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On Hochschild invariants of Landau-Ginzburg orbifolds

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with an Appendix by A. Basalaev, and D. Shklyarov

We develop an approach to calculating the cup and cap products on Hochschild cohomology and homology of curved algebras associated with polynomials and their finite abelian symmetry groups. For polynomials with isolated critical points, the approach yields a complete description of the products. We also reformulate the result for the corresponding categories of equivariant matrix factorizations. In an Appendix written jointly with Alexey Basalaev, we apply the formulas to calculate the Hochschild cohomology of a simple but non-trivial class of so-called invertible LG orbifold models. The resulting algebras turn out to be isomorphic to what has already appeared in the literature on LG mirror symmetry under the name of twisted or orbifolded Milnor/Jacobian algebras. We conjecture that this holds true for all invertible LG models. In the second part of the Appendix, the formulas are applied to a different class of LG orbifolds which have appeared in the context of homological mirror symmetry for varieties of general type as mirror partners of surfaces of genus 2 and higher. In combination with a homological mirror symmetry theorem for the surfaces, our calculation yields a new proof of the fact that the Hochschild cohomology of the Fukaya category of a surface is isomorphic, as an algebra, to the cohomology of the surface.

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1. Introduction

The problem of extending the classical Hochschild-Kostant-Rosenberg theorem to various classes of spaces has long been a popular topic of research. In any geometric setting, the ultimate goal of this activity is to describe — in geometric terms — the Hochschild homology and cohomology of spaces in question as completely as possible, including all parts of the rich structure that the Hochschild invariants carry: the cup and cap products, the Gerstenhaber bracket, pairings, etc. This work is devoted to HKR like isomorphisms in the setting of *affine Landau-Ginzburg orbifolds*, i.e. triples (X, W, G) where X is a (say complex) smooth affine variety, $W \in \mathbb{C}[X]$, and G is a finite group of automorphisms of (X, W). Our results can be summarized as follows: For a class of such orbifolds we present a complete description of the Hochschild cohomology and homology together with their cup and cap products.

We should clarify what kind of Hochschild invariants of LG orbifolds we have in mind. We adopt here the definition from [10, Sect.6]: The Hochschild (co)homology of (X, W, G) is the Hochschild (co)homology of the *curved algebra* $(\mathbb{C}[X] \rtimes G, W)$ where $\mathbb{C}[X] \rtimes G$ is the ordinary crossed product algebra associated with the G-action. A curved algebra [38] is pair (A, W) where A is an associative not necessarily commutative algebra and $W \in A$ is a central element ("curvature"). A variant of the classical Hochschild theory for such objects was developed in [10, 36].

Let us explain why the use of the physics terminology is legitimate here, or in other words, what the invariants we study have to do with the actual topological LG model associated with (X, W, G). According to a general philosophy [12, 22, 23, 29, 30], the truly useful Hochschild invariants associated with a topological string model are those of the corresponding D-brane category which in the LG case is the category $\mathrm{MF}_G(X, W)$ of G-equivariant matrix factorizations of W [2, 3, 34]. The point is that the Hochschild (co)homology of $(\mathbb{C}[X] \rtimes G, W)$ is *isomorphic* to the Hochschild (co)homology of $\mathrm{MF}_G(X, W)$. An explicit isomorphism was found in [36, 39]

and it turns out to preserve the cup and cap products (see Section 4.4). In particular, the Hochschild cohomology of $(\mathbb{C}[X] \rtimes G, W)$ is (or should be) isomorphic to the closed string algebra of the LG model. (We have to admit though that we do not know how to describe the topological metric on this algebra solely in terms of the Hochschild theory of $(\mathbb{C}[X] \rtimes G, W)$, whereas in terms of $\mathrm{MF}_G(X, W)$ it can be done [37, 41].)

As mentioned above, this work is about explicit formulas for the products on the Hochschild (co)homology of $(\mathbb{C}[X] \rtimes G, W)$ for special X, W, and G. Modulo minor details, those special triples look as follows:

1) X is $\mathbb{C}^N$, that is, $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_N]$;

2) W has isolated critical points;

3) G acts on $\mathbb{C}[X]$ by rescaling the variables x_i; in particular, G is abelian.

We would like to comment on the situation with more general cases. Firstly, we actually treat a slight generalization of $\mathbb{C}^N$ which looks exotic but turns out useful for applications (as demonstrated in Section A.2). However, abstract affine varieties are completely out of reach with our methods. Secondly, the problem with non-abelian groups is easy to explain: our HKR isomorphisms depend on the coordinate system $(x_1, \ldots, x_N)$ and are equivariant only with respect to rescalings. We do not yet know how to bring different coordinate systems under one roof in a way compatible with the cup and cap products. Lastly, the requirement that W have isolated singularities is more of a stylistic issue. In Section 4.2, while proving our theorem, we work out the general case. The problem is that our findings in the general case are not easy to formulate as a concise statement.

Let us outline the results of this paper in some more detail, focusing on the case of Hochschild cohomology HH^* and its cup product.

We assume from now on that X, W, and G are as in (1)–(3) above. Although we think of X as a variety, everything is linear in this setting and we will use the language of linear algebra and talk of subspaces of X instead of subvarieties, etc. For $g \in G$ we will denote by X^g the subspace of g-invariants in X. It has a unique g-invariant complement which we will denote by X_g. Since G is abelian, both X^g and X_g are G-stable. Let also $d_g := \dim X_g$. Finally, let W^g stand for the restriction of W to X^g and $M(W^g)$ for its Milnor algebra.

Let us first formulate an *additive* HKR theorem for the Hochschild cohomology. Consider the following $\mathbb{Z}/2\mathbb{Z}$-graded vector space:

$$\mathrm{M}^*(X,W,G) := \bigoplus_{g\in G} M(W^g)\otimes \det(X_g) \tag{1.1}$$

where $M(W^g)$ and $\det(X_g)$ are placed in degrees 0 and d_g mod 2, respectively. The space carries a degree preserving G-action coming from the G-action on $M(W^g)$ (induced by that on $\mathbb{C}[X^g]$) and the above-mentioned G-action on the subspace X_g (recall that it is G-stable). One has:

There is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded spaces
$\mathsf{HH}^*(\mathbb{C}[X]\rtimes G, W)\simeq \mathrm{M}^*(X,W,G)^G$.

The claim in fact holds for arbitrary polynomials but with $M(W^g) = \mathrm{H}^0(\wedge^* T_{X^g}, [W^g,\cdot])$ replaced by $\mathrm{H}^*(\wedge^* T_{X^g}, [W^g,\cdot])$ where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket. There are analogous results for the Hochschild homology. We prove all these claims in Section 4.2 (see Propositions 4.10, 4.12 and Section 4.2.4) but these facts seem to be known to the experts, in one version or another [4, 10, 37, 39].

The question we are really interested in is the following: *What is the product on* $\mathrm{M}^*(X,W,G)^G$ *that corresponds under the isomorphism to the cup product on the Hochschild cohomology?* There is another, even more interesting question, namely: *Is there a natural G-equivariant product on* $\mathrm{M}^*(X,W,G)$ *that gives rise to the correct product on the invariants?* In fact, such a product has to exist for very general reasons which we explain in Section 2.2. Moreover, we know some of its properties:

(1) It preserves the G-grading:

$$(M(W^g)\otimes\det(X_g))\cup\Big(M(W^h)\otimes\det(X_h)\Big)\subset\Big(M(W^{gh})\otimes\det(X_{gh})\Big).$$

(2) It is braided super-commutative:

$$v_g\cup v_h = (-1)^{|v_g||v_h|} v_h\cup h^{-1}(v_g) \tag{1.2}$$

for all $v_g\in M(W^g)\otimes\det(X_g)$ and $v_h\in M(W^h)\otimes\det(X_h)$ ($|\cdot|$ denotes the $\mathbb{Z}/2\mathbb{Z}$-degree). The induced product on the G-invariants is then super-commutative.

Let us make a guess as to what this product might look like. In the non-equivariant case the usual product on $M(W)$ can be interpreted as the

product on the 0th cohomology of the complex $(\wedge^* T_X, [W, \cdot])$ induced by the wedge product on polyvector fields. If we knew an equivariant analog of $\wedge^* T_X$ with its wedge product, we could try to predict the shape of the product on $\mathrm{M}^*(X, W, G)$. But, in fact, a good equivariant analog of $\wedge^* T_X$ *is* known: it is the Hochschild cohomology of the crossed product $\mathbb{C}[X] \rtimes G$ itself. As shown in [1, 35, 43], this Hochschild cohomology is isomorphic, as an algebra, to the G-invariant part of $\bigoplus_g \wedge^* T_{X^g} \otimes \det(X_g)$ equipped with the product

$$\begin{aligned} &(\mathfrak{X}_g \otimes \xi_g) \cup (\mathfrak{X}_h \otimes \xi_h) \\ &= \begin{cases} (-1)^{d_g \cdot |\mathfrak{X}_h|} (\mathfrak{X}_g|_{X^{gh}} \wedge \mathfrak{X}_h|_{X^{gh}}) \otimes (\xi_g \wedge \xi_h) & X^g \cap X^h = X^{gh} \ (\star) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\mathfrak{X}_g \in \wedge^* T_{X^g}$, $\xi_g \in \det(X_g)$, etc. The transversality condition $(\star)$ is equivalent to $X_g \oplus X_h = X_{gh}$ and the second wedge product on the right-hand side is simply the canonical isomorphism

$$\det(X_g) \otimes \det(X_h) \xrightarrow{\sim} \det(X_g \oplus X_h) = \det(X_{gh}).$$

So, here is our guess: The sought-after product on $\mathrm{M}^*(X, W, G)$ is given by the formula

(1.3)

$$(\varphi_g \otimes \xi_g) \cup (\varphi_h \otimes \xi_h) = \begin{cases} (\varphi_g|_{X^{gh}} \cdot \varphi_h|_{X^{gh}}) \otimes (\xi_g \wedge \xi_h) & X^g \cap X^h = X^{gh} \\ 0 & \text{otherwise} \end{cases}$$

where $\varphi_g \in M(W^g)$ and $\varphi_h \in M(W^h)$. This product is easily seen to satisfy all the properties we want: it is G-equivariant, G-graded, and braided super-commutative. (But, in fact, it is also super-commutative in the ordinary sense!) Let us consider a simple example.

Let $X = \mathbb{C}$, $W(x) = x^3$, and $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ where G acts by multiplication by the cubics roots of unity: $n \mapsto \zeta^n$ for ζ a primitive root. In this case $X_1 = X_2 = \mathbb{C}$ and therefore, as a $\mathbb{Z}/2\mathbb{Z}$-graded space,

$$\mathrm{M}^{\mathrm{even}}(X, W, G) = \mathbb{C}\xi_0 \oplus \mathbb{C}x\xi_0, \quad \mathrm{M}^{\mathrm{odd}}(X, W, G) = \mathbb{C}\xi_1 \oplus \mathbb{C}\xi_2$$

where ξ_i stands for a generator of $\det(X_i)$. The product (1.3) is quite boring in this case: ξ_0 is the unit and all other elements multiply to 0.

Actually, our guess is *wrong*: (1.3) is not the sought-after product on $\mathrm{M}^*(X, W, G)$ and it does not induce the right cup product on $\mathrm{M}^*(X, W, G)^G$. The relation between the product (1.3) and the actual one turns out to be

very similar to the relation between the classical and quantum cohomology of a symplectic manifold. Namely, $\mathrm{M}^*(X,W,G)$ with the above product is the limit of $\mathrm{M}^*(X,tW,G)$ with the true product as $t\to 0$.

The true product on $\mathrm{M}^*(X,tW,G)$ has the following form: For any fixed choice of generators $\{\xi_g\}_{g\in G}$ of the one-dimensional spaces $\det(X_g)$ there are elements $\sigma_{g,h}\in M(W^{gh})$ such that

$$(1.4)\qquad \begin{aligned}&(\varphi_g\otimes\xi_g)\cup(\varphi_h\otimes\xi_h)\\ &=\begin{cases} t^{\frac{d_g+d_h-d_{gh}}{2}}(\sigma_{g,h}\cdot\varphi_g|_{X^{gh}}\cdot\varphi_h|_{X^{gh}})\otimes\xi_{gh} & \frac{d_g+d_h-d_{gh}}{2}\in\mathbb{Z}_{\geq 0}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

As we mentioned above, the condition $(\star)$ in (1.3) is equivalent to $d_g+d_h-d_{gh}=0$. When the condition is satisfied, the power of t in (1.4) disappears and the corresponding products survive the limit $t\to 0$ (and, in fact, become what we had before). All other products tend to 0.

Our main result — Theorem 3.1 — gives explicit, though quite complicated, formula for $\sigma_{g,h}$. Together with (1.4) (for $t=1$) this provides a complete description of the product on $\mathrm{M}^*(X,W,G)$ and, consequently, a complete description of the product on $\mathrm{M}^*(X,W,G)^G\simeq\mathsf{HH}^*(\mathbb{C}[X]\rtimes G,W)$.

In the one-dimensional example we discussed above, the correct product on $\mathrm{M}^*(X,W,G)$ differs from the naive one in that the product of ξ_1 and ξ_2 is not 0 anymore: up to a renormalization of the generators ξ_i one has

$$(1.5)\qquad \xi_1\cup\xi_2=\frac{1}{\zeta-1}x\xi_0,\quad \xi_2\cup\xi_1=\frac{1}{\zeta^{-1}-1}x\xi_0.$$

Even in this simple example one can observe that the product on $\mathrm{M}^*(X,W,G)$ is indeed not super-commutative in general but braided super-commutative.

In fact, the formulas (1.5) can be compared with something that has already appeared in the literature. The point is that x^3 is (almost) the simplest example of an *invertible polynomial* [33], i.e. a quasi-homogeneous polynomial with an isolated critical point at the origin having the same number of variables and monomials. These polynomials have been studied quite extensively due to an important role they play in LG mirror symmetry and Fan-Jarvis-Ruan-Witten theory [7, 14, 32]. In particular, there already exists an analog of $\mathrm{M}^*(X,W,G)$ in this setting which was constructed "by hand" in [31] (building on pioneering ideas of [25–27]) and later used in [15] to prove LG mirror symmetry at the level of Frobenius algebras. In the recent work [6], a more systematic study of these algebras was undertaken and, in particular, an improved version of the original definition was

proposed (under the name of *the G-twisted Jacobian algebra* of W) which satisfies various expected properties, e.g. the braided super-commutativity. For $W(x) = x^3$ this G-twisted Jacobian algebra can easily be seen to be isomorphic to $\mathrm{M}^*(X, W, G)$. In Appendix A.1 the two algebras are compared in less trivial examples and turn out to be isomorphic there as well. We believe this holds true for all invertible polynomials.

There is yet another class of LG orbifolds against which our formulas can be tested, namely, certain orbifolded cusp singularities (in dimension 3) which are shown in [13, 40] to be homological mirror partners of two dimensional symplectic surfaces. In Appendix A.2 we calculate $\mathrm{M}^*(X, W, G)^G$ for these LG models and show that the resulting algebras are isomorphic to the cohomology of the corresponding surfaces, as it should be for certain general reasons [16–18].

We would like to conclude the Introduction by confessing that we do not yet understand the geometric meaning of our formulas for the above-mentioned "structure constants" $\sigma_{g,h}$. Perhaps, they could somehow be related to the Chern characters of matrix factorizations of $W(x) - W(y)$ but we have no evidence to support this idea. In any case, understanding what $\sigma_{g,h}$ mean geometrically should help to extend the results beyond the limited setting of the present work.

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2. Hochschild invariants of curved crossed product algebras

In this section, and throughout the paper, $\mathbb{K}$ is an arbitrary field of characteristic 0.

2.1. Curved Hochschild calculus

2.1.1. Outline. For any algebra A the Hochschild cohomology functor $\mathsf{HH}^*(A, -)$ from the category of A-bimodules to that of graded vector spaces carries a natural monoidal structure; the corresponding maps $\mathsf{HH}^*(A, M_1) \otimes$

$\mathsf{HH}^*(A, M_2) \to \mathsf{HH}^*(A, M_1 \otimes_A M_2)$ are usually referred to as "cup products". Also, the Hochschild homology functor $\mathsf{HH}_*(A, -)$ has a natural structure of an $\mathsf{HH}^*(A, -)$-module which is encoded in "cap products". The combination of these two structures is what is called *Hochschild calculus* in this paper. The aim of Section 2.1 is to discuss a counterpart of the Hochschild calculus for curved algebras. In fact, such a counterpart, in the much broader context of curved dg categories, already exists [36] (albeit without mentioning the products explicitly). In particular, one has a notion of Hochschild (co)homology "of the second kind" of a curved algebra with coefficients in a curved bimodule. However, for the purposes of this work the full power of the theory developed in [36] is not needed. We will only be interested in curved bimodules sitting in degree 0 (We call them simply bimodules, without the adjective "curved".) For such bimodules, the theory of [36] can be streamlined by implementing the language of mixed complexes.

2.1.2. Mixed complexes. Recall [24][1] that a mixed complex is a triple $(\mathcal{C}, b, B)$ where $\mathcal{C} = \oplus_{n\in\mathbb{Z}}\mathcal{C}_n$ is a $\mathbb{Z}$-graded vector space and b and B are operators on $\mathcal{C}$ of degrees 1 and -1, respectively, satisfying

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

The degree of $c \in \mathcal{C}$ will be denoted by $|c|$. A morphism $(\mathcal{C}, b, B) \to (\mathcal{C}', b', B')$ is by definition a degree preserving map $\mathcal{C} \to \mathcal{C}'$ commuting with both differentials. Such a morphism is a called a quasi-isomorphism if the induced morphism of complexes $(\mathcal{C}, b) \to (\mathcal{C}', b')$ is a quasi-isomorphism.

Mixed complexes form a tensor category under

$$(\mathcal{C}, b, B) \otimes (\mathcal{C}', b', B'); = (\mathcal{C} \otimes \mathcal{C}', b \otimes 1 + 1 \otimes b', B \otimes 1 + 1 \otimes B')$$

where $\otimes$ on the right-hand side is the usual tensor product in the category of graded vector spaces (namely, $(\mathcal{C} \otimes \mathcal{C}')_n = \oplus_{p+q=n}\mathcal{C}_p \otimes \mathcal{C}'_q$ and the new differentials pick up signs when applied to the elements of $\mathcal{C} \otimes \mathcal{C}'$, in agreement with the Koszul rule of signs).

A mixed complex $(\mathcal{C}, b, B)$ together with a morphism $(\mathcal{C}, b, B)^{\otimes 2} \to (\mathcal{C}, b, B)$ satisfying the associativity and unitality conditions will be called a mixed dg algebra. The notion of a mixed dg module over a mixed dg algebra is defined similarly.

Let us fix now a formal variable t of degree 2. Given a graded vector space $\mathcal{C} = \oplus_{n\in\mathbb{Z}}\mathcal{C}_n$, we will denote by $\mathcal{C}((t))$ the graded $\mathbb{K}[t^{\pm 1}]$-module spanned

[1]Unlike in [24], our mixed complexes are not necessarily left or right bounded.

(over $\mathbb{K}$) by homogeneous formal Laurent series in t with coefficients in $\mathcal{C}$:

$$\mathcal{C}(\!(t)\!)^n = \left\{ \sum_{i=i_0}^{\infty} c_i t^i \mid c_i \in \mathcal{C}_{n-2i} \right\}. \tag{2.1}$$

The correspondence $\mathcal{C} \mapsto \mathcal{C}(\!(t)\!)$ can be promoted to a functor from the category of mixed complexes to that of graded $\mathbb{K}[t^{\pm 1}]$-modules, namely:

$$(\mathcal{C}, b, B) \mapsto \mathrm{H}^*(\mathcal{C}(\!(t)\!), b + tB). \tag{2.2}$$

We call $\mathrm{H}^*(\mathcal{C}(\!(t)\!), b + tB)$ the periodic cohomology of $(\mathcal{C}, b, B)$.

Let us point out two properties of the functor (2.2) which we will use in the future. Firstly, the functor is lax monoidal. In particular, it transforms mixed dg algebras into $\mathbb{K}[t^{\pm 1}]$-linear algebras and mixed dg modules over the former into $\mathbb{K}[t^{\pm 1}]$-linear modules over the latter. Secondly, the functor transforms quasi-isomorphisms into isomorphisms [20, Prop.2.4].

2.1.3. Hochschild mixed complexes. Recall that the bar resolution of an associative unital algebra A is the complex $(\mathcal{B}_*(A), \delta_{\mathrm{bar}})$ of A-bimodules with $\mathcal{B}_{-n}(A) := A \otimes A^{\otimes n} \otimes A$ $(n = 0, 1, \ldots)$ and

$$\begin{aligned}\delta_{\mathrm{bar}}(a_0[a_1|\cdots|a_n]a_{n+1}) = {} & a_0 a_1[a_2|\cdots|a_n]a_{n+1} \\ & + \sum_{i=1}^{n-1} (-1)^i a_0[a_1|\cdots|a_i a_{i+1}|\cdots|a_n]a_{n+1} \\ & + (-1)^n a_0[a_1|\cdots|a_{n-1}]a_n a_{n+1}.\end{aligned}$$

where $a_0[a_1|\cdots|a_n]a_{n+1}$ is shorthand for $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}$.

Let now $W \in A$ be a central element. Associated with W there is a degree -1 differential on $\mathcal{B}_*(A)$, namely

$$\delta_{\mathrm{curv}}(a_0[a_1|\cdots|a_n]a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0[a_1|\cdots|a_i|W|a_{i+1}|\cdots|a_n]a_{n+1}.$$

One has

$$\delta_{\mathrm{curv}}^2 = 0, \quad \delta_{\mathrm{bar}}\delta_{\mathrm{curv}} + \delta_{\mathrm{curv}}\delta_{\mathrm{bar}} = l_W - r_W \tag{2.3}$$

where l_W (resp. r_W) is the operator in $\mathcal{B}_*(A)$ of left (resp. right) multiplication with W.

Remark 2.1. *All* the results and conclusions in this paper remain valid if one start with the *normalized* version $(\overline{\mathcal{B}}_*(A), \delta_{\mathrm{bar}}, \delta_{\mathrm{curv}})$ where $\overline{\mathcal{B}}_{-n}(A) := A \otimes (A/\mathbb{K})^{\otimes n} \otimes A$.

Let M be an (A, W)-bimodule, i. e. an A-bimodule in which the operators l_W and r_W of left and right multiplications with W coincide:

$$l_W - r_W = 0. \tag{2.4}$$

Let $\mathcal{B}^*(A, M) := \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(\mathcal{B}_{-*}(A), M)$ where A^{op} denotes the opposite algebra. (Note that $\mathcal{B}^*(A, M)$ is non-negatively graded.) The pairing of an element $D \in \mathcal{B}^*(A, M)$ with an element $\underline{a} \in \mathcal{B}_*(A)$ will be written as $\langle \underline{a}, D\rangle \in M$; so $\langle a_0\, \underline{a}\, a_1, D\rangle = a_0 \langle \underline{a}, D\rangle a_1$ for $a_0, a_1 \in A$.

Remark 2.2. We will also use $\langle \cdot, \cdot \rangle$ in a more general sense, namely, to denote the natural pairing $X \otimes \mathrm{Hom}_{A\otimes A}(X, Y) \to Y$ for *any* A-bimodules X, Y.

The Hochschild cochain mixed complex of (A, W) with coefficients in M is defined by

$$\mathcal{HH}^*(A, W; M) := \big(\mathcal{B}^*(A, M),\ \partial_{\mathrm{Hoch}} := \delta_{\mathrm{bar}}^\vee,\ \partial_{\mathrm{curv}} := \delta_{\mathrm{curv}}^\vee\big)$$

where $\delta^\vee$ denotes the standard dual of δ defined by

$$\langle -, \delta^\vee(D)\rangle = (-1)^{|D|} \langle \delta(-), D\rangle.$$

(That it is indeed a mixed complex follows from (2.3) and (2.4).) We will denote its periodic cohomology by ${}^t\mathsf{HH}^*(A, W; M)$, or simply ${}^t\mathsf{HH}^*(A, W)$ when $M = A$.

Let $\mathcal{B}_*(A, M) := M \otimes_{A\otimes A^{\mathrm{op}}} \mathcal{B}_*(A)$ where the right-hand side is an abbreviation for

$$M \otimes \mathcal{B}_*(A)/\{a_0\, m\, a_1 \otimes \underline{a} - m \otimes a_1\, \underline{a}\, a_0 \,|\, a_0, a_1 \in A, m \in M, \underline{a} \in \mathcal{B}_*(A)\}.$$

(Note that $\mathcal{B}_*(A, M)$ is non-positively graded.) The Hochschild chain mixed complex of (A, W) with coefficients in M is defined as the mixed complex

$$\mathcal{HH}_*(A, W; M) := (\mathcal{B}_*(A, M),\ b_{\mathrm{Hoch}} := 1 \otimes \delta_{\mathrm{bar}},\ b_{\mathrm{curv}} := 1 \otimes \delta_{\mathrm{curv}})$$

Its periodic cohomology will be denoted by ${}^t\mathsf{HH}_*(A, W; M)$ or ${}^t\mathsf{HH}_*(A, W)$ when $M = A$.

Remark 2.3. As we have already mentioned in Section 2.1.1, "our" Hochschild (co)homology is nothing but a special case of the Hochschild (co)homology of the second kind of (A, W) with coefficients in a *curved* (A, W)-bimodule introduced in [36]. However, the reader familiar with [36] will notice that even in the special case our definition does not match the one in [36] because the variable t does not appear there. The (co)homology $\mathsf{HH}^{\mathrm{II},*}(A, W; M)$ and $\mathsf{HH}^{\mathrm{II}}_*(A, W; M)$ of [36] are the $\mathbb{Z}/2\mathbb{Z}$-*graded* spaces defined as the cohomology of the $\mathbb{Z}/2\mathbb{Z}$-graded complexes

$$\left(\bigoplus_{i \text{ even}} \mathcal{B}^i(A, M) \oplus \bigoplus_{i \text{ odd}} \mathcal{B}^i(A, M),\ \partial_{\mathrm{Hoch}} + \partial_{\mathrm{curv}}\right)$$

and

$$\left(\prod_{i \text{ even}} \mathcal{B}_i(A, M) \oplus \prod_{i \text{ odd}} \mathcal{B}_i(A, M),\ b_{\mathrm{Hoch}} + b_{\mathrm{curv}}\right),$$

respectively. It is easy to see that there is a straightforward relation between the two definitions, namely, ${}^t\mathsf{HH}$ is just a 2-periodic $\mathbb{Z}$-graded version of $\mathsf{HH}^{\mathrm{II}}$:

$$\mathsf{HH}^{\mathrm{II,even}} \simeq {}^t\mathsf{HH}^{2n}, \quad \mathsf{HH}^{\mathrm{II,odd}} \simeq {}^t\mathsf{HH}^{2n+1} \qquad \forall n$$

and the same for homology. As a consequence, all the results we obtain in the present work have $\mathbb{Z}/2\mathbb{Z}$-graded counterparts for $\mathsf{HH}^{\mathrm{II}}$.

2.1.4. The cup and cap products. Let $\Delta = \Delta_{\mathrm{bar}} : \mathcal{B}_*(A) \to \mathcal{B}_*(A) \otimes_A \mathcal{B}_*(A)$ be the morphism of A-bimodules given by

$$\Delta(a_0[a_1|\cdots|a_n]a_{n+1}) = \sum_{i=0}^{n} (a_0[a_1|\cdots|a_i]1) \otimes (1[a_{i+1}|\cdots|a_n]a_{n+1}). \tag{2.5}$$

This morphism is coassociative, i. e. $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, and is easily seen to be compatible with the differentials δ_{bar} and δ_{curv}:

$$\Delta \cdot \delta_{\mathrm{bar}} = (\delta_{\mathrm{bar}} \otimes 1 + 1 \otimes \delta_{\mathrm{bar}}) \cdot \Delta, \quad \Delta \cdot \delta_{\mathrm{curv}} = (\delta_{\mathrm{curv}} \otimes 1 + 1 \otimes \delta_{\mathrm{curv}}).$$

Given two (A, W)-bimodules M_1 and M_2, Δ induces the *cup product*

$$\begin{aligned} &\cup = \cup_{\mathrm{Hoch}} : \mathcal{B}^*(A, M_1) \otimes \mathcal{B}^*(A, M_2) \to \mathcal{B}^*(A, M_1 \otimes_A M_2), \\ &\langle \underline{a}, D_1 \cup D_2 \rangle := \langle \Delta(\underline{a}), D_1 \boxtimes D_2 \rangle \\ &\qquad = (-1)^{|D_1||\underline{a}_{(2)}|} \langle \underline{a}_{(1)}, D_1 \rangle \otimes \langle \underline{a}_{(2)}, D_2 \rangle \quad (\forall\, \underline{a} \in \mathcal{B}_*(A)), \end{aligned} \tag{2.6}$$

as well as the *cap product*

$$\begin{aligned} &\cap = \cap_{\mathrm{Hoch}} : \mathcal{B}_*(A, M_1) \otimes \mathcal{B}^*(A, M_2) \to \mathcal{B}_*(A, M_1 \otimes_A M_2), \\ (2.7)\quad &(m \otimes \underline{a}) \cap D := m \otimes \langle \Delta(\underline{a}), D \boxtimes \mathrm{id}_{\mathcal{B}_*(A)} \rangle \\ &\qquad = (-1)^{|D||\underline{a}_{(2)}|} (m \otimes \langle \underline{a}_{(1)}, D \rangle) \otimes \underline{a}_{(2)} \end{aligned}$$

where $\underline{a}_{(1)} \otimes \underline{a}_{(2)} := \Delta(\underline{a})$ (Sweedler's notation) and $\boxtimes$ denotes the natural map

$$\begin{aligned} (2.8)\qquad &\mathrm{Hom}_{A\otimes A}(X_1, Y_1) \otimes \mathrm{Hom}_{A\otimes A}(X_2, Y_2) \\ &\to \mathrm{Hom}_{A\otimes A}(X_1 \otimes_A X_2, Y_1 \otimes_A Y_2) \end{aligned}$$

for A-bimodules X_i, Y_i. The following facts follow easily from the definitions:

Proposition 2.4. *(1) The cup product is a morphism of mixed complexes and hence yields a product*

$$\cup : {}^t\mathsf{HH}^*(A, W; M_1) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{HH}^*(A, W; M_2) \to {}^t\mathsf{HH}^*(A, W; M_1 \otimes_A M_2)$$

(2) For any (A, W)-bimodules M_i, $i = 1, 2, 3$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B}^*(A, M_1) \otimes \mathcal{B}^*(A, M_2) \otimes \mathcal{B}^*(A, M_3) & \xrightarrow{\mathrm{id}\otimes\cup} & \mathcal{B}^*(A, M_1) \otimes \mathcal{B}^*(A, M_2 \otimes_A M_3) \\ \Big\downarrow \cup\otimes\mathrm{id} & & \Big\downarrow \cup \\ \mathcal{B}^*(A, M_1 \otimes_A M_2) \otimes \mathcal{B}^*(A, M_3) & \xrightarrow{\cup} & \mathcal{B}^*(A, M_1 \otimes_A M_2 \otimes_A M_3) \end{array}$$

In particular, $\mathcal{HH}^*(A, W)$ is an associative mixed dg algebra[2] *and for any (A, W)-bimodule M $\mathcal{HH}^*(A, W; M)$ is a mixed dg bimodule over $\mathcal{HH}^*(A, W)$. As a consequence, ${}^t\mathsf{HH}^*(A, W)$ is an associative $\mathbb{K}[t^{\pm 1}]$-linear algebra and ${}^t\mathsf{HH}^*(A, W; M)$ is an ${}^t\mathsf{HH}^*(A, W)$-bimodule.*

(3) The cap product is a morphism of mixed complexes and therefore yields a product

$$\cap : {}^t\mathsf{HH}_*(A, W; M_1) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{HH}^*(A, W; M_2) \to {}^t\mathsf{HH}_*(A, W; M_1 \otimes_A M_2)$$

[2]The unit is the element of $\mathrm{Hom}\,(\mathbb{K}, A)$ (0-cochain) sending the unit of $\mathbb{K}$ to the unit of A.

(4) For M_i as above, the diagram

$$\begin{array}{ccc}
\mathcal{B}_*(A,M_1)\otimes\mathcal{B}^*(A,M_2)\otimes\mathcal{B}^*(A,M_3) & \xrightarrow{\mathrm{id}\otimes\cup} & \mathcal{B}_*(A,M_1)\otimes\mathcal{B}^*(A,M_2\otimes_A M_3) \\
\Big\downarrow{\scriptstyle\cap\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{B}_*(A,M_1\otimes_A M_2)\otimes\mathcal{B}^*(A,M_3) & \xrightarrow{\cap} & \mathcal{B}_*(A,M_1\otimes_A M_2\otimes_A M_3)
\end{array}$$

is commutative. In particular, $\mathcal{HH}_*(A,W;M)$, for any (A,W)-bimodule M, is a right mixed dg module over $\mathcal{HH}^*(A,W)$ and ${}^t\mathsf{HH}_*(A,W;M)$ is a right ${}^t\mathsf{HH}^*(A,W)$-module.

Remark 2.5. Just as in the non-curved case [19] ${}^t\mathsf{HH}^*(A,W)$ turns out to be super-commutative. This is a special case of a stronger result which we will discuss in Section 2.2.4.

2.1.5. Künneth isomorphisms for Hochschild calculi. Let (A,W) and (A',W') be two curved algebras. Their tensor product is defined as the curved algebra $(A\otimes A', W\otimes 1+1\otimes W')$. Observe that for an (A,W)-bimodule M and an (A',W')-bimodule M' the $A\otimes A'$-bimodule $M\otimes M'$ is actually an $(A\otimes A', W\otimes 1+1\otimes W')$-bimodule. Thus, it is natural to ask if, and how, the Hochschild calculus of $(A\otimes A', W\otimes 1+1\otimes W')$ can be "calculated" in terms of the Hochschild calculi of (A,W) and (A',W'). It is indeed possible, at least for a class of algebras, namely,

Recall [33] that an algebra A is called (homologically) smooth if, as a bimodule over itself, it admits a bounded below resolution by finitely generated projective A-bimodules.

Proposition 2.6. *If A and A' are smooth then there are natural isomorphisms*

$$\begin{aligned}
&{}^t\mathsf{HH}^*(A,W;M)\otimes_{\mathbb{K}[t^{\pm1}]}{}^t\mathsf{HH}^*(A',W';M') \\
\simeq\ &{}^t\mathsf{HH}^*(A\otimes A', W\otimes 1+1\otimes W'; M\otimes M'), \\
&{}^t\mathsf{HH}_*(A,W;M)\otimes_{\mathbb{K}[t^{\pm1}]}{}^t\mathsf{HH}_*(A',W';M') \\
\simeq\ &{}^t\mathsf{HH}_*(A\otimes A', W\otimes 1+1\otimes W'; M\otimes M'),
\end{aligned} \tag{2.9}$$

compatible with the cup and cap products.

Proof is given in Appendix B (page 238).

2.2. Hochschild (co)homology of equivariant curved algebras

2.2.1. Outline. Suppose A is acted upon by a finite symmetry group G (i.e. one has a group homomorphism $G \to \mathrm{Aut}(A)$). Recall that the crossed product $A \rtimes G$ is defined as the algebra

$$A \otimes \mathbb{K}[G] = \bigoplus_{g \in G} A \otimes g, \quad (a \otimes g) \cdot (b \otimes h) := ag(b) \otimes gh, \quad a, b \in A,\ g, h \in G.$$

A G-invariant element $W \in A$ gives rise to the curved algebra $(A \rtimes G, W)$. Our aim in this section is to develop an analog for $(A \rtimes G, W)$ of some standard technique [1, 5, 9, 21, 35, 43, ...] that allows one to calculate the (co)homology of $A \rtimes G$ in terms of the Hochschild calculus of A with coefficients in $A \rtimes G$.

2.2.2. G-twisted Hochschild (co)homology. The multiplication on $A \rtimes G$ endows each subspace $A \otimes g \subset A \rtimes G$ with an (A, W)-bimodule structure and induces isomorphisms of (A, W)-bimodules

$$(A \otimes g) \otimes_A (A \otimes h) \to A \otimes gh. \tag{2.10}$$

Then Proposition 2.4 implies that

$$\mathcal{HH}^*(A, W; A \rtimes G) = \bigoplus_{g \in G} \mathcal{HH}^*(A, W; A \otimes g),$$
$$\mathcal{HH}_*(A, W; A \rtimes G) = \bigoplus_{g \in G} \mathcal{HH}_*(A, W; A \otimes g)$$

have natural structures of a mixed dg algebra and a right mixed dg module over the dg algebra, respectively. Notice that the additional G-grading on both of them is compatible with the cup and cap products. (Perhaps, it is worthwhile mentioning explicitly that the G-grading is *non-homological*, in the sense that it does not affect the Leibniz rule.) Consequently,

$${}^t\mathsf{HH}^*(A, W; A \rtimes G) = \bigoplus_{g \in G} {}^t\mathsf{HH}^*(A, W; A \otimes g),$$
$${}^t\mathsf{HH}_*(A, W; A \rtimes G) = \bigoplus_{g \in G} {}^t\mathsf{HH}_*(A, W; A \otimes g)$$

have structures of an associative $(\mathbb{Z} \times G)$-graded $\mathbb{K}[t^{\pm 1}]$-linear algebra and a right $(\mathbb{Z} \times G)$-graded module over this algebra, respectively.

Furthermore, observe that G acts on $A \rtimes G$ by conjugation: $h(a \otimes g) = h(a) \otimes hgh^{-1}$ $(h \in G)$, and this action induces isomorphisms (of vector spaces) $h : A \otimes g \to A \otimes hgh^{-1}$ which are compatible with the isomorphisms (2.10). Also, G acts, in the obvious manner, on the bar resolution $\mathcal{B}_*(A)$, and this action respects the differentials δ_{bar} and δ_{curv}, as well as the coproduct Δ_{bar}. Combining the above two actions one gets well-defined G-actions on $\mathcal{B}^*(A, A \rtimes G)$ and $\mathcal{B}_*(A, A \rtimes G)$, namely

(2.11)
$$\mathcal{B}^*(A, A \otimes g) \ni D \mapsto h(D) := h \circ D \circ h^{-1} \in \mathcal{B}^*(A, A \otimes hgh^{-1}),$$
$$\mathcal{B}_*(A, A \otimes g) \ni m \otimes \underline{a} \mapsto h(m \otimes \underline{a}) := h(m) \otimes h(\underline{a}) \in \mathcal{B}_*(A, A \otimes hgh^{-1}).$$

These G-actions are easily seen to be compatible with all the structures we are interested in:

(1) They preserve the $\mathbb{Z}$-gradings and commute with the differentials, thereby inducing G-actions on the mixed complexes

$$\mathcal{HH}^*(A, W; A \rtimes G) \quad \text{and} \quad \mathcal{HH}_*(A, W; A \rtimes G).$$

(2) They commute with the cup and cap products:

$$k(D_1 \cup D_2) = k(D_1) \cup k(D_2), \quad k(\omega \cap D) = k(\omega) \cap k(D) \tag{2.12}$$

for all $k \in G$, $D_1 \in \mathcal{B}^*(A, A \otimes g)$, $D, D_2 \in \mathcal{B}^*(A, A \otimes h)$, and $\omega \in \mathcal{B}_*(A, A \otimes g)$.

2.2.3. Hochschild (co)homology of the curved crossed product. It follows from (2.12) that the cup and cap products on $\mathcal{HH}^*(A, W; A \rtimes G)$ and $\mathcal{HH}_*(A, W; A \rtimes G)$ descend to well-defined cup and cap products on the mixed complexes $\mathcal{HH}^*(A, W; A \rtimes G)^G$ and $\mathcal{HH}_*(A, W; A \rtimes G)_G$ where $(\,)^G$ resp. $(\,)_G$ denote the space of G-invariants resp. G-coinvariants. One has:

Proposition 2.7. *There are isomorphisms of $\mathbb{Z}$-graded $\mathbb{K}[t^{\pm 1}]$-modules*

$$^t\mathsf{HH}^*(A \rtimes G, W) \simeq {}^t\mathsf{HH}^*(A, W; A \rtimes G)^G,$$
$$^t\mathsf{HH}_*(A \rtimes G, W) \simeq {}^t\mathsf{HH}_*(A, W; A \rtimes G)_G$$

compatible with the cup and cap products.

Proof is given in Appendix B (page 242).

2.2.4. Braided (a.k.a. G-twisted) commutativity. We conclude this section by formulating an equivariant curved analog of the classical result of [19] on the super-commutativity of the Hochschild cohomology of ordinary algebras.

Proposition 2.8. *The algebra* ${}^t\mathsf{HH}^*(A, W; A \rtimes G)$ *is* braided super-commutative*: For all homogeneous classes* $[D_1] \in {}^t\mathsf{HH}^*(A, W; A \otimes g)$*,* $[D_2] \in {}^t\mathsf{HH}^*(A, W; A \otimes h)$

$$[D_1] \cup [D_2] = (-1)^{|[D_1]||[D_2]|}[D_2] \cup h^{-1}([D_1]). \tag{2.13}$$

Proof is given in Appendix B (page 243).

When G is trivial, the braided super-commutativity is just the ordinary super-commutativity, so

Corollary 2.9. *For any* (A, W) *the algebra* ${}^t\mathsf{HH}^*(A, W)$ *is super-commutative.*

3. Main results

3.1. Setting and notation

From now on, we focus on the following case:

- $A = \mathbb{K}[X] := \mathbb{K}[x_1, x_2, \ldots, x_N][S^{-1}]$ where S is the multiplicative set generated by a finite (or empty) set of polynomials of the form $(x_i - \lambda)$ with $\lambda \in \mathbb{K}^*$.
- The "curvature" is just a regular function $W = W(x) \in \mathbb{K}[X]$ $(x := (x_1, \ldots, x_N))$. In the statement of the main result we will require W to have isolated critical points by which we understand the condition that the cohomology of the complex $(\Omega^*_X, dW \wedge \cdot)$ vanishes in degrees less than N.
- The group G acts on $\mathbb{K}[X]$ by rescaling the variables:

$$(\mathbb{K}^*)^N \ni g = (g_1, \ldots, g_N) : (x_1, \ldots, x_N) \mapsto (g_1 x_1, \ldots, g_N x_N) =: g(x). \tag{3.1}$$

Let us introduce some notation which will be used in the statement of the main theorem, as well as throughout the proof.

3.1.1. Difference derivatives. We will write the elements of $\mathbb{K}[X]^{\otimes 2}$ as functions of two sets of N variables where the second set is $y = (y_1, \dots, y_N)$. Similarly, the elements of $\mathbb{K}[X]^{\otimes 3}$ will be written as $f(x, y, z)$.

Observe that because of the special form of the above multiplicative set S, we have the following well-defined maps:

$$\begin{aligned} &\nabla_i = \nabla_i^{x\to(x,y)} : \mathbb{K}[X] \to \mathbb{K}[X]^{\otimes 2} \quad (i = 1, \dots, N), \\ &\nabla_i(f) := \frac{l_i(f) - l_{i+1}(f)}{x_i - y_i}. \end{aligned} \tag{3.2}$$

where $l_i(f) := f(y_1, \dots, y_{i-1}, x_i, \dots, x_N)$ for $i = 1, \dots, N+1$. (In particular, $l_1 f = f \otimes 1 = f(x)$ and $l_{N+1} f = 1 \otimes f = f(y)$.) Note that

$$\sum_{i=1}^{N} (x_i - y_i) \nabla_i(f) = f(x) - f(y). \tag{3.3}$$

The symbol $\nabla^{y\to(y,z)} \nabla^{x\to(x,y)}$ will have the following meaning: $\nabla^{x\to(x,y)}$ is applied to a function of x and produces a function of (x, y); then $\nabla^{y\to(y,z)}$ is applied to this new function viewed as a function of y, with x "frozen"; the result is a function of (x, y, z).

The symbol $\nabla^{x\to(x,\psi(x))}$, where $\psi(x)$ is some function of x, will have the following meaning: $\nabla^{x\to(x,y)}$ is applied to a function of x and then $\psi(x)$ is substituted for y; the result is a new function of x. The symbol $\nabla^{x\to(x,\psi'(x))} \nabla^{x\to(x,\psi''(x))}$ is the composition of two operations of this type for two different functions of x; the result is again a function of x.

3.1.2. Clifford algebra. We will denote by Cl_N the Nth $\mathbb{Z}$-graded Clifford algebra:

$$\mathrm{Cl}_N = \mathbb{K}[\theta_1, \dots, \theta_N, \partial_{\theta_1}, \dots, \partial_{\theta_N}], \quad |\theta_i| = -1, \quad |\partial_{\theta_i}| = 1 \quad \forall i$$

modulo the relations

$$\theta_i \theta_j = -\theta_j \theta_i, \quad \partial_{\theta_i} \partial_{\theta_j} = -\partial_{\theta_j} \partial_{\theta_i}, \quad \partial_{\theta_i} \theta_j = -\theta_j \partial_{\theta_i} + \delta_{ij}.$$

For $I \subset \{1, \dots, N\}$ we write $\partial_{\theta_I} := \prod_{i\in I} \partial_{\theta_i}$, $\theta_I := \prod_{i\in I} \theta_i$ where in both cases the multipliers are taken in increasing order of the indices.

We endow the subspaces $\mathbb{K}[\theta]=\mathbb{K}[\theta_1,\dots,\theta_N]$ and $\mathbb{K}[\partial_\theta]=\mathbb{K}[\partial_{\theta_1},\dots,\partial_{\theta_N}]$ with left $\mathbb{Z}$-graded Cl_N-module structures via the isomorphisms

$$\mathbb{K}[\theta]\simeq \mathrm{Cl}_N/\mathrm{Cl}_N\cdot\langle\partial_{\theta_1},\dots,\partial_{\theta_N}\rangle,\quad \mathbb{K}[\partial_\theta]\simeq \mathrm{Cl}_N/\mathrm{Cl}_N\cdot\langle\theta_1,\dots,\theta_N\rangle.$$

Given a $\mathbb{Z}$-graded space $\mathcal{C}$, the spaces $\mathcal{C}[\theta]:=\mathcal{C}\otimes\mathbb{K}[\theta]$ and $\mathcal{C}[\partial_\theta]:=\mathcal{C}\otimes\mathbb{K}[\partial_\theta]$ are endowed with Cl_N-module structures by (graded) $\mathcal{C}$-linearity.

3.1.3. Notation related to the group action. For an element $g\in G$ as in (3.1) we define

$$I^g:=\{\,i\,|\,g_i=1\,\},\quad I_g:=\{1,\dots,N\}\setminus I^g,\quad d_g:=|I_g|.$$

We denote by $\mathbb{K}[X^g]$ the quotient of $\mathbb{K}[X]$ by the ideal generated by x_i with $i\in I_g$ and by res^g the projection $\mathbb{K}[X]\to\mathbb{K}[X^g]$. We write x^g instead of $\mathsf{res}^g(x)$, i.e. $x_i^g=x_i$ if $i\in I^g$ and 0 if $i\in I_g$.

3.2. The main theorem

Let $W^g:=\mathsf{res}^g(W)$ and $M(W^g):=\mathbb{K}[X^g]/(\partial_{x_i}W^g)_{i\in I^g}$. Consider the following two $(\mathbb{Z}\times G)$-graded $\mathbb{K}[t^{\pm1}]$-modules:

$$\begin{aligned}{}^t\mathrm{M}^*(X,W,G)&:=\bigoplus_{g\in G}M(W^g)[t^{\pm1}]\cdot\xi_g,\\ {}^t\Omega^*(X,W,G)&:=\bigoplus_{g\in G}M(W^g)[t^{\pm1}]\cdot\omega_g\end{aligned}\tag{3.4}$$

where the elements of $M(W^g)$ have degree 0, ξ_g is a formal generator of degree d_g, and ω_g is a formal generator of degree d_g-N. We equip (3.4) with degree preserving $\mathbb{K}[t^{\pm1}]$-linear G-actions by requiring that G act on $M(W^g)$ in the natural way and on ξ_g and ω_g as follows:

$$G\ni h=(h_1,\dots,h_N):\quad \xi_g\mapsto\prod_{i\in I_g}h_i^{-1}\cdot\xi_g,\quad \omega_g\mapsto\prod_{i\in I^g}h_i\cdot\omega_g.\tag{3.5}$$

Furthermore, for $g,h\in G$ we define $\sigma_{g,h}\in M(W^{gh})$ as the coefficient at $\partial_{\theta_{I_{gh}}}$ in the expression

$$\begin{aligned}\frac{1}{d_{g,h}!}\,\Upsilon\Big(\Big(&\lfloor\mathrm{H}_W(x,g(x),x)\rfloor_{gh}+\lfloor\mathrm{H}_{W,g}(x)\rfloor_{gh}\otimes1\\ &+1\otimes\lfloor\mathrm{H}_{W,h}(g(x))\rfloor_{gh}\Big)^{d_{g,h}}\otimes\partial_{\theta_{I_g}}\otimes\partial_{\theta_{I_h}}\Big)\end{aligned}\tag{3.6}$$

where

(1) $\mathrm{H}_W(x, g(x), x)$ is the degree -2 element of $\mathbb{K}[X] \otimes \mathbb{K}[\theta]^{\otimes 2}$ defined as the restriction to the set $\{y = g(x),\ z = x\}$ of the following degree -2 element of $\mathbb{K}[X]^{\otimes 3} \otimes \mathbb{K}[\theta]^{\otimes 2}$

$$\mathrm{H}_W = \mathrm{H}_W(x, y, z) := \sum_{1 \le j \le i \le N} \nabla_j^{y \to (y,z)} \nabla_i^{x \to (x,y)}(W)\, \theta_i \otimes \theta_j; \tag{3.7}$$

(2) $\mathrm{H}_{W,g} \in \mathbb{K}[X] \otimes \mathbb{K}[\theta]$ is the degree -2 element of $\mathbb{K}[X][\theta]$ given by the formula

$$\mathrm{H}_{W,g} = \mathrm{H}_{W,g}(x) := \sum_{i,j \in I_g,\ j<i} \frac{1}{1 - g_j} \nabla_j^{x \to (x, x^g)} \nabla_i^{x \to (x, g(x))}(W)\, \theta_j\, \theta_i; \tag{3.8}$$

(3) $\lfloor f \rfloor_g$ for $f \in \mathbb{K}[X]$ denotes the class of $\mathsf{res}^g(f) \in \mathbb{K}[X^g]$ in $M(W^g)$; we extend this operator to an operator $\mathbb{K}[X] \otimes V \to M(W^g) \otimes V$ by V-linearity;

(4) $d_{g,h} := \frac{d_g + d_h - d_{gh}}{2}$ and the $d_{g,h}$-th power in (3.6) is computed with respect to the natural product on $\mathbb{K}[X] \otimes \mathbb{K}[\theta] \otimes \mathbb{K}[\theta]$; we set

$$\sigma_{g,h} = 0 \textit{ if } d_{g,h} \textit{ is not a non-negative integer}; \tag{3.9}$$

(5) Υ is the $\mathbb{K}[X]$-linear extension of the degree 0 map $\mathbb{K}[\theta]^{\otimes 2} \otimes \mathbb{K}[\partial_\theta]^{\otimes 2} \to \mathbb{K}[\partial_\theta]$ defined by

$$p_1(\theta) \otimes p_2(\theta) \otimes q_1(\partial_\theta) \otimes q_2(\partial_\theta) \mapsto (-1)^{|q_1||p_2|} p_1(q_1) \cdot p_2(q_2) \tag{3.10}$$

where $p_i(q_i)$ denotes the action of $p_i(\theta)$ on $q_i(\partial_\theta)$ via the Cl_N-module structure on $\mathbb{K}[\partial_\theta]$ introduced in Section 3.1.2 and $\cdot$ is the natural product in $\mathbb{K}[\partial_\theta]$.

Theorem 3.1. *Assume W has only* **isolated critical points**. *Then*

(1) For all $g, h \in G$ the $\mathbb{K}[t^{\pm 1}]$-linear maps

$$\begin{aligned} \cup : M(W^g)[t^{\pm 1}] \cdot \xi_g \otimes_{\mathbb{K}[t^{\pm 1}]} M(W^h)[t^{\pm 1}] \cdot \xi_h &\to M(W^{gh})[t^{\pm 1}] \cdot \xi_{gh}, \\ \lfloor f_1 \rfloor_g \cdot \xi_g \otimes \lfloor f_2 \rfloor_h \cdot \xi_h &\mapsto t^{d_{g,h}} \cdot \lfloor f_1 f_2 \rfloor_{gh} \cdot \sigma_{g,h} \cdot \xi_{gh}, \end{aligned} \tag{3.11}$$

$$\cap : M(W^g)[t^{\pm 1}] \cdot \omega_g \otimes_{\mathbb{K}[t^{\pm 1}]} M(W^h)[t^{\pm 1}] \cdot \xi_h \to M(W^{gh})[t^{\pm 1}] \cdot \omega_{gh},$$
$$\lfloor f_1 \rfloor_g \cdot \omega_g \otimes \lfloor f_2 \rfloor_h \cdot \xi_h \mapsto t^{d_{g,h}} \cdot \lfloor f_1 f_2 \rfloor_{gh} \cdot \sigma_{g,h} \cdot \omega_{gh}$$
$$(\forall f_1, f_2 \in \mathbb{K}[X])$$

are well-defined and make ${}^t\mathrm{M}^*(X,W,G)$ *into a* $(\mathbb{Z}\times G)$*-graded* $\mathbb{K}[t^{\pm 1}]$*-linear* G*-equivariant braided super-commutative associative algebra* (*with unit* ξ_e) *and* ${}^t\Omega^*(X,W,G)$ *into a* $(\mathbb{Z}\times G)$*-graded* $\mathbb{K}[t^{\pm 1}]$*-linear* G*-equivariant free rank 1 right* ${}^t\mathrm{M}^*(X,W,G)$*-module* (*with generator* ω_e).

(2) There is a G*-equivariant isomorphism*

$$\big({}^t\mathsf{HH}^*(\mathbb{K}[X],W;\mathbb{K}[X]\rtimes G),\cup,\, {}^t\mathsf{HH}_*(\mathbb{K}[X],W;\mathbb{K}[X]\rtimes G),\cap\big)$$
$$\simeq \big({}^t\mathrm{M}^*(X,W,G),\cup,\, {}^t\Omega^*(X,W,G),\cap\big)\,.$$

Applying Proposition 2.7, we obtain

Corollary 3.2.

$$\big({}^t\mathsf{HH}^*(\mathbb{K}[X]\rtimes G,W),\cup,\, {}^t\mathsf{HH}_*(\mathbb{K}[X]\rtimes G,W),\cap\big)$$
$$\simeq \big({}^t\mathrm{M}^*(X,W,G)^G,\cup,\, {}^t\Omega^*(X,W,G)_G,\cap\big)\,.$$

Let us formulate a variant of the above results for the Hochschild homology of the second kind (see Remark 2.3). Let $\mathrm{M}^*(X,W,G)$ and $\Omega^*(X,W,G)$ stand for the algebra and the module over the algebra defined just as ${}^t\mathrm{M}^*(X,W,G)$ and ${}^t\Omega^*(X,W,G)$ but with $\mathbb{K}[t^{\pm 1}]$ replaced by $\mathbb{K}$ and the $\mathbb{Z}$-grading reduced to a $\mathbb{Z}/2\mathbb{Z}$-grading.

Corollary 3.3. *There is a* G*-equivariant isomorphism*

$$\big(\mathsf{HH}^{\mathrm{II},*}(\mathbb{K}[X],W;\mathbb{K}[X]\rtimes G),\cup,\mathsf{HH}^{\mathrm{II}}_*(\mathbb{K}[X],W;\mathbb{K}[X]\rtimes G),\cap\big)$$
$$\simeq (\mathrm{M}^*(X,W,G),\cup,\Omega^*(X,W,G),\cap)\,.$$

Consequently,

$$\big(\mathsf{HH}^{\mathrm{II},*}(\mathbb{K}[X]\rtimes G,W),\cup,\mathsf{HH}^{\mathrm{II}}_*(\mathbb{K}[X]\rtimes G,W),\cap\big)$$
$$\simeq \big(\mathrm{M}^*(X,W,G)^G,\cup,\Omega^*(X,W,G)_G,\cap\big)\,.$$

3.3. Implications for equivariant matrix factorization categories

Associated with any curved algebra (A,W) there is a differential $\mathbb{Z}/2\mathbb{Z}$-graded (dg) category of *curved dg* (A,W)*-modules* [36] whose objects are the

pairs (E, δ_E) where $E = E^{\text{even}} \oplus E^{\text{odd}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded finitely generated projective A-module and $\delta_E : E \to E$ is an odd morphism satisfying $\delta_E^2 = W \cdot \mathrm{id}_E$. When $(A, W) = (\mathbb{K}[X] \rtimes G, W)$, the category is precisely what we denoted by $\mathrm{MF}_G(X, W)$ in the Introduction.

The aim of this short section is to present a HKR like theorem for the Hochschild invariants $\mathrm{MF}_G(X, W)$. Although we do not discuss the Hochschild theory of dg categories in this paper, this material is much more standard than in the curved case and is much better represented in the literature; in particular, [36] can also serve as a reference.

Theorem 3.4. *Assume, in addition, that the only critical value of W is 0. Then*

$$\begin{aligned}&(\mathsf{HH}^*(\mathrm{MF}_G(X, W)), \cup, \mathsf{HH}_*(\mathrm{MF}_G(X, W)), \cap)\\ \simeq &\left(\mathrm{M}^*(X, W, G)^G, \cup, \Omega^*(X, W, G)_G, \cap\right).\end{aligned}$$

As will be explained in Section 4.4, this theorem is essentially a reformulation of Corollary 3.3.

4. Proofs

4.1. A multiplicative HKR isomorphism for the Hochschild calculus of LG models

4.1.1. Outline. The aim of this section is to establish an isomorphism of two calculi associated with $(\mathbb{K}[X], W)$:

$$\begin{aligned}&\left({}^t\mathsf{HH}^*(\mathbb{K}[X], W; -), \cup, {}^t\mathsf{HH}_*(\mathbb{K}[X], W; -), \cap\right)\\ \simeq &\left({}^t\mathsf{Kos}^*(\mathbb{K}[X], W; -), \cup, {}^t\mathsf{Kos}_*(\mathbb{K}[X], W; -), \cap\right)\end{aligned}$$

where ${}^t\mathsf{Kos}$ is the periodic cohomology of what we call *Koszul mixed complexes* and $\cup$ and $\cap$ are certain explicit products on ${}^t\mathsf{Kos}$.

In this section, W is arbitrary (e.g. $W = 0$ is allowed). We use freely the notation and conventions established Section 3.1.1, 3.1.2, 3.1.3.

4.1.2. Koszul mixed complexes. The Koszul resolution of $\mathbb{K}[X]$ is the complex $(\mathcal{K}_*(\mathbb{K}[X]), \delta_{\text{Kos}})$ of $\mathbb{K}[X]$-bimodules with $\mathcal{K}_*(\mathbb{K}[X]) = \mathbb{K}[X]^{\otimes 2}[\theta]$

and

$$\delta_{\mathrm{Kos}} := \sum_{i=1}^{N} (x_i - y_i) \cdot \partial_{\theta_i}.$$

Associated with W there is a degree -1 differential on $\mathcal{K}_*(\mathbb{K}[X])$, namely

$$\delta_{\mathrm{curv}} := \sum_{i=1}^{N} \nabla_i(W) \cdot \theta_i,$$

and it follows from the commutation relations in Cl_N and (3.3) that

$$\delta_{\mathrm{curv}}^2 = 0, \quad \delta_{\mathrm{Kos}}\delta_{\mathrm{curv}} + \delta_{\mathrm{curv}}\delta_{\mathrm{Kos}} = W(x) - W(y).$$

Let M be a $(\mathbb{K}[X], W)$-bimodule. We set

$$\begin{aligned} \mathcal{K}^*(\mathbb{K}[X], M) &:= \mathrm{Hom}_{\mathbb{K}[X]^{\otimes 2}}(\mathcal{K}_{-*}(\mathbb{K}[X]), M), \\ \mathcal{K}_*(\mathbb{K}[X], M) &:= M \otimes_{\mathbb{K}[X]^{\otimes 2}} \mathcal{K}_*(\mathbb{K}[X]). \end{aligned}$$

As in the abstract context, M gives rise to two mixed complexes

$$\begin{aligned} \mathcal{K}os^*(\mathbb{K}[X], W; M) &= \left(\mathcal{K}^*(\mathbb{K}[X], M),\ \partial_{\mathrm{Kos}} := \delta_{\mathrm{Kos}}^{\vee},\ \partial_{\mathrm{curv}} := \delta_{\mathrm{curv}}^{\vee}\right), \\ \mathcal{K}os_*(\mathbb{K}[X], W; M) &= \left(\mathcal{K}_*(\mathbb{K}[X], M),\ b_{\mathrm{Kos}} := 1 \otimes \delta_{\mathrm{Kos}},\ b_{\mathrm{curv}} := 1 \otimes \delta_{\mathrm{curv}}\right). \end{aligned}$$

Their periodic cohomology will be denoted by ${}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M)$ and ${}^t\mathsf{Kos}_*(\mathbb{K}[X], W; M)$.

4.1.3. Koszul vs Hochschild complexes. Consider the morphism of $\mathbb{K}[X]$-bimodules

$$\Psi : \mathcal{B}_*(\mathbb{K}[X]) \to \mathcal{K}_*(\mathbb{K}[X])$$

defined by $\Psi(f_0[]f_1) = l_1(f_0)l_{N+1}(f_1) = f_0(x)f_1(y)$ and

$$\begin{aligned} &\Psi(f_0[f_1|\cdots|f_n]f_{n+1}) \\ &= l_1(f_0)\left[\sum_{1\le j_1<\cdots<j_n\le N} \nabla_{j_1}(f_1)\theta_{j_1}\nabla_{j_2}(f_2)\theta_{j_2}\cdots\nabla_{j_n}(f_n)\theta_{j_n}\right] l_{N+1}(f_{n+1}) \end{aligned} \tag{4.1}$$

for $n \ge 1$ where the product on the right-hand side is taken in $\mathbb{K}[X]^{\otimes 2}[\theta]$.

Proposition 4.1. *One has*

$$\Psi \cdot \delta_{\mathrm{bar}} = \delta_{\mathrm{Kos}} \cdot \Psi, \quad \Psi \cdot \delta_{\mathrm{curv}} = \delta_{\mathrm{curv}} \cdot \Psi. \tag{4.2}$$

Moreover, $\Psi : (\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}}) \to (\mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}})$ *is a quasi-isomorphism.*

Proof is given in Appendix B (page 246).

Remark 4.2. After having discovered the formula (4.1), we looked for similar results in the literature and found them. We suspect that for $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_N]$ the map (4.1) coincides with the map Ψ_B from [42, Sect.4]. The description given in *loc.cit.* is quite combinatorial and we will not claim this as fact.

The previous proposition and the fact that both $(\mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}})$ and $(\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}})$ are K-projective complexes of $\mathbb{K}[X]$-bimodules, and hence K-flat [8, Sect.10.12], imply

Corollary 4.3. *For any* $(\mathbb{K}[X], W)$*-bimodule* M *the morphisms of mixed complexes*

$$\Psi^* := \Psi^\vee : \mathcal{K}os^*(\mathbb{K}[X], W; M) \to \mathcal{HH}^*(\mathbb{K}[X], W; M)$$

($\Psi^\vee$ *stands for the dual of* Ψ) *and*

$$\Psi_* := 1 \otimes \Psi : \mathcal{HH}_*(\mathbb{K}[X], W; M) \to \mathcal{K}os_*(\mathbb{K}[X], W; M)$$

are quasi-isomorphisms.

4.1.4. The cup and cap products on Koszul complexes. Our next goal is to "transfer" the cup and cap products to the Koszul mixed complexes. The naive idea that there exists a morphism of bimodules $\Delta : \mathcal{K}_*(\mathbb{K}[X]) \to \mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X])$ — an analog of (2.5) — that is compatible with δ_{Kos} and δ_{curv} and matches (2.5) under the quasi-isomorphism Ψ from the preceding section does not work. Calculations show that such a map does not exist even in the one-dimensional case ($N = 1$). As we will see in this and the next sections, the situation is more complicated.

Consider the sequence of morphisms of $\mathbb{K}[X]$-bimodules

$$\Delta_{-2l} : \mathbb{K}[X]^{\otimes 2}[\theta] \to \mathbb{K}[X]^{\otimes 3} \otimes \mathbb{K}[\theta]^{\otimes 2}, \quad l = 0, \dots, N$$

defined in terms of the (x, y, z)-coordinates (see Section 3.1.1) as follows:

$$\begin{aligned} (4.3) \qquad & \Delta_{-2l} : f(x,y) \cdot p(\theta_1, \dots, \theta_N) \\ & \mapsto \frac{1}{l!} f(x,z) \cdot \mathrm{H}_W(x,y,z)^l \cdot p(\theta_1 \otimes 1 + 1 \otimes \theta_1, \dots, \theta_N \otimes 1 + 1 \otimes \theta_N) \end{aligned}$$

where H_W is the element (3.7).

Note that we can also view these maps as maps to $\mathbb{K}[X]^{\otimes 2}[\theta] \otimes_{\mathbb{K}[X]} \mathbb{K}[X]^{\otimes 2}[\theta]$ where we identify the latter with $\mathbb{K}[X]^{\otimes 3} \otimes \mathbb{K}[\theta]^{\otimes 2}$ via

$$f_1(x,y)p_1(\theta) \otimes f_2(y,z)p_2(\theta) \mapsto f_1(x,y)f_2(y,z) \otimes p_1(\theta) \otimes p_2(\theta).$$

Keeping this in mind, one has

Proposition 4.4. *The morphism of* $\mathbb{K}[X][t^{\pm 1}]$*-bimodules*

$$\Delta = \Delta_{\mathrm{Kos}} : \mathbb{K}[X]^{\otimes 2}[t^{\pm 1}][\theta] \to \mathbb{K}[X]^{\otimes 2}[t^{\pm 1}][\theta] \otimes_{\mathbb{K}[X][t^{\pm 1}]} \mathbb{K}[X]^{\otimes 2}[t^{\pm 1}][\theta]$$

given by the formula

$$\Delta = \sum_{l=0}^{N} \Delta_{-2l} t^l = e^{t\mathrm{H}_W} \cdot \Delta_0$$

is compatible with the operator $(\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}})$*:*

$$\Delta \cdot (\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}}) = ((\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}}) \otimes 1 + 1 \otimes (\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}})) \cdot \Delta.$$

Proof is given in Appendix B (page 250).

It is the above operator Δ_{Kos} that will be playing the role of Δ_{bar} (2.5) in the "Koszul theory". Just as in the abstract setting, Δ_{Kos} gives rise to cup and cap products which one defines exactly as in (2.6) and (2.7): For

two $(\mathbb{K}[X], W)$-bimodules M_1 and M_2

$$\begin{aligned}\cup = \cup_{\mathrm{Kos}} : \mathcal{K}^*(\mathbb{K}[X], M_1)[t^{\pm 1}] \otimes_{\mathbb{K}[t^{\pm 1}]} \mathcal{K}^*(\mathbb{K}[X], M_2)[t^{\pm 1}] \\ \to \mathcal{K}^*(\mathbb{K}[X], M_1 \otimes_{\mathbb{K}[X]} M_2)[t^{\pm 1}],\end{aligned}$$

is the map determined by

$$\begin{aligned}(4.4) \quad \forall \omega \in \mathcal{K}_*(\mathbb{K}[X]) : \langle \omega, \xi_1 \cup_{\mathrm{Kos}} \xi_2 \rangle &:= \langle \Delta_{\mathrm{Kos}}(\omega), \xi_1 \boxtimes \xi_2 \rangle \\ &= (-1)^{|\xi_1||\omega_{(2)}|} \langle \omega_{(1)}, \xi_1 \rangle \otimes \langle \omega_{(2)}, \xi_2 \rangle\end{aligned}$$

and

$$\begin{aligned}\cap = \cap_{\mathrm{Kos}} : \mathcal{K}_*(\mathbb{K}[X], M_1)[t^{\pm 1}] \otimes_{\mathbb{K}[t^{\pm 1}]} \mathcal{K}^*(\mathbb{K}[X], M_2)[t^{\pm 1}] \\ \to \mathcal{K}_*(\mathbb{K}[X], M_1 \otimes_{\mathbb{K}[X]} M_2)[t^{\pm 1}],\end{aligned}$$

is the map defined by

$$\begin{aligned}(m \otimes \omega) \cap_{\mathrm{Kos}} \xi &:= m \otimes \langle \Delta_{\mathrm{Kos}}(\omega), \xi \boxtimes \mathrm{id}_{\mathcal{K}_*(\mathbb{K}[X])} \rangle \\ &= (-1)^{|\xi||\omega_{(2)}|} (m \otimes \langle \omega_{(1)}, \xi \rangle) \otimes \omega_{(2)}.\end{aligned}$$

In the above formulas, $\langle \cdot, \cdot \rangle$ and $\boxtimes$ have the same meaning as before, that is, as in Remark 2.2 and in (2.8), respectively. (Formally speaking, $\langle \cdot, \cdot \rangle$ above is a $\mathbb{K}[t^{\pm 1}]$-linear extension of the previous definition.)

Proposition 4.4 yields

Corollary 4.5. *The maps $\cup_{\mathrm{Kos}}$ and $\cap_{\mathrm{Kos}}$ descend to products on the periodic cohomology*

$$\begin{aligned}&{}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_1) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_2) \\ \to\ &{}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_1 \otimes_{\mathbb{K}[X]} M_2), \\ &{}^t\mathsf{Kos}_*(\mathbb{K}[X], W; M_1) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_2) \\ \to\ &{}^t\mathsf{Kos}_*(\mathbb{K}[X], W; M_1 \otimes_{\mathbb{K}[X]} M_2).\end{aligned}$$

Remark 4.6. Note that, unlike $\cup_{\mathrm{Hoch}}$ and $\cap_{\mathrm{Hoch}}$, the products $\cup_{\mathrm{Kos}}$ and $\cap_{\mathrm{Kos}}$ are *not associative* on the cochain level. However, as we will see shortly, the induced products on the periodic cohomology are associative.

4.1.5. Comparing the products on the Hochschild and Koszul complexes. The diagram

$$\begin{array}{ccc}
\mathcal{B}_*(\mathbb{K}[X])(\!(t)\!) & \xrightarrow{\Delta_{\mathrm{bar}}} & \big(\mathcal{B}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{B}_*(\mathbb{K}[X])\big)(\!(t)\!) \\
\big\downarrow{\scriptstyle \Psi} & & \big\downarrow{\scriptstyle \Psi\otimes\Psi} \\
\mathcal{K}_*(\mathbb{K}[X])[t^{\pm 1}] & \xrightarrow{\Delta_{\mathrm{Kos}}} & \big(\mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X])\big)[t^{\pm 1}]
\end{array}$$

is not commutative (e. g. because only Δ_{Kos} "depends" on t). Nevertheless, one has

Proposition 4.7. *The diagram is commutative up to homotopy: There exist morphisms of $\mathbb{K}[X]$-bimodules*

$$h_i : \mathcal{B}_*(\mathbb{K}[X]) \to \mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]), \quad |h_i| = -2i-1, \quad i = 0, 1, \ldots$$

such that

$$\begin{aligned}
&(\Psi \otimes \Psi) \cdot \Delta_{\mathrm{bar}} - \Delta_{\mathrm{Kos}} \cdot \Psi \\
&= ((\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}}) \otimes 1 + 1 \otimes (\delta_{\mathrm{Kos}} + t\delta_{\mathrm{curv}})) \\
&\quad \times \sum_{i=0}^{\infty} h_i t^i + \sum_{i=0}^{\infty} h_i t^i \cdot (\delta_{\mathrm{bar}} + t\delta_{\mathrm{curv}}).
\end{aligned} \tag{4.5}$$

Proof is given in Appendix B (page 251).

Corollary 4.8. *The isomorphisms*

$$\begin{aligned}
\Psi^* &: {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M) \to {}^t\mathsf{HH}^*(\mathbb{K}[X], W; M), \\
\Psi_* &: {}^t\mathsf{HH}_*(\mathbb{K}[X], W; M) \to {}^t\mathsf{Kos}_*(\mathbb{K}[X], W; M),
\end{aligned}$$

induced by the quasi-isomorphisms from Corollary 4.3, are compatible with the cup and cap products:

$$\begin{aligned}
&\Psi^*(\xi_1 \cup_{\mathrm{Kos}} \xi_2) = \Psi^*(\xi_1) \cup_{\mathrm{Hoch}} \Psi^*(\xi_2), \\
&\quad \forall\, \xi_i \in {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_i), \\
&\Psi_*(\omega) \cap_{\mathrm{Kos}} \xi = \Psi_*(\omega \cap_{\mathrm{Hoch}} \Psi^*(\xi)) \\
&\quad \forall\, \omega \in {}^t\mathsf{HH}_*(\mathbb{K}[X], W; M_1), \xi \in {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; M_2).
\end{aligned}$$

4.1.6. A more explicit description of the Koszul calculus. Our aim in this section is to rewrite all the structures in the Koszul calculus in a slightly more explicit way. We start by working out a more convenient description of the mixed Koszul complexes.

Let $\star : \mathrm{Cl}_N \to \mathrm{Cl}_N$ be the algebra involution (i.e., a $\mathbb{K}$-linear degree 0 involutive map satisfying $(\xi_1\xi_2)^\star = (-1)^{|\xi_1||\xi_2|}\xi_2^\star\xi_1^\star$) determined by

$$\theta_i^\star = \theta_i, \quad \partial_{\theta_i}^\star = -\partial_{\theta_i}, \quad \forall\, i = 1, \dots, N.$$

Consider the following perfect pairing between $\mathbb{K}[\theta]$ and $\mathbb{K}[\partial_\theta]$:

$$\begin{aligned} &\{\cdot,\cdot\} : \mathbb{K}[\theta] \otimes \mathbb{K}[\partial_\theta] \to \mathbb{K}, \\ &\{p(\theta), q(\partial_\theta)\} := (-1)^{|p(\theta)||q(\partial_\theta)|}\ \mathrm{CT}\,(q(\partial_\theta)^\star(p(\theta))) \end{aligned} \tag{4.6}$$

where on the right-hand side we apply the "differential operator" $q(\partial_\theta)^\star$ to the polynomial $p(\theta)$ and take the constant term of the resulting polynomial. One can easily check that

$$\begin{aligned} &\{\xi(p(\theta)), q(\partial_\theta)\} = (-1)^{|p(\theta)||\xi|}\{p(\theta), \xi^\star(q(\partial_\theta))\} \\ &\forall \xi \in \mathrm{Cl}_N,\ p(\theta) \in \mathbb{K}[\theta],\ q(\partial_\theta) \in \mathbb{K}[\partial_\theta] \end{aligned} \tag{4.7}$$

where ξ acts on $\mathbb{K}[\theta]$ and $\mathbb{K}[\partial_\theta]$ as described in Section 3.1.2.

Let M be a $(\mathbb{K}[X], W)$-bimodule. The above pairing gives rise to the pairing

$$\begin{aligned} &\mathcal{K}_*(\mathbb{K}[X]) \otimes M[\partial_\theta] \to M, \\ &\{f_0(x)f_1(y)p(\theta), mq(\partial_\theta)\} := (f_0 m f_1)\{p(\theta), q(\partial_\theta)\} \end{aligned} \tag{4.8}$$

which induces an isomorphism

$$M[\partial_\theta] \to \mathcal{K}^*(\mathbb{K}[X], M). \tag{4.9}$$

Under this isomorphism the differentials $\partial_{\mathrm{Kos}} = \delta_{\mathrm{Kos}}^\vee$ and $\partial_{\mathrm{curv}} = \delta_{\mathrm{curv}}^\vee$ on the right-hand side correspond to the differentials (denoted by the same symbols)

$$\partial_{\mathrm{Kos}} := \sum_{i=1}^N (x_i - y_i)\cdot\partial_{\theta_i}, \quad \partial_{\mathrm{curv}} := -\sum_{i=1}^N \nabla_i^{x\to(x,y)}(W)\cdot\theta_i$$

on $M[\partial_\theta]$ where the x-variables act on M from the left and the y-variables act from the right. Thus,

$$\mathcal{K}os^*(\mathbb{K}[X], W; M) \simeq (M[\partial_\theta], \partial_{\mathrm{Kos}}, \partial_{\mathrm{curv}})\,. \tag{4.10}$$

There is a similar description of $\mathcal{K}os_*(\mathbb{K}[X], W; M)$. Namely, observe that

$$\mathcal{K}_*(\mathbb{K}[X], M) \to M[\theta], \quad m \otimes f_0(x) f_1(y) p(\theta) \mapsto (f_1 m f_0) p(\theta) \tag{4.11}$$

is an isomorphism. It transforms the differentials $b_{\mathrm{Kos}} = 1 \otimes \delta_{\mathrm{Kos}}$ and $b_{\mathrm{curv}} = 1 \otimes \delta_{\mathrm{curv}}$ on the left-hand side into the differentials

$$b_{\mathrm{Kos}} := \sum_{i=1}^{N} (x_i - y_i) \cdot \partial_{\theta_i}, \quad b_{\mathrm{curv}} := \sum_{i=1}^{N} \nabla_i^{x \to (x,y)}(W) \cdot \theta_i$$

on $M[\theta]$ where now the x-variables act on M from the *right* and the y-variables act from the *left*. Thus,

$$\mathcal{K}os_*(\mathbb{K}[X], W; M) \simeq (M[\theta], b_{\mathrm{Kos}}, b_{\mathrm{curv}})\,. \tag{4.12}$$

Our next goal is to describe $\cup_{\mathrm{Kos}}$ and $\cap_{\mathrm{Kos}}$ in terms of the mixed complexes on the right-hand side of (4.10) and (4.12). In what follows, M_1 and M_2 are two $(\mathbb{K}[X], W)$-bimodules. We start with the cup product.

Take arbitrary two elements $\xi_i \in M_i[\partial_\theta]$ $(i = 1, 2)$ and let $\widehat{\xi_i} \in \mathcal{K}_*(\mathbb{K}[X]), M_i)$ denote their images under (4.9). Take also any $p(\theta) \in \mathbb{K}[\theta] \subset \mathbb{K}[X]^{\otimes 2}[\theta] = \mathcal{K}_*(\mathbb{K}[X])$. Then, by our definitions (see (4.4))

$$\{p(\theta), \xi_1 \cup_{\mathrm{Kos}} \xi_2\} = \langle p(\theta), \widehat{\xi_1} \cup_{\mathrm{Kos}} \widehat{\xi_2} \rangle = \langle \Delta_{\mathrm{Kos}}(p(\theta)), \widehat{\xi_1} \boxtimes \widehat{\xi_2} \rangle.$$

The pairing $\langle \cdot, \cdot \rangle$ in the last term is the $\mathbb{K}[t^{\pm 1}]$-linear extension of the pairing

$$\begin{aligned} \mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]) \otimes \mathrm{Hom}_{\mathbb{K}[X]^{\otimes 2}}(\mathcal{K}_*(\mathbb{K}[X]) \\ \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]), M_1 \otimes_{\mathbb{K}[X]} M_2) \to M_1 \otimes_{\mathbb{K}[X]} M_2. \end{aligned}$$

Under the isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{K}[X]^{\otimes 2}}(\mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]), M_1 \otimes_{\mathbb{K}[X]} M_2) \\ &\simeq M_1[\partial_\theta] \otimes_{\mathbb{K}[X]} M_2[\partial_\theta] \end{aligned}$$

$\widehat{\xi_1} \boxtimes \widehat{\xi_2}$ is just $\xi_1 \otimes \xi_2$ and the above pairing is nothing but the pairing (4.6) for $\mathrm{Cl}_N \otimes \mathrm{Cl}_N = \mathrm{Cl}_{2N}$ (or, rather, its extension analogous to (4.8)). Let us

denote it by $\{\{\cdot,\cdot\}\}$. Putting everything together, we obtain

$$\{p(\theta), \xi_1 \cup_{\mathrm{Kos}} \xi_2\} = \{\{e^{t\mathrm{H}_W}\Delta_0(p(\theta)), \xi_1 \otimes \xi_2\}\}.$$

Furthermore,

$$\begin{aligned}\{\{e^{t\mathrm{H}_W}\Delta_0(p(\theta)), \xi_1 \otimes \xi_2\}\} &= \{\{\Delta_0(p(\theta)), e^{t\mathrm{H}_W^\star}(\xi_1 \otimes \xi_2)\}\} \\ &= \{\{\Delta_0(p(\theta)), e^{t\mathrm{H}_W}(\xi_1 \otimes \xi_2)\}\}\end{aligned}$$

where we use that H_W is even (so no signs pop up upon using (4.7)) and also that $\mathrm{H}_W^\star = \mathrm{H}_W$. To conclude the calculation of the cup product, we observe that

$$\{\{\Delta_0(p(\theta)), q_1(\partial_\theta) \otimes q_2(\partial_\theta)\}\} = \{p(\theta), q_1(\partial_\theta)q_2(\partial_\theta)\} \quad \forall\, p, q_1, q_2$$

where on the right-hand side we multiply $q_1(\partial_\theta)$ and $q_2(\partial_\theta)$ just as elements of $\mathbb{K}[\partial_\theta]$. The formula is easily verified by substituting monomials for p, q_1 and q_2.

Summarizing the above calculations, we obtain the following result. Let us write symbolically

$$e^{t\mathrm{H}_W} = \sum_{h_i, p_i, l} (h_1 \otimes h_2 \otimes h_3)(p_1 \otimes p_2)t^l, \quad h_i \in \mathbb{K}[X],\ p_i \in \mathbb{K}[\theta].$$

Then for $m_1q_1(\partial_\theta) \in M_1[\partial_\theta]$ and $m_2q_2(\partial_\theta) \in M_2[\partial_\theta]$

$$m_1q_1(\partial_\theta) \cup m_2q_2(\partial_\theta) = \Upsilon(e^{t\mathrm{H}_W} \otimes m_1q_1(\partial_\theta) \otimes m_2q_2(\partial_\theta)) \tag{4.13}$$

where the right-hand side is an element in $(M_1 \otimes_{\mathbb{K}[X]} M_2)[\partial_\theta][t^{\pm 1}]$ given by the formula

$$\Upsilon(e^{t\mathrm{H}_W} \otimes m_1q_1 \otimes m_2q_2) = \sum (h_1m_1h_2 \otimes m_2h_3)\, \Upsilon(p_1 \otimes p_2 \otimes q_1 \otimes q_2)\, t^l$$

with Υ on the right-hand side being the map (3.10).

There is a similar formula for $\cap_{\mathrm{Kos}}$, namely:

$$m_1p(\theta) \cap m_2q(\partial_\theta) = \Upsilon^\dagger(e^{t\mathrm{H}_W} \otimes m_1p(\theta) \otimes m_2q(\partial_\theta)) \tag{4.14}$$

where the right-hand side is an element in $(M_1 \otimes_{\mathbb{K}[X]} M_2)[\theta][t^{\pm 1}]$ given by the formula

$$\Upsilon^\dagger(e^{t\mathrm{H}_W} \otimes m_1p \otimes m_2q) = \sum (h_3m_1h_1 \otimes m_2h_2)\, \Upsilon^\dagger(p_1 \otimes p_2 \otimes p \otimes q)\, t^l$$

with $\Upsilon^\dagger : \mathbb{K}[\theta] \otimes \mathbb{K}[\theta] \otimes \mathbb{K}[\theta] \otimes \mathbb{K}[\partial_\theta] \to \mathbb{K}[\theta]$ being the unique map such that

$$\begin{aligned} &\{\Upsilon^\dagger(p_1 \otimes p_2 \otimes p \otimes q), q'\} \\ &= (-1)^{|p|(|p_1|+|p_2|)}\{p, \Upsilon(p_1 \otimes p_2 \otimes q \otimes q')\} \quad \forall q' \in \mathbb{K}[\partial_\theta]. \end{aligned} \tag{4.15}$$

The derivation of (4.14) is analogous to that of (4.13) and is left to the reader.

In the remainder of this section, we will identify $\mathcal{K}os^*(\mathbb{K}[X], W; M)$ *with* $(M[\partial_\theta], \partial_{\mathrm{Kos}}, \partial_{\mathrm{curv}})$ *and* $\mathcal{K}os_*(\mathbb{K}[X], W; M)$ *with* $(M[\theta], b_{\mathrm{Kos}}, b_{\mathrm{curv}})$.

4.2. A multiplicative HKR isomorphism for the Hochschild (co)homology of LG orbifolds

4.2.1. Outline. In this section, $W \in \mathbb{K}[X]$ is again arbitrary (e.g. 0) and we also fix an abelian group G of symmetries of (X, W) of the form specified in Section 3.1.

The results of the previous section yield an explicit isomorphism between the tuples

$$\left({}^t\mathsf{HH}^*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G), \cup_{\mathrm{Hoch}}, {}^t\mathsf{HH}_*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G), \cap_{\mathrm{Hoch}}\right)$$

and

$$\left({}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G), \cup_{\mathrm{Kos}}, {}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G), \cap_{\mathrm{Kos}}\right).$$

Our aim in this section is to derive a more detailed description of the latter tuple and also a description of the G-action on the Koszul (co)homology that corresponds to the G-action on the Hochschild (co)homology (Section 2.2.2) under the isomorphism.

4.2.2. G-twisted Koszul (co)homology. Let g be an arbitrary element of G. Our first goal is to "calculate" the Koszul (co)homology

$${}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \quad \text{and} \quad {}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g).$$

We start with the cohomology.

The isomorphism $\mathbb{K}[X] \otimes g \to \mathbb{K}[x]$, $f \otimes g \mapsto f$ induces an isomorphism of mixed complexes

$$((\mathbb{K}[X] \otimes g)[\partial_\theta], \partial_{\mathrm{Kos}}, \partial_{\mathrm{curv}}) \simeq (\mathbb{K}[X][\partial_\theta], \partial_{\mathrm{Kos}}(g), \partial_{\mathrm{curv}}(g)) \tag{4.16}$$

where

$$\partial_{\mathrm{Kos}}(g) := \sum_{i=1}^{N} (x_i - g(x_i))\, \partial_{\theta_i}, \quad \partial_{\mathrm{curv}}(g) := -\sum_{i=1}^{N} \nabla_i^{x \to (x,y)}(W)|_{y=g(x)}\, \theta_i.$$

Observe that $\partial_{\mathrm{Kos}}(g) = \sum_{i \in I_g} (1 - g_i) x_i\, \partial_{\theta_i}$ and $\nabla_i^{x \to (x,y)}(W)|_{y=g(x)} = l_i^{x \to (x,g(x))}(\partial_{x_i} W(x))$ for $i \in I^g$. The latter observation suggests one to split $\partial_{\mathrm{curv}}(g)$ into two components, namely

$$\partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g) = \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)' + t\partial_{\mathrm{curv}}(g)''$$

with

$$\partial_{\mathrm{curv}}(g)' := -\sum_{i \in I_g} \nabla_i^{x \to (x,g(x))}(W)\, \theta_i,$$
$$\partial_{\mathrm{curv}}(g)'' := -\sum_{i \in I^g} l_i^{x \to (x,g(x))}(\partial_{x_i} W(x))\, \theta_i.$$

Lemma 4.9. *Let* $\mathrm{H}_{W,g}$ *be as in (3.8). Then*

$$\partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g) = e^{t\mathrm{H}_{W,g}} \cdot \left(\partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)''\right) \cdot e^{-t\mathrm{H}_{W,g}} \tag{4.17}$$

viewed as elements of $\mathbb{K}[X][t^{\pm 1}] \otimes \mathrm{Cl}_N$.

Proof is given in Appendix B (page 253).

By this lemma the map $e^{t\mathrm{H}_{W,g}} : \mathbb{K}[X][\partial_\theta][t^{\pm 1}] \to \mathbb{K}[X][\partial_\theta][t^{\pm 1}]$ induces a quasi-isomorphism

$$\begin{aligned} &(\mathbb{K}[X][\partial_\theta][t^{\pm 1}], \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)'') \\ &\xrightarrow{\sim} (\mathbb{K}[X][\partial_\theta][t^{\pm 1}], \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)). \end{aligned} \tag{4.18}$$

Let us calculate the cohomology of the complex on the left-hand side.

Let $\mathbb{K}[\partial_\theta^g]$ denote the subalgebra in Cl_N generated by $\{\partial_{\theta_i}\}_{i \in I^g}$. Consider the projection $\mathbb{K}[\partial_\theta] \to \mathbb{K}[\partial_\theta^g] \cdot \partial_{\theta_{I_g}}$ which annihilates monomials containing

less than d_g elements ∂_{θ_i}, $i \in I_g$. This projection together with the restriction $\mathsf{res}^g : \mathbb{K}[X] \to \mathbb{K}[X^g]$ (Section 3.1.3) give rise to a morphism of mixed complexes

$$(\mathbb{K}[X][\partial_\theta],\, \partial_{\mathrm{Kos}}(g),\, \partial_{\mathrm{curv}}(g)'') \to (\mathbb{K}[X^g][\partial_\theta^g] \cdot \partial_{\theta_{I_g}},\, 0,\, \sum_{i \in I^g} (\partial_{x_i} W^g)\theta_i)$$

which clearly is a quasi-isomorphism. In particular,

$$\begin{aligned} &\mathrm{H}^*(\mathbb{K}[X][\partial_\theta][t^{\pm 1}], \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)'') \\ \simeq\ &\mathrm{H}^{*-d_g}\left(\mathbb{K}[X^g][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}},\, t\sum_{i \in I^g} (\partial_{x_i} W^g)\theta_i\right) \\ \simeq\ &{}^t\mathsf{Kos}^{*-d_g}(\mathbb{K}[X^g], W^g). \end{aligned}$$

Combining this observation with (4.18), we obtain:

Proposition 4.10. *As a $\mathbb{Z}$-graded $\mathbb{K}[t^{\pm 1}]$-module,*

$$\begin{aligned} &{}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \\ \simeq\ &{}^t\mathsf{Kos}^{*-d_g}(\mathbb{K}[X^g], W^g) (\simeq \mathrm{H}^{*-d_g}(\wedge^* T_{X^g}, [W^g, \cdot])[t^{\pm 1}]). \end{aligned}$$

Remark 4.11. For the purpose of calculation of the cup products it is important to have explicit representatives of classes in ${}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$. Let us therefore formulate the above observations more carefully. The natural embedding $\mathbb{K}[X][\partial_\theta^g] \cdot \partial_{\theta_{I_g}} \hookrightarrow \mathbb{K}[X][\partial_\theta]$ induces a map

$$\begin{aligned} &\mathrm{Ker}(t\partial_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}}) \\ &\to \mathrm{H}^*(\mathbb{K}[X][\partial_\theta][t^{\pm 1}],\, \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)'') \end{aligned} \tag{4.19}$$

which by the preceeding discussion is surjective. Consequently, representatives of classes in ${}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$ can be obtained by applying the map (4.18) to elements of the space $\mathrm{Ker}(t\partial_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}})$. To complete this description, we need to understand what the kernel of (4.19) looks like. It is easy to describe: it contains $\mathrm{Im}(t\partial_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}})$, as well as the closed elements annihilated by res^g. Thus, the kernel equals

$$\begin{aligned} &\mathrm{Im}(t\partial_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}}) + \mathrm{Ker}(t\partial_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}}) \\ &\qquad\qquad \cap \sum_{i \in I_g} x_i \cdot \mathbb{K}[X][\partial_\theta^g][t^{\pm 1}] \cdot \partial_{\theta_{I_g}}. \end{aligned}$$

Let us derive an analogous description for ${}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$. Since the argument is very similar, we will only sketch it and omit details.

There is a natural isomorphism of mixed complexes

$$\begin{aligned}(4.20)\qquad & ((\mathbb{K}[X] \otimes g)[\theta], b_{\mathrm{Kos}}, b_{\mathrm{curv}}) \\ & \simeq \big(\mathbb{K}[X][\theta], b_{\mathrm{Kos}}(g), b_{\mathrm{curv}}(g) = b_{\mathrm{curv}}(g)' + b_{\mathrm{curv}}(g)''\big)\end{aligned}$$

where $b_{\mathrm{Kos}}(g) := -\sum_{i \in I_g}(1 - g_i)x_i\,\partial_{\theta_i}$,

$$\begin{aligned} b_{\mathrm{curv}}(g)' &:= \sum_{i \in I_g} \nabla_i^{x \to (g(x),x)}(W)\,\theta_i, \\ b_{\mathrm{curv}}(g)'' &:= \sum_{i \in I^g} l_i^{x \to (g(x),x)}(\partial_{x_i} W(x))\,\theta_i. \end{aligned}$$

By analogy with the previous case, we have a quasi-isomorphism

$$\begin{aligned}(4.21)\qquad & e^{t\mathrm{H}^\dagger_{W,g}} : (\mathbb{K}[X][\theta][t^{\pm 1}], b_{\mathrm{Kos}}(g) + t b_{\mathrm{curv}}(g)'') \\ & \xrightarrow{\sim} (\mathbb{K}[X][\theta][t^{\pm 1}], b_{\mathrm{Kos}}(g) + t b_{\mathrm{curv}}(g))\end{aligned}$$

where this time

$$\mathrm{H}^\dagger_{W,g} = \mathrm{H}^\dagger_{W,g}(x) := \sum_{i,j \in I_g,\ j<i} \frac{1}{1 - g_j} \nabla_j^{x \to (x,x^g)} \nabla_i^{x \to (g(x),x)}(W)\,\theta_j\,\theta_i.$$

Let $\mathbb{K}[\theta^g]$ stand for the subalgebra in Cl_N generated by $\{\theta_i\}_{i \in I^g}$. Consider the projection $\mathbb{K}[\theta] \to \mathbb{K}[\theta^g]$ that annihilates monomials containing θ_i, $i \in I_g$. This projection and the homomorphism res^g give rise to a quasi-isomorphism of mixed complexes

$$(\mathbb{K}[X][\theta],\, b_{\mathrm{Kos}}(g),\, b_{\mathrm{curv}}(g)'') \to \left(\mathbb{K}[X^g][\theta^g],\, 0,\, \sum_{i \in I^g} (\partial_{x_i} W^g)\theta_i\right),$$

which we combine with (4.21) to obtain

Proposition 4.12. *As a $\mathbb{Z}$-graded $\mathbb{K}[t^{\pm 1}]$-module,*

$${}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \simeq {}^t\mathsf{Kos}_*(\mathbb{K}[X^g], W^g)(\simeq \mathrm{H}^*(\Omega^*_{X^g}, dW^g \wedge \cdot)[t^{\pm 1}]).$$

Remark 4.13. Note that the embedding $\mathbb{K}[X][\theta^g] \hookrightarrow \mathbb{K}[X][\theta]$ induces a surjective map

$$\mathrm{Ker}(tb_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\theta^g][t^{\pm 1}]}) \to \mathrm{H}^*(\mathbb{K}[X][\theta][t^{\pm 1}],\, b_{\mathrm{Kos}}(g) + tb_{\mathrm{curv}}(g)'') \tag{4.22}$$

and so explicit representatives of classes in ${}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$ can be obtained by applying (4.21) to elements of $\mathrm{Ker}(tb_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\theta^g][t^{\pm 1}]})$. The kernel of (4.22) equals

$$\mathrm{Im}(tb_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\theta^g][t^{\pm 1}]}) + \mathrm{Ker}(tb_{\mathrm{curv}}(g)''|_{\mathbb{K}[X][\theta^g][t^{\pm 1}]}) \cap \sum_{i \in I_g} x_i \mathbb{K}[X][\theta^g][t^{\pm 1}].$$

Remark 4.14. Part of the above picture can be generalized to not necessarily abelian subgroups $G \subset GL_N(\mathbb{K})$. Namely, even if g is not diagonal but *can* be diagonalized (say when $\mathbb{K} = \mathbb{C}$), Propositions 4.10 and 4.12 hold true for the simple reason that neither the Hochschild (co)homology ${}^t\mathsf{HH}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$, ${}^t\mathsf{HH}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g)$ nor the geometrically defined cohomology $\mathrm{H}^*(\wedge^* T_{X^g}, [W^g, \cdot])$ and $\mathrm{H}^*(\Omega^*_{X^g}, dW^g \wedge \cdot)$ depend on any coordinate systems. Furthermore, even though the explicit description of the cohomology classes that we have derived in Remarks 4.11, 4.13 does not apply to non-diagonal elements g, the underlying idea — namely, splitting the differentials into two parts and using exponential twists to simplify complexes — seems quite universal. In order to calculate the g-twisted Koszul (co)homology in a concrete example, one can write the differentials $\partial_{\mathrm{Kos}}(g)$, $\partial_{\mathrm{curv}}(g)$, $b_{\mathrm{Kos}}(g)$, $b_{\mathrm{curv}}(g)$ in linear coordinates in which g is diagonal, apply the above idea to do calculations, and write the result in terms of the original coordinates.

4.2.3. Products. Our next goal is to describe the products

$$\begin{aligned}
&{}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes h) \\
&\xrightarrow{\cup} {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes gh), \\
&{}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{Kos}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes h) \\
&\xrightarrow{\cap} {}^t\mathsf{Kos}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes gh).
\end{aligned}$$

The following proposition is an immediate consequence of (4.13) and (4.14).

Proposition 4.15. *Under the isomorphism (4.16), (4.20) the above products transform into*

$$f_1 q_1(\partial_\theta) \cup f_2 q_2(\partial_\theta) = f_1 \cdot g(f_2) \cdot \Upsilon(e^{t\mathrm{H}_W(x,g(x),gh(x))} \otimes q_1(\partial_\theta) \otimes q_2(\partial_\theta)),$$
$$f_1 p(\theta) \cap f_2 q(\partial_\theta) = f_1 \cdot g(f_2) \cdot \Upsilon^\dagger(e^{t\mathrm{H}_W(g(x),gh(x),x)} \otimes p(\theta) \otimes q(\partial_\theta)),$$

respectively, where $f_1, f_2 \in \mathbb{K}[X]$ and Υ, $\Upsilon^\dagger$ are the $\mathbb{K}[X]$-linear extensions of (3.10) and (4.15).

Remark 4.16. Note that these formulas are valid for non-abelian groups as well.

4.2.4. *G*-actions. Our final goal in this section is to transfer the G-actions on the Hochschild (co)homology

$${}^t\mathsf{HH}^*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G), \quad {}^t\mathsf{HH}_*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G)$$

(see (2.11)) to the corresponding Koszul (co)homology groups. According to Proposition 2.7, the (co)invariants of the resulting G-actions are isomorphic to ${}^t\mathsf{HH}^*(\mathbb{K}[X] \rtimes G, W)$ and ${}^t\mathsf{HH}_*(\mathbb{K}[X] \rtimes G, W)$.

Since we are dealing with an abelian group, the G-actions on

$$\mathcal{HH}^*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G) \quad \text{and} \quad \mathcal{HH}_*(\mathbb{K}[X], W; \mathbb{K}[X] \rtimes G)$$

preserve the mixed subcomplexes

$$\mathcal{HH}^*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g) \quad \text{and} \quad \mathcal{HH}_*(\mathbb{K}[X], W; \mathbb{K}[X] \otimes g),$$

for all g. Let us introduce the following G-action on the algebra Cl_N:

$$G \ni g = (g_1, \dots, g_N): \quad \theta_i \mapsto g_i \theta_i, \quad \partial_{\theta_i} \mapsto g_i^{-1} \partial_{\theta_i}. \tag{4.23}$$

Note it is compatible with the Cl_N-module structures on $\mathbb{K}[\partial_\theta]$ and $\mathbb{K}[\theta]$ from Section 3.1.2. The combination of the G-actions on $\mathbb{K}[X]$ and Cl_N yields G-actions on various spaces we have been studying, e.g. on $\mathbb{K}[X][\partial_\theta]$ and $\mathbb{K}[X][\theta]$, and also on the underlying space $\mathcal{K}_*(\mathbb{K}[X])$ of the Koszul resolution of $\mathbb{K}[X]$. The latter action commutes with the differentials δ_{Kos} and δ_{curv}. This follows from the observation that δ_{Kos} and δ_{curv} are G-invariant and from the above-mentioned compatibility of the Cl_N- and G-actions on $\mathbb{K}[\theta]$.

As a result, we obtain G-actions on the mixed complexes

$$\begin{aligned} &\mathcal{K}os^*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g) = (\mathcal{K}^*(\mathbb{K}[X], \mathbb{K}[X] \otimes g),\ \partial_{\mathrm{Kos}},\ \partial_{\mathrm{curv}})\,, \\ &\mathcal{K}os_*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g) = (\mathcal{K}_*(\mathbb{K}[X], \mathbb{K}[X] \otimes g),\ b_{\mathrm{Kos}},\ b_{\mathrm{curv}})\,. \end{aligned} \tag{4.24}$$

defined on the underlying spaces of the complexes by analogy with (2.11).

Proposition 4.17. *(a) The quasi-isomorphisms (see Corollary 4.3)*

$$\begin{aligned} &\Psi^* : \mathcal{K}os^*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g) \to \mathcal{HH}^*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g), \\ &\Psi_* : \mathcal{HH}_*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g) \to \mathcal{K}os_*(\mathbb{K}[X], w; \mathbb{K}[X] \otimes g) \end{aligned}$$

are G-equivariant.

(b) The isomorphisms (4.10) and (4.12) between the mixed complexes on the right-hand sides of (4.24) and the mixed complexes

$$\begin{aligned} &((\mathbb{K}[X] \otimes g)[\partial_\theta], \partial_{\mathrm{Kos}}, \partial_{\mathrm{curv}})\ (\simeq (\mathbb{K}[X][\partial_\theta], \partial_{\mathrm{Kos}}(g), \partial_{\mathrm{curv}}(g))), \\ &((\mathbb{K}[X] \otimes g)[\theta], b_{\mathrm{Kos}}, b_{\mathrm{curv}})\ (\simeq (\mathbb{K}[X][\theta], b_{\mathrm{Kos}}(g), b_{\mathrm{curv}}(g))) \end{aligned} \tag{4.25}$$

are also G-equivariant.

Part (a) follows from the observation that the morphism (4.1) intertwines the G-actions on the bar and Koszul resolutions. Part (b) is obvious for the second complex; for the first one the claim follows from the fact that the pairing (4.6) is G-invariant.

Let us also point out that the G-actions on (4.25) are compatible with the explicit description of the cohomology classes that we worked out in Remarks 4.11, 4.13. More precisely, the splitting of the differentials $\partial_{\mathrm{curv}}(g) = \partial_{\mathrm{curv}}(g)' + \partial_{\mathrm{curv}}(g)''$ and $b_{\mathrm{curv}}(g) = b_{\mathrm{curv}}(g)' + b_{\mathrm{curv}}(g)''$ is G-invariant and the isomorphisms (4.18) and (4.21) intertwine the G-actions.

Remark 4.18. Unlike the results of the previous two sections (cf. Remarks 4.14, 4.16), the above picture relies very heavily on the fact that G acts by rescaling the variables. The major problem with other groups is that the morphism (4.1) will not be equivariant anymore, even if $G \subset GL_N(\mathbb{K})$.

4.3. Proof of Theorem 3.1

In this section, we assume that W has only isolated critical points. Then each $W^g \in \mathbb{K}[X^g]$ also has isolated critical points. Indeed (cf. [37, Lem.2.5.3]), the

equality $W(g(x)) = W(x)$ implies (by differentiating both sides)

$$\mathsf{res}^g(\partial_{x_i} W) = 0 \quad \forall\, i \in I_g, \qquad \mathsf{res}^g(\partial_{x_i} W) = \partial_{x_i} W^g \quad \forall\, i \in I^g$$

which in turn implies that res^g induces a surjective map $\mathrm{H}^*(\Omega^*_X, dW \wedge \cdot) \to \mathrm{H}^*(\Omega^*_{X^g}, dW^g \wedge \cdot)$.

Let us start by writing down explicit isomorphisms

$$\begin{aligned} {}^t\mathrm{M}^*(X,W,G) &\simeq \bigoplus_{g\in G} \mathrm{H}^*(\mathbb{K}[X][\partial_\theta][t^{\pm 1}], \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)), \\ {}^t\Omega^*(X,W,G) &\simeq \bigoplus_{g\in G} \mathrm{H}^*(\mathbb{K}[X][\theta][t^{\pm 1}], b_{\mathrm{Kos}}(g) + tb_{\mathrm{curv}}(g)) \end{aligned} \tag{4.26}$$

as $(\mathbb{Z}\times G)$-graded G-equivariant $\mathbb{K}[t^{\pm 1}]$-modules. Since all W^g have isolated critical points, Propositions 4.10, 4.12 and Remarks 4.11, 4.13 imply that $e^{t\mathrm{H}_{W,g}} : \mathbb{K}[X][\partial_\theta][t^{\pm 1}] \to \mathbb{K}[X][\partial_\theta][t^{\pm 1}]$ induces an isomorphism

$$e^{t\mathrm{H}_{W,g}} : M(W^g)[t^{\pm 1}]\, \partial_{\theta_{I_g}} \simeq \mathrm{H}^*(\mathbb{K}[X][\partial_\theta][t^{\pm 1}], \partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)) \tag{4.27}$$

and $e^{t\mathrm{H}^\dagger_{W,g}} : \mathbb{K}[X][\theta][t^{\pm 1}] \to \mathbb{K}[X][\theta][t^{\pm 1}]$ induces an isomorphism

$$e^{t\mathrm{H}^\dagger_{W,g}} : M(W^g)[t^{\pm 1}]\, \theta_{I^g} \simeq \mathrm{H}^*(\mathbb{K}[X][\theta][t^{\pm 1}], b_{\mathrm{Kos}}(g) + tb_{\mathrm{curv}}(g)). \tag{4.28}$$

The specific isomorphisms (4.26) we are interested in are the $\mathbb{K}[t^{\pm 1}]$-linear extensions of the maps

$$\begin{aligned} &\lfloor f \rfloor_g \cdot \xi_g \mapsto \lfloor f \rfloor_g \cdot [e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})], \\ &\lfloor f \rfloor_g \cdot \omega_g \mapsto (-1)^{Nd_g + ||I_g||}\, \lfloor f \rfloor_g \cdot [e^{t\mathrm{H}^\dagger_{W,g}}(\theta_{I^g})] \end{aligned} \tag{4.29}$$

($f \in \mathbb{K}[X]$) where $[\ ]$ denotes the cohomology class and $||(i_1, i_2, \ldots, i_k)|| := i_1 + i_2 + \cdots + i_k$. Obviously, these isomorphisms preserve the gradings and respect the G-actions (cf. (3.5), (4.23)).

Let us now calculate the cup products between elements of the form $\lfloor f \rfloor_g \cdot [e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]$. To begin with, the existence of $\sigma_{g,h} \in M(W^{gh})$ such that

$$[e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})] \cup [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})] = t^{d_{g,h}} \cdot \sigma_{g,h} \cdot [e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})]$$

is an immediate consequence of (4.27). That $\sigma_{g,h}$ is given by the formula (3.6) is a consequence of the formula for $\cup$ from Proposition 4.15 since (3.6) is nothing but the class in $M(W^{gh})$ of the coefficient at $t^{d_{g,h}}\partial_{\theta_{I_{gh}}}$ in

$e^{-t\mathrm{H}_{W,gh}}\left(e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})\cup e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})\right)$. (Note that according to the formulas in Proposition 4.15 we should be using $\mathrm{H}_W(x,g(x),gh(x))$ instead of $\mathrm{H}_W(x,g(x),x)$ in (3.6). But, obviously, after applying $\lfloor\,\rfloor_{gh}$ the result is the same. Note also that $\mathrm{H}_{W,gh}$ does not appear in (3.6) because for any $k\in G$ the element $\partial_{\theta_{I_k}}$ is not contained in the image of $\mathrm{H}_{W,k}$; see (3.8).)

Furthermore, by Proposition 4.15 the elements $\sigma_{g,h}$ determine the products on the entire cohomology groups, namely

$$\begin{aligned}&\lfloor f_1\rfloor_g\cdot[e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\cup\lfloor f_2\rfloor_h\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\\ =\;&t^{d_{g,h}}\cdot\lfloor f_1g(f_2)\rfloor_{gh}\cdot\sigma_{g,h}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})].\end{aligned}$$

for all $f_1,f_2\in\mathbb{K}[X]$. (We implicitly use the obvious fact that the operator $\lfloor\,\rfloor_g$ commutes with the G-actions on $\mathbb{K}[X]$ and $M(W^g)$.) Note the difference between the right-hand side of the latter formula and that of (3.11): In (3.11) f_2 is not twisted by g. We claim that the twist by g is not needed, i.e. $\lfloor f_1g(f_2)\rfloor_{gh}\cdot\sigma_{g,h}=\lfloor f_1f_2\rfloor_{gh}\cdot\sigma_{g,h}$ as elements of $M(W^{gh})$. This is a consequence of the braided super-commutativity (2.13) of the cup product. Indeed, by the braided super-commutativity

$$\begin{aligned}t^{d_{g,h}}\cdot\sigma_{g,h}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})]&=[e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\cup[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\\ &=(-1)^{d_gd_h}\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\cup h^{-1}\left([e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\right)\\ &=\alpha\cdot(-1)^{d_gd_h}\cdot t^{d_{g,h}}\cdot\sigma_{h,g}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})]\end{aligned}\tag{4.30}$$

where the constant α is defined by $h^{-1}(\partial_{\theta_{I_g}})=\alpha\cdot\partial_{\theta_{I_g}}$. Hence

$$\begin{aligned}&t^{d_{g,h}}\cdot\lfloor f_1g(f_2)\rfloor_{gh}\cdot\sigma_{g,h}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})]\\ =\;&\lfloor f_1\rfloor_g\cdot[e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\cup\lfloor f_2\rfloor_h\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\\ =\;&(-1)^{d_gd_h}\cdot\lfloor f_2\rfloor_h\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\cup h^{-1}\left(\lfloor f_1\rfloor_g\right)\cdot h^{-1}\left([e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\right)\\ =\;&(-1)^{d_gd_h}\cdot\lfloor f_2\rfloor_h\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\cup\lfloor h^{-1}(f_1)\rfloor_g\cdot h^{-1}\left([e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\right)\\ =\;&\alpha\cdot(-1)^{d_gd_h}\cdot\lfloor f_2\rfloor_h\cdot[e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\cup\lfloor h^{-1}(f_1)\rfloor_g\cdot[e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\\ =\;&\alpha\cdot(-1)^{d_gd_h}\cdot t^{d_{g,h}}\cdot\lfloor f_2f_1\rfloor_{gh}\cdot\sigma_{h,g}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})]\\ \overset{(4.30)}{=}\;&t^{d_{g,h}}\cdot\lfloor f_1f_2\rfloor_{gh}\cdot\sigma_{g,h}\cdot[e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})].\end{aligned}$$

This completes the proof of the first half of Theorem 3.1. The second half is proved similarly, so we only sketch the argument.

It follows from (4.28) that there exist elements $\widetilde{\sigma}_{g,h} \in M(W^{gh})$ such that

$$[e^{t\mathrm{H}^{\dagger}_{W,g}}(\theta_{I^g})] \cap [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})] = t^{d_{g,h}} \cdot \widetilde{\sigma}_{g,h} \cdot [e^{t\mathrm{H}^{\dagger}_{W,gh}}(\theta_{I^{gh}})]$$

We claim that $\widetilde{\sigma}_{g,h} = (-1)^{N(d_g+d_{gh})+||I_g||+||I_{gh}||}\sigma_{g,h}$ (cf. (4.29)). Indeed, one can show, using (4.14), that $\widetilde{\sigma}_{e,g} = (-1)^{Nd_g+||I_g||}$. Hence

$$\begin{aligned}
&[e^{t\mathrm{H}^{\dagger}_{W,g}}(\theta_{I^g})] \cap [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})] \\
&= (-1)^{Nd_g+||I_g||}\left([\theta_{I^e}] \cap [e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})]\right) \cap [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})] \\
&= (-1)^{Nd_g+||I_g||} \cdot [\theta_{I^e}] \cap \left([e^{t\mathrm{H}_{W,g}}(\partial_{\theta_{I_g}})] \cup [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})]\right) \\
&= (-1)^{Nd_g+||I_g||} \cdot t^{d_{g,h}} \cdot \sigma_{g,h} \cdot [\theta_{I^e}] \cap [e^{t\mathrm{H}_{W,gh}}(\partial_{\theta_{I_{gh}}})] \\
&= (-1)^{Nd_g+||I_g||} \cdot (-1)^{Nd_{gh}+||I_{gh}||} \cdot t^{d_{g,h}} \cdot \sigma_{g,h} \cdot [e^{t\mathrm{H}^{\dagger}_{W,gh}}(\theta_{I^{gh}})].
\end{aligned}$$

Thus,

$$\begin{aligned}
&(-1)^{Nd_g+||I_g||} \cdot [e^{t\mathrm{H}^{\dagger}_{W,g}}(\theta_{I^g})] \cap [e^{t\mathrm{H}_{W,h}}(\partial_{\theta_{I_h}})] \\
&= t^{d_{g,h}} \cdot \sigma_{g,h} \cdot (-1)^{Nd_{gh}+||I_{gh}||} \cdot [e^{t\mathrm{H}^{\dagger}_{W,gh}}(\theta_{I^{gh}})]
\end{aligned}$$

which proves that the second map in (4.29) respects the cap products on the generators. The extension to the entire (co)homology groups is completely parallel to the case of the cup product and is left to the reader.

Finally, the claims that ξ_e is the unit of ${}^t\mathrm{M}^*(X,W,G)$ and ω_e is a free generator of ${}^t\Omega^*(X,W,G)$ both follow from the obvious fact that $\sigma_{e,g} = 1$ for any $g \in G$.

4.4. Proof of Theorem 3.4

The proof will be outlined very schematically since it involves notions and results of the theory of curved dg categories and their Hochschild invariants [36] which are far beyond the scope of the present work. (Unfortunately, we have to assume familiarity with the subject.) Also, we only sketch the proof of the isomorphism

$$(\mathsf{HH}^*(\mathrm{MF}_G(X,W)), \cup) \simeq \left(\mathsf{HH}^{\mathrm{II},*}(\mathbb{K}[X] \rtimes G, W), \cup\right). \tag{4.31}$$

The proof of the other half is just a straightforward extension of the argument given below (for the homology all the arrows below should be reversed).

To start with, we want to replace the ordinary Hochschild cohomology of the matrix factorizations with their Hochschild cohomology of the second kind. More precisely, there is a natural algebra homomorphism [36, Sect.2.4, (24)]

$$\mathsf{HH}^{\mathrm{II},*}(\mathrm{MF}_G(X,W)) \to \mathsf{HH}^*(\mathrm{MF}_G(X,W)) \tag{4.32}$$

which is an isomorphism in our case. It is this point where the absence of critical points outside of $W^{-1}(0)$ is needed; see [36, Sect.4.8,4.10]. (Formally speaking, this is spelled out in *loc.cit.* only in the non-equivariant setting but the same argument, when combined with the discussion in [37, Sect.2.5], applies in the equivariant case.)

Furthermore, there is a diagram of morphisms

$$\mathsf{HH}^{\mathrm{II},*}(\mathrm{MF}_G(X,W)) \xleftarrow{I_1} \mathsf{HH}^{\mathrm{II},*}(\mathfrak{A}) \xrightarrow{I_2} \mathsf{HH}^{\mathrm{II},*}(\mathbb{K}[X]\rtimes G, W) \tag{4.33}$$

where

(1) $\mathfrak{A}$ is the curved dg category whose objects are the $\mathbb{Z}/2\mathbb{Z}$-graded finitely generated projective $(\mathbb{K}[X]\rtimes G)$-modules $E = E^{\mathrm{even}} \oplus E^{\mathrm{odd}}$, the curvature of every object E is $W\cdot \mathrm{id}_E$, and $\mathrm{Hom}_{\mathfrak{A}}(E_1,E_2)$ is the $\mathbb{Z}/2\mathbb{Z}$-graded space of all $(\mathbb{K}[X]\rtimes G)$-linear maps from E_1 to E_2; we endow this space with the trivial differential.

(2) I_2 is induced by the embedding of curved dg categories $(\mathbb{K}[X]\rtimes G, W) \to \mathfrak{A}$ (we view the former as a curved dg category with a single object) sending the unique object of $(\mathbb{K}[X]\rtimes G, W)$ to the object with $E^{\mathrm{even}} = A$ and $E^{\mathrm{odd}} = 0$.

(3) I_1 is the dual to the map $I_1^\vee$ from the bar resolution of the second kind of $\mathrm{MF}_G(X,W)$ to the bar resolution of the second kind of $\mathfrak{A}$ given by the following explicit formula (see [36, Sect.2.4, (18)] and [39, Sect.2.3]):

$$I_1^\vee = \exp\,(\text{“insert } \delta\text{”})$$

where

$$\begin{aligned}&\text{“insert } \delta\text{”} : \phi_{E_0E_1}\ [\ \phi_{E_1E_2}\ |\ \cdots |\ \phi_{E_{n-1}E_n}\ ]\ \phi_{E_nE_{n+1}}\\ &\mapsto \sum_{i=1}^{n} \phi_{E_0E_1}\ [\ \cdots |\ \phi_{E_{i-1}E_i} |\ \delta_{E_i}\ |\ \phi_{E_iE_{i+1}}\ | \cdots\ ]\ \phi_{E_nE_{n+1}}.\end{aligned}$$

On the left-hand side, (E_i, δ_{E_i}) are matrix factorizations and $\phi_{E_iE_{i+1}}$ are morphisms in the category $\mathrm{MF}_G(X,W)$; on the right-hand side, E_i, $\phi_{E_iE_{i+1}}$, and δ_{E_i} are viewed as objects/morphisms in $\mathfrak{A}$.

The proof of (4.31) is completed by noticing that, firstly, both I_1 and I_2 are isomorphisms (this is obvious for I_1 and follows from [36, Sect.2.6,(45)] for I_2) and, secondly, both I_1 and I_2 are morphisms of algebras (this is obvious for I_2; for I_1 this follows the fact that "insert δ" is a coderivation with respect to the coproduct Δ_{bar} (2.5)).

Appendix A. Examples and applications
(*by A. Basalaev and D. Shklyarov*)

In Appendix A, the ground field is $\mathbb{C}$.

A.1. Example: Invertible polynomials

The aim of this section is to discuss in more detail the (isomorphism class of the) algebra $\mathrm{M}^*(X, W, G)$ from Corollary 3.3 in the case when $X = \mathbb{C}^N$ and $W \in \mathbb{C}[X]$ is an *invertible* polynomial [32, 33].

Instead of giving a formal definition of invertible polynomials, we recall the only fact that matters to us, namely, the classification of such polynomials obtained in [33] which says that up to a natural equivalence, any invertible polynomial is the Thom-Sebastiani sum of polynomials of the following *atomic types*:

Ferma type: $x_1^{a_1}$,

Chain type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}$,

Loop type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}x_1$

where $a_k \in \mathbb{N}_{\geq 2}$ and $N \geq 2$. (The Thom-Sebastiani sum of two functions (X_1, W_1) and (X_2, W_2) is the function $(X_1 \times X_2, W_1 \boxplus W_2)$ with $(W_1 \boxplus W_2)(x_1, x_2) := W_1(x_1) + W_2(x_2)$.)

Note that every invertible polynomial has an isolated critical point at the origin and, due to its quasi-homogeneity, no other critical points. Thus, the main results of the present work are applicable and yield a description of the Hochschild cohomology, i.e. of the "closed string algebra" of any invertible LG model. But, as was already mentioned in the Introduction, there already exists [6] a good candidate for this role which, like $\mathrm{M}^*(\mathbb{C}^N, W, G)^G$, is constructed as the subalgebra of G-invariants of a $(\mathbb{Z}/2\mathbb{Z} \times G)$-graded, G-equivariant, and braided super-commutative algebra. This latter algebra is called the G-twisted Jacobian algebra of W and denoted by $\mathrm{Jac}'(W, G)$. Let us recall its description.

It suffices to describe $\mathrm{Jac}'(W,G)$ for each atomic polynomial since the algebra has the following "Künneth property": If $W = W_1 \boxplus \cdots \boxplus W_l$, where each W_i is a polynomial of one of the three atomic types, and $G = G_1 \times \cdots \times G_l$, where G_i is an abelian group of symmetries of W_i, then

$$\mathrm{Jac}'(W,G) = \mathrm{Jac}'(W_1,G_1) \otimes \cdots \otimes \mathrm{Jac}'(W_l,G_l). \tag{A.1}$$

Let W be an atomic polynomial and G be its abelian symmetry group. The algebra $\mathrm{Jac}'(W,G)$ looks exactly the same as $\mathrm{M}^*(\mathbb{C}^N,W,G)$ — that is, it is isomorphic to $\mathrm{M}^*(\mathbb{C}^N,W,G)$ as a $(\mathbb{Z}/2\mathbb{Z}\times G)$-graded G-equivariant $M(W)$-module — but it has different "structure constants" $\sigma'_{g,h}$ in the products (3.11), namely

$$\sigma'_{e,g} = \sigma'_{g,e} = 1, \quad \sigma'_{g,g^{-1}} = e^{-\pi\sqrt{-1}\,\mathrm{age}(g)} \lfloor \det(\partial_{x_i}\partial_{x_j} W)_{i,j\in I_g} \rfloor_e, \quad \forall g \in G$$

and $\sigma'_{g,h} = 0$ otherwise. In the above formula $\lfloor\,\rfloor_e$, we recall, denotes the class of an element in $M(W = W^e)$ and $\mathrm{age}(g) := \sum_{i=1}^N q_i$ with q_i being the rational numbers satisfying $0 \le q_i < 1$ and $g = \mathrm{diag}(e^{2\pi\sqrt{-1}q_i})$. (The actual structure constants of $\mathrm{Jac}'(W,G)$, as defined in [6], differ from $\sigma'_{g,h}$ by complex factors which we ignore here since they do not affect the isomorphism class.) As explained in [6, Sect.4], the map $\xi_g \otimes \xi_h \mapsto \sigma'_{g,h}\xi_{gh}$ does extend by $M(W)$-linearity to a well-defined associative braided super-commutative product on the whole of $\mathrm{Jac}'(W,G)$.

We propose the following conjecture:

Conjecture. *For any invertible W and any abelian symmetry group G there is an isomorphism of $(\mathbb{Z}/2\mathbb{Z}\times G)$-graded G-equivariant algebras $\mathrm{M}^*(\mathbb{C}^N,W,G) \simeq \mathrm{Jac}'(W,G)$.*

Remark A.1. (1) Note that Proposition 2.6 implies that $\mathrm{M}^*(\mathbb{C}^N,W,G)$ has the property (A.1) as well. Thus, it would suffice to prove the conjecture for polynomials of the three atomic types. The Ferma case is an easy exercise, the conjecture is interesting only for the other two types.
(2) There is also an analog of $\Omega^*(\mathbb{C}^N,W,G)$ in [6] which the authors denote by $\Omega'_{W,G}$. Its structure is completely analogous to that of $\Omega^*(\mathbb{C}^N,W,G)$: it is a free rank 1 $\mathrm{Jac}'(W,G)$-module spanned by a generator of degree $-N$. It is not included in the conjecture for the simple reason that it is defined in *loc. cit.* as a *left* module.

In the remainder of this section we check the conjecture for a generic chain type polynomial in two variables and its maximal (hence any) symmetry group:

$$W = x_1^{a_1}x_2 + x_2^{a_2} \quad (a_2 \geq 3), \quad G := \{(\zeta_1, \zeta_2) \in (\mathbb{C}^*)^2 \mid \zeta_1^{a_1}\zeta_2 = 1,\ \zeta_2^{a_2} = 1\}$$

Our strategy is as follows: we are going to show that

A: $\sigma_{g,h} = 0$ provided neither of g, h, gh is the unit of G.

B: $\sigma_{g,g^{-1}} = \alpha_g \cdot \sigma'_{g,g^{-1}}$ for some $\alpha_g \in \mathbb{C}^*$.

(It is worthwhile noting that $\sigma'_{g,g^{-1}} \in M(W)$ is always non-zero; see [6, Prop.30].) Let us explain why checking these conditions suffices to verify the claim of the conjecture. It follows from the braided super-commutativity of both $\mathrm{M}^*(\mathbb{C}^2, W, G)$ and $\mathrm{Jac}'(W, G)$ that

$$\sigma_{g^{-1},g} = \det(g)\sigma_{g,g^{-1}}, \quad \sigma'_{g^{-1},g} = \det(g)\sigma'_{g,g^{-1}}.$$

Together with **B** this implies that $\alpha_g = \alpha_{g^{-1}}$ which, in turn, implies that the assignment

$$\xi_g^{\mathrm{M}^*} \mapsto \sqrt{\alpha}_g \cdot \xi_g^{\mathrm{Jac}'}$$

— for any choice of the square root of the function $G \to \mathbb{C}^*, g \mapsto \alpha_g$ satisfying $\sqrt{\alpha}_g = \sqrt{\alpha}_{g^{-1}}$ — extends by $M(W)$-linearity to an algebra isomorphism $\mathrm{M}^*(\mathbb{C}^2, W, G) \to \mathrm{Jac}'(W, G)$.

Let us calculate all the ingredients for the formula (3.6) in our case. One has

$$\begin{aligned}\mathrm{H}_W(x, y, z) = &\left(\frac{x_1^{a_1} - y_1^{a_1}}{x_1 - y_1} - \frac{x_1^{a_1} - z_1^{a_1}}{x_1 - z_1}\right)\frac{x_2}{y_1 - z_1}\,\theta_1 \otimes \theta_1 \\ &+ \left(\frac{x_2^{a_2} - y_2^{a_2}}{x_2 - y_2} - \frac{x_2^{a_2} - z_2^{a_2}}{x_2 - z_2}\right)\frac{1}{y_2 - z_2}\,\theta_2 \otimes \theta_2 \\ &+ \frac{y_1^{a_1} - z_1^{a_1}}{y_1 - z_1}\,\theta_2 \otimes \theta_1\end{aligned}$$

which implies that for $g = (\zeta_1, \zeta_2) \neq e$ ($\Leftrightarrow\ \zeta_1 \neq 1$)

$$\begin{aligned}\text{(A.2)} \qquad \mathrm{H}_W(x, g(x), x) = &\left(\frac{a_1}{1 - \zeta_1} - \frac{1 - \zeta_1^{a_1}}{(1 - \zeta_1)^2}\right) x_1^{a_1 - 2}x_2\,\theta_1 \otimes \theta_1 \\ &+ \frac{a_2}{1 - \zeta_2}x_2^{a_2 - 2}\,\theta_2 \otimes \theta_2 + \frac{1 - \zeta_1^{a_1}}{1 - \zeta_1}x_1^{a_1 - 1}\,\theta_2 \otimes \theta_1,\end{aligned}$$

if $\zeta_2 \neq 1$ ($\Leftrightarrow$ $d_g = 2$), and

$$\text{(A.3)}\quad \mathrm{H}_W(x, g(x), x) = \frac{a_1}{1-\zeta_1} x_1^{a_1-2} x_2\, \theta_1 \otimes \theta_1 + \frac{a_2(a_2-1)}{2} x_2^{a_2-2}\, \theta_2 \otimes \theta_2 + \frac{1-\zeta_1^{a_1}}{1-\zeta_1} x_1^{a_1-1}\, \theta_2 \otimes \theta_1,$$

if $\zeta_2 = 1$ ($\Leftrightarrow$ $d_g = 1$). Also,

$$\text{(A.4)}\qquad \mathrm{H}_{W,g}(x) = \frac{\zeta_1^{a_1}}{1-\zeta_1} x_1^{a_1-1}\, \theta_1\theta_2 \quad \text{if } \zeta_2 \neq 1 \quad \text{and} \quad \mathrm{H}_{W,g}(x) = 0 \quad \text{if } \zeta_2 = 1.$$

We are in a position now to check the above conditions **A** and **B**.

A: Let us fix $g, h \in G$ such that neither of g, h, gh is the unit. We have to consider the following possibilities:

1) $d_g = d_h = d_{gh} = 1$;
2) $d_g = 1$, $d_h = d_{gh} = 2$ or $d_h = 1$, $d_g = d_{gh} = 2$;
3) $d_g = d_h = d_{gh} = 2$.

However, because of (3.9), only the last possibility is non-trivial, so we assume $d_g = d_h = d_{gh} = 2$ (consequently, $d_{g,h} = 1$).

By Theorem 3.1 $\sigma_{g,h}$ is the coefficient at $\partial_{\theta_1}\partial_{\theta_2}$ in the expression

$$\Upsilon\Big((\lfloor \mathrm{H}_W(x, g(x), x)\rfloor_{gh} + \lfloor \mathrm{H}_{W,g}(x)\rfloor_{gh} \otimes 1 + 1 \otimes \lfloor \mathrm{H}_{W,h}(g(x))\rfloor_{gh}) \otimes \partial_{\theta_1}\partial_{\theta_2} \otimes \partial_{\theta_1}\partial_{\theta_2}\Big).$$

Since $d_{gh} = 2$, the operator $\lfloor\ \rfloor_{gh}$ sets both x_1 and x_2 equal to 0. Thus, by (A.2) and (A.4) the above expression is 0.

B: We fix now $g = (\zeta_1, \zeta_2) \neq e$ and consider two cases:

<u>$\zeta_2 \neq 1$ ($\Leftrightarrow$ $d_g = 2$):</u> In this case $\sigma_{g,g^{-1}}$ is the constant coefficient in the expression

$$\frac{1}{2}\Upsilon\Big(\big(\lfloor \mathrm{H}_W(x, g(x), x)\rfloor_e + \lfloor \mathrm{H}_{W,g}(x)\rfloor_e \otimes 1 + 1 \otimes \lfloor \mathrm{H}_{W,g^{-1}}(g(x))\rfloor_e\big)^2 \otimes \partial_{\theta_1}\partial_{\theta_2} \otimes \partial_{\theta_1}\partial_{\theta_2}\Big)$$

for $\mathrm{H}_W(x, g(x), x)$ given by (A.2). Denoting

$$\mathrm{H}_W(x, g(x), x) = A_1\,\theta_1 \otimes \theta_1 + A_2\,\theta_2 \otimes \theta_2 + A_3\,\theta_2 \otimes \theta_1, \tag{A.5}$$

$$\mathrm{H}_{W,g}(x) = B\,\theta_1\theta_2, \quad \mathrm{H}_{W,g^{-1}}(g(x)) = C\,\theta_1\theta_2, \tag{A.6}$$

one easily checks that $\sigma_{g,g^{-1}} = \lfloor B \cdot C - A_1 \cdot A_2 \rfloor_e$, that is

$$\sigma_{g,g^{-1}} = \left\lfloor -\frac{\zeta_1^{a_1}}{(1-\zeta_1)^2} x_1^{2a_1-2} - \left(\frac{a_1}{1-\zeta_1} - \frac{1-\zeta_1^{a_1}}{(1-\zeta_1)^2}\right) \frac{a_2}{1-\zeta_2} x_1^{a_1-2} x_2^{a_2-1} \right\rfloor_e$$

or, taking account of the facts that

$$\zeta_1^{a_1} = \zeta_2^{-1} \quad \text{and} \quad \lfloor x_1^{2a_1-2} \rfloor_e = \lfloor -a_2 x_1^{a_1-2} x_2^{a_2-1} \rfloor_e,$$

$$\sigma_{g,g^{-1}} = -\frac{a_1 a_2}{(1-\zeta_1)(1-\zeta_2)} \left\lfloor x_1^{a_1-2} x_2^{a_2-1} \right\rfloor_e .$$

The reader is invited to check that the class of the Hessian of W in $M(W)$ is proportional to the above element which completes the verification of **B** in the present case.

<u>$\zeta_2 = 1$ ($\Leftrightarrow$ $d_g = 1$):</u> This time $\sigma_{g,g^{-1}}$ is the constant coefficient in the expression

$$\Upsilon\left(\lfloor \mathrm{H}_W(x, g(x), x) \rfloor_e \otimes \partial_{\theta_1} \otimes \partial_{\theta_1}\right)$$

for $\mathrm{H}_W(x, g(x), x)$ given by (A.3). Using the notation (A.5), one sees that

$$\sigma_{g,g^{-1}} = \lfloor -A_1 \rfloor_e = -\frac{a_1}{1-\zeta_1} \lfloor x_1^{a_1-2} x_2 \rfloor_e.$$

On the other hand, $\partial_{x_1}^2 W = a_1(a_1-1)x_1^{a_1-2}x_2$, so **B** holds in this case as well.

A.2. Application: Hochschild cohomology of Fukaya categories of surfaces

Let us fix an integer $\mathsf{g} \geq 2$ and a symplectic compact connected oriented surface S of genus g. Let $\mathcal{F}(S)$ denote the ($\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear) Fukaya A_∞ category of S as defined in [40]. Our aim in this section is to combine Corollary 3.4 with the homological mirror symmetry theorem for surfaces established in [40] ($\mathsf{g} = 2$) and [13] ($\mathsf{g} \geq 3$) in order to prove the following claim:

Theorem A.2. *There is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras* $\mathsf{HH}^*(\mathcal{F}(S)) \simeq \mathrm{H}^*(S, \mathbb{C})$.

We should emphasize that the theorem itself is not new; we only present a new proof. The claim can also be deduced — again in combination with the theorems of [13, 40] which imply that $\mathcal{F}(S)$ is homologically smooth — from results of [16–18] (see, in particular, Corollary 7 in [16, Sect.1.2]). In fact, the latter approach yields more than the mere existence of an isomorphism of algebras. It shows that a *specific* map $\mathrm{H}^*(S, \mathbb{C}) \to \mathsf{HH}^*(\mathcal{F}(S))$, the so-called closed-open map, is an isomorphism.

Let us proceed to the proof. Of course, the first step is to apply the mirror symmetry theorem of [13, 40] and thereby convert Theorem A.2 into a claim about Hochschild invariants of specific LG models. Namely, the category $\mathcal{F}(S)$ is shown in [13, 40] to be derived Morita equivalent to the dg category $\mathrm{MF}_G(\mathbb{C}^3, W)$ of equivariant matrix factorizations associated with the pair

$$\begin{aligned} W &:= x_1^{2\mathsf{g}+1} + x_2^{2\mathsf{g}+1} + x_3^{2\mathsf{g}+1} - x_1x_2x_3, \\ G &:= \{(\zeta, \zeta, \zeta^{-2}) \in (\mathbb{C}^*)^3 \mid \zeta^{2\mathsf{g}+1} = 1\}. \end{aligned} \tag{A.7}$$

It follows, according to [28], that $\mathsf{HH}^*(\mathcal{F}(S)) \simeq \mathsf{HH}^*(\mathrm{MF}_G(\mathbb{C}^3, W))$ and so proving the theorem reduces to showing the existence of an algebra isomorphism $\mathsf{HH}^*(\mathrm{MF}_G(\mathbb{C}^3, W)) \simeq \mathrm{H}^*(S, \mathbb{C})$.

The problem now is that the above W does not satisfy the condition of Corollary 3.4. In addition to the origin $\underline{0} \in \mathbb{C}^3$, W has $(2\mathsf{g}+1)^2\ (2\mathsf{g}-2)$ other isolated critical points where the critical values are different from 0. To apply Corollary 3.4, we need to replace $\mathbb{C}^3$ by an open affine subset $X \subset \mathbb{C}^3$ containing $\underline{0}$ but none of those extra critical points. (Note that upon restricting the domain in this way we do not alter the Hochschild homology of the matrix factorizations since the natural dg functor $\mathrm{MF}_G(\mathbb{C}^3, W) \to \mathrm{MF}_G(X, W)$ is known to be a dg Morita equivalence.)

The subtlety is that our new domain X has to be as specified in Section 3.1, i.e. within the range of applicability of the main results of this paper. That is, X should be of the form $\mathbb{C}^3 \setminus \bigcup_{i,j}\{x_i = \lambda_i^j\}$ for some $\lambda_i^j \neq 0$. The union of hyperplanes that we remove should be G-stable and contain all the critical points we want to get rid of. Such a configuration is easy to construct: we take as λ_i^j all the non-zero coordinates of all those redundant critical points. The G-invariance of W implies that this configuration is G-stable.

Corollary 3.4 is applicable now and reduces the theorem to the following claim:

Proposition A.3. *There is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras* $\mathrm{M}^*(X,W,G)^G \simeq \mathrm{H}^*(S,\mathbb{C})$.

The remainder of this appendix is devoted to the proof of the proposition.

<u>$\mathrm{M}^*(X,W,G)^G$ *as a $\mathbb{Z}/2\mathbb{Z}$-graded space:*</u> Let $\widehat{\zeta} = (\zeta,\zeta,\zeta^{-2})$ be any cyclic generator of G. Let us rename the generators ξ_g of the "twisted sectors" in $\mathrm{M}^*(X,W,G)$ as follows:

$$\xi_k^+ := \xi_{\widehat{\zeta}^k}, \quad \xi_k^- := \xi_{\widehat{\zeta}^{-k}}, \qquad k = 1,\dots,\mathsf{g}.$$

Since the g-fixed locus of any $e \neq g \in G$ is the origin $\underline{0}$, $d_g = 3$ and $M(W^g) = \mathbb{C}$ in this case, so

$$\mathrm{M}^{\mathrm{even}}(X,W,G) = M(W)\xi_e \quad \mathrm{M}^{\mathrm{odd}}(X,W,G) = \bigoplus_{k=1}^{\mathsf{g}} \left(\mathbb{C}\xi_k^+ \oplus \mathbb{C}\xi_k^-\right).$$

It is easy to see that all the $\xi_k^\pm$ are G-invariant (see (3.5) for the definition of the G-action), hence

$$\mathrm{M}^{\mathrm{even}}(X,W,G)^G = M(W)^G\xi_e \quad \mathrm{M}^{\mathrm{odd}}(X,W,G)^G = \bigoplus_{k=1}^{\mathsf{g}} \left(\mathbb{C}\xi_k^+ \oplus \mathbb{C}\xi_k^-\right).$$

Let us compute $M(W)^G$. An important technical aspect is that we can now treat W as a *local germ* at $\underline{0}$ rather than a global function, since $\underline{0}$ is the only critical point of W in X. One can show that $M(W)$ is spanned by the classes of 1, $x_1x_2x_3$, and x_i^l for $i = 1,2,3$, $l = 1,\dots,2\mathsf{g}$. Hence $M(W)^G = \mathbb{C} \oplus \mathbb{C}\varphi$ where φ is the class of $x_1x_2x_3$. Thus, finally,

$$\mathrm{M}^{\mathrm{even}}(X,W,G)^G = \mathbb{C}\xi_e \oplus \mathbb{C}\varphi\xi_e \quad \mathrm{M}^{\mathrm{odd}}(X,W,G)^G = \bigoplus_{k=1}^{\mathsf{g}} \left(\mathbb{C}\xi_k^+ \oplus \mathbb{C}\xi_k^-\right).$$

<u>$\mathrm{M}^*(X,W,G)^G$ *as an algebra:*</u> Being the Hochschild cohomology of something, $\mathrm{M}^*(X,W,G)^G$ is automatically super-commutative (Corollary 2.9). Furthermore, $\varphi^2 = 0$ since φ^2 is G-invariant and so lies in $\mathbb{C} \oplus \mathbb{C}\cdot\varphi$ but it has to be nilpotent (recall that we are dealing with the Milnor algebra of an

isolated local singularity). Hence

$$\varphi\xi_e \cup \varphi\xi_e = 0. \tag{A.8}$$

Also, the class of $x_1x_2x_3$ vanishes in $M(W^g) = \mathbb{C}$ for $g \neq e$, hence

$$\varphi\xi_e \cup \xi_k^{\pm} = 0 \quad \forall k. \tag{A.9}$$

Next, since $d_g = 3$ for all $g \neq e$, (3.9) implies that

$$\xi_k^+ \cup \xi_l^+ = \xi_k^- \cup \xi_l^- = 0 \quad \forall k, l; \qquad \xi_k^+ \cup \xi_l^- = 0 \quad k \neq l. \tag{A.10}$$

Hence the only interesting products are

$$\xi_k^+ \cup \xi_k^- = \sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}} \cdot \xi_e, \quad \sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}} \in M(W)^G. \tag{A.11}$$

The shape of the formula (3.6) — namely, the fact that the formula contains only the second partial difference derivatives of W — suggests that $\sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}}$ will have no constant term, i.e.

$$\sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}} = c_k\varphi, \quad c_k \in \mathbb{C}.$$

We compute $\sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}}$ below and, in particular, show that $c_k \neq 0$ for all k. This suffices to conclude the proof of the proposition. Indeed, as an abstract super-commutative algebra, $\mathrm{H}^*(S, \mathbb{C})$ has the form:

$$\mathrm{H}^{\mathrm{even}}(S, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}\gamma, \quad \mathrm{H}^{\mathrm{odd}}(S, \mathbb{C}) = \bigoplus_{k=1}^{\mathbf{g}} (\mathbb{C} \cdot \alpha_k \oplus \mathbb{C} \cdot \beta_k)$$

with

$$\gamma \cdot \gamma = 0, \qquad \gamma \cdot \alpha_k = \gamma \cdot \beta_k = 0 \quad \forall k$$
$$\alpha_k \cdot \alpha_l = \beta_k \cdot \beta_l = 0 \quad \forall k, l; \qquad \alpha_k \cdot \beta_l = 0 \quad \forall k \neq l; \qquad \alpha_k \cdot \beta_k = \gamma \quad \forall k.$$

Thus, by (A.8), (A.9), (A.10), and (A.11) the assignment

$$\xi_e \mapsto 1, \quad \xi_k^+ \mapsto \alpha_k, \quad \xi_k^- \mapsto c_k\beta_k, \quad \varphi\xi_e \mapsto \gamma$$

extends to an isomorphism $\mathrm{M}^*(X, W, G)^G \to \mathrm{H}^*(S, \mathbb{C})$.

So the last step is to prove

Lemma A.4. $\sigma_{\widehat{\zeta}^k, \widehat{\zeta}^{-k}} = \frac{1}{(1-\zeta^k)^2 \cdot (1-\zeta^{-2k})}\, \varphi.$

We have

$$\begin{aligned}\mathrm{H}_W(x,y,z) = &\left(\frac{x_1^{2\mathsf{g}+1}-y_1^{2\mathsf{g}+1}}{x_1-y_1} - \frac{x_1^{2\mathsf{g}+1}-z_1^{2\mathsf{g}+1}}{x_1-z_1}\right)\frac{1}{y_1-z_1}\theta_1\otimes\theta_1\\ &+\left(\frac{x_2^{2\mathsf{g}+1}-y_2^{2\mathsf{g}+1}}{x_2-y_2} - \frac{x_2^{2\mathsf{g}+1}-z_2^{2\mathsf{g}+1}}{x_2-z_2}\right)\frac{1}{y_2-z_2}\theta_2\otimes\theta_2\\ &+\left(\frac{x_3^{2\mathsf{g}+1}-y_3^{2\mathsf{g}+1}}{x_3-y_3} - \frac{x_3^{2\mathsf{g}+1}-z_3^{2\mathsf{g}+1}}{x_3-z_3}\right)\frac{1}{y_3-z_3}\theta_3\otimes\theta_3\\ &- x_3\ \theta_2\otimes\theta_1 - y_2\ \theta_3\otimes\theta_1 - z_1\theta_3\otimes\theta_2\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{H}_W(x,\widehat{\zeta}^k(x),x) = &\frac{2\mathsf{g}+1}{1-\zeta^k}x_1^{2\mathsf{g}-1}\theta_1\otimes\theta_1 + \frac{2\mathsf{g}+1}{1-\zeta^k}x_2^{2\mathsf{g}-1}\theta_2\otimes\theta_2\\ &+\frac{2\mathsf{g}+1}{1-\zeta^{-2k}}x_3^{2\mathsf{g}-1}\theta_3\otimes\theta_3 - x_3\ \theta_2\otimes\theta_1\\ &-\zeta^k x_2\ \theta_3\otimes\theta_1 - x_1\theta_3\otimes\theta_2.\end{aligned}$$

Also

$$\begin{aligned}\mathrm{H}_{W,\widehat{\zeta}^k}(x) &= -\frac{\zeta^k}{1-\zeta^k}x_3\ \theta_1\theta_2 - \frac{\zeta^{2k}}{1-\zeta^k}x_2\ \theta_1\theta_3,\\ \mathrm{H}_{W,\widehat{\zeta}^{-k}}(\widehat{\zeta}^k(x)) &= \frac{\zeta^{-2k}}{1-\zeta^k}x_3\ \theta_1\theta_2 + \frac{1}{1-\zeta^k}x_2\ \theta_1\theta_3.\end{aligned}$$

We are looking for $\sigma_{\widehat{\zeta}^k,\widehat{\zeta}^{-k}}$ which is the constant coefficient in the expression

$$\frac{1}{6}\Upsilon\Big(\Big(\lfloor \mathrm{H}_W(x,\widehat{\zeta}^k(x),x)\rfloor_e + \lfloor \mathrm{H}_{W,\widehat{\zeta}^k}(x)\rfloor_e\otimes 1 + 1\otimes\lfloor \mathrm{H}_{W,\widehat{\zeta}^{-k}}(\widehat{\zeta}^k(x))\rfloor_e\Big)^3 \otimes\partial_{\theta_1}\partial_{\theta_2}\partial_{\theta_3}\otimes\partial_{\theta_1}\partial_{\theta_2}\partial_{\theta_3}\Big).$$

Denoting

$$\begin{aligned}\mathrm{H}_W(x,\widehat{\zeta}^k(x),x) = &A_{11}\,\theta_1\otimes\theta_1 + A_{22}\,\theta_2\otimes\theta_2 + A_{33}\,\theta_3\otimes\theta_3\\ &+ A_{21}\,\theta_2\otimes\theta_1 + A_{31}\,\theta_3\otimes\theta_1 + A_{32}\,\theta_3\otimes\theta_2,\end{aligned}$$

and

$$\mathrm{H}_{W,\widehat{\zeta}^k}(x) = B_{12}\,\theta_1\theta_2 + B_{13}\,\theta_1\theta_3,\quad \mathrm{H}_{W,\widehat{\zeta}^{-k}}(\widehat{\zeta}^k(x)) = C_{12}\,\theta_1\theta_2 + C_{13}\,\theta_1\theta_3,$$

one checks that

$$\begin{aligned}\sigma_{\widehat{\zeta}^k,\widehat{\zeta}^{-k}} &= \lfloor A_{11}A_{22}A_{33} - A_{22}B_{13}C_{13} - A_{33}B_{12}C_{12} + A_{32}B_{12}C_{13}\rfloor_e \\ &= \frac{(2\mathsf{g}+1)^3}{(1-\zeta^k)^2(1-\zeta^{-2k})}\lfloor x_1^{2\mathsf{g}-1}x_2^{2\mathsf{g}-1}x_3^{2\mathsf{g}-1}\rfloor_e + \frac{(2\mathsf{g}+1)\zeta^{2k}}{(1-\zeta^k)^3}\lfloor x_2^{2\mathsf{g}+1}\rfloor_e \\ &\quad + \frac{(2\mathsf{g}+1)\zeta^{-k}}{(1-\zeta^k)^2(1-\zeta^{-2k})}\lfloor x_3^{2\mathsf{g}+1}\rfloor_e + \frac{\zeta^k}{(1-\zeta^k)^2}\lfloor x_1x_2x_3\rfloor_e.\end{aligned}$$

The first term equals $\varphi^{2\mathsf{g}-1}$ and hence vanishes. Also,

$$\begin{aligned}&(2\mathsf{g}+1)\lfloor x_1^{2\mathsf{g}}\rfloor_e = \lfloor x_2x_3\rfloor_e, \quad (2\mathsf{g}+1)\lfloor x_2^{2\mathsf{g}}\rfloor_e = \lfloor x_1x_3\rfloor_e, \\ &(2\mathsf{g}+1)\lfloor x_3^{2\mathsf{g}}\rfloor_e = \lfloor x_1x_2\rfloor_e\end{aligned}$$

imply $\lfloor (2\mathsf{g}+1)x_i^{2\mathsf{g}+1}\rfloor_e = \varphi$. Thus,

$$\begin{aligned}\sigma_{\widehat{\zeta}^k,\widehat{\zeta}^{-k}} &= \frac{\zeta^{2k}}{(1-\zeta^k)^3}\varphi + \frac{\zeta^{-k}}{(1-\zeta^k)^2(1-\zeta^{-2k})}\varphi + \frac{\zeta^k}{(1-\zeta^k)^2}\varphi \\ &= \frac{1}{(1-\zeta^k)^2(1-\zeta^{-2k})}\ \varphi.\end{aligned}$$

Appendix B. Proofs of intermediate results

Proof of Proposition 2.6. Consider the $\mathbb{Z}$-graded $(A\otimes A')$-bimodule

$$\mathcal{D}_*(A\otimes A') := \mathcal{B}_*(A)\otimes\mathcal{B}_*(A')$$

and denote by $\delta^{\mathcal{D}}_{\mathrm{bar}}$ and $\delta^{\mathcal{D}}_{\mathrm{curv}}$ the differentials $\delta_{\mathrm{bar}}\otimes 1 + 1\otimes\delta_{\mathrm{bar}}$ and $\delta_{\mathrm{curv}}\otimes 1 + 1\otimes\delta_{\mathrm{curv}}$ thereon. Repeating the definitions from Section 2.1.3 for the triple $(\mathcal{D}_*(A\otimes A'),\delta^{\mathcal{D}}_{\mathrm{bar}},\delta^{\mathcal{D}}_{\mathrm{curv}})$ instead of $(\mathcal{B}_*(A\otimes A'),\delta_{\mathrm{bar}},\delta_{\mathrm{curv}})$, one obtains functors

$${}^t\mathsf{HD}^*(A\otimes A', W\otimes 1 + 1\otimes W';-), \quad {}^t\mathsf{HD}_*(A\otimes A', W\otimes 1 + 1\otimes W';-)$$

on the category of $(A\otimes A', W\otimes 1 + 1\otimes W')$-bimodules which share all the properties of the Hochschild (co)homology. In particular, the new functors come with their cup and cap products defined just as $\cup_{\mathrm{Hoch}}$ and $\cap_{\mathrm{Hoch}}$ using

the coproduct

$$\Delta_{\mathcal{D}} : \mathcal{D}_*(A \otimes A') \to \mathcal{D}_*(A \otimes A') \otimes \mathcal{D}_*(A \otimes A'),$$
$$\Delta_{\mathcal{D}}(\underline{a}' \otimes \underline{a}'') = (-1)^{|\underline{a}'_{(2)}||\underline{a}''_{(1)}|}(\underline{a}'_{(1)} \otimes \underline{a}''_{(1)}) \otimes (\underline{a}'_{(2)} \otimes \underline{a}''_{(2)})$$

where $\underline{a}' \in \mathcal{B}_*(A)$, $\underline{a}'' \in \mathcal{B}_*(A')$, and $\underline{a}'_{(1)} \otimes \underline{a}'_{(2)} := \Delta_{\mathrm{bar}}(\underline{a}')$, $\underline{a}''_{(1)} \otimes \underline{a}''_{(2)} := \Delta_{\mathrm{bar}}(\underline{a}'')$.

One can formulate a version of Proposition 2.6 with ${}^t\mathsf{HD}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$ and ${}^t\mathsf{HD}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$ on the right-hand sides of (2.9) instead of the Hochschild (co)homology. We claim that this version of the assertion does hold: The isomorphisms in this case are induced by the obvious maps

$$\begin{aligned} &\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(\mathcal{B}_*(A), M) \otimes \mathrm{Hom}_{A'\otimes A'^{\mathrm{op}}}(\mathcal{B}_*(A'), M') \\ &\to \mathrm{Hom}_{(A\otimes A')\otimes(A\otimes A')^{\mathrm{op}}}(\mathcal{D}_*(A \otimes A'), M \otimes M'), \\ &(M \otimes_{A\otimes A^{\mathrm{op}}} \mathcal{B}_*(A)) \otimes (M' \otimes_{A'\otimes A'^{\mathrm{op}}} \mathcal{B}_*(A')) \\ &\to (M \otimes M') \otimes_{(A\otimes A')\otimes(A\otimes A')^{\mathrm{op}}} \mathcal{D}_*(A \otimes A'). \end{aligned} \tag{B.1}$$

Despite the seeming simplicity of the claim, there are two subtleties that require clarification. (In fact, it is this part of the proof that relies on the *smoothness* assumption.)

Firstly, we need to explain why (B.1) induce quasi-isomorphisms of the corresponding mixed complexes. For the second map it is straightforward: the map is an isomorphism even on the cochain level. The first map, on the other hand, is in general only an inclusion. However, as explained in the proof of Theorem 3.1 on page 210 in [11], the assumption that A and A' admit resolutions by finitely generated projective bimodules suffices to claim that the induced map is still a quasi-isomorphism.

Secondly, we should explain why the quasi-isomorphisms of the mixed complexes induced by (B.1) yield isomorphisms of $\mathbb{K}[t^{\pm 1}]$-modules

$$\begin{aligned} &{}^t\mathsf{HH}^*(A, W; M) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{HH}^*(A', W'; M') \\ &\simeq {}^t\mathsf{HD}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; M \otimes M'), \\ &{}^t\mathsf{HH}_*(A, W; M) \otimes_{\mathbb{K}[t^{\pm 1}]} {}^t\mathsf{HH}_*(A', W'; M') \\ &\simeq {}^t\mathsf{HD}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; M \otimes M'). \end{aligned} \tag{B.2}$$

In general, the functor $(\mathcal{C}, b, B) \mapsto \mathrm{H}^*(\mathcal{C}(\!(t)\!), b + tB)$ is only *lax* monoidal, i. e. for generic mixed complexes $(\mathcal{C}, b, B)$ and $(\mathcal{C}', b', B')$ the natural map

$$\begin{aligned} &\mathrm{H}^*(\mathcal{C}(\!(t)\!), b + tB) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathrm{H}^*(\mathcal{C}'(\!(t)\!), b' + tB') \\ &\to \mathrm{H}^*((\mathcal{C} \otimes \mathcal{C}')(\!(t)\!), (b + tB) \otimes 1 + 1 \otimes (b' + tB')) \end{aligned} \tag{B.3}$$

need not to be an isomorphism.

Lemma. *If* $\mathrm{H}^n(\mathcal{C}, b) = 0$ *and* $\mathrm{H}^n(\mathcal{C}', b') = 0$ *for* $n \ll 0$ *then (B.3) is an isomorphism.*

To prove the lemma, we pick n_0 so that $\mathrm{H}^n(\mathcal{C}, b) = 0$ and $\mathrm{H}^n(\mathcal{C}', b') = 0$ vanish in degrees $< n_0$ and consider the corresponding truncated mixed complexes $(\tau\mathcal{C}, b, B)$ and $(\tau\mathcal{C}', b', B')$ where

$$\tau\mathcal{C}_n := \begin{cases} 0 & n < n_0 \\ \mathrm{Coker}(b : \mathcal{C}_{n_0 - 1} \to \mathcal{C}_{n_0}) & n = n_0 \\ \mathcal{C}_n & n > n_0 \end{cases}$$

and the same for $\tau\mathcal{C}'$. Since the canonical projections

$$p : (\mathcal{C}, b, B) \to (\tau\mathcal{C}, b, B), \quad p' : (\mathcal{C}', b', B') \to (\tau\mathcal{C}', b', B')$$

are quasi-isomorphisms, their tensor product

$$p \otimes p' : (\mathcal{C}, b, B) \otimes (\mathcal{C}', b', B') \to (\tau\mathcal{C}, b, B) \otimes (\tau\mathcal{C}', b', B')$$

is a quasi-isomorphism as well. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{C}(\!(t)\!), b + tB) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathrm{H}^*(\mathcal{C}'(\!(t)\!), b' + tB') & \longrightarrow & \mathrm{H}^*((\mathcal{C} \otimes \mathcal{C}')(\!(t)\!), (b + tB) \otimes 1 + 1 \otimes (b' + tB')) \\ \downarrow & & \downarrow \\ \mathrm{H}^*(\tau\mathcal{C}(\!(t)\!), b + tB) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathrm{H}^*(\tau\mathcal{C}'(\!(t)\!), b' + tB') & \longrightarrow & \mathrm{H}^*((\tau\mathcal{C} \otimes \tau\mathcal{C}')(\!(t)\!), (b + tB) \otimes 1 + 1 \otimes (b' + tB')) \end{array}$$

where the vertical maps are the isomorphisms induced by the above canonical projections and the horizontal maps come from the monoidal structure of the periodic cohomology functor. The claim of the lemma follows immediately from the observation that the lower horizontal map in the diagram is also an isomorphism — this is a consequence of the isomorphisms

$$\begin{aligned} &\tau\mathcal{C}(\!(t)\!) \simeq \tau\mathcal{C} \otimes \mathbb{K}[t^{\pm 1}], \quad \tau\mathcal{C}'(\!(t)\!) \simeq \tau\mathcal{C}' \otimes \mathbb{K}[t^{\pm 1}], \\ &(\tau\mathcal{C} \otimes \tau\mathcal{C}')(\!(t)\!) \simeq (\tau\mathcal{C} \otimes \tau\mathcal{C}') \otimes \mathbb{K}[t^{\pm 1}] \end{aligned}$$

which themselves follow from the fact that all the complexes involved are bounded below; see (2.1).

Note that the mixed complex $\mathcal{HH}^*(A, W; M)$ always satisfies the condition of the lemma and the mixed complex $\mathcal{HH}_*(A, W; M)$ satisfies this condition provided A is smooth. Thus, (B.2) are isomorphisms.

To conclude the proof of Proposition 2.6 it remains to establish a product-preserving equivalence between the (co)homology functors ${}^t\mathsf{HD}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$ and ${}^t\mathsf{HD}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$ and the ordinary Hochschild (co)homology functors ${}^t\mathsf{HH}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$ and ${}^t\mathsf{HH}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)$

Consider the map of $A \otimes A'$-bimodules $\mathsf{Sh} : \mathcal{D}_*(A \otimes A') = \mathcal{B}_*(A) \otimes \mathcal{B}_*(A') \to \mathcal{B}_*(A \otimes A')$ determined by

$$\begin{aligned}\mathsf{Sh} : &(1[a_1|\cdots|a_n]1) \otimes (1[a'_1|\cdots|a'_m]1)\\ &\mapsto (1 \otimes 1)\mathsf{sh}[a_1 \otimes 1|\cdots|a_n \otimes 1|1 \otimes a'_1|\cdots|1 \otimes a'_m](1 \otimes 1)\end{aligned}$$

where sh stands for the sum over all the permutations that shuffle the a's with the a''s while preserving the order within the two groups, and each summand is multiplied by the sign of the corresponding permutation. It is a classical fact (cf. Section 6, Chapter XI in [11]) that Sh is compatible with the differential δ_{bar} on the bar resolutions and induces a quasi-isomorphism of complexes. It also turns out to be compatible with δ_{curv}; this is a more straightforward observation and we leave the proof to the reader. Thus, Sh gives rise to isomorphisms of functors

$$\begin{aligned}\mathsf{Sh}^* : &\ {}^t\mathsf{HH}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)\\ &\to {}^t\mathsf{HD}^*(A \otimes A', W \otimes 1 + 1 \otimes W'; -),\\ \mathsf{Sh}_* : &\ {}^t\mathsf{HD}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; -)\\ &\to {}^t\mathsf{HH}_*(A \otimes A', W \otimes 1 + 1 \otimes W'; -).\end{aligned}$$

To show the compatibility of the isomorphisms with the products, it suffices to show that the diagram

$$\begin{tikzcd}\mathcal{D}_*(A \otimes A') \arrow[r, "\Delta_{\mathcal{D}}"] \arrow[d, "\mathsf{Sh}"] & \mathcal{D}_*(A \otimes A') \otimes \mathcal{D}_*(A \otimes A') \arrow[d, "\mathsf{Sh}\otimes\mathsf{Sh}"] \\ \mathcal{B}_*(A \otimes A') \arrow[r, "\Delta_{\mathrm{bar}}"] & \mathcal{B}_*(A \otimes A') \otimes \mathcal{B}_*(A \otimes A')\end{tikzcd}$$

is commutative or, denoting $x_i := a_i \otimes 1$, $y_j := 1 \otimes a'_j$, that

$$\begin{aligned}&\Delta_{\mathrm{bar}}\left(\mathsf{sh}[x_1|\cdots|x_n|y_1|\cdots|y_m]\right)\\ =&\sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{j(n-i)}\mathsf{sh}[x_1|\cdots|x_i|y_1|\cdots|y_j]\otimes\mathsf{sh}[x_{i+1}|\cdots|x_n|y_{j+1}|\cdots|y_m].\end{aligned}$$

The latter is an easy combinatorial exercise.

Proof of Proposition 2.7.. The claim is a straightforward generalization of well-known results in the non-curved setting, so we will only sketch the proof.

Consider the following two maps of graded spaces:

(1) $\Xi^* : \mathcal{B}^*(A, A \rtimes G)^G \to \mathcal{B}^*(A \rtimes G, A \rtimes G)$ defined as the restriction to the G-invariants of the map $\mathcal{B}^*(A, A \rtimes G) \to \mathcal{B}^*(A \rtimes G, A \rtimes G)$ that sends the element $1[a_1|a_2|\cdots|a_n]1 \mapsto \widetilde{D}(a_1|a_2|\cdots|a_n) \otimes g$ of $\mathcal{B}^*(A, A \rtimes G)$ to

$$\begin{aligned}&1[a_1 \otimes g_1|a_2 \otimes g_2|\cdots|a_n \otimes g_n]1\\ \mapsto\ &\widetilde{D}(a_1|g_1(a_2)|\cdots|g_1g_2\cdots g_{n-1}(a_n)) \otimes gg_1g_2\cdots g_n\end{aligned}$$

(2) $\Xi_* : \mathcal{B}_*(A \rtimes G, A \rtimes G) \to \mathcal{B}_*(A, A \rtimes G)_G$ induced by the map $\mathcal{B}_*(A \rtimes G, A \rtimes G) \to \mathcal{B}_*(A, A \rtimes G)$ given by

$$\begin{aligned}&(a \otimes g) \otimes (1[a_1 \otimes g_1|a_2 \otimes g_2|\cdots|a_n \otimes g_n]1)\\ \mapsto\ &(a \otimes gg_1g_2\cdots g_n) \otimes \left(1[g_n^{-1}\cdots g_1^{-1}(a_1)|g_n^{-1}\cdots g_2^{-1}(a_2)|\cdots|g_n^{-1}(a_n)]1\right)\end{aligned}$$

The maps Ξ^* and Ξ_* can be shown to define morphisms of mixed complexes

$$\begin{aligned}\Xi^* &: \mathcal{HH}^*(A, W; A \rtimes G)^G \to \mathcal{HH}^*(A \rtimes G, W),\\ \Xi_* &: \mathcal{HH}_*(A \rtimes G, W) \to \mathcal{HH}_*(A, W; A \rtimes G)_G,\end{aligned}$$

and the key observation is that they are quasi-isomorphisms; see [9, Thm.5.4] and [5, Prop.8]. Finally, Ξ^* and Ξ_* are easily seen to be compatible with the cup and cap products in the following sense:

$$\Xi^*(D_1 \cup D_2) = \Xi^*(D_1) \cup \Xi^*(D_2), \quad \forall\, D_i \in \mathcal{B}^*(A, A \rtimes G)^G,$$

$$\Xi_*(\omega) \cap D = \Xi_*(\omega \cap \Xi^*(D)) \quad \forall\, \omega \in \mathcal{B}_*(A \rtimes G, A \rtimes G), D \in \mathcal{B}^*(A, A \rtimes G)^G.$$

Proof of Proposition 2.8.. The idea is to show that the diagram of $\mathbb{K}[t^{\pm 1}]$-linear complexes

$$\begin{tikzcd}
\mathcal{B}^*(A, A\otimes g)(\!(t)\!) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathcal{B}^*(A, A\otimes h)(\!(t)\!) \arrow[d, "(\mathrm{id}\otimes h^{-1})\cdot\sigma"] \arrow[dr, "\cup"] & \\
\mathcal{B}^*(A, A\otimes h)(\!(t)\!) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathcal{B}^*(A, A\otimes h^{-1}gh)(\!(t)\!) \arrow[r, "\cup"] & \mathcal{B}^*(A, A\otimes gh)(\!(t)\!),
\end{tikzcd}$$

where σ denote the (graded) transposition of the terms, commutes *up to homotopy* or, in other words, that there exists a series $\sum_{i=i_0}^{\infty} \chi_i t^i$, where χ_i is a degree $-2i-1$ linear operator from $\mathcal{B}^*(A, A\otimes g)\otimes \mathcal{B}^*(A, A\otimes h)$ to $\mathcal{B}^*(A, A\otimes gh)$, such that

$$\begin{aligned}
\cup - \cup \cdot (\mathrm{id}\otimes h^{-1})\cdot\sigma = (\partial_{\mathrm{Hoch}} + t\partial_{\mathrm{curv}}) \sum_i \chi_i t^i \\
+ \sum_i \chi_i t^i ((\partial_{\mathrm{Hoch}} + t\partial_{\mathrm{curv}})\otimes 1 \\
+ 1\otimes(\partial_{\mathrm{Hoch}} + t\partial_{\mathrm{curv}})).
\end{aligned}$$

In fact, we will show that there exists such a series $\sum_i \chi_i t^i$ with $\chi_i = 0$ for $i \neq 0$, i.e. we will construct a degree -1 operator $\chi = \chi_0 : \mathcal{B}^*(A, A\otimes g)\otimes \mathcal{B}^*(A, A\otimes h) \to \mathcal{B}^*(A, A\otimes gh)$ such that

$$\begin{aligned}
\cup - \cup \cdot (\mathrm{id}\otimes h^{-1})\cdot\sigma &= \partial_{\mathrm{Hoch}}\cdot\chi + \chi\cdot(\partial_{\mathrm{Hoch}}\otimes 1 + 1\otimes\partial_{\mathrm{Hoch}}), \\
0 &= \partial_{\mathrm{curv}}\cdot\chi + \chi\cdot(\partial_{\mathrm{curv}}\otimes 1 + 1\otimes\partial_{\mathrm{curv}}).
\end{aligned} \tag{B.4}$$

The homotopy χ is given by a slight modification of M. Gerstenhaber's well-known formula in the non-equivariant setting [19, Thm.3]. Namely, let us fix $D_1 \in \mathcal{B}^l(A, A\otimes g)$ and $D_2 \in \mathcal{B}^m(A, A\otimes h)$. So,

(B.5)

$$D_1(a_0[a_1|\cdots|a_n]a_{n+1}) = \begin{cases} a_0\cdot\widetilde{D}_1(a_1|\cdots|a_l)\cdot g(a_{l+1})\otimes g, & n = l \\ 0, & \text{otherwise} \end{cases},$$

$$D_2(a_0[a_1|\cdots|a_n]a_{n+1}) = \begin{cases} a_0\cdot\widetilde{D}_2(a_1|\cdots|a_m)\cdot h(a_{m+1})\otimes h, & n = m \\ 0, & \text{otherwise} \end{cases}$$

for some linear maps $\widetilde{D}_1 : A^{\otimes l} \to A$ and $\widetilde{D}_2 : A^{\otimes m} \to A$. (This notation will be used throughout the proof: $\widetilde{D}$ will stand for the map $A^{\otimes *} \to A$ associated with a Hochschild cochain $D \in \mathcal{B}^*(A, A\otimes -)$ by the above rule.) We define

$\chi(D_1, D_2) \in \mathcal{B}^{l+m-1}(A, A \otimes gh)$ by

$$\begin{aligned}
&\chi(\widetilde{D_1, D_2})(a_1|\cdots|a_{l+m-1}) \\
&= \sum_{i=1}^{l} (-1)^{(i-1)(m-1)+l} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m-1})).
\end{aligned}$$

Let us compute $\partial\chi(D_1, D_2) + \chi(\partial D_1, D_2) + (-1)^l \chi(D_1, \partial D_2)$ for $\partial := \partial_{\mathrm{Hoch}}$. We have

$$\begin{aligned}
&\partial\chi(\widetilde{D_1, D_2})(a_1|\cdots|a_{l+m}) \\
&= (-1)^{l+m-1} a_1 \cdot \chi(\widetilde{D_1, D_2})(a_2|\cdots|a_{l+m}) \\
&\quad - \chi(\widetilde{D_1, D_2})(a_1|\cdots|a_{l+m-1}) \cdot gh(a_{l+m}) \\
&\quad + \sum_{j=1}^{l+m-1} (-1)^{j+l+m-1} \chi(\widetilde{D_1, D_2})(a_1|\cdots|a_j a_{j+1}|\cdots|a_{l+m}).
\end{aligned}$$

Unfolding the definitions,

$$\begin{aligned}
&(-1)^{l+m-1} a_1 \cdot \chi(\widetilde{D_1, D_2})(a_2|\cdots|a_{l+m}) \\
&= \sum_{i=1}^{l} (-1)^{i(m-1)} a_1 \\
&\quad \times \widetilde{D}_1(a_2|\cdots|a_i|\widetilde{D}_2(a_{i+1}|\cdots|a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m})),
\end{aligned}$$

$$\begin{aligned}
&- \chi(\widetilde{D_1, D_2})(a_1|\cdots|a_{l+m-1}) \cdot gh(a_{l+m}) \\
&= - \sum_{i=1}^{l} (-1)^{(i-1)(m-1)+l} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m-1})) \cdot gh(a_{l+m})
\end{aligned}$$

and

$$\sum_{j=1}^{l+m-1} (-1)^{j+l+m-1} \chi(\widetilde{D_1, D_2})(a_1|\cdots|a_j a_{j+1}|\cdots|a_{l+m}) =$$

$$
\begin{aligned}
&= \sum_{j=m+1}^{l+m-1} \sum_{i=1}^{j-m} (-1)^{i(m-1)+j} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|\cdots|h(a_j a_{j+1})|\cdots|h(a_{l+m})) \\
&+ \sum_{j=m}^{l+m-1} \sum_{i=j-m+1}^{j} (-1)^{i(m-1)+j} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_j a_{j+1}|\cdots|a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m})) \\
&+ \sum_{j=1}^{l-2} \sum_{i=j+2}^{l} (-1)^{(i-1)(m-1)+j} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_j a_{j+1}|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m})).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\chi(\widetilde{\partial D_1, D_2})(a_1|\cdots|a_{l+m}) \\
&= \sum_{i=1}^{l+1} (-1)^{(i-1)(m-1)+l+1} \\
&\quad \times \widetilde{\partial D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m})) \\
&= -\widetilde{D}_2(a_1|\cdots|a_m) \cdot \widetilde{D}_1(h(a_{1+m})|\cdots|h(a_{l+m})) + (-1)^{lm} \widetilde{D}_1(a_1|\cdots|a_l) \\
&\quad \times g\left(\widetilde{D}_2(a_{l+1}|\cdots|a_{l+m})\right) \\
&\quad - \sum_{i=2}^{l+1} (-1)^{(i-1)(m-1)} a_1 \\
&\quad \times \widetilde{D}_1(a_2|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m})) \\
&\quad + \sum_{i=1}^{l} (-1)^{(i-1)(m-1)+l} \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m-1})) \cdot gh(a_{l+m}) \\
&\quad - \sum_{i=3}^{l+1} \sum_{j=1}^{i-2} (-1)^{(i-1)(m-1)+j} (-1)^j \\
&\quad \times \widetilde{D}_1(a_1|\cdots|a_j a_{j+1}|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m})) \\
&\quad - \sum_{i=2}^{l+1} (-1)^{(i-1)m} \widetilde{D}_1(a_1|\cdots|a_{i-1} \cdot \widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_{l+m}))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{l}(-1)^{(i-1)m}\widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})\cdot h(a_{i+m})|\cdots|h(a_{l+m}))\\
&-\sum_{i=1}^{l-1}\sum_{j=i+m}^{l+m-1}(-1)^{i(m-1)+j}\\
&\quad\times\widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})|h(a_{i+m})|\cdots|h(a_ja_{j+1})|\cdots|h(a_{l+m})).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&(-1)^l\chi(\widetilde{D_1,\partial D_2})(a_1|\cdots|a_{l+m})\\
&=\sum_{i=1}^{l}(-1)^{(i-1)m}\widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{\partial D}_2(a_i|\cdots|a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m}))\\
&=\sum_{i=1}^{l}(-1)^{im}\widetilde{D}_1(a_1|\cdots|a_{i-1}|a_i\cdot\widetilde{D}_2(a_{i+1}|\cdots|a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m}))\\
&\quad-\sum_{i=1}^{l}(-1)^{(i-1)m}\\
&\qquad\times\widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_{i+m-1})\cdot h(a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m}))\\
&\quad-\sum_{i=1}^{l}\sum_{j=i}^{i+m-1}(-1)^{i(m-1)+j}\\
&\qquad\times\widetilde{D}_1(a_1|\cdots|a_{i-1}|\widetilde{D}_2(a_i|\cdots|a_ja_{j+1}|\cdots|a_{i+m})|h(a_{i+m+1})|\cdots|h(a_{l+m}))
\end{aligned}
$$

Summing all the above equalities results in

$$
\begin{aligned}
&\partial\chi(\widetilde{D_1,D_2})(a_1|\cdots|a_{l+m})+\chi(\widetilde{\partial D_1,D_2})(a_1|\cdots|a_{l+m})\\
&\quad+(-1)^l\chi(\widetilde{D_1,\partial D_2})(a_1|\cdots|a_{l+m})\\
&=-\widetilde{D}_2(a_1|\cdots|a_m)\cdot\widetilde{D}_1(h(a_{1+m})|\cdots|h(a_{l+m}))\\
&\quad+(-1)^{lm}\widetilde{D}_1(a_1|\cdots|a_l)\cdot g\left(\widetilde{D}_2(a_{l+1}|\cdots|a_{l+m})\right)
\end{aligned}
$$

which is equivalent to the first equality in (B.4). The proof of the second equality is easier and is left to the reader.

Proof of Proposition 4.1.. Both equalities in (4.2) are proven by a direct calculation. To avoid *too* long formulas, we are including calculations in a special case, hoping that the reader will see the patterns.

Let us calculate $\delta_{\mathrm{Kos}}(\Psi(f_0[f_1|f_2|f_3]f_4))$. To begin with, since $\theta_i^2 = 0$ for all i, we can replace the strict inequalities in the index set in the right-hand side of (4.1) by the non-strict ones, that is

$$\begin{aligned}
&\delta_{\mathrm{Kos}}(\Psi(f_0[f_1|f_2|f_3]f_4)) \\
&= \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\,\delta_{\mathrm{Kos}}\left(\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}\right)\,l_{N+1}(f_4)
\end{aligned}$$

The commutation relations in Cl_N imply

$$\begin{aligned}
&\delta_{\mathrm{Kos}}(\Psi(f_0[f_1|f_2|f_3]f_4)) \\
&= \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)(x_{j_1}-y_{j_1})\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3} l_{N+1}(f_4) \\
&\quad - \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)(x_{j_2}-y_{j_2})\,\nabla_{j_3}(f_3)\theta_{j_3} l_{N+1}(f_4) \\
&\quad + \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)(x_{j_3}-y_{j_3}) l_{N+1}(f_4)
\end{aligned}$$

which by (3.2) equals

$$\begin{aligned}
&\sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)(l_{j_1}(f_1)-l_{j_1+1}(f_1))\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3} l_{N+1}(f_4) \\
&- \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,(l_{j_2}(f_2)-l_{j_2+1}(f_2))\,\nabla_{j_3}(f_3)\theta_{j_3} l_{N+1}(f_4) \\
&+ \sum_{1\le j_1\le j_2\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,(l_{j_3}(f_3)-l_{j_3+1}(f_3)) l_{N+1}(f_4).
\end{aligned}$$

Observe that

$$\sum_{j=\alpha}^{\beta}(l_j(f)-l_{j+1}(f)) = l_\alpha(f) - l_{\beta+1}(f), \tag{B.6}$$

so the previous expression equals

$$\begin{aligned}
&\sum_{1\le j_2\le j_3\le N} l_1(f_0)(l_1(f_1)-l_{j_2+1}(f_1))\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4)\\
&-\sum_{1\le j_1\le j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,(l_{j_1}(f_2)-l_{j_3+1}(f_2))\,\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4)\\
&+\sum_{1\le j_1\le j_2\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,(l_{j_2}(f_3)-l_{N+1}(f_3))l_{N+1}(f_4)
\end{aligned}$$

or, renaming the indices,

$$\begin{aligned}
&\sum_{1\le\alpha\le\beta\le N} l_1(f_0)(l_1(f_1)-l_{\alpha+1}(f_1))\,\nabla_{\alpha}(f_2)\theta_{\alpha}\,\nabla_{\beta}(f_3)\theta_{\beta}l_{N+1}(f_4)\\
&-\sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_{\alpha}(f_1)\theta_{\alpha}\,(l_{\alpha}(f_2)-l_{\beta+1}(f_2))\,\nabla_{\beta}(f_3)\theta_{\beta}l_{N+1}(f_4)\\
&+\sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_{\alpha}(f_1)\theta_{\alpha}\,\nabla_{\beta}(f_2)\theta_{\beta}\,(l_{\beta}(f_3)-l_{N+1}(f_3))l_{N+1}(f_4).
\end{aligned}$$

The latter can be simplified by regrouping the summands

$$\begin{aligned}
=&\sum_{1\le\alpha\le\beta\le N} l_1(f_0)l_1(f_1)\,\nabla_{\alpha}(f_2)\theta_{\alpha}\,\nabla_{\beta}(f_3)\theta_{\beta}l_{N+1}(f_4)\\
&-\sum_{1\le\alpha\le\beta\le N} l_1(f_0)(l_{\alpha+1}(f_1)\,\nabla_{\alpha}(f_2)+\nabla_{\alpha}(f_1)\,l_{\alpha}(f_2))\theta_{\alpha}\,\nabla_{\beta}(f_3)\theta_{\beta}l_{N+1}(f_4)\\
&+\sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_{\alpha}(f_1)\theta_{\alpha}\,(l_{\beta+1}(f_2)\,\nabla_{\beta}(f_3)+\nabla_{\beta}(f_2)\,l_{\beta}(f_3))\theta_{\beta}l_{N+1}(f_4)\\
&-\sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_{\alpha}(f_1)\theta_{\alpha}\,\nabla_{\beta}(f_2)\theta_{\beta}\,l_{N+1}(f_3)l_{N+1}(f_4)
\end{aligned}$$

and employing the obvious equality

$$\nabla_i(fg)=\nabla_i(f)l_i(g)+l_{i+1}(f)\nabla_i(g) \tag{B.7}$$

which yields

$$
\begin{aligned}
&= \sum_{1\le\alpha\le\beta\le N} l_1(f_0f_1)\,\nabla_\alpha(f_2)\theta_\alpha\,\nabla_\beta(f_3)\theta_\beta l_{N+1}(f_4)\\
&\quad - \sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_\alpha(f_1f_2)\theta_\alpha\,\nabla_\beta(f_3)\theta_\beta l_{N+1}(f_4)\\
&\quad + \sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_\alpha(f_1)\theta_\alpha\,\nabla_\beta(f_2f_3)\theta_\beta l_{N+1}(f_4)\\
&\quad - \sum_{1\le\alpha\le\beta\le N} l_1(f_0)\nabla_\alpha(f_1)\theta_\alpha\,\nabla_\beta(f_2)\theta_\beta\, l_{N+1}(f_3f_4)
\end{aligned}
$$

which is precisely $\Psi(\delta_{\mathrm{bar}}(f_0[f_1|f_2|f_3]f_4))$.

The second equality in (4.2) is easier to prove. Let us again demonstrate it in a special case:

$$
\begin{aligned}
&\delta_{\mathrm{curv}}(\Psi(f_0[f_1|f_2|f_3]f_4))\\
&= \sum_{1\le j_1<j_2<j_3\le N} l_1(f_0)\delta_{\mathrm{curv}}\left(\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}\right) l_{N+1}(f_4)\\
&= \sum_{1\le j_1<j_2<j_3\le N}\sum_{j=1}^{N} l_1(f_0)\nabla_j(W)\theta_j\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4).
\end{aligned}
$$

The sum over j splits into $\sum_{1\le j<j_1}+\sum_{j_1<j<j_2}+\sum_{j_2<j<j_3}+\sum_{j_3<j\le N}$ and we get

$$
\begin{aligned}
&\sum_{1\le j<j_1<j_2<j_3\le N} l_1(f_0)\nabla_j(W)\theta_j\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4)\\
&- \sum_{1\le j_1<j<j_2<j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_j(W)\theta_j\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4)\\
&+ \sum_{1\le j_1<j_2<j<j_3\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_j(W)\theta_j\nabla_{j_3}(f_3)\theta_{j_3}l_{N+1}(f_4)\\
&- \sum_{1\le j_1<j_2<j_3<j\le N} l_1(f_0)\nabla_{j_1}(f_1)\theta_{j_1}\,\nabla_{j_2}(f_2)\theta_{j_2}\,\nabla_{j_3}(f_3)\theta_{j_3}\nabla_j(W)\theta_j l_{N+1}(f_4)
\end{aligned}
$$

which is precisely $\Psi(\delta_{\mathrm{curv}}(f_0[f_1|f_2|f_3]f_4))$.

Finally, to prove the last claim in the proposition, we recall that there is the following well-known quasi-isomorphism of complexes of bimodules

$i : (\mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}}) \to (\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}})$

$$i : f_0(x)f_1(y)\theta_{k_1}\cdots\theta_{k_n} \mapsto \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\cdot f_0[x_{k_{\sigma(1)}}|\cdots|x_{k_{\sigma(n)}}]f_1. \tag{B.8}$$

It remains to notice that $\Psi\cdot i = \mathrm{id}_{\mathcal{K}_*(\mathbb{K}[X])}$.

Proof of Proposition 4.4.. To shorten formulas, we will identify $\mathrm{Cl}_N^{\otimes 2}$ with $\mathrm{Cl}_{2N} = \mathbb{K}\langle\theta_i,\eta_i,\partial_{\theta_i},\partial_{\eta_i}\rangle$ via $\theta_i\otimes 1\mapsto\theta_i$, $1\otimes\theta_i\mapsto\eta_i$, etc. In the computation below we use the fact that

$$\mathrm{H}_W(x,y,z) = \sum_{i,j=1}^{N} \nabla_j^{y\to(y,z)}\nabla_i^{x\to(x,y)}(W)\,\theta_i\eta_j$$

because $\nabla_j^{y\to(y,z)}\nabla_i^{x\to(x,y)} = 0$ for $i\le j$.

Abbreviating $(\delta\otimes 1 + 1\otimes\delta)\cdot\Delta - \Delta\cdot\delta$ to $[\delta,\Delta]$, the claim is that

$$[\delta_{\mathrm{Kos}},\Delta_0] = 0, \quad [\delta_{\mathrm{curv}},\Delta_{-2l}] + [\delta_{\mathrm{Kos}},\Delta_{-2l-2}] = 0, \quad \forall\, l \tag{B.9}$$

According to our notation, $\delta_{\mathrm{Kos}} = \sum_{i=1}^N (x_i - z_i)\partial_{\theta_i}$ and

$$\delta_{\mathrm{Kos}}\otimes 1 + 1\otimes\delta_{\mathrm{Kos}} = \sum_{i=1}^{N}\left((x_i-y_i)\partial_{\theta_i} + (y_i-z_i)\partial_{\eta_i}\right).$$

The first equality in (B.9) follows from these formulas and the following obvious relation:

$$((x_i-y_i)\partial_{\theta_i} + (y_i-z_i)\partial_{\eta_i})\cdot\Delta_0 = \Delta_0\cdot(x_i-z_i)\partial_{\theta_i} \quad \forall\, i.$$

Furthermore, $\delta_{\mathrm{curv}} = \sum_{i=1}^N \nabla_i^{x\to(x,z)}(W)\cdot\theta_i$ and therefore $\Delta_0\cdot\delta_{\mathrm{curv}} = (\delta'_{\mathrm{curv}}\otimes 1 + 1\otimes\delta'_{\mathrm{curv}})\cdot\Delta_0$ where

$$\delta'_{\mathrm{curv}}\otimes 1 + 1\otimes\delta'_{\mathrm{curv}} = \sum_{i=1}^{N}\left(\nabla_i^{x\to(x,z)}(W)\cdot\theta_i + \nabla_i^{x\to(x,z)}(W)\eta_i\right).$$

This observation, together with the first equality in (B.9) and the fact that H_W is even (hence central in $\mathbb{K}[X]^{\otimes 3}[\theta,\eta]$), reduces the proof of the remaining relations in (B.9) to proving the following equality:

$$\begin{aligned}&(\delta_{\mathrm{curv}}\otimes 1 + 1\otimes\delta_{\mathrm{curv}}) - (\delta'_{\mathrm{curv}}\otimes 1 + 1\otimes\delta'_{\mathrm{curv}})\\ &= -(\delta_{\mathrm{Kos}}\otimes 1 + 1\otimes\delta_{\mathrm{Kos}})(\mathrm{H}_W)\end{aligned}$$

or, in our new notation,

(B.10)
$$\sum_{i=1}^{N}\Big((\nabla_i^{x\to(x,y)}(W)-\nabla_i^{x\to(x,z)}(W))\theta_i+(\nabla_i^{y\to(y,z)}(W)-\nabla_i^{x\to(x,z)}(W))\eta_i\Big)$$
$$=-\sum_{j=1}^{N}\big((x_j-y_j)\partial_{\theta_j}+(y_j-z_j)\partial_{\eta_j}\big)\,(\mathrm{H}_W).$$

Obviously, for any collection of polynomials $\{W_{\alpha\beta}\}$

$$\sum_{j=1}^{N}\big((x_j-y_j)\partial_{\theta_j}+(y_j-z_j)\partial_{\eta_j}\big)\sum_{\alpha,\beta=1}^{N}W_{\alpha\beta}\theta_\alpha\eta_\beta$$
$$=\sum_{j,\beta=1}^{N}(x_j-y_j)W_{j\beta}\eta_\beta-\sum_{j,\alpha=1}^{N}(y_j-z_j)W_{\alpha j}\theta_\alpha.$$

Thus, setting $W_{\alpha\beta}=\nabla_\beta^{y\to(y,z)}\nabla_\alpha^{x\to(x,y)}(W)$, (B.10) amounts to

$$\nabla_i^{x\to(x,y)}(W)-\nabla_i^{x\to(x,z)}(W)=\sum_{j=1}^{N}(y_j-z_j)\nabla_j^{y\to(y,z)}\nabla_i^{x\to(x,y)}(W),$$
$$\nabla_i^{x\to(x,z)}(W)-\nabla_i^{y\to(y,z)}(W)=\sum_{j=1}^{N}(x_j-y_j)\nabla_i^{y\to(y,z)}\nabla_j^{x\to(x,y)}(W).$$

The first equality follows from (3.3). The second one will also be a consequence of (3.3) once we use the formula

$$\nabla_i^{y\to(y,z)}\nabla_j^{x\to(x,y)}=\nabla_j^{x\to(x,y)}\nabla_i^{x\to(x,z)}$$

which is easy to check by applying both hand sides to monomials in x.

Proof of Proposition 4.7.. The left-hand side of (4.5) is a series (in fact, a polynomial) in t. Let us denote the coefficients of this series by ψ_i:

$$(\Psi\otimes\Psi)\cdot\Delta_{\mathrm{bar}}-\Delta_{\mathrm{Kos}}\cdot\Psi=:\sum_{i=0}^{\infty}\psi_i t^i,\quad |\psi_i|=-2i.$$

We will view the ψ_is as elements of the complex of morphisms of complexes of $\mathbb{K}[X]$-bimodules:
(B.11)
$$\mathrm{Hom}^*\left((\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}}) \, , \, (\mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}} \otimes 1 + 1 \otimes \delta_{\mathrm{Kos}}) \right).$$

Since $\mathrm{H}^*(\mathcal{K}_*(\mathbb{K}[X]) \otimes_{\mathbb{K}[X]} \mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}} \otimes 1 + 1 \otimes \delta_{\mathrm{Kos}}) \simeq \mathbb{K}[X]$, the complex (B.11) is quasi-isomorphic to the complex

$$\mathrm{Hom}^* \left((\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}}), \mathbb{K}[X] \right),$$

i. e. to the Hochschild cochain complex of $\mathbb{K}[X]$. Thus, (B.11) has non-trivial cohomology groups only in non-negative degrees. This observation and the fact that the degress of the ψ_is are non-positive imply that in order to prove the existence of the h_is in (4.5), it would suffice to show that the "constant term" ψ_0 defines a trivial class in the cohomology of (B.11). Let us explain why this would be enough.

Let us denote the differential in (4.5) by $\widehat{\delta}$

$$\widehat{\delta}(\psi) = (\delta_{\mathrm{Kos}} \otimes 1 + 1 \otimes \delta_{\mathrm{Kos}}) \cdot \psi - (-1)^{|\psi|} \psi \cdot \delta_{\mathrm{bar}}.$$

and the second — associated with δ_{curv} — differential (of degree -1) on this complex by $\widehat{\delta}_{\mathrm{curv}}$:

$$\widehat{\delta}_{\mathrm{curv}}(\psi) = (\delta_{\mathrm{curv}} \otimes 1 + 1 \otimes \delta_{\mathrm{curv}}) \cdot \psi - (-1)^{|\psi|} \psi \cdot \delta_{\mathrm{curv}}.$$

Note that the two differential anti-commute, so $\widehat{\delta} + t\widehat{\delta}_{\mathrm{curv}}$ squares to 0.

By Propositions 4.1 and 4.4

$$(\widehat{\delta} + t\widehat{\delta}_{\mathrm{curv}}) \left(\sum_{i=0}^{\infty} \psi_i t^i \right) = 0.$$

We are looking for a series $\sum_{i=0}^{\infty} h_i t^i$ such that

$$\sum_{i=0}^{\infty} \psi_i t^i = (\widehat{\delta} + t\widehat{\delta}_{\mathrm{curv}}) \left(\sum_{i=0}^{\infty} h_i t^i \right).$$

Assume we can find a degree -1 bimodule map h_0 such that $\psi_0 = \widehat{\delta}(h_0)$ and consider the series

$$\sum_{i=1}^{\infty} \psi_i' \, t^i := \sum_{i=0}^{\infty} \psi_i t^i - (\widehat{\delta} + t\widehat{\delta}_{\mathrm{curv}})(h_0).$$

It also satisfies $(\widehat{\delta} + t\widehat{\delta}_{\mathrm{curv}})(\sum_i \psi_i' t^i) = 0$ but $|\psi_1'| = -2$ so, by the above observation, ψ_1' defines a trivial class in the cohomology of (B.11), i. e. there exists a degree -3 bimodule map h_1' such that $\psi_1' = \widehat{\delta}(h_1')$, etc.

So, to finish the proof of the proposition, it remains to prove that the class of ψ_0 is trivial, i. e. that there exists an h_0 as above. By Proposition 4.1 $\Psi : (\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}}) \to (\mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}})$ is a quasi-isomorphism of complexes of $\mathbb{K}[X]$-bimodules. Since both complexes are K-projective, Ψ is a homotopy equivalence. Moreover, we already know its right homotopy inverse: it is the quasi-isomorphism $i : (\mathcal{K}_*(\mathbb{K}[X]), \delta_{\mathrm{Kos}}) \to (\mathcal{B}_*(\mathbb{K}[X]), \delta_{\mathrm{bar}})$ defined in (B.8). By a standard general argument, i is also a left homotopy inverse of Ψ, i. e. $i \cdot \Psi = \mathrm{id}_{\mathcal{B}_*(\mathbb{K}[X])} - \delta_{\mathrm{bar}} \cdot h - h \cdot \delta_{\mathrm{bar}}$ for some degree -1 $\mathbb{K}[X]$-bimodule map $h : \mathcal{B}_*(\mathbb{K}[X]) \to \mathcal{B}_*(\mathbb{K}[X])$.

It is easy to show that $\Delta_{\mathrm{bar}} \cdot i = (i \otimes i) \cdot \Delta_0$ where Δ_0 is the morphism defined in (4.3). Since $\psi_0 = (\Psi \otimes \Psi) \cdot \Delta_{\mathrm{bar}} - \Delta_0 \cdot \Psi$, we get

$$\psi_0 \cdot i = (\Psi \otimes \Psi) \cdot \Delta_{\mathrm{bar}} \cdot i - \Delta_0 \cdot \Psi \cdot i = (\Psi \otimes \Psi) \cdot (i \otimes i) \cdot \Delta_0 - \Delta_0 = 0.$$

Then

$$\begin{aligned}
\psi_0 = \psi_0 \cdot \mathrm{id}_{\mathcal{B}_*(\mathbb{K}[X])} &= \psi_0 \cdot (i \cdot \Psi + \delta_{\mathrm{bar}} \cdot h + h \cdot \delta_{\mathrm{bar}}) \\
&= (\psi_0 \cdot i) \cdot \Psi + \psi_0 \cdot \delta_{\mathrm{bar}} \cdot h + \psi_0 \cdot h \cdot \delta_{\mathrm{bar}} \\
&= (\delta_{\mathrm{bar}} \otimes 1 + 1 \otimes \delta_{\mathrm{bar}}) \cdot (\psi_0 \cdot h) + (\psi_0 \cdot h) \cdot \delta_{\mathrm{bar}} \\
&= \widehat{\delta}(\psi_0 \cdot h).
\end{aligned}$$

Proof of Lemma 4.9.. The condition (4.17) is equivalent to

$$\partial_{\mathrm{Kos}}(g) + t\partial_{\mathrm{curv}}(g)' = e^{t\mathrm{H}_{W,g}} \cdot \partial_{\mathrm{Kos}}(g) \cdot e^{-t\cdot \mathrm{H}_{W,g}}$$

which, in turn, is easily seen to be equivalent to

$$\sum_{i \in I_g} (1 - g_i) x_i \, \partial_{\theta_i}(\mathrm{H}_{W,g}) = \sum_{i \in I_g} \nabla_i^{x \to (x, g(x))}(W) \, \theta_i \tag{B.12}$$

where both hand sides are viewed as elements of $\mathbb{K}[X][\theta]$. Let us show that the element (3.8) satisfies the latter condition.

In the following calculation $W_i := \nabla_i^{x\to(x,g(x))}(W)$, the indices α, i, j belong to I_g, and l_i, ∇_i stand for $l_i^{x\to(x,x^g)}$, $\nabla_i^{x\to(x,x^g)}$, respectively. We have:

$$\begin{aligned}\sum_\alpha (1-g_\alpha)x_\alpha\,\partial_{\theta_\alpha}(\mathrm{H}_{W,g}) &= \sum_\alpha\sum_{j<i}\frac{1-g_\alpha}{1-g_j}x_\alpha\nabla_j(W_i)\partial_{\theta_\alpha}(\theta_j\,\theta_i)\\ &= \sum_{j<i}x_j\nabla_j(W_i)\theta_i - \sum_{j<i}\frac{1-g_i}{1-g_j}x_i\nabla_j(W_i)\theta_j.\end{aligned}$$

Note that

$$\sum_{j<i}x_j\nabla_j(W_i)\theta_i = \sum_{j<i}(l_j(W_i)-l_{j+1}(W_i))\theta_i = \sum_i(W_i - l_i(W_i))\theta_i$$

where we have used the fact that $l_{j+1}^{x\to(x,x^g)} = l_{j'}^{x\to(x,x^g)}$ for any pair $j<j'$ of *consecutive* elements of I_g. Thus,

$$\begin{aligned}\sum_\alpha (1-g_\alpha)x_\alpha\,\partial_{\theta_\alpha}(\mathrm{H}_{W,g}) &= \sum_i W_i\theta_i - \sum_i l_i(W_i)\theta_i\\ &\quad - \sum_{j<i}\frac{1-g_i}{1-g_j}x_i\nabla_j(W_i)\theta_j.\end{aligned} \tag{B.13}$$

Furthermore,

$$l_j(x_i) = \begin{cases} x_i & j\le i\\ 0 & \text{otherwise}\end{cases} \tag{B.14}$$

and by (B.7)

$$\begin{aligned}\sum_{j<i}\frac{1-g_i}{1-g_j}x_i\nabla_j(W_i)\theta_j &= \sum_{j<i}\frac{1-g_i}{1-g_j}l_{j+1}(x_i)\nabla_j(W_i)\theta_j\\ = \sum_{j<i}\frac{1-g_i}{1-g_j}\nabla_j(x_iW_i)\theta_j &- \sum_{j<i}\frac{1-g_i}{1-g_j}\nabla_j(x_i)l_j(W_i)\theta_j\\ = \sum_{j<i}\frac{1-g_i}{1-g_j}\nabla_j(x_iW_i)\theta_j &= -\sum_{j\ge i}\frac{1-g_i}{1-g_j}\nabla_j(x_iW_i)\theta_j.\end{aligned}$$

where the last equality follows from $\sum_i(1-g_i)x_iW_i = 0$. Because of (B.14) $\nabla_j(x_iW_i) = 0$ for $j>i$. Using this fact, together with (B.7) and (B.14), we

obtain:

$$\sum_{j<i} \frac{1-g_i}{1-g_j} x_i \nabla_j(W_i)\theta_j = -\sum_i \nabla_i(x_i W_i)\theta_i$$
$$= -\sum_i \left(\nabla_i(x_i) l_i(W_i) + l_{i+1}(x_i)\nabla_i(W_i)\right)\theta_i = -\sum_i l_i(W_i)\theta_i.$$

Substituting this result into (B.13) we get

$$\sum_\alpha (1-g_\alpha) x_\alpha \, \partial_{\theta_\alpha}(\mathrm{H}_{W,g}) = \sum_i W_i \theta_i.$$

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