar{\Omega}_\mu{}^a$ describe AdS_3 vacuum. We can take $\bar{h} = \beta\bar{e}$ and $\bar{f} = \gamma\bar{e}$, where β and γ are just constant parameters, and then background is a solution of equations of motion (58)–(61) provided that equations (64)–(66) are satisfied. Therefore, we have

$$(72)\qquad D(\bar{\Omega})\bar{e} = 0, \qquad \mathcal{R}(\bar{\Omega}) + \frac{1}{l^2}\bar{e}\wedge\bar{e} = 0.$$

[4]Refer to [10, 32] to see such analysis for minimal massive gravity, and new version of generalized zwei-dreibein gravity, respectively. See also [14, 15] for spin-3 topologically massive gravity.

We now expand e, ω, h and f about the background as follows:

$$
\begin{aligned}
e &= \bar{e} + \hat{\varepsilon}u, & \Omega &= \bar{\Omega} + \hat{\varepsilon}v \\
h &= \beta(\bar{e} + \hat{\varepsilon}u) + \hat{\varepsilon}w, & f &= \gamma(\bar{e} + \hat{\varepsilon}u) + \hat{\varepsilon}z
\end{aligned} \tag{73}
$$

where $\hat{\varepsilon}$ is a small expansion parameter. By substituting these expressions into the Lagrangian (71) and using (64)–(66), one can show that linear term in Lagrangian expansion vanishes and quadratic Lagrangian for the fluctuations u, v, w and z is given by

$$
\begin{aligned}
L^{(2)} = tr\Bigg\{ & \frac{1}{2\mu l^2} u \wedge \bar{D}u + \frac{1}{2\mu} v \wedge \bar{D}v - \left(\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2}\right) u \wedge \bar{D}v \\
& + \frac{\alpha^2}{2\mu} w \wedge \bar{D}w + \left(1 + \alpha\sigma + \frac{\alpha^2\beta}{\mu} + \frac{\alpha\gamma}{m^2}\right) u \wedge \bar{D}w \\
& - \frac{\alpha}{\mu} v \wedge \bar{D}w - \frac{1}{m^2} v \wedge \bar{D}z + \frac{\alpha\beta}{m^2} u \wedge \bar{D}z \\
& + \frac{\alpha}{m^2} z \wedge \bar{D}w - \left(\frac{\sigma}{l^2} + \frac{\alpha\beta}{\mu l^2} + \frac{2\gamma^2}{m^2}\right) \bar{e} \wedge u \wedge u \\
& - \left(\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2}\right) \bar{e} \wedge v \wedge v + \frac{1}{\mu l^2} \left(\bar{e} \wedge u \wedge v + \bar{e} \wedge v \wedge u\right) \\
& - \alpha \left(1 + \alpha\sigma + \frac{\alpha^2\beta}{\mu} + \frac{\alpha\gamma}{m^2}\right) \bar{e} \wedge w \wedge w \\
& - \frac{\alpha}{\mu l^2} \left(\bar{e} \wedge u \wedge w + \bar{e} \wedge w \wedge u\right) \\
& + \left(1 + \alpha\sigma + \frac{\alpha^2\beta}{\mu} + \frac{\alpha\gamma}{m^2}\right) \left(\bar{e} \wedge v \wedge w + \bar{e} \wedge w \wedge v\right) \\
& + \frac{\alpha\beta}{m^2} \left(\bar{e} \wedge v \wedge z + \bar{e} \wedge z \wedge v\right) \\
& - \frac{1}{m^2 l^2} \left(\bar{e} \wedge u \wedge z + \bar{e} \wedge z \wedge u\right) \Bigg\}
\end{aligned} \tag{74}
$$

where $\bar{D} = D(\bar{\Omega})$. One can extract the linearized equations of motion from the Lagrangian (74), or equivalently, from linearization of the equations of motion (58)–(61). Then, linearized equations of motion are

$$\begin{aligned}
&\bar{D}u + (\bar{e} \wedge v + v \wedge \bar{e}) = 0, \\
&\bar{D}v + \frac{1}{l^2}(\bar{e} \wedge u + u \wedge \bar{e}) + \left[1 + \alpha\sigma - \frac{\alpha\gamma}{m^2}\right](\bar{e} \wedge z + z \wedge \bar{e}) = 0, \\
&\bar{D}w - \alpha\beta(\bar{e} \wedge w + w \wedge \bar{e}) + \left[\sigma - \frac{\gamma}{m^2}\right](\bar{e} \wedge z + z \wedge \bar{e}) = 0, \\
&\bar{D}z + \left[\frac{m^2}{\mu} - \alpha\beta\right](\bar{e} \wedge z + z \wedge \bar{e}) - m^2\left[1 + \alpha\sigma + \frac{\alpha\gamma}{m^2}\right](\bar{e} \wedge w + w \wedge \bar{e}) = 0.
\end{aligned} \tag{75}$$

Now, we introduce a transformation from (u, v, w, z) to new Lie algebra valued one-form fluctuations (q_+, q_-, q_1, q_2):

$$\begin{aligned}
u =& B_+q_+ + B_-q_- + B_1q_1 + B_2q_2, \\
v =& m_+B_+q_+ + m_-B_-q_- + m_1B_1q_1 + m_2B_2q_2, \\
w =& C_1B_1q_1 + C_2B_2q_2, \\
z =& F_1B_1q_1 + F_2B_2q_2,
\end{aligned} \tag{76}$$

where $\{B_+, B_-, B_1, B_2\}$ are arbitrary constants and $\{m_+, m_-, m_1, m_2\}$ are given by

$$\begin{aligned}
&m_+ = \frac{1}{l}, \qquad m_- = -\frac{1}{l} \\
&m_1 = \frac{m^2}{2\mu} - \alpha\beta + \left[\frac{m^4}{4\mu^2} + \gamma + \frac{\alpha\gamma^2}{m^2} - (1 + \alpha\sigma)m^2\sigma\right]^{\frac{1}{2}}, \\
&m_2 = \frac{m^2}{2\mu} - \alpha\beta - \left[\frac{m^4}{4\mu^2} + \gamma + \frac{\alpha\gamma^2}{m^2} - (1 + \alpha\sigma)m^2\sigma\right]^{\frac{1}{2}},
\end{aligned} \tag{77}$$

also,

$$\begin{aligned}
&C_1 = \frac{\left(m_1^2 - \frac{1}{l^2}\right)(m_2 + \alpha\beta)}{m^2\left[(1 + \alpha\sigma)^2 - \left(\frac{\alpha\gamma}{m^2}\right)^2\right]}, \qquad C_2 = \frac{\left(m_2^2 - \frac{1}{l^2}\right)(m_1 + \alpha\beta)}{m^2\left[(1 + \alpha\sigma)^2 - \left(\frac{\alpha\gamma}{m^2}\right)^2\right]}, \\
&F_1 = \frac{\left(m_1^2 - \frac{1}{l^2}\right)}{\left[1 + \alpha\sigma - \frac{\alpha\gamma}{m^2}\right]}, \qquad F_2 = \frac{\left(m_2^2 - \frac{1}{l^2}\right)}{\left[1 + \alpha\sigma - \frac{\alpha\gamma}{m^2}\right]}.
\end{aligned} \tag{78}$$

By using Eq.(76) we can diagonalize the linearized equations (75) as follows

$$
\begin{aligned}
&\bar{D}q_+ + m_+ \left(\bar{e} \wedge q_+ + q_+ \wedge \bar{e}\right) = 0,\\
&\bar{D}q_- + m_- \left(\bar{e} \wedge q_- + q_- \wedge \bar{e}\right) = 0,\\
&\bar{D}q_1 + m_1 \left(\bar{e} \wedge q_1 + q_1 \wedge \bar{e}\right) = 0,\\
&\bar{D}q_2 + m_2 \left(\bar{e} \wedge q_2 + q_2 \wedge \bar{e}\right) = 0.
\end{aligned} \tag{79}
$$

In this manner, we expect that the introduced transformation, Eq.(76), diagonalize the Lagrangian (74). So we can rewrite the Lagrangian (74) in the diagonalized form, in terms of new Lie algebra valued 1-form fields,

(80)

$$
\begin{aligned}
-L^{(2)} = tr\Big\{ &\left[\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2} - \frac{1}{\mu l}\right] B_+^2 m_+ \left(q_+ \wedge \bar{D}q_+ + m_+ \bar{e} \wedge q_+ \wedge q_+\right)\\
&+ \left[\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2} + \frac{1}{\mu l}\right] B_-^2 m_- \left(q_- \wedge \bar{D}q_- + m_- \bar{e} \wedge q_- \wedge q_-\right)\\
&+ B_1^2 \tilde{B}_1 m_1 \left(q_1 \wedge \bar{D}q_1 + m_1 \bar{e} \wedge q_1 \wedge q_1\right)\\
&+ B_2^2 \tilde{B}_2 m_2 \left(q_2 \wedge \bar{D}q_2 + m_1 \bar{e} \wedge q_2 \wedge q_2\right)\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{B}_1 =& -\frac{1}{2\mu} m_1 + \left(\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2}\right) + \frac{\alpha C_1}{\mu} + \frac{F_1}{m^2} - \frac{1}{2\mu l^2 m_1} - \frac{\alpha^2 C_1^2}{2\mu m_1}\\
&- \left(1 + \alpha\sigma + \frac{\alpha^2\beta}{\mu} + \frac{\alpha\gamma}{m^2}\right)\frac{C_1}{m_1} - \frac{\alpha\beta F_1}{m^2 m_1} - \frac{\alpha F_1 C_1}{m^2 m_1},\\
\tilde{B}_2 =& -\frac{1}{2\mu} m_2 + \left(\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2}\right) + \frac{\alpha C_2}{\mu} + \frac{F_2}{m^2} - \frac{1}{2\mu l^2 m_2} - \frac{\alpha^2 C_2^2}{2\mu m_2}\\
&- \left(1 + \alpha\sigma + \frac{\alpha^2\beta}{\mu} + \frac{\alpha\gamma}{m^2}\right)\frac{C_2}{m_2} - \frac{\alpha\beta F_2}{m^2 m_2} - \frac{\alpha F_2 C_2}{m^2 m_2}.
\end{aligned} \tag{81}
$$

Two first terms in Lagrangian (80) can be written as difference of two linearized $SL(3,R)$ Chern-Simons 3-forms and then the $q_\pm$ fields have no local degrees of freedom for spin-2 and spin-3 fields. Each of two last terms in Lagrangian (80) describes a single spin-2 massive mode and a single trace-less spin-3 massive mode. Therefore spin-3 GMMG model propagate 2 massive graviton modes and 2 massive spin-3 modes. The massive modes are not ghosts as long as $\tilde{B}_1$ and $\tilde{B}_2$ are both positive definite.

Now, we want to find the mass of spin-2 and spin-3 modes which are appeared in this model. To this end, we use Eq.(73) and Eq.(76) to write

$$e = \bar{e} + \hat{\varepsilon} \sum_I B_I q_I, \qquad (I = +, -, 1, 2). \tag{82}$$

where q_I is a $SL(3, R)$ Lie algebra valued 1-form

$$q_{I\mu} = q_{I\mu}{}^a J_a + q_{I\mu}{}^{ab} T_{ab}, \tag{83}$$

with $q_{I\mu a}{}^a = 0$, that is $q_{I\mu}{}^{ab}$ is trace-less in Lorentz indices. In this way, equations (79) can be written as

$$\bar{D} q_I + m_I \left(\bar{e} \wedge q_I + q_I \wedge \bar{e} \right) = 0, \tag{84}$$

where $I = +, -, 1, 2$. By substituting Eq.(82) into Eq.(5), we find metric fluctuations about AdS$_3$ metric $\bar{g}_{\mu\nu}$ as

$$\mathbf{h}_{I\mu\nu} = \frac{1}{2} tr \left\{ \bar{e}_\mu q_{I\nu} \right\} = \bar{e}_{\mu a} q_{I\nu}{}^a, \tag{85}$$

where $tr \left\{ \bar{e}_{[\mu} q_{I\nu]} \right\} = 0$ was assumed. Let $\bar{\nabla}_\mu$ denote covariant derivative with respect to the connection $\bar{\Gamma}^\alpha_{\mu\nu}$ compatible with background metric $\bar{g}_{\mu\nu}$ and $\bar{\epsilon}_{\mu\nu\lambda} = \varepsilon_{abc} \bar{e}_\mu{}^a \bar{e}_\nu{}^b \bar{e}_\lambda{}^c$. Let $\bar{D}^{(T)}_\mu$ denote total derivative compatible with the background dreibein $\bar{e}_\mu{}^A$, i.e. $\bar{D}^{(T)}_\mu \bar{e}_\nu = 0$ [5]. Therefore, we can write

$$\bar{\epsilon}^{\alpha\beta\nu} \bar{\nabla}_\beta \mathbf{h}_{I\mu\nu} = \frac{1}{2} \bar{\epsilon}^{\alpha\beta\nu} \bar{D}^{(T)}_\beta \left[tr \left\{ \bar{e}_\mu q_{I\nu} \right\} \right] = \frac{1}{2} \bar{\epsilon}^{\alpha\beta\nu} tr \left\{ \bar{e}_\mu \bar{D}_\beta q_{I\nu} \right\}. \tag{86}$$

By substituting Eq.(84) into Eq.(86) and using trace-less condition $\mathbf{h}_I{}^\alpha{}_\alpha = 0$, we find a first-order equation for $\mathbf{h}_I$:

$$\mathcal{D}[m_I]^\mu{}_\alpha \mathbf{h}_I{}^\alpha{}_\nu = 0, \tag{87}$$

where

$$\mathcal{D}[m_I]^\mu{}_\nu = m_I \delta^\mu{}_\nu + \bar{\epsilon}^{\mu\beta}{}_\nu \bar{\nabla}_\beta. \tag{88}$$

In order to find the second-order equation, we apply the operator $\mathcal{D}[-m_I]$ on the equation (87), that is

$$\mathcal{D}[-m_I]^\mu{}_\alpha \mathcal{D}[m_I]^\alpha{}_\beta \mathbf{h}_I{}^\beta{}_\nu = 0. \tag{89}$$

---

[5] $\bar{D}^{(T)}_\mu \bar{e}_\nu = \partial_\mu \bar{e}_\nu + [\bar{\Omega}_\mu, \bar{e}_\nu] - \bar{\Gamma}^\lambda_{\mu\nu} \bar{e}_\lambda$.

In this way, one obtains a second-order equation

$$\left(\bar{\square} - \mathcal{M}_I^2 + \frac{2}{l^2}\right) \mathbf{h}_{I\mu\nu} = 0, \tag{90}$$

with

$$\mathcal{M}_I^2 = m_I^2 - \frac{1}{l^2}, \tag{91}$$

where transverse condition $\bar{\nabla}_\alpha \mathbf{h}_I{}^\alpha{}_\nu = 0$ was used. It is clear that, for $I = 1$ and 2, Eq.(90) is Fierz-Pauli spin-2 field equation in AdS_3 for a spin-2 field $\mathbf{h}_{I\mu\nu}$ of mass $\mathcal{M}_I$ [6].

Now, by substituting Eq.(82) into Eq.(6), we obtain spin-3 fluctuations

$$\mathbf{H}_{I\mu\nu\lambda} = \frac{1}{2} tr \left\{\bar{e}_\mu \bar{e}_\nu q_{I\lambda}\right\} = \bar{e}_{\mu a} \bar{e}_{\nu b} q_{I\lambda}{}^{ab}, \tag{92}$$

where $tr\left\{\bar{e}_{[\mu} \bar{e}_\nu q_{I\lambda]}\right\} = 0$ was assumed. Also, it should be noted that $\bar{\varphi}_{\mu\nu\lambda} = \frac{1}{3!} tr(\bar{e}_\mu \bar{e}_\nu \bar{e}_\lambda) = 0$. Therefore, we can write

$$\bar{\epsilon}^{\alpha\beta\lambda} \bar{\nabla}_\beta \mathbf{H}_{I\mu\nu\lambda} = \frac{1}{2} \bar{\epsilon}^{\alpha\beta\lambda} tr\left\{\bar{e}_\mu \bar{e}_\nu \bar{D}_\beta q_{I\lambda}\right\}. \tag{93}$$

By substituting Eq.(84) into Eq.(93), we find a first-order equation for $\mathbf{H}_I$:

$$\tilde{\mathcal{D}}[m_I]^\mu{}_\alpha \mathbf{H}_I{}^\alpha{}_{\nu\lambda} = 0, \tag{94}$$

with

$$\tilde{\mathcal{D}}[m_I]^\mu{}_\nu = 2m_I \delta^\mu{}_\nu + \bar{\epsilon}^{\mu\beta}{}_\nu \bar{\nabla}_\beta. \tag{95}$$

where trace-less condition $\mathbf{H}_I{}^\alpha{}_{\alpha\mu\nu} = 0$ was used. Now, we apply the operator $\tilde{\mathcal{D}}[-m_I]$ on the equation (94), that is

$$\tilde{\mathcal{D}}[-m_I]^\mu{}_\alpha \tilde{\mathcal{D}}[m_I]^\alpha{}_\beta \mathbf{H}_I{}^\beta{}_{\nu\lambda} = 0. \tag{96}$$

Then, one obtains a second-order equation

$$\left(\bar{\square} - \tilde{\mathcal{M}}_I^2\right) \mathbf{H}_{I\mu\nu\lambda} = 0, \tag{97}$$

[6]The massless graviton in three dimensions has no degrees of freedom, which is why people call Einstein-Hilbert three dimensional gravity a topological theory. But in the higher-derivative theories one gets in addition to the massless graviton (which is pure gauge), several massive gravitons which all have 2 degrees of freedom each.

with

$$\tilde{\mathcal{M}}_I^2 = 4\mathcal{M}_I^2 = 4\left(m_I^2 - \frac{1}{l^2}\right), \tag{98}$$

where transverse condition $\bar{\nabla}_\alpha \mathbf{H}_I{}^\alpha{}_{\mu\nu} = 0$ was used. For $I = 1$ and 2, Eq.(97) is the spin-3 field equation in AdS$_3$ for a spin-3 field $\mathbf{H}_{I\mu\nu\lambda}$ of mass $\tilde{\mathcal{M}}_I$. It is clear from Eq.(91) and Eq.(98) that "no-tachyon" conditions can be written as $|lm_I| \geq 1$. Because $q_\pm$ fields have no local degrees of freedom, then $\mathbf{H}_{+\mu\nu\lambda}$ and $\mathbf{H}_{-\mu\nu\lambda}$ are not propagating modes. Thus, we have two massive propagating modes $\mathbf{H}_{1\mu\nu\lambda}$ and $\mathbf{H}_{2\mu\nu\lambda}$.

7. Example

In this section we consider the Spin-3 GMMG, then we find energy, angular momentum and entropy of a special black hole solution. In higher spin theories the metric is gauge dependent, so may be one say that the notion of an event horizon for a black hole is not well defined concept [33]. However different black hole solutions in higher spin gravity have been obtained [17, 18, 22, 29, 30, 33–42]. In some of these papers also different methods for deriving entropy of higher spin black holes have been presented [18, 22, 29, 40]. In spin-3 gravity in contrast to the ordinary gravity, where black hole entropy is given by the area of the horizon, since the metric is gauge dependent, this statement should be replaced with a gauge invariant criterion [33]. Until now there is not a general formula for entropy of all black hole solutions in higher spin gravity models. Also there are some discrepancy between assumptions and results of different methods which have been presented for deriving the entropy of higher spin black holes. Here we present a formula for obtaining entropy of higher spin black holes.

In order to relate Chern-Simons gauge theory (1) to the first order formalism based on dreibein and spin-connection, we introduce two independent connection one-forms

$$A^\pm = \omega \pm \frac{1}{l} e. \tag{99}$$

Consider following gauge connections which solve the spin-3 gravity field equations [24, 29]

$$\begin{aligned} A^+ &= b(\rho)^{-1} a^+(x^+) b(\rho) + b(\rho)^{-1} db(\rho) \\ A^- &= b(\rho) a^-(x^-) b(\rho)^{-1} + b(\rho) db(\rho)^{-1}, \end{aligned} \tag{100}$$

where $b(\rho) = \exp(\rho L_0)$ and $x^{\pm} = t/l \pm \phi$. Also, $a^{\pm}(x^{\pm})$ are given by

$$a^{+}(x^{+}) = \big(L_1 - \mathcal{L}^{+}(x^{+})L_{-1} - \mathcal{W}^{+}(x^{+})W_{-2}\big)\, dx^{+}$$
$$a^{-}(x^{-}) = \big(L_{-1} - \mathcal{L}^{-}(x^{-})L_{1} + \mathcal{W}^{-}(x^{-})W_{2}\big)\, dx^{-}. \tag{101}$$

where four functions $\mathcal{L}^{\pm}(x^{\pm})$ and $\mathcal{W}^{\pm}(x^{\pm})$ transform under gauge transformations which preserve the asymptotic conditions (see [24] for details). By substituting the given gauge connections (100) into Eq.(99) we can find generalized dreibein as

$$\begin{aligned} e_t &= \frac{1}{2}\big((e^{\rho} - \mathcal{L}^{-}e^{-\rho})L_1 + (e^{\rho} - \mathcal{L}^{+}e^{-\rho})L_{-1} \\ &\qquad + \mathcal{W}^{-}e^{-2\rho}W_2 - \mathcal{W}^{+}e^{-2\rho}W_{-2}\big) \\ e_\phi &= \frac{l}{2}\big((e^{\rho} + \mathcal{L}^{-}e^{-\rho})L_1 - (e^{\rho} + \mathcal{L}^{+}e^{-\rho})L_{-1} \\ &\qquad - \mathcal{W}^{-}e^{-2\rho}W_2 - \mathcal{W}^{+}e^{-2\rho}W_{-2}\big) \\ e_\rho &= lL_0. \end{aligned} \tag{102}$$

Similarly, space-time components of generalized spin-connection can be find as

$$\omega_t = \frac{1}{l^2}e_\phi, \qquad \omega_\rho = 0, \qquad \omega_\phi = e_t. \tag{103}$$

By using Eq.(5), one can extract metric,

$$\begin{aligned} ds^2 = &-\big(\mathcal{L}^{+}\mathcal{L}^{-}e^{-2\rho} + 4\mathcal{W}^{+}\mathcal{W}^{-}e^{-4\rho} - \mathcal{L}^{+} - \mathcal{L}^{-} + e^{2\rho}\big)\, dt^2 \\ &+ l^2 d\rho^2 + 2l(\mathcal{L}^{+} - \mathcal{L}^{-})dtd\phi \\ &+ l^2\big(\mathcal{L}^{+}\mathcal{L}^{-}e^{-2\rho} + 4\mathcal{W}^{+}\mathcal{W}^{-}e^{-4\rho} + \mathcal{L}^{+} + \mathcal{L}^{-} + e^{2\rho}\big)\, d\phi^2. \end{aligned} \tag{104}$$

which is of the form

$$ds^2 = g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi. \tag{105}$$

We have $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$ on the Killing horizon H which leads to the following equation

$$\mathcal{L}^{+}\mathcal{L}^{-}e^{-2\rho_H} + 4\mathcal{W}^{+}\mathcal{W}^{-}e^{-4\rho_H} + e^{2\rho_H} = 2\sqrt{\mathcal{L}^{+}\mathcal{L}^{-}}. \tag{106}$$

Here we assume that Killing horizon is located at $\rho = \rho_H$. In previous paper [16] we have shown that ρ_H is a real positive-definite root of Eq.(106).

However we are not sure that the metric (104) for non-zero $\mathcal{W}^{\pm}$ can describe a black hole solution of higher-spin gravity. But if we extend $a^{\pm}(x^{\pm})$, given by Eq.(101), so that they contain chemical potential conjugate to the W charges, then we have a black hole with spin-3 charge.[7] Also we should note that an important property of higher-spin gravity is that the gauge transformation of higher-spin field acts nontrivially on the metric. Due to this feature of the theory, the event horizon of black hole become gauge dependent [31]. The authors of [31] have discussed on the solutions of spin-3 gravity previously introduced in [29], and have shown that those solutions describe a traversable wormhole connecting two asymptotic region, instead a black hole. Then they have shown that under a higher spin gauge transformation these solutions can be transformed to describe black holes with manifestly smooth event horizons.

Generalized off-shell ADT conserved charge of the Spin-3 GMMG can be obtained as

$$(107)\quad \tilde{\mathcal{Q}}_{ADT}(a,\delta a;\xi) = tr\Big\{ -\left(\sigma i_\xi e + \frac{1}{m^2} i_\xi f + \frac{\alpha}{\mu} i_\xi h - \frac{1}{\mu}(i_\xi \Omega - \lambda_\xi)\right)\delta\Omega - (i_\xi\Omega - \lambda_\xi)\left(\sigma\delta e + \frac{1}{m^2}\delta f + \frac{\alpha}{\mu}\delta h\right) + \frac{\alpha^2}{\mu} i_\xi h \delta h + (1+\alpha\sigma)(i_\xi e\delta h + i_\xi h \delta e) + \frac{\alpha}{m^2}(i_\xi f\delta h + i_\xi h \delta f)\Big\}.$$

and, for h and f given by ansatz (62), it reduces to

$$(108)\quad \tilde{\mathcal{Q}}_{ADT}(a,\delta a;\xi) = -tr\Big\{ \left(\left(\sigma + \frac{\gamma}{m^2} + \frac{\alpha\beta}{\mu}\right) i_\xi e - \frac{1}{\mu}(i_\xi\Omega - \lambda_\xi)\right)\delta\Omega + \left(\left(\sigma + \frac{\gamma}{m^2} + \frac{\alpha\beta}{\mu}\right)(i_\xi\Omega - \lambda_\xi) - \frac{1}{\mu l^2} i_\xi e\right)\delta e\Big\}.$$

[7]The classical solutions of higher spin generalization of TMG, include AdS pp-wave (with higher spin hair), the spacelike, timelike, null warped AdS_3 spacetimes and also spacelike warped AdS_3 black hole have been obtained in [41]. Also the higher spin black holes in a truncated version of higher spin gravity in AdS_3 have been studied in [42].

Now, we take AdS_3 spacetime as background solution, where

$$e_t = \frac{1}{2}e^{\rho}\left(L_1 + L_{-1}\right), \qquad e_\phi = \frac{l}{2}e^{\rho}\left(L_1 - L_{-1}\right), \qquad e_\rho = lL_0$$
$$\Omega_t = \frac{1}{l^2}e_\phi, \qquad \Omega_\rho = 0, \qquad \Omega_\phi = e_t \tag{109}$$

and its perturbation is given by [24]

$$\delta e_t = -\frac{1}{2}e^{-\rho}\left(\delta\mathcal{L}^- L_1 + \delta\mathcal{L}^+ L_{-1} + e^{-\rho}\delta\mathcal{W}^- W_2 + e^{-\rho}\mathcal{W}^+ W_{-2}\right),$$
$$\delta e_\phi = -\frac{l}{2}e^{-\rho}\left(-\delta\mathcal{L}^- L_1 + \delta\mathcal{L}^+ L_{-1} - e^{-\rho}\delta\mathcal{W}^- W_2 + e^{-\rho}\delta\mathcal{W}^+ W_{-2}\right),$$
$$\delta e_\rho = 0, \tag{110}$$

Energy corresponds to the Killing vector $\xi_{(t)} = -\partial_t$, and for that Killing vector we have

$$\lambda_{\xi_{(t)}} - i_{\xi_{(t)}}\Omega = \frac{1}{l^2}e_\phi. \tag{111}$$

Therefore, Eq.(108) becomes

$$\tilde{\mathcal{Q}}_{ADT}(-\partial_t) = 2\left[\left(\sigma + \frac{\gamma}{m^2} + \frac{\alpha\beta}{\mu}\right)\left(\delta\mathcal{L}^+ + \delta\mathcal{L}^-\right) + \frac{1}{\mu l}(\delta\mathcal{L}^- - \delta\mathcal{L}^+)\right]d\phi. \tag{112}$$

We take Σ as a circle of arbitrary radii, then by substituting Eq.(112) into Eq.(47) we find energy of the considered solution as

$$E = \frac{1}{4\pi}\int_0^{2\pi} d\phi\left[\left(\sigma + \frac{\gamma}{m^2} + \frac{\alpha\beta}{\mu}\right)\left(\mathcal{L}^+ + \mathcal{L}^-\right) + \frac{1}{\mu l}(\mathcal{L}^- - \mathcal{L}^+)\right]. \tag{113}$$

Angular momentum corresponds to the Killing vector $\xi_{(\phi)} = \partial_\phi$ and for that Killing vector we have

$$\lambda_{\xi_{(\phi)}} - i_{\xi_{(\phi)}}\Omega = e_t. \tag{114}$$

Therefore, for this case, Eq.(108) becomes

$$\tilde{\mathcal{Q}}_{ADT}(\partial_\phi) = 2l\left[\left(\sigma+\frac{\gamma}{m^2}+\frac{\alpha\beta}{\mu}\right)\left(\delta\mathcal{L}^- - \delta\mathcal{L}^+\right) + \frac{1}{\mu l}(\delta\mathcal{L}^+ + \delta\mathcal{L}^-)\right]d\phi. \tag{115}$$

By substituting Eq.(115) into Eq.(47) we find angular momentum of the considered solution as

$$j=\frac{l}{4\pi}\int_0^{2\pi}d\phi\left[\left(\sigma+\frac{\gamma}{m^2}+\frac{\alpha\beta}{\mu}\right)(\mathcal{L}^- - \mathcal{L}^+)+\frac{1}{\mu l}(\mathcal{L}^+ + \mathcal{L}^-)\right], \tag{116}$$

Now, we apply the formula (54) to calculate the entropy of the considered black hole solution. Since the components of the flavour metric are given by Eq.(55) and $\Omega=\omega-\alpha h$, then

$$S=\int_{\mathcal{B}}\frac{d\phi}{\sqrt{g_{\phi\phi}}}tr\left\{\sigma e_\phi e_\phi+\frac{1}{m^2}e_\phi f_\phi-\frac{1}{\mu}e_\phi\Omega_\phi+\frac{\alpha}{\mu}e_\phi h_\phi\right\}. \tag{117}$$

By substituting f and h from Eq.(62) into above equation we find that

$$S=2\int_0^{2\pi}\frac{d\phi}{\sqrt{g_{\phi\phi}}}\left[\left(\sigma+\frac{\gamma}{m^2}+\frac{\alpha\beta}{\mu}\right)g_{\phi\phi}-\frac{1}{\mu}g_{t\phi}\right]. \tag{118}$$

where, in the last step of calculations, Eq.(5) was used. If we set ${e_\mu}^{ab}={\omega_\mu}^{ab}=0$, then Eq.(117) reduces to the entropy formula in the ordinary GMMG which was obtained in the paper [21]. One can substitute $g_{\phi\phi}$ and $g_{t\phi}$ from Eq.(104) into Eq.(118) then

$$S=2\int_0^{2\pi}\left[\left(\sigma+\frac{\gamma}{m^2}+\frac{\alpha\beta}{\mu}\right)(\sqrt{\mathcal{L}^-}+\sqrt{\mathcal{L}^+})-\frac{1}{\mu}(\sqrt{\mathcal{L}^-}-\sqrt{\mathcal{L}^+})\right]d\phi. \tag{119}$$

In the limit where Spin-3 GMMG model reduces to the spin-3 TMG model, i.e. $\alpha\to 0$ and $m^2\to\infty$, energy, angular momentum and entropy expressions obtained in Eqs. (113), (116) and (119) will reduce to the corresponding results in the spin-3 TMG [16]. Also as we have shown in [16], in the limiting case $\frac{1}{\mu}\to 0$, where TMG action reduces to the Einstein-Hilbert action, our conserved charge will reduce to the result presented in papers [17, 18]. This coincidence of results confirm that the quasi-local formalism works correctly for these types of theories.

Now, we want to compare Eq.(119) with entropy of the BTZ black hole [43] in ordinary MMG model (see Ref.[44]). The ordinary MMG can be seen as a limiting case of ordinary GMMG model. To find corresponding result in that model, we set ${e_\mu}^{ab}={\omega_\mu}^{ab}=0$ and $m^2\to\infty$. In this limit Eq.(64) and Eq.(65) become

$$\Lambda_0-\alpha(1+\alpha\sigma)\beta^2=-\frac{\sigma}{l^2}, \tag{120}$$

$$\alpha^2\beta^2 + 2\mu(1+\alpha\sigma)\beta = \frac{1}{l^2}. \tag{121}$$

In this case we do not consider Eq.(66) because it was derived from the variation of Lagrangian of spin-3 GMMG with respect to auxiliary field f and we lose dependence on f as m^2 tends to infinity (see Eq.(71)). From equations (120) and (121), one finds

$$\beta = \frac{1-\alpha\Lambda_0 l^2}{2\mu l^2\left(1+\alpha\sigma\right)^2}. \tag{122}$$

For the BTZ black hole we have

$$\mathcal{L}^{\pm} = \left(\frac{r_+ \pm r_-}{2l}\right)^2, \tag{123}$$

where r_+ and r_- are outer and inner horizon radiuses, respectively. By substituting Eq.(122) and (123) into the Eq.(119) and taking $m^2 \to \infty$ we will obtain entropy of the BTZ black hole in ordinary MMG which has been calculated in [44]. The same arguments are held for energy and angular momentum of the BTZ black hole.

8. Conclusion

In this paper, we considered the spin-3 gravity in the first order formalism, then we generalized that to the Spin-3 CSLTG. It should be noted that the Spin-3 TMG [14, 15] is an example of such theories. We provided a general formula to compute conserved charges and entropy of solutions in these theories which are generalizations of the standard three-dimensional higher spin gravity.

We found the off-shell ADT current (42) associated to a vector field ξ for the Spin-3 CSLTG. This current is conserved when ξ is a Killing vector field. We defined the generalized off-shell ADT current by (44), so that it becomes conserved for an asymptotically Killing vector field as well as a Killing vector field admitted by spacetime everywhere. The generalized off-sell ADT current reduces to the ordinary one when ξ is a Killing vector field admitted by spacetime everywhere and to the simplectic current when equations of motion and linearized equations of motion are satisfied. We defined the generalized off-shell ADT conserved charge (46) through the generalized off-sell ADT current. We used the generalized off-shell ADT conserved charge in order to define quasi-local conserved charge Eq.(48). The obtained quasi-local conserved charge (48) is associated to an asymptotically Killing vector field ξ and the integration surface Σ can be chosen arbitrarily. The advantage of this formalism is that it is applicable to solutions that are not asymptotically (A)dS. In Section 5, we found a general formula for entropy of black holes in the context of Spin-3 CSLTG (54). In Section 6, we considered the Spin-3 GMMG as an example of the Spin-3 CSLTG and we have obtained a class of solutions for that model. We have found the quadratic Lagrangian for the fluctuations u, v, w and z. Then, by introduction of a transformation from (u, v, w, z) to new Lie algebra valued one-form fluctuations (q_+, q_-, q_1, q_2), we were able to diagonalize quadratic Lagrangian. The two first terms in the diagonalized quadratic Lagrangian (80) can be written as difference of two linearized $SL(3, R)$ Chern-Simons 3-forms. Then, $q_\pm$ fields have no local degrees of freedom for spin-2 and spin-3 fields, as we expected. Each of two last terms in the diagonalized quadratic Lagrangian (80) describes a single spin-2 massive mode and a single trace-less spin-3 massive mode. So spin-3 GMMG model propagates 2 massive graviton modes and 2 massive spin-3 modes. The massive modes are not ghosts as long as $\tilde{B}_1$ and $\tilde{B}_2$ are both positive definite. We have shown that the spin-2 fluctuations $\mathbf{h}_{I\mu\nu}$ satisfy the Fierz-Pauli spin-2 field equation (90) in AdS_3 space of mass $\mathcal{M}_I$ and, the spin-3 fluctuations $\mathbf{H}_{I\mu\nu\lambda}$ satisfy the spin-3 field equation (97) in AdS_3 space of mass $\tilde{\mathcal{M}}_I$. Also, we deduced that no-tachyon conditions are $|lm_I| \geq 1$. Eventually, in Section 7, we obtained energy, angular momentum and entropy of a special black hole solution in the context of the Spin-3 CSLTG.

In Subsection 6.2, we analysed spin-3 fluctuations. We focused on its traceless part. There is actually a trace part of spin-3 fluctuations. Such a problem has been studied in the context of spin-3 TMG [15]. The spin-3 GMMG Lagrangian will reduce to spin-3 TMG one when we set $m^2 \to \infty$ and $\alpha = 0$. In that case the diagonalized quadratic Lagrangian (80) just contains one massive mode Lagrangian (i.e. in the spin-3 TMG limit, one of the coefficients $\tilde{B}_1$ or $\tilde{B}_2$ will vanish.). In the spin-3 TMG model, massive trace mode has zero energy and becomes pure gauge at the chiral point. In a similar way, in the context of spin-3 GMMG, we expect that massive trace modes become pure gauges at the chiral point $\sigma + \frac{\alpha\beta}{\mu} + \frac{\gamma}{m^2} = \frac{1}{\mu l}$ when two operators $\mathcal{D}[m_1]$ and $\mathcal{D}[m_2]$ are degenerate (namely, when $m_1 = m_2$). Therefore, our conclusions are unaffected for chiral spin-3 GMMG with $m_1 = m_2$.

It would be interesting to study a Hamiltonian analysis of Spin-3 CSLTG as the authors of [12] have done for ordinary CSLTG. For example a Hamiltonian analysis shows that the GMMG model has no Boulware-Deser ghosts and this model propagates only two physical modes [11]. GMMG also avoids the aforementioned "bulk-boundary unitarity clash". So it is a semi-classical quantum gravity model in $2+1$ dimensions which in both bulk and boundary is unitary. Therefore it is clear that study of Spin-3 GMMG following what have done in the papers [11, 14, 15] is interesting request. We let this study for future work.

Before the end, we would like to discuss the relation between our analysis and other results in the literature. Three dimensional spin-3 gravity theory has been investigated in [6, 7]. Generalization of TMG to higher spins has been done in [14, 15]. Such a generalization motivated us to extend ordinary CSLTG to spin-3 one. As an example of such theories, we have introduced spin-3 GMMG and analized it at linearized level. Although linearization in metric-like formalism has been used in spin-3 TMG [14] but it could not be used here. Instead, in order to have such an analysis, we have used a method which is appropriate in frame-like formalism. Such a extension of ordinary CSLTG to spin-3 one allows us to investigate other spin-3 versions of gravity theories in three dimensions, like exotic massive 3D gravity [45] etc. The concept of conserved charges is a very important matter in gravity theories as well as in other physical theories. Therefore, we found an expression for conserved charges and a formula for black hole entropy in spin-3 CSLTG. These two formulae should be examined for other solutions, which have been obtained in [17, 18, 22, 29, 30, 33–42]. For black holes containing higher spin charges, if a modification is required, we just need to fix gauge in another way, i.e. we need to find a new expression for λ_ξ rather than (52), without changing the arguments that lead to the formulae.

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Appendix A. $SL(3,\mathbb{R})$ generators

The algebra $sl(3,\mathbb{R})$ have 3 generators J_a and 5 generators T_{ab} with the following commutation relations[8]

$$\begin{aligned} &[J_a, J_b] = \varepsilon_{ab}{}^{c} J_c, \qquad [J_a, T_{bc}] = 2\varepsilon^{d}{}_{a(b} T_{c)d}, \\ &[T_{ab}, T_{cd}] = -2\left(\eta_{a(c}\varepsilon_{d)b}{}^{e} + \eta_{b(c}\varepsilon_{d)a}{}^{e}\right) J_e, \end{aligned} \tag{A.1}$$

[8] In this paper, we use the ordinary symmetrization by a pair of parentheses, for instance $A_{(\mu}B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$, i.e. we divide by the number of terms in the symmetrization.

where T_{ab} is symmetric and trace-less in the Lorentz indices. Here η_{ab} and ε_{abc} are Minkowski metric and Levi-Civita symbol, respectively. In this representation, the inner product of the generators are [30]

$$\begin{aligned} &tr(J_a) = 0, \qquad tr(T_{ab}) = 0, \qquad tr(J_a T_{bc}) = 0, \\ &tr(J_a J_b) = 2\eta_{ab}, \qquad tr(T_{ab}T_{cd}) = 4\eta_{a(c}\eta_{d)b} - \frac{4}{3}\eta_{ab}\eta_{cd}. \end{aligned} \tag{A.2}$$

In this paper, we use the following basis of $SL(3,\mathbb{R})$ generators [24, 29]

(A.3)

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad W_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are related to the previous one via the isomorphism

$$\begin{aligned} &J_0 = \frac{1}{2}(L_1 + L_{-1}), \qquad J_1 = \frac{1}{2}(L_1 - L_{-1}), \qquad J_2 = L_0, \\ &T_{00} = \frac{1}{4}(W_2 + W_{-2} + 2W_0), \qquad T_{01} = \frac{1}{4}(W_2 - W_{-2}), \\ &T_{11} = \frac{1}{4}(W_2 + W_{-2} - 2W_0), \qquad T_{02} = \frac{1}{2}(W_1 + W_{-1}), \\ &T_{22} = W_0, \qquad T_{12} = \frac{1}{2}(W_1 - W_{-1}). \end{aligned} \tag{A.4}$$

In other words, we take $J_A = \{L_0, L_{\pm1}, W_0, W_{\pm1}, W_{\pm2}\}$.These generators obey the following commutation relations

$$\begin{aligned} [L_i, L_j] &= (i-j)L_{i+j} \\ [L_i, W_m] &= (2i-m)W_{i+m} \\ [W_m, W_n] &= -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}. \end{aligned} \tag{A.5}$$

where $-1 \leq i, j \leq 1$ and $-2 \leq m, n \leq 2$. Also, the non-zero traces are

$$\begin{aligned} &tr(L_0L_0) = 2, \quad tr(L_1L_{-1}) = -4 \\ &tr(W_0W_0) = \frac{8}{3}, \quad tr(W_1W_{-1}) = -4, \quad tr(W_2W_{-2}) = 16. \end{aligned} \tag{A.6}$$

The Killing form in the fundamental representation of $sl(3, \mathbb{R})$ is defined as

$$K_{AB} = \frac{1}{2} tr(J_A J_B), \tag{A.7}$$

and anti-symmetric and symmetric structure constants of the Lie algebra are given by

$$f_{ABC} = \frac{1}{2} tr([J_A, J_B]J_C), \qquad d_{ABC} = \frac{1}{2} tr(\{J_A, J_B\}J_C). \tag{A.8}$$

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Department of Science, University of Kurdistan
Sanandaj, Kurdistan Province, Iran
E-mail address: rezakord@ipm.ir
E-mail address: hamed.adami@yahoo.com