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Projections, modules and connections for the noncommutative cylinder

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We initiate a study of projections and modules over a noncommutative cylinder, a simple example of a noncompact noncommutative manifold. Since its algebraic structure turns out to have many similarities with the noncommutative torus, one can develop several concepts in a close analogy with the latter. In particular, we exhibit a countable number of nontrivial projections in the algebra of the noncommutative cylinder itself, and show that they provide concrete representatives for each class in the corresponding K_0 group. We also construct a class of bimodules endowed with connections of constant curvature. Furthermore, with the noncommutative cylinder considered from the perspective of pseudo-Riemannian calculi, we derive an explicit expression for the Levi-Civita connection and compute the Gaussian curvature.

1. Introduction

In the rapidly developing and conceptually growing field of noncommutative geometry it has been of paramount importance to have at least one tractable example exhibiting many of the nontrivial subtleties of the theory. In this respect, the noncommutative torus is perhaps the most studied object in noncommutative geometry, and it has served as a inspirational source (as well as testing ground) for many results and concepts in more general situations. However, to explore the notion of noncompact manifolds, the torus is not equally well suited.

In this paper, we set out to study the noncommutative cylinder as a simple manageable example of a noncompact noncommutative manifold which still exhibits nontrivial features. Inspired by the algebraic similarities with the torus, we follow the same lines of thought in order to see to what extent known concepts apply in this noncompact situation as well.

Starting from a known description in terms of Fourier transforms, we choose a particular presentation of the noncommutative cylinder and introduce a (commuting) set of hermitian derivations as well as a trace. After providing basic results about these structures, we proceed to construct a class of projections in the algebra itself, and show that they are classified by the integers. Moreover, by showing that the corresponding projective modules respect the group structure of the integers, we conclude that these projections provide concrete representatives for each class in the K_0 group of the noncommutative cylinder (which is known to be $\mathbb{Z}$). A corresponding "Chern number" can be computed for each projective module by evaluating the projections against a cyclic 2-cocycle.

Next, in analogy with the torus modules defined by Connes and Rieffel [Con80, Rie81], we find a class of bimodules for the noncommutative cylinder, on which connections of constant curvature are defined. Interestingly, these modules turn out to be isomorphic to copies of the algebra itself. Although the details of the bimodule structure depend on a choice of parameters, it is the case that the curvature only depends on the deformation parameters $\hbar$ and $\hbar'$ defining the left and right algebras, respectively.

Finally, we recall the framework of pseudo-Riemannian calculi, and show that for a given choice of metric, there exists a calculus over the noncommutative cylinder with a unique torsion-free and metric connection, for which one may explicitly compute the Gaussian curvature. Moreover, we illustrate a Gauss-Bonnet type theorem where the total curvature (that is, the integral of the Gaussian curvature with respect to the Riemannian volume form) is shown to be independent of a class of metric perturbations.

2. The algebra of the noncommutative cylinder

Let us start by recalling the definition of the algebra of the noncommutative cylinder. Let $\mathcal{S}(\mathbb{R} \times S^1)$ denote the space of Schwartz functions on $\mathbb{R} \times S^1$.

Every $f \in \mathcal{S}(\mathbb{R} \times S^1)$ may be written as

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) e^{2\pi i n t}, \tag{2.1}$$

with $f_n \in \mathcal{S}(\mathbb{R})$ and we introduce the Fourier transform of the coefficients f_n as

$$\hat{f}_n(x) = \int_{\mathbb{R}} f_n(u) e^{-2\pi i u x} du.$$

Thus any function as in (2.1), is written as

$$f(u,t) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_n(x) e^{2\pi i (nt+ux)} dx.$$

Following the general strategy of [Rie93], we define a twisted convolution product on $\mathcal{S}(\mathbb{R} \times S^1)$ via

$$\widehat{(f \bullet_\hbar g)}_n(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_k(y) \hat{g}_{n-k}(x-y) \sigma_\hbar(\vec{y}, \vec{x}-\vec{y}) dy \tag{2.2}$$

where $\vec{x} = (x,n)$, $\vec{y} = (y,k)$ and $\sigma_\hbar$ is a cocycle fulfilling the condition

$$\sigma_\hbar(\vec{x},\vec{y})\sigma_\hbar(\vec{x}+\vec{y},\vec{z}) = \sigma_\hbar(\vec{x},\vec{y}+\vec{z})\sigma_\hbar(\vec{y},\vec{z}),$$

ensuring associativity of the product. For our purposes we will choose a particular cocycle given by

$$\sigma_\hbar\big((x,n),(y,k)\big) = e^{2\pi i \hbar y n}. \tag{2.3}$$

Note that this cocycle is cohomologous to its antisymmetrization

$$\sigma_\hbar(\vec{x},\vec{y}) = e^{\pi i \hbar (yn - xk)},$$

giving the corresponding twisted convolution as defined in [vS04]; the two corresponding algebras are thus isomorphic.

Definition 2.1. Let $\mathcal{C}^\infty_\hbar = (\mathcal{S}(\mathbb{R} \times S^1), \bullet_\hbar)$ be the algebra defined by the vector space $\mathcal{S}(\mathbb{R} \times S^1)$ together with the product $\bullet_\hbar$ in (2.2) for the cocycle

$$\sigma_\hbar\big((x,n),(y,k)\big) = e^{2\pi i \hbar y n}.$$

As the product in $\mathcal{C}^\infty_\hbar$ is defined on the level of Fourier transforms, let us derive a more explicit expression in the following form.

Proposition 2.2. *Let $f, g \in \mathcal{C}^\infty_\hbar$ be such that*

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u)e^{2\pi int} \qquad \text{and} \qquad g(u,t) = \sum_{n\in\mathbb{Z}} g_n(u)e^{2\pi int}.$$

Then

$$(f \bullet_\hbar g)(u,t) = \sum_{n\in\mathbb{Z}} \Big[\sum_{k\in\mathbb{Z}} f_k(u)g_{n-k}(u+k\hbar)\Big]e^{2\pi int}.$$

Proof. The proof consists of a straight-forward computation:

$$\begin{aligned}
(f \bullet_\hbar g)_n(u) &= \int_{\mathbb{R}} \widehat{f \bullet_\hbar g}_n(x)e^{2\pi ixu}dx \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}_k(y)\hat{g}_{n-k}(x-y)e^{2\pi i\hbar(x-y)k}e^{2\pi ixu}dydx \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_k(y)\Big[\int_{\mathbb{R}} \hat{g}_{n-k}(x-y)e^{2\pi ix(u+k\hbar)}dx\Big]e^{-2\pi iyk\hbar}dy \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_k(y)e^{2\pi iy(u+k\hbar)}\Big[\int_{\mathbb{R}} \hat{g}_{n-k}(x)e^{2\pi ix(u+k\hbar)}dx\Big]e^{-2\pi iyk\hbar}dy \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_k(y)g_{n-k}(u+k\hbar)e^{2\pi iyu}dy = \sum_{k\in\mathbb{Z}} f_k(u)g_{n-k}(u+k\hbar).
\end{aligned}$$

□

From Proposition 2.2 one infers the simple commutation rule

$$f(u)e^{2\pi int} \bullet_\hbar g(u) = f(u)g(u+n\hbar) \bullet_\hbar e^{2\pi int} \tag{2.4}$$

which we shall often use in the following. To slightly simplify the notation, let us introduce $W = e^{2\pi it}$ such that every $f \in \mathcal{S}(\mathbb{R}\times S^1)$ may be written as

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u)W^n.$$

In particular, (2.4) now reads

$$W^n \bullet_\hbar f(u) = f(u+n\hbar) \bullet_\hbar W^n. \tag{2.5}$$

Remark 2.3. As a side remark, we note that the relation (2.4) can formally be derived from the canonical commutation relation $[u,t] = i\hbar/2\pi$ via

$$\begin{aligned} e^{2\pi it}u &= \sum_{n\geq 0}\frac{(2\pi it)^n u}{n!} = \sum_{n\geq 0}\frac{(2\pi i)^n(ut^n - in\hbar t^{n-1}/2\pi)}{n!} \\ &= ue^{2\pi it} + \hbar\sum_{k\geq 1}\frac{(2\pi it)^{n-1}}{(n-1)!} = (u+\hbar)e^{2\pi it}, \end{aligned}$$

which is in close analogy with the noncommutative catenoid defined in [AH18].

One may readily introduce a $*$-algebra structure on $\mathcal{C}_\hbar^\infty$.

Proposition 2.4. *For* $f = \sum_{n\in\mathbb{Z}} f_n(u)W^n \in \mathcal{C}_\hbar^\infty$*, set*

$$f^* = \sum_{n\in\mathbb{Z}}\overline{f_n(u-n\hbar)}W^{-n} = \sum_{n\in\mathbb{Z}}\overline{f_{-n}(u+n\hbar)}W^n.$$

Then it follows that $(f^*)^* = f$ *and* $(f \bullet_\hbar g)^* = g^* \bullet_\hbar f^*$*.*

Proof. Just compute

$$\begin{aligned} (f^*)^* &= \sum_{n\in\mathbb{Z}}\overline{f^*_{-n}(u+n\hbar)}W^n \\ &= \sum_{n\in\mathbb{Z}}\overline{\overline{f_{-(-n)}(u+n\hbar-n\hbar)}}W^n = \sum_{n\in\mathbb{Z}} f_n(u)W^n = f. \end{aligned}$$

Next, consider

$$f = \sum_{n\in\mathbb{Z}} f_n(u)W^n \quad \text{and} \quad g = \sum_{n\in\mathbb{Z}} g_n(u)W^n$$

and compute

$$
\begin{aligned}
(g \bullet_\hbar f)^* &= \sum_{n\in\mathbb{Z}} \overline{(g \bullet_\hbar f)_{-n}(u+n\hbar)} W^n \\
&= \sum_{n,k\in\mathbb{Z}} \overline{g_k(u+n\hbar)}\,\overline{f_{-n-k}(u+(n+k)\hbar)} W^n \\
&= \sum_{n,l\in\mathbb{Z}} \overline{f_{-l}(u+l\hbar)}\,\overline{g_{-(n-l)}(u+n\hbar)} W^n \\
&= \sum_{n,l\in\mathbb{Z}} \overline{f_{-l}(u+l\hbar)}\,\overline{g_{-(n-l)}(u+l\hbar+(n-l)\hbar)} W^n \\
&= \sum_{n,l\in\mathbb{Z}} f_l^*(u) g_{n-l}^*(u+l\hbar) W^n = f^* \bullet_\hbar g^*,
\end{aligned}
$$

which proves the second statement. □

Thus, with respect to the involution defined in Proposition 2.4 the algebra $\mathcal{C}_\hbar^\infty$ is a $*$-algebra. By representing $\mathcal{C}_\hbar^\infty$ as multiplication operators on $L^2(\mathbb{R}\times S^1)$, i.e. $T_f\psi(u,t) = f(u,t)\psi(u,t)$ for $f\in\mathcal{C}_\hbar^\infty$, one may complete $\mathcal{C}_\hbar^\infty$ in the operator norm to a C^*-algebra which we shall denote by $\mathcal{C}_\hbar$ (cf. [vS04]).

Next, let us introduce a set of commuting derivations.

Proposition 2.5. *For $f\in\mathcal{C}_\hbar^\infty$ with*

$$
f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) W^n
$$

define

$$
\partial_1 f = \sum_{n\in\mathbb{Z}} f_n'(u) W^n \quad \text{and} \quad \partial_2 f = 2\pi i \sum_{n\in\mathbb{Z}} n f_n(u) W^n.
$$

Then ∂_1 and ∂_2 are hermitian derivations of $\mathcal{C}_\hbar^\infty$ such that $[\partial_1,\partial_2]=0$.

Proof. It is clear that ∂_1 and ∂_2 are linear maps; let us show that they satisfy Leibniz rule. One obtains

$$
\begin{aligned}
\partial_1(f \bullet_\hbar g) &= \sum_{n,k\in\mathbb{Z}} \Big(f_k'(u) g_{n-k}(u+\hbar k) + f_k(u) g_{n-k}'(u+k\hbar)\Big) W^n \\
&= \sum_{n,k\in\mathbb{Z}} f_k'(u) g_{n-k}(u+\hbar k) W^n + \sum_{n,k\in\mathbb{Z}} f_k(u) g_{n-k}'(u+k\hbar) W^n \\
&= (\partial_1 f) \bullet_\hbar g + f \bullet_\hbar (\partial_1 g),
\end{aligned}
$$

and

$$\begin{aligned}\partial_2(f \bullet_\hbar g) &= 2\pi i \sum_{n,k\in\mathbb{Z}} n f_k(u) g_{n-k}(u+k\hbar)W^n \\ &= 2\pi i \sum_{n,k\in\mathbb{Z}} k f_k(u) g_{n-k}(u+k\hbar)W^n \\ &\quad + 2\pi i \sum_{n,k\in\mathbb{Z}} f_k(u)(n-k) g_{n-k}(u+k\hbar)W^n \\ &= (\partial_2 f) \bullet_\hbar g + f \bullet_\hbar (\partial_2 g),\end{aligned}$$

showing that ∂_1, ∂_2 are indeed derivations of $\mathcal{C}^\infty_\hbar$. Furthermore, it is easy to see that

$$[\partial_1, \partial_2](f) = 2\pi i \partial_1 \sum_{n\in\mathbb{Z}} n f_n(u) W^n - \partial_2 \sum_{n\in\mathbb{Z}} f_n'(u) W^n = 0.$$

Finally, let us show that ∂_1 and ∂_2 are hermitian derivations. One computes

$$\begin{aligned}\big(\partial_1(f)\big)^* &= \Big(\sum_{n\in\mathbb{Z}} f_n'(u)W^n\Big)^* = \sum_{n\in\mathbb{Z}} \overline{f_{-n}'(u+n\hbar)}W^n \\ &= \partial_1 \sum_{n\in\mathbb{Z}} \overline{f_{-n}(u+n\hbar)}W^n = \partial_1(f^*)\end{aligned}$$

as well as

$$\begin{aligned}\big(\partial_2(f)\big)^* &= \Big(2\pi i \sum_{n\in\mathbb{Z}} n f_n(u)W^n\Big)^* = -2\pi i \sum_{n\in\mathbb{Z}} \overline{(-n) f_{-n}(u+n\hbar)}W^n \\ &= 2\pi i \sum_{n\in\mathbb{Z}} n\overline{f_{-n}(u+n\hbar)}W^n = \partial_2 \sum_{n\in\mathbb{Z}} \overline{f_{-n}(u+n\hbar)}W^n = \partial_2(f^*)\end{aligned}$$

which proves that ∂_1, ∂_2 are hermitian. □

Remark 2.6. Clearly the function u does not belong to the algebra $\mathcal{C}^\infty_\hbar$. In spite of this a direct computation shows that one can formally obtain a commutation expression for the derivation ∂_2, that is

$$\partial_2 f = \frac{2\pi i}{\hbar}\,(fu - uf) \tag{2.6}$$

for any $f \in \mathcal{C}^\infty_\hbar$.

On the algebra $\mathcal{C}^\infty_\hbar$ we have a trace as well.

Definition 2.7. For $f \in \mathcal{C}^\infty_\hbar$ with

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) W^n$$

we set

$$\tau(f) = \int_{-\infty}^{\infty} f_0(u)du. \tag{2.7}$$

It is clear from the definition that τ is a linear map.

Proposition 2.8. *The map τ is a positive invariant trace; that is, it has the properties*

1) $\tau(f^*) = \overline{\tau(f)}$,
2) $\tau(f^* \bullet_\hbar f) \geq 0$,
3) $\tau(f \bullet_\hbar g) = \tau(g \bullet_\hbar f)$,
4) $\tau(\partial_1 f) = \tau(\partial_2 f) = 0$,

for all $f, g \in \mathcal{C}^\infty_\hbar$.

Proof. It is immediate to see that $\tau(f^*) = \overline{\tau(f)}$. A direct computation yields

$$\begin{aligned}\tau(f \bullet_\hbar g) &= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} f_k(u) g_{-k}(u+k\hbar)du = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} g_k(u-k\hbar) f_{-k}(u)du \\ &= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} g_k(v) f_{-k}(v+k\hbar)dv = \tau(g \bullet_\hbar f).\end{aligned}$$

Furthermore, one finds that

$$\tau(\partial_1 f) = \int_{\mathbb{R}} f_0'(u)du = \big[f_0(u)\big]_{-\infty}^{\infty} = 0$$

as well as

$$\tau(\partial_2 f) = \tau\bigg(\sum_{n\in\mathbb{Z}} n f_n(u) W^n\bigg) = \int_{\mathbb{R}} 0 \cdot f_0(u)du = 0.$$

Finally, we check that

$$\tau(f^* \bullet_\hbar f) = \sum_{k\in\mathbb{Z}} f_k^* f_{-k}(u+k\hbar) = \sum_{k\in\mathbb{Z}} |f_{-k}(u+k\hbar)|^2 \geq 0,$$

which completes the proof of the statements. $\square$

From now on we shall drop the cumbersome notation $f \bullet_\hbar g$ and simply write fg when no confusion can arise.

For the noncommutative torus, there exists a convenient cyclic 2-cocycle which can be evaluated on 2-forms. For the noncommutative cylinder, one can make use of a similar construction. The cyclic 2-cocycle below will be used in the next section in order to compute "Chern numbers" of a class of projective modules.

Proposition 2.9. *For $f_0, f_1, f_2 \in \mathcal{C}^\infty_\hbar$ we set*

$$\Psi(f_0, f_1, f_2) = \frac{1}{2\pi i}\tau\big(f_0(\partial_1 f_1)(\partial_2 f_2) - f_0(\partial_2 f_1)(\partial_1 f_2)\big).$$

Then Ψ is a cyclic,

$$\Psi(f_2, f_0, f_1) = \Psi(f_0, f_1, f_2),$$

Hochschild 2-cocycle,

$$\Psi(f_0f_1, f_2, f_3) - \Psi(f_0, f_1f_2, f_3) + \Psi(f_0, f_1, f_2f_3) - \Psi(f_3f_0, f_1, f_2) = 0,$$

for all $f_0, f_1, f_2, f_3 \in \mathcal{C}^\infty_\hbar$.

Proof. Let us first show that Ψ is cyclic. By using $\tau(fg) = \tau(gf)$ one finds that

$$\begin{aligned}
2\pi i\Psi(f_2, f_0, f_1) &= \tau\Big[f_2(\partial_1 f_0)(\partial_2 f_1) - f_2(\partial_2 f_0)(\partial_1 f_1)\Big]\\
&= \tau\Big[(\partial_1 f_2 f_0)(\partial_2 f_1) - (\partial_2 f_2 f_0)(\partial_1 f_1) - (\partial_1 f_2) f_0(\partial_2 f_1) + (\partial_2 f_2) f_0(\partial_1 f_1)\Big]\\
&= \tau\Big[(\partial_1 f_2 f_0)(\partial_2 f_1) - (\partial_2 f_2 f_0)(\partial_1 f_1)\Big] + 2\pi i\Psi(f_0, f_1, f_2),
\end{aligned}$$

and since $\tau(\partial_1 f) = \tau(\partial_2 f) = 0$ (by Proposition 2.8) it follows that

$$\begin{aligned}
2\pi i\Psi(f_2, f_0, f_1) &= 2\pi i\Psi(f_0, f_1, f_2) - \tau\Big[f_2 f_0(\partial_1\partial_2 f_1 - \partial_2\partial_1 f_1)\Big]\\
&= 2\pi i\Psi(f_0, f_1, f_2)
\end{aligned}$$

since $[\partial_1, \partial_2] = 0$. To show that Ψ is a cocycle, i.e.

$$\Psi(f_0f_1, f_2, f_3) - \Psi(f_0, f_1f_2, f_3) + \Psi(f_0, f_1, f_2f_3) - \Psi(f_3f_0, f_1, f_2) = 0,$$

is a straight-forward computation where one expands all derivatives of products of functions, and uses the fact that $\tau(fg) = \tau(gf)$. □

3. Projections in the algebra

For the noncommutative torus, it is well known that its algebra, in contrast to the commutative case, contains nontrivial projections which one may explicitly describe [Rie81]. In this section, we will show that a similar construction can be carried out for the noncommutative cylinder. Namely, we shall construct projections $p \in \mathcal{C}_\hbar^\infty$ of the following form:

$$p = g(u+\hbar)W + f(u) + g(u)W^{-1}.$$

Proposition 3.1. *Let $f, g \in \mathcal{S}(\mathbb{R})$ be real-valued functions, and set*

$$p = g(u+\hbar)W + f(u) + g(u)W^{-1}.$$

Then $p^* = p$. Moreover $p^2 = p$ if the functions f and g satisfy

$$g(u)g(u+\hbar) = 0 \tag{3.1}$$

$$g(u)\big(1 - f(u) - f(u-\hbar)\big) = 0 \tag{3.2}$$

$$g(u)^2 + g(u+\hbar)^2 = f(u) - f(u)^2 \tag{3.3}$$

for all $u \in \mathbb{R}$.

Proof. Since f and g are real-valued, using (2.5) one immediately obtains

$$p^* = W^{-1}g(u+\hbar) + f(u) + Wg(u) = g(u)W^{-1} + f(u) + g(u+\hbar)W = p.$$

Then, a straight-forward computation of p^2 gives

$$\begin{aligned} p^2 = g(u)g(u-\hbar)W^{-2} &+ \big(f(u)g(u) + g(u)f(u-\hbar)\big)W^{-1} \\ &+ \big(f(u)g(u+\hbar) + g(u+\hbar)f(u+\hbar)\big)W + g(u+\hbar)g(u+2\hbar)W^2 \\ &+ g(u+\hbar)^2 + f(u)^2 + g(u)^2 \end{aligned}$$

which indeed equals p by using (3.1)–(3.3). □

Let us now construct a particular class of projections satisfying the requirements of Proposition 3.1. Let f_0 be a function increasing smoothly from 0 to 1 on the interval $[0, \hbar]$, and define $f, g : \mathbb{R} \to \mathbb{R}$ as

$$f(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ or } u \geq 2\hbar \\ f_0(u) & \text{if } 0 \leq u \leq \hbar \\ 1 - f_0(u - \hbar) & \text{if } \hbar \leq u \leq 2\hbar \end{cases} \tag{3.4}$$

$$g(u) = \begin{cases} 0 & \text{if } u \leq \hbar \text{ or } u \geq 2\hbar \\ \sqrt{f(u) - f(u)^2} & \text{if } \hbar \leq u \leq 2\hbar. \end{cases} \tag{3.5}$$

Next, for $n \geq 1$ we set

$$f_n(u) = \sum_{k=1}^{n} W^{-2k} f(u) W^{2k}$$
$$g_n(u) = \sum_{k=1}^{n} W^{-2k} g(u) W^{2k},$$

resulting in n shifted copies of the original functions, as depicted in Figure 1. Note that f_n and g_n have compact support being defined on $[0, 2\hbar n]$, where they are $2\hbar$-periodic by construction.

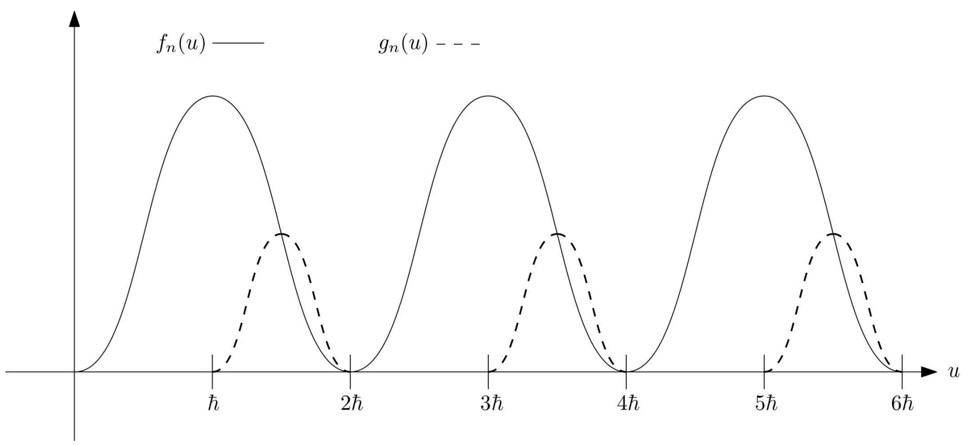

Figure 1: The functions f_n and g_n as constructed from (3.4) and (3.5).

It is straightforward to check that f_n and g_n satisfy (3.1), (3.2) and (3.3). For instance, for $u \in [0, \hbar]$ it is immediate that (3.1) and (3.2) holds since

$g(u) = 0$. Moreover,

$$
\begin{aligned}
g(u+\hbar)^2 &= f(u+\hbar) - f(u+\hbar)^2 = 1 - f_0(u) - (1-f_0(u))^2 \\
&= f_0(u) - f_0(u)^2 = f(u) - f(u)^2,
\end{aligned}
$$

showing that (3.3) is satisfied as well. Thus, one may conclude from Proposition 3.1 that

$$
p_n = g_n(u+\hbar)W + f_n(u) + g_n(u)W^{-1} \tag{3.6}
$$

is indeed a projection in $\mathcal{C}^\infty_\hbar$. Next, let us compute the trace of these projections.

Proposition 3.2. *Let p_n be defined as above. Then $\tau(p_n) = n\hbar$.*

Proof. Since f_n is supported on $[0, 2\hbar n]$, where it is $2\hbar$-periodic, it follows that

$$
\tau(p_n) = \tau\Big(g_n(u+\hbar)W + f_n(u) + g_n(u)W^{-1}\Big) = n\int_0^{2\hbar} f_n(u)du,
$$

and from the definition of f_n one obtains

$$
\begin{aligned}
\tau(p_n) &= n\int_0^{\hbar} f_0(u)du + n\int_{\hbar}^{2\hbar}\big(1 - f_0(u-\hbar)\big)du \\
&= n\int_{\hbar}^{2\hbar} du = n\hbar.
\end{aligned}
$$

□

The curvature 2-form related to the projection p_n is given by $F_n = p_n dp_n dp_n$, which may be evaluated against the cyclic 2-cocycle defined in Proposition 2.9.

Proposition 3.3. *For any projection p_n as in* (3.6)*, one has*

$$
\Psi(p_n, p_n, p_n) = n.
$$

Proof. As

$$
\Psi(p_n, p_n, p_n) = \frac{1}{2\pi i}\tau\big(p_n(\partial_1 p_n)(\partial_2 p_n) - p_n(\partial_2 p_n)(\partial_1 p_n)\big)
$$

we compute

$$\begin{aligned} p_n &= g_n(u+\hbar)W + f_n(u) + g_n(u)W^{-1} \\ \partial_1 p_n &= g_n'(u+\hbar)W + f_n'(u) + g_n'(u)W^{-1} \\ \partial_2 p_n &= 2\pi i g_n(u+\hbar)W - 2\pi i g_n(u)W^{-1}. \end{aligned}$$

Writing

$$p_n(\partial_1 p_n)(\partial_2 p_n) - p_n(\partial_2 p_n)(\partial_1 p_n) = \sum_{n\in\mathbb{Z}} A_n(u)W^n$$

one finds that

$$\begin{aligned} -\frac{1}{2\pi i}A_0 &= f_n(u)\Big(g_n(u+\hbar)Wg_n'(u)W^{-1} - g_n(u)W^{-1}g_n'(u+\hbar)W\Big) \\ &\quad - g_n(u+\hbar)Wg_n(u)W^{-1}f_n'(u) + g_n(u)W^{-1}g_n(u+\hbar)Wf_n'(u) \\ &\quad - f_n(u)\Big(- g_n'(u+\hbar)Wg_n(u)W^{-1} + g_n'(u)W^{-1}g_n(u+\hbar)W\Big) \\ &\quad + g_n(u+\hbar)Wf_n'(u)g_n(u)W^{-1} - g_n(u)W^{-1}f_n'(u)g_n(u+\hbar)W \\ &= f_n(u)\Big(2g_n(u+\hbar)g_n'(u+\hbar) - 2g_n(u)g_n'(u)\Big) - g_n(u+\hbar)^2 f_n'(u) \\ &\quad + g_n(u)^2 f_n'(u) + g_n'(u+\hbar)^2 f_n'(u+\hbar) - g_n(u)^2 f_n'(u-\hbar) \\ &= \big(f_n(u)g_n(u+\hbar)^2 - f_n(u)g_n(u)^2\big)_u' - 2g_n(u+\hbar)^2 f_n'(u) \\ &\quad + 2g_n(u)^2 f_n'(u) + g_n(u+\hbar)^2 f_n'(u+\hbar) - g_n(u)^2 f_n'(u-\hbar), \end{aligned}$$

giving

$$\begin{aligned} \Psi(p_n,p_n,p_n) = \frac{1}{2\pi i}\tau(A_0) &= \tau\Big[2g_n(u+\hbar)^2 f_n'(u) + g_n(u)^2 f_n'(u-\hbar) \\ &\qquad - 2g_n(u)^2 f_n'(u) - g_n(u+\hbar)^2 f_n'(u+\hbar)\Big] \\ &= 3\int_{-\infty}^{\infty} g_n(u)^2 f_n'(u-\hbar)du - 3\int_{-\infty}^{\infty} g_n(u)^2 f_n'(u)du. \end{aligned}$$

Since $g_n(u) = 0$ for all $u \in [2\pi k, 2\pi k + \hbar]$ and $f_n(u) = 1 - f_n(u-\hbar)$ for all $u \in [2k\hbar + \hbar, 2k\hbar + 2\hbar]$ for $k = 0, \ldots, n-1$,

$$\int_{-\infty}^{\infty} g_n(u)^2 f_n'(u)du = -\int_{-\infty}^{\infty} g_n(u) f_n'(u-\hbar)du,$$

and it follows that

$$\Psi(p_n, p_n, p_n) = 6\int_{-\infty}^{\infty} g_n(u)^2 f_n'(u-\hbar)du = 6n\int_{\hbar}^{2\hbar} g(u)^2 f'(u-\hbar)du.$$

Noting that for $u \in [\hbar, 2\hbar]$

$$\begin{aligned} g(u)^2 = f(u) - f(u)^2 &= 1 - f(u-\hbar) - (1 - f(u-\hbar))^2 \\ &= f(u-\hbar) - f(u-\hbar)^2 \end{aligned}$$

one computes

$$\begin{aligned} \Psi(p_n, p_n, p_n) &= 6n\int_{\hbar}^{2\hbar} \big(f(u-\hbar) - f(u-\hbar)^2\big) f'(u-\hbar)du \\ &= 6n\int_0^1 (s - s^2)ds = 6n\left[\frac{1}{2} - \frac{1}{3}\right] = n, \end{aligned}$$

which proves the statement. □

Considering the construction of the projection p_n, and the results in Proposition 3.2 and Proposition 3.3, it is natural to ask how the direct sum of the projective modules defined by p_n and p_m is related to the module defined by p_{m+n}. The next result shows that they are indeed isomorphic.

Proposition 3.4. *Let n, m be integers with $n, m \geq 1$. Then*

$$p_n\mathcal{C}_\hbar^\infty \oplus p_m\mathcal{C}_\hbar^\infty \simeq p_{n+m}\mathcal{C}_\hbar^\infty$$

as (right) $\mathcal{C}_\hbar^\infty$-modules.

Proof. Let p_n and p_m be given as

$$\begin{aligned} p_n &= g_n(u+\hbar)W + f_n(u) + g_n(u)W^{-1} \\ p_m &= g_m(u+\hbar)W + f_m(u) + g_m(u)W^{-1} \end{aligned}$$

and introduce

$$\begin{aligned} \tilde{p}_m = W^{-2n}p_mW^{2n} = g_m(&u+\hbar-2n\hbar)W \\ &+ f_m(u-2n\hbar) + g_m(u-2n\hbar)W^{-1}. \end{aligned}$$

Since $\tilde{p}_m$ is unitarily equivalent to p_m, the modules $p_m\mathcal{C}_\hbar^\infty$ and $\tilde{p}_m\mathcal{C}_\hbar^\infty$ are isomorphic and, furthermore, it is clear that $p_n + \tilde{p}_m = p_{n+m}$. Next, let us show that p_n and $\tilde{p}_m$ are orthogonal; i.e. that $p_n\tilde{p}_m = 0$. Introduce

$$\begin{aligned}\tilde{g}_m(u) &= g_m(u - 2n\hbar)\\ \tilde{f}_m(u) &= f_m(u - 2n\hbar)\end{aligned}$$

and note that

$$g_n(u)\tilde{g}_m(u) = f_n(u)\tilde{f}_m(u) = g_n(u)\tilde{f}_m(u) = f_n(u)\tilde{g}_m(u) = 0$$

since $\operatorname{supp}(f_n, g_n) \subseteq (0, 2n\hbar)$ and $\operatorname{supp}(\tilde{f}_m, \tilde{g}_m) \subseteq (2n\hbar, 2(n+m)\hbar)$ are disjoint. Using these facts, one finds that

$$\begin{aligned}p_n\tilde{p}_m = g_n(u+\hbar)\tilde{g}_m(u+2\hbar)W^2 + f_n(u)\tilde{g}_m(u+\hbar)W\\ + g_n(u)\tilde{f}_m(u-\hbar)W^{-1} + g_n(u)\tilde{g}_m(u-\hbar)W^{-2}.\end{aligned}$$

First of all, it is clear that $g_n(u)\tilde{f}_m(u-\hbar) = 0$ and $g_n(u)\tilde{g}_m(u-\hbar) = 0$ since $u - \hbar < 2n\hbar$ whenever $u \in \operatorname{supp} g_n \subseteq (0, 2n\hbar)$. Furthermore, it also follows that $g_n(u+\hbar)\tilde{g}_m(u+2\hbar) = 0$ and $f_n(u)\tilde{g}_m(u+\hbar) = 0$ since $\tilde{g}_m(u) = 0$ for $u \in [2n\hbar, (2n+1)\hbar]$. Thus, we conclude that $p_n\tilde{p}_m = 0$. For orthogonal projections,

$$p_n\mathcal{C}_\hbar^\infty \oplus \tilde{p}_m\mathcal{C}_\hbar^\infty \simeq (p_n + \tilde{p}_m)\mathcal{C}_\hbar^\infty,$$

and in combination with the previous arguments one obtains

$$p_n\mathcal{C}_\hbar^\infty \oplus p_m\mathcal{C}_\hbar^\infty \simeq p_n\mathcal{C}_\hbar^\infty \oplus \tilde{p}_m\mathcal{C}_\hbar^\infty \simeq (p_n + \tilde{p}_m)\mathcal{C}_\hbar^\infty \simeq p_{n+m}\mathcal{C}_\hbar^\infty,$$

which proves the desired result. □

Let us discuss these results from the perspective of K-theory. In [vS04], K_0 of the noncommutative cylinder was shown to be isomorphic to $\mathbb{Z}$. Since the algebra is nonunital, one then expects that there exists a countable class of nontrivial projections . In this section we have constructed projections p_n (for each $n \geq 1$), and Proposition 3.2 shows that if $m \neq n$ then p_n and p_m are not equivalent. Moreover, from Proposition 3.4 it follows that the map $p_n \mapsto n$ respects the group structure of the integers, and we conclude that p_n represents the K_0 class labeled by n. In this sense, one may consider the projection p_1 to be a generator of K_0.

4. Bimodules

We now construct $\mathcal{C}^\infty_\hbar$-modules on the space of Schwartz functions in one discrete and one real variable in analogy with the noncommutative torus. We show that one may construct left and right $\mathcal{C}^\infty_\hbar$-modules, as well as bimodules, depending on a set of parameters. Furthermore, it turns out that these modules are in fact isomorphic to a number of copies of the algebra itself. To begin with, for $\xi,\eta\in\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ set

$$(\xi,\eta)_L=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\xi(x,k)\overline{\eta(x,k)}dx$$
$$(\xi,\eta)_R=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\overline{\xi(x,k)}\eta(x,k)dx=\overline{(\xi,\eta)}_L.$$

The corresponding left module structure is given in the following result.

Proposition 4.1. *Let $\lambda_0,\lambda_1,\varepsilon,\hbar\in\mathbb{R}$ and $r\in\mathbb{Z}$ be such that $\lambda_0\varepsilon+\lambda_1 r=-\hbar$. For $f=\sum_{n\in\mathbb{Z}}f_n(u)W^n$ set*

$$(f\xi)(x,k)=\sum_{n\in\mathbb{Z}}f_n(\lambda_0x+\lambda_1k)\xi(x-n\varepsilon,k-nr) \tag{4.1}$$

for $\xi\in\mathcal{S}(\mathbb{R}\times\mathbb{Z})$. Then $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ is a left $\mathcal{C}^\infty_\hbar$-module such that

$$(f\xi,\eta)_L=(\xi,f^*\eta)_L$$

for all $f\in\mathcal{C}^\infty_\hbar$.

Proof. In order for (4.1) to define a module action, one has to check that it respects the relations in the algebra; i.e. $\big((fg)\xi\big)(x,k)=\big(f(g\xi)\big)(x,k)$. One finds that

$$\begin{aligned}
\big(f(g\xi)\big)(x,k)&=\sum_{n\mathbb{Z}}f_n(\lambda_0x+\lambda_1k)(g\xi)(x-n\varepsilon,k-nr)\\
=\sum_{n,m\in\mathbb{Z}}&f_n(\lambda_0x+\lambda_1k)g_m\big(\lambda_0(x-n\varepsilon)+\lambda_1(k-nr)\big)\\
&\times\xi\big(x-(n+m)\varepsilon,k-(n+m)r\big)\\
=\sum_{n,l\in\mathbb{Z}}&f_n(\lambda_0x+\lambda_1k)g_{l-n}\big(\lambda_0x+\lambda_1k-n(\lambda_0\varepsilon+\lambda_1r)\big)\xi\big(x-l\varepsilon,k-lr\big)\\
=\sum_{n,l\in\mathbb{Z}}&f_n(\lambda_0x+\lambda_1k)g_{l-n}\big(\lambda_0x+\lambda_1k+n\hbar\big)\xi\big(x-l\varepsilon,k-lr\big),
\end{aligned}$$

by using that $\lambda_0\varepsilon + \lambda_1 r = -\hbar$. On the other hand

$$\big((fg)\xi\big)(x,k) = \bigg(\sum_{n,m\in\mathbb{Z}} f_m(u)g_{n-m}(u+m\hbar)W^n\xi\bigg)(x,k)$$
$$= \sum_{n,m\in\mathbb{Z}} f_m(\lambda_0 x+\lambda_1 k)g_{n-m}(\lambda_0 x+\lambda_1 k+m\hbar)\xi(x-m\varepsilon,k-mr),$$

which is seen to equal $\big(f(g\xi)\big)(x,k)$. Next, let us show that $(f\xi,\eta)_L = (\xi,f^*\eta)_L$ by first computing

$$\begin{aligned}(f\xi,\eta)_L &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\bigg(\sum_{n\in\mathbb{Z}} f_n(u)W^n\xi\bigg)(x,k)\overline{\eta(x,k)}dx\\ &= \sum_{k,n\in\mathbb{Z}}\int_{\mathbb{R}} f_n(\lambda_0 x+\lambda_1 k)\xi(x-n\varepsilon,k-nr)\overline{\eta(x,k)}dx.\end{aligned}$$

Let us compare this with

$$\begin{aligned}(\xi,f^*\eta)_L &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\xi(x,k)\overline{\bigg(\sum_{n\in\mathbb{Z}}\overline{f_{-n}(u+n\hbar)}W^n\eta\bigg)(x,k)}dx\\ &= \sum_{k,n\in\mathbb{Z}}\int_{\mathbb{R}}\xi(x,k)f_{-n}(\lambda_0 x+\lambda_1 k+n\hbar)\overline{\eta(x-n\varepsilon,k-nr)}dx.\end{aligned}$$

Setting $m=-n$ gives

$$(\xi,f^*\eta)_L = \sum_{k,m\in\mathbb{Z}}\int_{\mathbb{R}}\xi(x,k)f_m(\lambda_0 x+\lambda_1 k-m\hbar)\overline{\eta(x+m\varepsilon,k+mr)}dx$$

and changing the integration variable to $y = x+m\varepsilon$ yields

$$(\xi,f^*\eta)_L = \sum_{k,m\in\mathbb{Z}}\int_{\mathbb{R}}\xi(y-m\varepsilon,k)f_m(\lambda_0 y+\lambda_1 k-\lambda_0 m\varepsilon-m\hbar)\overline{\eta(y,k+mr)}dy.$$

Finally, we set $l = k+mr$ and use that $\lambda_0\varepsilon+\lambda_1 r = -\hbar$ to obtain

$$(\xi,f^*\eta)_L = \sum_{l,m\in\mathbb{Z}}\int_{\mathbb{R}}\xi(y-m\varepsilon,l-nr)f_m(\lambda_0 y+\lambda_1 l)\overline{\eta(y,l)}dy,$$

which equals $(f\xi,\eta)_L$. □

In the same way, one may construct a right module structure on $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$. The proof is analogous to that of Proposition 4.1.

Proposition 4.2. *Let $\mu_0, \mu_1, \varepsilon', \hbar' \in \mathbb{R}$ and $r' \in \mathbb{Z}$ such that $\mu_0\varepsilon' + \mu_1 r' = \hbar'$. For $f = \sum_{n\in\mathbb{Z}} f_n(u)W^n$ set*

$$(\xi f)(x,k) = \sum_{n\in\mathbb{Z}} f_n(\mu_0 x + \mu_1 k - n\hbar')\xi(x - n\varepsilon', k - nr') \tag{4.2}$$

for $\xi \in \mathcal{S}(\mathbb{R}\times\mathbb{Z})$. Then $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ is a right $\mathcal{C}^\infty_{\hbar'}$-module such that

$$(\xi f, \eta)_R = (\xi, \eta f^*)_R$$

for all $f \in \mathcal{C}^\infty_{\hbar'}$.

A (left or right) $\mathcal{C}^\infty_\hbar$-module constructed as above will be denoted by $\mathcal{E}_\hbar$ with a suitable choice of parameters implicitly assumed. If the parameters of the left and right module structures are compatible, $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ becomes a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule.

Proposition 4.3. *Let $\lambda_0, \mu_0, \lambda_1, \mu_1, \varepsilon, \varepsilon', \hbar, \hbar' \in \mathbb{R}$ and $r, r' \in \mathbb{Z}$ such that*

$$\begin{aligned} \lambda_0\varepsilon + \lambda_1 r &= -\hbar & \mu_0\varepsilon' + \mu_1 r' &= \hbar' \\ \lambda_0\varepsilon' + \lambda_1 r' &= 0 & \mu_0\varepsilon + \mu_1 r &= 0. \end{aligned}$$

Then $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ is a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule with respect to the left and right actions defined in Proposition 4.1 and Proposition 4.2.

Proof. In order for $\mathcal{S}(\mathbb{R}\times\mathbb{Z})$ to be a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule, it must hold that

$$\big(g(\xi f)\big)(x,k) = \big((g\xi)f\big)(x,k)$$

for all $\xi \in \mathcal{S}(\mathbb{R}\times\mathbb{Z})$, $g \in \mathcal{C}^\infty_\hbar$ and $f \in \mathcal{C}^\infty_{\hbar'}$. One finds that

$$\begin{aligned} \big(g(\xi f)\big)(x,k) &= \sum_{n\in\mathbb{Z}} g_n(\lambda_0 x + \lambda_1 k)(\xi f)(x - n\varepsilon, k - nr) \\ &= \sum_{n,m\in\mathbb{Z}} g_n(\lambda_0 x + \lambda_1 k) f_m\big(\mu_0(x - n\varepsilon) + \mu_1(k - nr) - m\hbar'\big) \\ &\qquad \times \xi(x - n\varepsilon - m\varepsilon', k - nr - mr') \\ &= \sum_{n,m\in\mathbb{Z}} g_n(\lambda_0 x + \lambda_1 k) f_m(\mu_0 x + \mu_1 k - m\hbar')\xi(x - n\varepsilon - m\varepsilon', k - nr - mr'), \end{aligned}$$

by using that $\mu_0\varepsilon + \mu_1 r = 0$. On the other hand

$$\begin{aligned}\big((g\xi)f\big)(x,k) &= \sum_{m\in\mathbb{Z}} f_m(\mu_0 x + \mu_1 k - m\hbar')(g\xi)(x - m\varepsilon', k - mr')\\ &= \sum_{m,n\in\mathbb{Z}} f_m(\mu_0 x + \mu_1 k - m\hbar') g_n\big(\lambda_0(x - m\varepsilon') + \lambda_1(k - mr')\big)\\ &\qquad\qquad \times \xi(x - m\varepsilon' - n\varepsilon, k - mr' - nr)\\ &= \sum_{m,n\in\mathbb{Z}} g_n\big(\lambda_0 x + \lambda_1 k\big) f_m(\mu_0 x + \mu_1 k - m\hbar')\\ &\qquad\qquad \times \xi(x - m\varepsilon' - n\varepsilon, k - mr' - nr),\end{aligned}$$

since $\lambda_0\varepsilon' + \lambda_1 r' = 0$. We conclude that $\big(g(\xi f)\big)(x,k) = \big((g\xi)f\big)(x,k)$. □

A bimodule defined as in Proposition 4.3 will be denoted by $\mathcal{E}_{\hbar,\hbar'}$, again tacitly assuming a choice of the parameters $\lambda_0, \lambda_1, \mu_0, \mu_1, \varepsilon, \varepsilon' \in \mathbb{R}$ and $r, r' \in \mathbb{Z}$ satisfying the requirements in Proposition 4.3. Note that one can construct bimodules for arbitrary choices of $\hbar, \hbar'$ by choosing e.g. $\varepsilon, \varepsilon', r, r'$ such that $\varepsilon r' \neq \varepsilon' r$ and setting

$$\begin{aligned}\lambda_0 &= -\frac{\hbar r'}{\varepsilon r' - \varepsilon' r} & \lambda_1 &= \frac{\hbar\varepsilon'}{\varepsilon r' - \varepsilon' r}\\ \mu_0 &= -\frac{\hbar' r}{\varepsilon r' - \varepsilon' r} & \mu_1 &= \frac{\hbar'\varepsilon}{\varepsilon r' - \varepsilon' r}.\end{aligned}$$

Let us now point out some obvious isomorphisms between modules defined by different sets of parameters. To simplify the description, we make the following definition.

Definition 4.4. The vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n$ are called 0-*compatible* if either $x_i = y_i = 0$ or $x_i, y_i \neq 0$ for $i = 1, \ldots, n$.

Proposition 4.5. *Let $\mathcal{E}_\hbar$ and $\tilde{\mathcal{E}}_\hbar$ be left $C^\infty_\hbar$-modules (as in Proposition 4.1) defined by the parameters $(\lambda_0, \varepsilon, \lambda_1, r)$ and $(\tilde\lambda_0, \tilde\varepsilon, \lambda_1, r)$, respectively. If (λ_0, ε) and $(\tilde\lambda_0, \tilde\varepsilon)$ are 0-compatible then $\mathcal{E}_\hbar \simeq \tilde{\mathcal{E}}_\hbar$ as left $C^\infty_\hbar$-modules.*

Proof. Note that since $\lambda_0\varepsilon + \lambda_1 r = -\hbar = \tilde\lambda_0\tilde\varepsilon + \lambda_1 r$, it follows that $\lambda_0\varepsilon = \tilde\lambda_0\tilde\varepsilon$. We shall proceed by defining module homomorphisms $\phi_\tau : \mathcal{E}_\hbar \to \tilde{\mathcal{E}}_\hbar$ as

$$\phi_\tau(\xi)(x,k) = \xi(\tau x, k)$$

for $\tau \in \mathbb{R}$. Note that ϕ_τ is a linear map and if $\tau \neq 0$ then ϕ_τ is invertible. Now, let us derive conditions for ϕ_τ to be a module homomorphism; thus, we demand that $\phi_\tau(f\xi) = f\phi_\tau(\xi)$ for all $f \in \mathcal{C}^\infty_\hbar$. To this end one computes

$$\begin{aligned}\phi_\tau(f\xi)(x,k) &= \sum_{n\in\mathbb{Z}} f_n(\lambda_0\tau x + \lambda_1 k)\xi(\tau x - n\varepsilon, k - nr)\\ \big(f\phi_\tau(\xi)\big)(x,k) &= \sum_{n\in\mathbb{Z}} f_n(\tilde\lambda_0 x + \lambda_1 k)\phi_\tau(\xi)(x - n\tilde\varepsilon, k - nr)\\ &= \sum_{n\in\mathbb{Z}} f_n(\tilde\lambda_0 x + \lambda_1 k)\xi\big(\tau(x - n\tilde\varepsilon), k - nr\big).\end{aligned}$$

The above expressions are equal if

$$\lambda_0\tau = \tilde\lambda_0 \quad \text{and} \quad \varepsilon = \tau\tilde\varepsilon. \tag{4.3}$$

Note that since (λ_0, ε) and $(\tilde\lambda_0, \tilde\varepsilon)$ are 0-compatible, either both sides of each equation are zero (i.e. trivially giving a solution) or both sides are non-zero (as long as $\tau \neq 0$). Thus, if $\lambda_0 = \tilde\lambda_0 = 0$ and $\varepsilon = \tilde\varepsilon = 0$, then ϕ_τ is an isomorphism for any $\tau \neq 0$. If $\lambda_0, \tilde\lambda_0 \neq 0$ and $\varepsilon = \tilde\varepsilon = 0$, one can set $\tau = \tilde\lambda_0/\lambda_0$, solving (4.3) (and similarly in the case when $\varepsilon, \tilde\varepsilon \neq 0$ but $\lambda_0 = \tilde\lambda_0 = 0$). Now, for the case when $\lambda_0, \tilde\lambda_0, \varepsilon, \tilde\varepsilon \neq 0$ one sets $\tau = \tilde\lambda_0/\lambda_0$ and notes that

$$\tau\tilde\varepsilon = \frac{\tilde\lambda_0\tilde\varepsilon}{\lambda_0} = \frac{\lambda_0\varepsilon}{\lambda_0} = \varepsilon$$

giving a solution of (4.3). Thus, it follows that under the assumptions given in the proposition, there exists a module isomorphism $\phi_\tau : \mathcal{E}_\hbar \to \tilde{\mathcal{E}}_\hbar$. □

Somewhat surprisingly, it turns out that these modules are in fact isomorphic (as modules) to copies of the algebra itself. More precisely, we formulate the statement as follows.

Proposition 4.6. *Let $\mathcal{E}_\hbar$ be a left $\mathcal{C}^\infty_\hbar$-module defined by the parameters $\lambda_0, \lambda_1, \varepsilon, r$ such that $\lambda_0\varepsilon + \lambda_1 r = -\hbar$ and $\lambda_0 \neq 0$. For arbitrary $F \in (\mathcal{C}^\infty_\hbar)^r$ (considered as a left $\mathcal{C}^\infty_\hbar$-module) we write $F = (F^0, F^1, \ldots, F^{r-1})$ with $F^k \in \mathcal{C}^\infty_\hbar$ for $k = 0, 1, \ldots, r-1$, and introduce the components $F^k_n(u)$ via*

$$F^k = \sum_{n\in\mathbb{Z}} F^k_n(u) W^n.$$

Furthermore, for an integer k we let $k_0 \in \mathbb{Z}$ and $0 \leq k_1 \leq r-1$ be defined by $k = k_0 r + k_1$. Then the map $\phi : \big(\mathcal{C}^\infty_\hbar\big)^r \to \mathcal{E}_\hbar$, defined as

$$\phi(F)(x,k) = F^{k_1}_{k_0}(\lambda_0 x + \lambda_1 k),$$

is an isomorphism of left $\mathcal{C}_\hbar^\infty$-modules.

Proof. First of all, it is clear that $\phi(F+F')=\phi(F)+\phi(F')$. Furthermore, one finds that

$$\begin{aligned}\phi(fF)(x,k)&=(fF)^{k_1}_{k_0}(\lambda_0x+\lambda_1k)\\&=\sum_{n\in\mathbb{Z}}f_n(\lambda_0x+\lambda_1k)F^{k_1}_{k_0-n}(\lambda_0x+\lambda_1k+n\hbar)\end{aligned}$$

as well as

$$\begin{aligned}\big(f\phi(F)\big)(x,k)&=\sum_{n\in\mathbb{Z}}f(\lambda_0x+\lambda_1k)\phi(F)(x-n\varepsilon,k-nr)\\&=\sum_{n\in\mathbb{Z}}f(\lambda_0x+\lambda_1k)\phi(F)(x-n\varepsilon,(k_0-n)r+k_1)\\&=\sum_{n\in\mathbb{Z}}f(\lambda_0x+\lambda_1k)F^{k_1}_{k_0-n}\big(\lambda_0(x-n\varepsilon)+\lambda_1(k_0r+k_1)-n\lambda_1r\big)\\&=\sum_{n\in\mathbb{Z}}f(\lambda_0x+\lambda_1k)F^{k_1}_{k_0-n}\big(\lambda_0x+\lambda_1k+n\hbar\big)=\phi(fF)(x,k)\end{aligned}$$

by using that $-n(\lambda_0\varepsilon+\lambda_1r)=n\hbar$. Hence, ϕ is a left module homomorphism. One may readily construct the inverse

$$\phi^{-1}(\xi)=(F^0,\dots,F^{r-1}):\quad F^k=\sum_{n\in\mathbb{Z}}\xi\big(\tfrac{1}{\lambda_0}(u-\lambda_1(nr+k)),nr+k\big)W^n$$

and check that

$$\begin{aligned}\phi\big(\phi^{-1}(\xi)\big)(x,k)&=\big(\phi^{-1}(\xi)\big)^{k_1}_{k_0}(\lambda_0x+\lambda_1k)\\&=\xi\big(x-\lambda_1(k_0r+k_1)+\lambda_1k,k_0r+k_1\big)=\xi(x,k).\end{aligned}$$

We conclude that ϕ is indeed a left module isomorphism. □

In the case when $r=1$ and, moreover, $\mathcal{E}_\hbar$ is a $\mathcal{C}_\hbar^\infty$-bimodule, one can strengthen the result to obtain a bimodule isomorphism.

Proposition 4.7. *For $\hbar>0$, let $\mathcal{E}_{\hbar,\hbar}$ be a $\mathcal{C}_\hbar^\infty$-bimodule (as in Proposition 4.3) with $\lambda_0\neq0$ and $r=r'=1$. The map $\phi:\mathcal{C}_\hbar^\infty\to\mathcal{E}_{\hbar,\hbar}$, defined as*

$$\phi(f)(x,k)=f_k\big(\lambda_0x+\lambda_1k\big)\tag{4.4}$$

for $f = \sum_{n\in\mathbb{Z}} f_n(u)W^n$*, is a bimodule isomorphism with inverse*

$$\phi^{-1}(\xi) = \sum_{n\in\mathbb{Z}} \xi\big(\tfrac{1}{\lambda_0}(u-\lambda_1 n), n\big)W^n.$$

for $\xi \in \mathcal{E}_{\hbar,\hbar}$*.*

Proof. The fact that ϕ is a left module isomorphism with

$$\phi^{-1}(\xi) = \sum_{n\in\mathbb{Z}} \xi\big(\tfrac{1}{\lambda_0}(u-\lambda_1 n), n\big)W^n$$

follows immediately from Proposition 4.6 (and its proof). Before showing that ϕ is a also a right $\mathcal{C}^\infty_\hbar$-module homomorphism, let us derive a few properties of the parameters defining the module. For a bimodule with $\hbar = \hbar' > 0$ and $r = r' = 1$ one necessarily has $\varepsilon \neq \varepsilon'$ (otherwise implying $\hbar = 0$). Hence, one may solve for λ_0, μ_0 and λ_1, μ_1 to obtain

$$\lambda_0 = \mu_0 = \frac{\hbar}{\varepsilon' - \varepsilon} \qquad \lambda_1 = -\frac{\hbar\varepsilon'}{\varepsilon' - \varepsilon} \qquad \mu_1 = -\frac{\hbar\varepsilon}{\varepsilon' - \varepsilon}$$

and we note that

$$\mu_1 - \hbar = \frac{-\hbar\varepsilon - (\varepsilon' - \varepsilon)\hbar}{\varepsilon' - \varepsilon} = -\frac{\hbar\varepsilon'}{\varepsilon' - \varepsilon} = \lambda_1. \tag{4.5}$$

Now, one computes for $f, g \in \mathcal{C}^\infty_\hbar$

$$\begin{aligned}\phi(fg)(x,k) &= (fg)_k(\lambda_0 x + \lambda_1 k)\\ &= \sum_{n\in\mathbb{Z}} f_n(\lambda_0 x + \lambda_1 k) g_{k-n}(\lambda_0 x + \lambda_1 k + n\hbar)\end{aligned}$$

and

$$\begin{aligned}\big(\phi(f)g\big)(x,k) &= \sum_{l\in\mathbb{Z}} g_l(\mu_0 x + \mu_1 k - l\hbar)\phi(f)\big(x - l\varepsilon', k - l\big)\\ &= \sum_{l\in\mathbb{Z}} f_{k-l}\big(\lambda_0(x - l\varepsilon') + \lambda_1(k-l)\big) g_l(\mu_0 x + \mu_1 k - l\hbar)\\ &= \sum_{n\in\mathbb{Z}} f_{k-l}\big(\lambda_0 x + \lambda_1 k - l(\lambda_0\varepsilon' + \lambda_1)\big) g_l(\mu_0 x + \mu_1 k - l\hbar)\\ &= \sum_{n\in\mathbb{Z}} f_{k-l}\big(\lambda_0 x + \lambda_1 k\big) g_l(\mu_0 x + \mu_1 k - l\hbar)\end{aligned}$$

since $0 = \lambda_0\varepsilon' + \lambda_1 r' = \lambda_0\varepsilon' + \lambda_1$. Moreover, changing the summation index to $n = k - l$ gives

$$\begin{aligned}\big(\phi(f)g\big)(x,k) &= \sum_{n\in\mathbb{Z}} f_n\big(\lambda_0 x + \lambda_1 k\big) g_{k-n}(\mu_0 x + (\mu_1 - \hbar)k + n\hbar) \\ &= \sum_{n\in\mathbb{Z}} f_n\big(\lambda_0 x + \lambda_1 k\big) g_{k-n}(\lambda_0 x + \lambda_1 k + n\hbar) = \phi(fg)(x,k),\end{aligned}$$

by using that $\lambda_0 = \mu_0$ and $\mu_1 - \hbar = \lambda_1$, as shown in (4.5). □

4.1. Hermitian structures

Continuing the analogy with the noncommutative torus, we show there exist hermitian structures on the $(\mathcal{C}_\hbar, \mathcal{C}_{\hbar'})$-bimodule $\mathcal{E}_{\hbar,\hbar'}$.

Proposition 4.8. *Let $\mathcal{E}_{\hbar,\hbar'}$ be a $(\mathcal{C}_\hbar, \mathcal{C}_{\hbar'})$-bimodule, as in Proposition 4.3, and define ${}_\bullet\langle\cdot,\cdot\rangle : \mathcal{E}_{\hbar,\hbar'} \times \mathcal{E}_{\hbar,\hbar'} \to \mathcal{C}_\hbar$ and $\langle\cdot,\cdot\rangle_\bullet : \mathcal{E}_{\hbar,\hbar'} \times \mathcal{E}_{\hbar,\hbar'} \to \mathcal{C}_{\hbar'}$ as*

$$\begin{aligned}{}_\bullet\langle\xi,\eta\rangle &= \sum_{n\in\mathbb{Z}} \left[\int_{\mathbb{R}} \big(\xi, e^{2\pi i\hat{x}u} W^n \eta\big)_L e^{2\pi i\hat{x}u} d\hat{x}\right] W^n \\ \langle\xi,\eta\rangle_\bullet &= \sum_{n\in\mathbb{Z}} \left[\int_{\mathbb{R}} \big(\xi e^{2\pi i\hat{x}u} W^n, \eta\big)_R e^{2\pi i\hat{x}u} d\hat{x}\right] W^n.\end{aligned}$$

Then it follows that

$${}_\bullet\langle a\xi,\eta\rangle = a\big({}_\bullet\langle\xi,\eta\rangle\big) \qquad \langle\xi,\eta b\rangle_\bullet = \big(\langle\xi,\eta\rangle_\bullet\big)b$$

as well as the compatibility condition

$${}_\bullet\langle\xi,\eta\rangle\psi = \xi\langle\eta,\psi\rangle_\bullet$$

for $a \in \mathcal{C}_\hbar$, $b \in \mathcal{C}_{\hbar'}$ and $\xi,\eta,\psi \in \mathcal{E}_{\hbar,\hbar'}$.

Remark 4.9. Strictly speaking, $e^{2\pi i\hat{x}u} W^n \eta$ is not defined since $e^{2\pi i\hat{x}u}$ does not decay as $u \to \infty$ (and, hence, does not belong to the algebra). However, having in mind the left action (4.1), we interpret the above expression as

$$(\xi, e^{2\pi i\hat{x}u} W^n \eta)_L = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \xi(x,k)\overline{\eta(x - n\varepsilon, k - nr)} e^{-2\pi i\hat{x}(\lambda_0 x + \lambda_1 k)} dx$$

and allow ourselves a formulation as in Proposition 4.8 since it more clearly reflects the idea behind the construction. Similarly for the right structure.

Proof. Let us start by showing that ${}_\bullet\langle a\xi,\eta\rangle = a({}_\bullet\langle\xi,\eta\rangle)$. Thus, we set

$$a = \sum_k a_k(u)W^k$$

and write

$$\begin{aligned}
{}_\bullet\langle a\xi,\eta\rangle &= \sum_{k,n}\int_{\mathbb{R}} \big(a_k(u)W^k\xi, e^{2\pi i\hat{x}u}W^n\eta\big)_L e^{2\pi i\hat{x}u}W^n d\hat{x} \\
&= \sum_{k,n}\int_{\mathbb{R}} \big(\xi, W^{-k}\overline{a_k(u)}e^{2\pi i\hat{x}u}W^n\eta\big)_L e^{2\pi i\hat{x}u}W^n d\hat{x} \\
&= \sum_{k,n}\int_{\mathbb{R}} \big(\xi, \overline{a_k(u-k\hbar)}e^{2\pi i\hat{x}(u-k\hbar)}W^{n-k}\eta\big)_L e^{2\pi i\hat{x}u}W^n d\hat{x}. \\
&= \sum_{k,n}\int_{\mathbb{R}} \big(\xi, \overline{a_k(u-k\hbar)}e^{2\pi i\hat{x}}W^{n-k}\eta\big)_L e^{2\pi i\hat{x}k\hbar}e^{2\pi i\hat{x}u}W^n d\hat{x}.
\end{aligned}$$

Now, let us replace $a_k(u-k\hbar)$ by its Fourier integral:

$$\begin{aligned}
{}_\bullet\langle a\xi,\eta\rangle =&\sum_{k,n}\iint_{\mathbb{R}^2} \big(\xi, \overline{\hat{a}_k(x)}e^{-2\pi ix(u-k\hbar)}e^{2\pi i\hat{x}u}W^{n-k}\eta\big)_L e^{2\pi i\hat{x}k\hbar}e^{2\pi i\hat{x}u}W^n dx d\hat{x} \\
=&\sum_{k,n}\iint_{\mathbb{R}^2} \big(\xi, e^{2\pi iu(\hat{x}-x)}W^{n-k}\eta\big)_L \hat{a}_k(x)e^{2\pi i(\hat{x}-x)k\hbar}e^{2\pi i\hat{x}u}W^n dx d\hat{x}.
\end{aligned}$$

By a change of variables we set $l = n-k$ and $y = \hat{x}-x$, giving

$$\begin{aligned}
{}_\bullet\langle a\xi,\eta\rangle &= \sum_{k,l}\iint_{\mathbb{R}^2} \big(\xi, e^{2\pi iuy}W^l\eta\big)_L \hat{a}_k(x)e^{2\pi iyk\hbar}e^{2\pi i(x+y)u}W^{k+l}dx dy \\
&= \sum_{k,l}\int_{\mathbb{R}} \hat{a}_k(x)e^{2\pi ixu}dx\int_{\mathbb{R}^2} \big(\xi, e^{2\pi iuy}W^l\eta\big)_L e^{2\pi iyk\hbar}e^{2\pi iyu}W^{k+l}dy \\
&= \sum_{k,l} a_k(u)W^k\int_{\mathbb{R}} \big(\xi, e^{2\pi iuy}W^l\eta\big)_L e^{2\pi iyu}W^l dy = a\big({}_\bullet\langle\xi,\eta\rangle\big).
\end{aligned}$$

The proof of the statement $\langle\xi,\eta a\rangle_\bullet = (\langle\xi,\eta\rangle_\bullet)a$ is completely analogous. Finally, we need to show compatibility of the two products; namely, that

$${}_\bullet\langle\xi,\eta\rangle\psi = \xi\langle\eta,\psi\rangle_\bullet$$

for $\xi, \eta, \psi \in \mathcal{E}_{\hbar,\hbar'}$. One writes

$$\begin{aligned}
({}_\bullet\langle \xi,\eta\rangle\psi)(x,k) &= \sum_n \int_{\mathbb{R}} \big(\xi, e^{2\pi i\hat{x}u} W^n \eta\big)_L e^{2\pi i \hat{x}(\lambda_0 x + \lambda_1 k)} \psi(x-n\varepsilon, k-nr) d\hat{x} \\
&= \sum_{n,l} \iint_{\mathbb{R}^2} \xi(y,l) e^{-2\pi i\hat{x}(\lambda_0 y + \lambda_1 l)} \overline{\eta(y-n\varepsilon, l-nr)} \\
&\qquad \times e^{2\pi i\hat{x}(\lambda_0 x+\lambda_1 k)} \psi(x-n\varepsilon, k-nr) dy d\hat{x} \\
&= \sum_{n,l} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i \hat{x}(\lambda_0(x-y)+\lambda_1(k-l))} \\
&\qquad \times \xi(y,l)\overline{\eta(y-n\varepsilon, l-nr)}\psi(x-n\varepsilon, k-nr) dy d\hat{x} \\
&= \sum_{n,l} \int_{\mathbb{R}} \delta(\lambda_0(x-y)+\lambda_1(k-l))\xi(y,l) \\
&\qquad \times \overline{\eta(y-n\varepsilon, l-nr)}\psi(x-n\varepsilon, k-nr) dy \\
&= \sum_{n,l} \xi\big(x+(k-l)\lambda_1/\lambda_0, l\big) \\
&\qquad \times \overline{\eta\big(x+(k-l)\lambda_1/\lambda_0 - n\varepsilon, l-nr\big)}\psi(x-n\varepsilon, k-nr).
\end{aligned}$$

A similar computation for $\xi\langle \eta,\psi\rangle_\bullet$ gives

$$\begin{aligned}
\big(\xi\langle\eta,\psi\rangle_\bullet\big)(x,k) = &\sum_{n',l'} \xi(x-n'\varepsilon', k-n'r') \\
&\qquad \times \overline{\eta\big(x+(k-l')\mu_1/\mu_0 - n'\varepsilon', l'-n'r'\big)} \\
&\qquad \times \psi\big(x+(k-l')\mu_1/\mu_0, l'\big)
\end{aligned}$$

To prove that the expressions for $({}_\bullet\langle\xi,\eta\rangle\psi)(x,k)$ and $\big(\xi\langle\eta,\psi\rangle_\bullet\big)(x,k)$ are equal for all x and k, we compare the sums term by term. Let us proceed as follows: We fix arbitrary n in the expression for $({}_\bullet\langle\xi,\eta\rangle\psi)(x,k)$ and arbitrary n' in the expression for $\big(\xi\langle\eta,\psi\rangle_\bullet\big)(x,k)$. Now, for every such choice we will prove that there exists l and l' such that the corresponding terms are equal. Setting

$$l = k - n'r' \quad \text{and} \quad l' = k - nr,$$

one finds (by comparing the arguments of ξ, η and ψ) that the corresponding terms in the two sums are equal if

$$\begin{aligned}&\frac{\lambda_1}{\lambda_0}(k-l) = -n'\varepsilon'\\&\frac{\lambda_1}{\lambda_0}(k-l) - n\varepsilon = \frac{\mu_1}{\mu_0}(k-l') - n'\varepsilon'\\&l - nr = l' - n'r'\\&-n\varepsilon = \frac{\mu_1}{\mu_0}(k-l')\end{aligned}$$

Inserting $l = k - n'r'$ and $l' = k - nr$ into these equations yields

$$\lambda_0\varepsilon' + \lambda_1 r' = 0 \quad \text{and} \quad \mu_0\varepsilon + \mu_1 r = 0$$

as the remaining conditions. However, these conditions are true due to the fact that $\mathcal{E}_{\hbar,\hbar'}$ is assumed to be a bimodule fulfilling the requirements of Proposition 4.3. □

4.2. Bimodule connections

Let us now turn to the question of finding connections ∇ on the bimodule $\mathcal{E}_{\hbar,\hbar'}$, of the form

$$\nabla_1\xi(x,k) = \alpha\xi'_x(x,k) \qquad \nabla_2\xi(x,k) = \beta x\xi(x,k) + \gamma k\xi(x,k), \tag{4.6}$$

for $\alpha, \beta, \gamma \in \mathbb{C}$, and we start by working out the conditions for ∇ to be a left connection; that is, such that

$$\nabla_k(f\xi) = f\nabla_k\xi + (\partial_k f)\xi$$

for $k = 1, 2$, $\xi \in \mathcal{E}_{\hbar,\hbar'}$ and $f \in C^\infty_\hbar$.

Proposition 4.10. *Let $\mathcal{E}_\hbar$ be a left $C^\infty_\hbar$-module with respect to a choice of $\lambda_0, \lambda_1, \varepsilon, r$ (as in Proposition 4.1) with $\lambda_0 \neq 0$. If $\alpha = 1/\lambda_0$ and $\beta\varepsilon + \gamma r = 2\pi i$ then ∇, as defined in* (4.6)*, is a left module connection.*

Proof. One finds that

$$\begin{aligned}\nabla_1(f\xi)(x,k) &= \alpha\big(f\xi'_x\big)(x,k) + \alpha\lambda_0\big((\partial_1 f)\xi\big)(x,k)\\&= (f\nabla_1\xi)(x,k) + \big((\partial_1 f)\xi\big)(x,k)\end{aligned}$$

since $\alpha\lambda_0 = 1$. Furthermore,

$$\begin{aligned}\nabla_2(f\xi)(x,k) &= \big(f(\nabla_2\xi)\big)(x,k) + \frac{\beta\varepsilon+\gamma r}{2\pi i}\big((\partial_2 f)\xi\big)(x,k)\\ &= \big(f(\nabla_2\xi)\big)(x,k) + \big((\partial_2 f)\xi\big)(x,k)\end{aligned}$$

since $\beta\varepsilon + \gamma r = 2\pi i$, showing that ∇ is a left module connection. □

Remark 4.11. By using the isomorphism $\phi : \mathcal{C}_\hbar^\infty \to \mathcal{E}_\hbar$ in (4.4) one may induce connections on the algebra itself via

$$\widetilde{\nabla}_k f = \phi^{-1}\big(\nabla_k \phi(f)\big)$$

for $f \in \mathcal{C}_\hbar^\infty$, giving

$$\widetilde{\nabla}_1 f = \partial_1 f$$

and

$$\widetilde{\nabla}_2 f = \left(\frac{\alpha\beta\hbar}{2\pi i} + 1\right)\partial_2 f + \alpha\beta\, uf = \partial_2 f + \alpha\beta f u\,,$$

using the expression (2.6) for ∂_2.

It is straightforward to compute the curvature of ∇ in (4.6):

$$F_{12}\xi(x,k) = (\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi(x,k) = \alpha\beta\,\xi(x,k)$$

showing that ∇ has a constant curvature equal to $\alpha\beta$. Note that the curvature does not depend on γ, which implies that one may construct connections with arbitrary constant curvature. Namely, let $R \in \mathbb{C}$ and set

$$\alpha = \frac{1}{\lambda_0} \qquad \beta = \lambda_0 R \qquad \gamma = \frac{2\pi i - \lambda_0\varepsilon R}{r}$$

clearly fulfilling the requirements of Proposition 4.10, giving a connection of constant curvature $\alpha\beta = R$.

Moreover, one easily shows that the curvature is $\mathcal{C}_\hbar^\infty$-linear, that is

$$F_{12}(f\xi(x,k)) = f(F_{12}\xi(x,k))$$

for $\xi \in \mathcal{E}_\hbar$ and $f \in \mathcal{C}_\hbar^\infty$, implying that the curvature F_{12} is an element of

$$\mathrm{End}_{\mathcal{C}_\hbar^\infty}(\mathcal{E}_\hbar) = \{T : \mathcal{E}_\hbar \to \mathcal{E}_\hbar) \mid T(f\xi(x,k)) = f(T\xi(x,k))\}$$

for $\xi \in \mathcal{E}_\hbar$ and $f \in \mathcal{C}_\hbar^\infty$; this is an algebra under composition.

Given a left connection on $\mathcal{E}_\hbar$, one can define associated derivations on $\mathrm{End}_{\mathcal{C}^\infty_\hbar}(\mathcal{E}_\hbar)$ from the commutators

$$\delta_k(T) = \nabla_k \circ T - T \circ \nabla_k \tag{4.7}$$

for $k = 1, 2$ and $T \in \mathrm{End}_{\mathcal{C}^\infty_\hbar}(\mathcal{E}_\hbar)$. If $\mathcal{E}_\hbar$ is also a right $\mathcal{C}^\infty_{\hbar'}$-module, then $\mathcal{C}^\infty_{\hbar'}$ (acting on the right) is a subalgebra of $\mathrm{End}_{\mathcal{C}^\infty_\hbar}(\mathcal{E}_\hbar)$. On the algebra $\mathcal{C}^\infty_{\hbar'}$, with the connection ∇ as in (4.6) one recovers a rescaling of the natural derivations from (4.7), as stated in the following result.

Proposition 4.12. *Let $\mathcal{E}_{\hbar,\hbar'}$ be a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule with and let ∇ be a left module connection as defined in* (4.6)*. Then on $f \in \mathcal{C}^\infty_{\hbar'}$ the derivations in* (4.7) *are given by*

$$\delta_1(f) = (\alpha\mu_0)\partial_1 f \qquad \delta_2(f) = \frac{\beta\varepsilon' + \gamma r'}{2\pi i}\partial_2 f.$$

Proof. With the right structure as in (4.2), one computes

$$\begin{aligned}(\xi\delta_1 f)(x,k) &= \nabla_1(\xi f)(x,k) - ((\nabla_1\xi)f)(x,k)\\ &= (\alpha\mu_0)\sum_{n\in\mathbb{Z}} f_n'(\mu_0 x + \mu_1 k - n\hbar')\xi(x - n\varepsilon', x - nr')\\ &= (\alpha\mu_0)(\xi\partial_1 f)(x,k),\end{aligned}$$

as well as

$$\begin{aligned}(\xi\delta_2 f)(x,k) &= \nabla_2(\xi f)(x,k) - ((\nabla_2\xi)f)(x,k)\\ &= (\beta\varepsilon' + \gamma r')\sum_{n\in\mathbb{Z}} n f_n(\mu_0 x + \mu_1 k - n\hbar')\xi(x - n\varepsilon', x - nr')\\ &= \frac{\beta\varepsilon' + \gamma r'}{2\pi i}(\xi\partial_2 f)(x,k),\end{aligned}$$

which establish the results. □

As a corollary we have the condition for ∇ to be a right module connection.

Proposition 4.13. *Let $\mathcal{E}_{\hbar,\hbar'}$ be a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule with and let ∇ be a left module connection as defined in* (4.6)*. If $\alpha = 1/\mu_0$ and $\beta\varepsilon' + \gamma r' = 2\pi i$ then ∇ is a right module connection; that is,*

$$\nabla_k(\xi f) = (\nabla_k\xi)f + \xi(\partial_k f)$$

for $k = 1, 2$, $\xi \in \mathcal{E}_{\hbar,\hbar'}$ and $f \in \mathcal{C}^\infty_{\hbar'}$.

Proof. With the restriction on the parameters one has $\delta_k(f) = \partial_k(f)$ for $k = 1, 2$ and $f \in \mathcal{C}^\infty_{\hbar'}$. With $T = f$ (acting on the right), (4.7) becomes

$$\xi(\partial_k f) = \nabla_k(\xi f) - (\nabla_k \xi) f,$$

showing that ∇ is indeed a (right) connection. □

In order for ∇ to be a bimodule connection on $\mathcal{E}_{\hbar,\hbar'}$ the parameters, apart from satisfying the requirements of a bimodule, need to satisfy the requirements of Proposition 4.10 and Proposition 4.13. Let us work out what this implies for the parameters.

Proposition 4.14. *For $\lambda_0 \neq 0$ and $\hbar, \hbar' > 0$, let $\mathcal{E}_{\hbar,\hbar'}$ be a $(\mathcal{C}^\infty_\hbar, \mathcal{C}^\infty_{\hbar'})$-bimodule with parameters $\lambda_0, \lambda_1, \varepsilon, r$ and $\lambda_0, \mu_1, \varepsilon', r'$ respectively, satisfying the requirements of Proposition 4.3, and assume that $\beta, \gamma \in \mathbb{R}$ are such that*

$$\beta\varepsilon + \gamma r = \beta\varepsilon' + \gamma r' = 2\pi i.$$

1) If $\hbar = \hbar'$ then $\beta = 0$, $r = r' \neq 0$, $\varepsilon \neq \varepsilon'$ and

$$\lambda_0 = \frac{\hbar}{\varepsilon' - \varepsilon} \quad \lambda_1 = -\frac{\hbar\varepsilon'}{r(\varepsilon' - \varepsilon)} \quad \mu_1 = -\frac{\hbar\varepsilon}{r(\varepsilon' - \varepsilon)} \quad \gamma = \frac{2\pi i}{r}.$$

2) If $\hbar \neq \hbar'$ then $\hbar/\hbar' = r/r' \in \mathbb{Q}$ and

$$\mu_1 = \frac{\hbar' - \hbar}{r' - r} + \lambda_1 \quad \varepsilon = -\frac{1}{\lambda_0}(\hbar + \lambda_1 r) \quad \varepsilon' = -\frac{\lambda_1 r'}{\lambda_0}$$
$$\beta = 2\pi i\lambda_0 \frac{\hbar - \hbar'}{\hbar\hbar'} \quad \gamma = \frac{2\pi i}{r'} + 2\pi i\lambda_1 \frac{\hbar - \hbar'}{\hbar\hbar'}$$

Proof. The requirements on the parameters can be summarized as:

$$\lambda_0\varepsilon + \lambda_1 r = -\hbar \tag{4.8}$$
$$\lambda_0\varepsilon' + \lambda_1 r' = 0 \tag{4.9}$$
$$\lambda_0\varepsilon' + \mu_1 r' = \hbar' \tag{4.10}$$
$$\lambda_0\varepsilon + \mu_1 r = 0 \tag{4.11}$$
$$\beta\varepsilon + \gamma r = \beta\varepsilon' + \gamma r' = 2\pi i, \tag{4.12}$$

and we note that (4.8) and (4.11) imply $(\lambda_1 - \mu_1)r = -\hbar$, and (4.9) and (4.10) imply that $(\lambda_1 - \mu_1)r' = -\hbar'$, giving

$$(\lambda_1 - \mu_1)(r - r') = \hbar' - \hbar. \tag{4.13}$$

Let us first consider the case when $\hbar = \hbar'$, which implies by (4.13) that $\lambda_1 = \mu_1$ or $r = r'$. However, if $\lambda_1 = \mu_1$ then (4.8) and (4.11) imply that $\hbar = 0$, which contradicts the assumption that $\hbar > 0$. Hence, one must have that $r = r'$. Next, we note that $r = r' \neq 0$ since otherwise (4.8) and (4.11) imply that $\hbar = 0$. The same kind argument shows that $\varepsilon' \neq \varepsilon$ if $r = r'$ (otherwise (4.8) and (4.9) imply that $\hbar = 0$). Thus, the following equations remain to be solved:

$$\begin{aligned}
&\lambda_0\varepsilon + \lambda_1 r = -\hbar\\
&\lambda_0\varepsilon' + \lambda_1 r = 0\\
&\lambda_0\varepsilon' + \mu_1 r = \hbar\\
&\lambda_0\varepsilon + \mu_1 r = 0\\
&\beta\varepsilon + \gamma r = \beta\varepsilon' + \gamma r = 2\pi i.
\end{aligned}$$

The last equation implies that $\beta(\varepsilon' - \varepsilon) = 0$ which gives $\beta = 0$ since $\varepsilon \neq \varepsilon'$. Solving the above system for the variables $\lambda_0, \lambda_1, \mu_1, \gamma$ immediately gives the claimed result.

Next, let us consider the case when $\hbar \neq \hbar'$. Equation (4.13) then implies that $\lambda_1 \neq \mu_1$ and $r \neq r'$, giving

$$\mu_1 = \frac{\hbar' - \hbar}{r - r'} + \lambda_1.$$

Equations (4.8) and (4.11) imply that $\mu_1 r - \lambda_1 r = \hbar$ which, by inserting the above expression, gives

$$\frac{\hbar'}{\hbar} = \frac{r'}{r} \tag{4.14}$$

which in particular implies that $\hbar'\hbar \in \mathbb{Q}$. With $\mu_1 = (\hbar' - \hbar)/(r - r') + \lambda_1$ and $\hbar'/\hbar = r'/r$, (4.8)–(4.12) are equivalent to

$$\begin{aligned}
&\lambda_0\varepsilon + \lambda_1 r = -\hbar\\
&\lambda_0\varepsilon' + \lambda_1 r' = 0\\
&\beta\varepsilon + \gamma r = \beta\varepsilon' + \gamma r' = 2\pi i.
\end{aligned}$$

Solving the first to equations for $\varepsilon, \varepsilon'$ gives

$$\varepsilon = -\frac{1}{\lambda_0}\big(\hbar + \lambda_1 r\big) \quad \text{and} \quad \varepsilon' = -\frac{\lambda_1 r'}{\lambda_0}$$

which, furthermore, gives

$$\beta(\varepsilon - \varepsilon') = \gamma(r' - r) \quad \Rightarrow \quad \gamma = \frac{\beta}{\lambda_0}\left(\lambda_1 - \frac{\hbar}{r' - r}\right).$$

Inserting the expression for γ into $\beta\varepsilon + \gamma r = 2\pi i$ gives

$$\beta = 2\pi i \frac{\lambda_0(r - r')}{\hbar r'} = 2\pi i \lambda_0 \frac{r/r' - 1}{\hbar} = 2\pi i \lambda_0 \frac{\hbar - \hbar'}{\hbar\hbar'}$$

(using $\hbar'/\hbar = r'/r$) and finally

$$\gamma = \frac{2\pi i}{r'} + 2\pi i \lambda_1 \frac{\hbar - \hbar'}{\hbar\hbar'},$$

completing the proof. □

Thus, if $\hbar = \hbar'$ then $\beta = 0$ which implies that a $(\mathcal{C}_\hbar^\infty, \mathcal{C}_\hbar^\infty)$-bimodule connection of the type (4.6) has zero curvature

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi = \alpha\beta\,\xi = 0.$$

In the situation when $\hbar \neq \hbar'$, Proposition 4.14 shows that in order for such a bimodule connection to exist, the ratio between $\hbar$ and $\hbar'$ has to be rational (note that, they need not themselves be rational). Moreover, in this case one may always find module parameters guaranteeing that ∇ is a bimodule connection. The curvature then becomes

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi = \alpha\beta\,\xi = 2\pi i\,\frac{\hbar - \hbar'}{\hbar\hbar'}\,\xi. \tag{4.15}$$

It is noteworthy that the curvature is independent of the particular parameters defining the bimodule structure, since it only depends on $\hbar$ and $\hbar'$. Moreover, since $\hbar/\hbar'$ is rational from (4.14), the curvature is a rational multiple of $2\pi i$.

5. A pseudo-Riemannian calculus

In [AW17a, AW17b] the concept of *pseudo-Riemannian calculi* was introduced in order to discuss Levi-Civita connections on vector bundles over noncommutative manifolds. In this particular setting, one can prove that there exists at most one metric and torsion-free connection on a vector bundle which is equipped with a soldering map; that is a linear map which maps derivations into sections of the bundle. In this section, we shall construct a pseudo-Riemannian calculus for the noncommutative cylinder and explicitly compute the Levi-Civita connection as well as the corresponding curvature.

We first recall a few definitions. For the time being, we assume that $\mathcal{A}$ is an arbitrary $*$-algebra.

Definition 5.1. Let M be a (right) $\mathcal{A}$-module and let h be a non-degenerate $\mathcal{A}$-valued hermitian form on M. Furthermore, let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a real Lie algebra of hermitian derivations of $\mathcal{A}$ and let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map. The data $(M, h, \mathfrak{g}, \varphi)$ is called a *real metric calculus over* $\mathcal{A}$ if

1) the image $M_\varphi = \varphi(\mathfrak{g})$ generates M as an $\mathcal{A}$-module,
2) $h(E, E')^* = h(E, E')$ for all $E, E' \in M_\varphi$.

An affine connection ∇ on $(M, \mathfrak{g})$ is a map $\nabla : \mathfrak{g} \times M \to M$ such that (with the notation $\nabla(\partial, U) = \nabla_\partial(U)$), one has

1) $\nabla_\partial(U + V) = \nabla_\partial U + \nabla_\partial V$,
2) $\nabla_{\lambda\partial+\partial'} U = \lambda\nabla_\partial U + \nabla_{\partial'} U$,
3) $\nabla_\partial(Ua) = \big(\nabla_\partial U\big)a + U\partial(a)$,

for all $U, V \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{R}$.

Definition 5.2. Let $(M, h, \mathfrak{g}, \varphi)$ be a real metric calculus and let ∇ denote an affine connection on $(M, \mathfrak{g})$. If

$$h(\nabla_\partial E, E') = h(\nabla_\partial E, E')^*$$

for all $E, E' \in M_\varphi$ and $\partial \in \mathfrak{g}$ then $(M, h, \mathfrak{g}, \varphi, \nabla)$ is called a *real connection calculus.*

Definition 5.3. Let $(M, h, \mathfrak{g}, \varphi, \nabla)$ be a real connection calculus over M. The calculus is *metric* if

$$\partial\big(h(U, V)\big) = h\big(\nabla_\partial U, V\big) + h\big(U, \nabla_\partial V\big)$$

for all $\partial \in \mathfrak{g}$, $U, V \in M$, and *torsion-free* if

$$\nabla_\partial\varphi(\partial') - \nabla_{\partial'}\varphi(\partial) - \varphi\big([\partial, \partial']\big) = 0$$

for all $\partial, \partial' \in \mathfrak{g}$. A metric and torsion-free real connection calculus over $\mathcal{A}$ is called a *pseudo-Riemannian calculus over* $\mathcal{A}$.

Within this framework, the uniqueness of a metric and torsion-free connection can be stated in the following way.

Theorem 5.4 ([AW17b]). *Let $(M, h, \mathfrak{g}, \varphi)$ be a real metric calculus over $\mathcal{A}$. Then there exists at most one affine connection ∇ on $(M, \mathfrak{g})$, such that $(M, h, \mathfrak{g}, \varphi, \nabla)$ is a pseudo-Riemannian calculus.*

Let us now return to the noncommutative cylinder. The algebra $\mathcal{C}_\hbar^\infty$ consists of smooth functions on $\mathbb{R} \times S^1$ that fall off rapidly at infinity. Of course, there are many more smooth functions on $\mathbb{R} \times S^1$ and in this section we allow ourselves to consider a different algebra $\widehat{\mathcal{C}}_\hbar^\infty$ consisting of elements

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) e^{2\pi i n t}$$

where $f_n \in C^\infty(\mathbb{R})$ is such that $f_n \neq 0$ only for a finite number of terms. The product is defined as before as

$$(f \bullet_\hbar g)(u,t) = \sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} f_k(u) g_{n-k}(u + k\hbar) \right] e^{2\pi i n t}$$

and the $*$-algebra structure is again given as in Proposition 2.4. Note that whenever $f, g \in \mathcal{C}_\hbar^\infty \cap \widehat{\mathcal{C}}_\hbar^\infty$ the products of the two algebras coincide (as well as the involution). In what follows, we shall construct a pseudo-Riemannian calculus over $\widehat{\mathcal{C}}_\hbar^\infty$.

Let $\mathfrak{g}$ denote the real (abelian) Lie algebra generated by the hermitian derivations ∂_1, ∂_2, and let $T\widehat{\mathcal{C}}_\hbar^\infty$ denote the free (right) $\widehat{\mathcal{C}}_\hbar^\infty$-module of rank 2, with basis e_1, e_2. Moreover, we introduce the hermitian form $h : T\widehat{\mathcal{C}}_\hbar^\infty \times T\widehat{\mathcal{C}}_\hbar^\infty \to \widehat{\mathcal{C}}_\hbar^\infty$

$$h(X,Y) = (X^i)^* h_{ij} Y^j$$

for $X = X^i e_i$ and $Y = Y^i e_i$ and $h_{ij} \in \widehat{\mathcal{C}}_\hbar^\infty$ such that $h_{ij}^* = h_{ji} = h_{ij}$. (Here and in the following we sum over up-down repeated indices from 1 to 2.) We shall assume that h is invertible in the sense that there exists $h^{ij} \in \widehat{\mathcal{C}}_\hbar^\infty$ such that $h^{ij} h_{jk} = h_{kj} h^{ji} = \delta^i_k$. (For instance, one might choose $h_{ij} = e^{2k(u)} \delta_{ij}$ for arbitrary (real) $k(u) \in \widehat{\mathcal{C}}_\hbar^\infty$.) Clearly, h is nondegenerate in the sense that $h(X,Y) = 0$ for all $Y \in T\widehat{\mathcal{C}}_\hbar^\infty$ implies that $X = 0$.
Finally, we define $\varphi : \mathfrak{g} \to T\widehat{\mathcal{C}}_\hbar^\infty$ as $\varphi(\partial_k) = e_k$ (for $k = 1, 2$) extended linearly to all of $\mathfrak{g}$. It is immediate that the image of φ generate $T\widehat{\mathcal{C}}_\hbar^\infty$ and that $h(E, E')$ is hermitian for all E, E' in the image of φ. These considerations

imply that $(T\widehat{\mathcal{C}}^{\infty}_{\hbar}, h, \mathfrak{g}, \varphi)$ is a real metric calculus over $\widehat{\mathcal{C}}^{\infty}_{\hbar}$. The (unique) Levi-Civita connection can be computed via Koszul's formula (cf. [AW17a, AW17b]) which, since $[\partial_i, \partial_j] = 0$, becomes

$$h(\nabla_i e_j, e_k) = \frac{1}{2}\big(\partial_i h(e_j, e_k) + \partial_j h(e_k, e_i) - \partial_k h(e_i, e_j)\big),$$

where $\nabla_i = \nabla_{\partial_i}$. Writing $\nabla_i e_j = e_l \Gamma^l_{ij}$ gives

$$(\Gamma^l_{ij})^* h_{lk} = \frac{1}{2}\big(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij}\big),$$

and one finds that

$$\Gamma^l_{ij} = \frac{1}{2} h^{lk}\big(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij}\big).$$

Let us exemplify this construction for $h_{ij} = e^{2k(u)}\delta_{ij}$. In this case, one obtains

$$\begin{aligned}\Gamma^1_{11} &= \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = k'(u)\\ \Gamma^2_{11} &= \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = 0\end{aligned}$$

giving

$$\nabla_1 e_1 = e_1 k'(u) \qquad \nabla_1 e_2 = \nabla_2 e_1 = e_2 k'(u) \qquad \nabla_2 e_2 = -e_1 k'(u).$$

We immediately note that

$$h(\nabla_i e_j, \nabla_k e_l)^* = h(\nabla_i e_j, \nabla_k e_l)$$

since both $\nabla_i e_j$ and h_{ij} depend only on u. This implies that the curvature of the connection will have all the classical symmetries [AW17b]; hence, there is only one independent component of the curvature, and one finds

$$\begin{aligned}R(\partial_1, \partial_2)e_1 &= \nabla_1\nabla_2 e_1 - \nabla_2\nabla_1 e_1 = e_2 k''(u)\\ R(\partial_1, \partial_2)e_2 &= \nabla_1\nabla_2 e_2 - \nabla_2\nabla_1 e_2 = -e_1 k''(u)\\ R_{1212} &= h\big(e_1, R(\partial_1, \partial_2)e_2\big) = -e^{2k(u)} k''(u),\end{aligned}$$

giving the Gaussian curvature as

$$K = \frac{1}{2} h^{ij} R_{ikjl} h^{kl} = -e^{-2k(u)} k''(u).$$

For a metric of the above form, a natural integration measure corresponding to the volume form is given by $\tau_h(f) = \tau(f e^{2k(u)})$. For the sake of illustration, let us compute the total curvature (when it exists)

$$\tau_h(K) = -\int_{-\infty}^{\infty} e^{-2k(u)} k''(u) e^{2k(u)} du = -\int_{-\infty}^{\infty} k''(u) du$$
$$= \lim_{u\to-\infty} k'(u) - \lim_{u\to\infty} k'(u)$$

Here one notes a certain independence of the total curvature with respect to perturbations of the metric; i.e. for $\tilde{k}(u) = \delta(u) + k(u)$ one finds that $\tau_h(\tilde{K}) = \tau_h(K)$ whenever

$$\lim_{u\to\infty} \delta'(u) = \lim_{u\to-\infty} \delta'(u).$$

For instance, for $k(u) = \ln(\cosh(u))$, corresponding to the induced metric on the catenoid (cf. [AH18]), one obtains

$$\tau_h(K) = \lim_{u\to-\infty} \tanh(u) - \lim_{u\to\infty} \tanh(u) = -2$$

which, in the geometrical situation where the trace naturally gains a factor of 2π (from the integration along S^1), gives the expected value of -4π for the total curvature of the catenoid.

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