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A proposal for nonabelian (0,2) mirrors

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In this paper we give a proposal for mirrors to (0,2) supersymmetric gauged linear sigma models (GLSMs), for those (0,2) GLSMs which are deformations of (2,2) GLSMs. Specifically, we propose a construction of (0,2) mirrors for (0,2) GLSMs with E terms that are linear and diagonal, reducing to both the Hori-Vafa prescription as well as a recent (2,2) nonabelian mirrors proposal on the (2,2) locus. For the special case of abelian (0,2) GLSMs, two of the authors have previously proposed a systematic construction, which is both simplified and generalized by the proposal here.

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1. Introduction

One of the outstanding problems in heterotic string compactifications is to understand nonperturbative effects due to worldsheet instantons. For type II strings and (2,2) worldsheet theories, these effects are well-understood, and are encoded in quantum cohomology rings and Gromov-Witten theory. In principle, there are analogues of both for more general heterotic theories, but there are comparatively many open questions.

For example, in a heterotic $E_8 \times E_8$ compactification on a Calabi-Yau threefold with a rank three bundle, the low-energy theory contains states in the **27** and $\overline{\mathbf{27}}$ representations of E_6, with cubic couplings appearing as spacetime superpotential terms. On the (2,2) locus (the standard embedding, where the gauge bundle equals the tangent bundle), for the case of the quintic threefold, those couplings have the standard form [1, 2]

$$\overline{\mathbf{27}}^3 \;=\; 5 \;+\; \sum_{k=1}^{\infty} n_k \frac{k^3 q^k}{1-q^k} \;=\; 5 \;+\; 2875\, q \;+\; 4876875\, q^2 \;+\; \cdots\,, \tag{1.1}$$

where the n_k encode the Gromov-Witten invariants. These are computed by three-point functions in the A model topological field theory on the worldsheet. Off the (2,2) locus, for more general gauge bundles, these couplings have a closely analogous form: a classical contribution plus a sum of nonperturbative contributions, without any perturbative loop corrections [3–8]. As a result, we know that more general heterotic versions of the Gromov-Witten invariants exist, and from general holomorphy arguments, must be nontrivial.

In principle, off the (2,2) locus, heterotic Yukawa couplings such as those in equation (1.1) are computed by the A/2 and B/2 models, which on the (2,2) locus reduce to the ordinary A and B model topological field theories. For e.g. Fano spaces, both these heterotic Gromov-Witten invariants[1] as

[1]For three-point functions on S^2, only. More precisely, correlation function computations are understood in (analogues of) topological field theories, but correlation function computations in analogues of topological string theories – with the

well as heterotic versions of quantum cohomology rings (here, a quantum-corrected ring of sheaf cohomology groups [9] of the form $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$, introduced in [10–13]) are known for toric varieties [14–18], Grassmannians $G(k,n)$ [19, 20], and flag manifolds [21], all for the case that the gauge bundle is a deformation of the tangent bundle. (Cases involving more general gauge bundles are not currently understood.) See for example [22–25] for reviews.

For Calabi-Yau spaces, one can compute many correlation functions; however, it is not yet known how to extract the precise analogues of Gromov-Witten invariants from these computations, as one needs, for example, a heterotic analogue of the Picard-Fuchs equations to get a precise mirror map and vacuum normalization. Similarly, more recent methods applying supersymmetric localization [26] are also not applicable.

Historically, Gromov-Witten invariants in (2,2) supersymmetric theories were first computed using mirror symmetry, and so one might hope that a (0,2) supersymmetric version of mirror symmetry might aid in such developments. This is one of the motivations to understand (0,2) mirrors (see e.g. [27, 28] for some early work).

To date, there has been significant progress on understanding (0,2) mirror symmetry, but many results are still limited (and certainly heterotic Gromov-Witten invariants for Calabi-Yau's are not yet known). For example, for the case of reflexively-plain polytopes, and bundles that are deformations of the tangent bundle, a generalization of the Batyrev construction of ordinary Calabi-Yau mirrors exists, see [29–31].

In this paper, we shall propose what is ultimately a (0,2) analogue of the Hori-Vafa construction [32, 33], which is to say, a mirror construction for two-dimensional gauge theories, resulting in a Landau-Ginzburg model, which in our case will be defined for a special class of (0,2) deformations off of the (2,2) locus. For abelian theories, there has been nontrivial work in this area in the past [34–37]. This work has included ansatzes for various special cases of toric varieties [35, 36], as well as a more systematic proposal for abelian theories [37]. The proposal in this paper will both extend such constructions to nonabelian[2] (0,2) GLSMs, as well as give a simpler, more straightforward, presentation in abelian cases than that in [37]. We do not claim to have a proof of the construction, but we do show that the proposal satisfies a

exception of those that reduce to topological field theory computations – are still unknown.

[2]Specifically, on the (2,2) locus, this will reduce to the nonabelian mirrors proposal described in [38–40]. Other proposals have appeared in the math community in e.g. [41, 42], as reviewed in [38][section 4.9, appendix A].

number of general consistency tests consistent with (2,2)-supersymmetric gauge theory mirrors, for example:

- axial anomalies of the original theory are realized by classically-broken symmetries of the mirror, and can be restored by a shift of the mirror to the theta angle,
- quantum sheaf cohomology ring relations of the original gauge theory are realized classically in the mirror by critical locus relations,
- correlation functions match,
- integrating out matter fields from the mirror returns the one-loop effective superpotential of the original gauge theory on the Coulomb branch,

all just as happens in (2,2)-supersymmetric gauge theory mirrors [32, 38]. We also check the mirror construction in several concrete examples.

We begin in section 2 by reviewing mirrors to (2,2)-supersymmetric gauge theories, both abelian [32] and nonabelian [38]. We discuss the mirror constructions themselves and expected properties of mirrors to two-dimensional gauge theories.

In section 3 we describe our proposal for (0,2) mirrors for a special class of (0,2) deformations off the (2,2) locus. As many subtleties of nonabelian mirrors have already been extensively discussed in [38–40], here we focus solely on the novel aspects introduced by (0,2) supersymmetry. We also discuss mirrors to symmetries and their anomalies, and check that the quantum sheaf cohomology ring relations of the original theory are correctly reproduced in the mirror by classical critical locus relations.

In section 4, we give further general arguments checking this proposal. Specifically, we check that correlation functions match, and we demonstrate that integrating-out matter fields correctly reproduces the one-loop effective superpotential on the Coulomb branch, exactly as happens in (2,2)-supersymmetric gauge theory mirrors. (We do not, however, claim to have a proof.)

In section 5, we specialize to abelian theories. In particular, the ansatz here simplifies and generalizes the ansatz two of the authors previously discussed in [37].

In section 6 we compare to the previous systematic proposal for (0,2) mirrors to abelian theories by two of the authors [37]. The ansatz presented here is both more general and rather simpler, and we also argue that when we restrict to (0,2) deformations of the form considered in [37], our current proposal gives the same results as [37].

In the next several sections, we discuss concrete examples. We begin in section 7 by giving a detailed analysis of mirrors to $\mathbb{P}^n \times \mathbb{P}^m$. We verify correlation functions in the original theory, construct lower-energy Landau-Ginzburg theories in the style of (2,2) Toda duals to projective spaces, discussing subtleties that arise in their construction, explicitly verify correlation functions in those lower-energy theories, and also compare to previous (0,2) mirrors for these spaces in [35].

In section 8 we perform analogous analyses for (0,2) mirrors to Hirzebruch surfaces, constructing lower-energy theories and comparing to results in [36].

In section 9, we discuss our first nonabelian examples, GLSMs for (0,2) deformations of Grassmannians $G(k, N)$. These are two-dimensional $U(k)$ gauge theory with matter in copies of the fundamental representation. We construct lower-energy Landau-Ginzburg models, analogues of (2,2) Toda duals, that generalize the Grassmannian mirrors discussed in [38], and explicitly verify that quantum sheaf cohomology rings [19, 20] are reproduced. We also explicitly verify that correlation functions are correctly reproduced in a few tractable examples.

In section 10 we briefly discuss (0,2) deformations of flag manifolds, generalizations of Grassmannians that are also described by two-dimensional nonabelian gauge theories. We verify that quantum sheaf cohomology rings [21] are reproduced.

Finally, in section 11, we briefly discuss (0,2) mirrors to theories with hypersurfaces. The rest of the paper is concerned with mirrors to theories without a (0,2) superpotential; in this section, we discuss how the result is modified to take into account a (0,2) superpotential, and also discuss how the mirror ansatz reproduces some conjectures regarding hypersurface mirrors in [15].

2. Review of mirrors to (2,2) supersymmetric gauge theories

In this section we shall review results of [32, 38] on mirror symmetry for two-dimensional (2,2) supersymmetric gauge theories.

Briefly, in these papers, the mirror to a two-dimensional gauge theory is given as a Landau-Ginzburg model, whose classical physics encodes the quantum physics of the original gauge theory.

For abelian two-dimensional (2,2) supersymmetric gauge theories, mirrors were described in [32]. For a $U(1)^k$ gauge theory with n chiral superfields with charges encoded in charge matrix ρ_i^a ($a \in \{1, \cdots, k\}$, $i \in \{1, \cdots n\}$, and Fayet-Iliopoulos parameters t_a, the mirror Landau-Ginzburg model is described by twisted chiral multiplets σ_a, Y_i, and the superpotential

$$W = \sum_{a=1}^{k} \sigma_a \left(\sum_{i=1}^{n} \rho_i^a Y_i - t_a \right) + \sum_{i=1}^{n} \exp(-Y_i) . \tag{2.1}$$

Operators in the mirror and the original gauge theory are related by the operator mirror map

$$\exp(-Y_i) \leftrightarrow \sum_a \rho_i^a \sigma_a, \tag{2.2}$$

derived from the superpotential above (see e.g. [37, 38] for details).

This Landau-Ginzburg model is mirror in the sense that classical computations in the B-twisted Landau-Ginzburg model reproduce quantum computations in the A-twist of the gauge theory. For one example, the axial $U(1)_A$ anomaly of the original gauge theory appears as a classical obstruction to the existence of the corresponding symmetry in the mirror theory, specifically

$$Y_i \mapsto Y_i - 2i\alpha, \quad \sigma_a \mapsto \sigma_a \exp(+2i\alpha) , \tag{2.3}$$

where α parametrizes the symmetry, at the same time that the superspace coordinates θ get phase factors. This symmetry has a classical obstruction which can be resolved if one shifts the θ angle. For another example, the quantum cohomology ring relations of the original gauge theory are encoded in the classical critical locus of the mirror Landau-Ginzburg model. For a third example, integrating out the Y fields returns the twisted one-loop effective superpotential of the original A-twisted gauge theory. More systematically, all correlation functions of the original gauge theory, including quantum effects, are reproduced from classical contributions to correlation functions in the mirror B-twisted Landau-Ginzburg model. In fact, more can be said – for example, open string sectors mirror in the expected fashion – but in this paper we focus on computations that have heterotic analogues.

We have only described the mirror in the case that the original gauge theory has no superpotential, but this description is straightforward to modify in the presence of a superpotential. Specifically, if the original theory has a superpotential, then some of the chiral superfields in the original gauge theory have nonzero R-charges. In such a case, we take the corresponding fundamental field in the mirror to be not Y but instead $\exp(-(r/2)Y)$, where

r denotes the r charge, and the mirror also has a $\mathbb{Z}_{2/r}$ orbifold acting by phases on that field. (Twistability of the original theory restricts allowed R-charges to $r \in \{0, 1, 2\}$, as discussed in [38, section 2].)

The nonabelian extension proposed in [38] followed the same pattern, proposing a (B-twisted) Landau-Ginzburg mirror to (nonabelian) A-twisted two-dimensional (2,2) supersymmetric gauge theories, in which quantum effects in the A-twisted theory are realized classically in the B-twisted mirror, which reduces to [32] for abelian gauge theories. Briefly, for a G-gauge theory with n chiral superfields in some (typically reducible) representation of G, and Fayet-Iliopoulos parameters t_a, $a \in \{1, \cdots, \operatorname{rank} G\}$, the mirror Landau-Ginzburg model is defined by (a Weyl-group orbifold of) twisted chiral superfields σ_a, Y_i ($i \in \{1, \cdots, n\}$), and $X_{\tilde{\mu}}$, the latter corresponding to nonzero roots of the Lie algebra $\mathfrak{g}$ of the gauge group G, and a superpotential

$$\begin{aligned} W \;=\; & \sum_{a=1}^{\operatorname{rank} G} \sigma_a \left(\sum_{i=1}^{n} \rho_i^a Y_i \;-\; \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} \;-\; t_a \right) \\ & + \sum_{i=1}^{n} \exp\left(-Y_i\right) \;+\; \sum_{\tilde{\mu}} X_{\tilde{\mu}}. \end{aligned} \tag{2.4}$$

Operators in the mirror and the original gauge theory are related by the operator mirror map

$$\exp\left(-Y_i\right) \;\leftrightarrow\; \sum_a \rho_i^a \sigma_a, \quad X_{\tilde{\mu}} \;\leftrightarrow\; \sum_a \alpha_{\tilde{\mu}}^a \sigma_a, \tag{2.5}$$

derived from the superpotential above (see e.g. [37, 38] for details). The new ingredients, relative to the abelian case, are the fields $X_{\tilde{\mu}}$, corresponding to nonzero roots of the Lie algebra $\mathfrak{g}$ of G, and the Weyl orbifold.

This proposal necessarily possesses all the same properties as the Hori-Vafa proposal, as well as some new ones. For one example, the axial anomaly of the original gauge theory is realized in the mirror again as an obstruction to a classical symmetry, specifically

$$Y_i \;\mapsto\; Y_i - 2i\alpha, \quad X_{\tilde{\mu}} \;\mapsto\; X_{\tilde{\mu}} \exp\left(+2i\alpha\right), \quad \sigma_a \;\mapsto\; \sigma_a \exp\left(+2i\alpha\right), \tag{2.6}$$

where α parametrizes the symmetry, and the superspace coordinates θ get phase factors. The classical obstruction to this symmetry can be cured with a theta angle shift, just as in the abelian case. For another example, quantum cohomology ring relations as well as Coulomb branch relations (analogues of quantum cohomology relations in cases where the Higgs branch has no weak-coupling limit, because of no continuously-variable Fayet-Iliopoulos parameter) arising from quantum corrections in the original gauge theory are realized classically in the mirror as critical locus relations. For a third example, integrating out the X and Y fields in the mirror reproduces the twisted one-loop effective superpotential of the original gauge theory. More systematically, all correlation functions of the original gauge theory, including quantum effects, are reproduced from classical contributions to correlation functions in the mirror B-twisted Landau-Ginzburg model.

Nonabelian cases also possess a few additional properties. For one example, Coulomb branch vacua in a nonabelian two-dimensional gauge theory are partly defined by 'excluded loci,' constraining the σ fields. For example, in a $U(k)$ gauge theory with fundamental-valued matter, for $a \neq b$, $\sigma_a \neq \sigma_b$. (One way to understand this is from supersymmetric localization, where the integration measure has a factor proportional to $(\sigma_a - \sigma_b)^2$, which removes contributions from coincident pairs of σ's.) One of the challenges in finding a nonabelian mirror, one of the constraints on a possible mirror, is to reproduce that excluded locus in the classical physics of the B-twisted Landau-Ginzburg model. Now, realizing a closed condition, such as a restriction to a subvariety, is relatively straightforward, following the pattern described in [43]. The excluded locus condition above, however, is an open condition, defining an open subset of the Coulomb branch. In the mirror proposal in [38], the excluded locus condition is mirror to poles in the mirror superpotential. For example, in the case of a $U(k)$ gauge theory with fundamentals, the mirror theory has a field $X_{\mu\nu}$ which is mirror to the difference $\sigma_\mu - \sigma_\nu$, and the superpotential has a pole where $X_{\mu\nu} = 0$, implying that σ_μ must be distinct from σ_ν. In more general cases, the excluded loci can be considerably more intricate, and one of the checks performed in [38] was to verify that the classical physics of the mirror did correctly reproduce those excluded loci.

Numerous other consistency tests have also been performed. For example, in the case of the two-dimensional gauge theory describing Grassmannians $G(k, n)$, integrating out the X fields reproduces a proposal of [32] for the mirror to a Grassmannian. In [32], the proposal had factors of the form

$$\prod_{a<b} (\sigma_a - \sigma_b)^2 \tag{2.7}$$

in the integration measure, whose possible origin in a local field theory was rather unclear, but becomes much more clear in the mirror of [38].

3. Proposal for (0,2) supersymmetric gauge theories

In this section, we will describe our ansatz for mirrors to (0,2) supersymmetric[3] GLSMs which are deformations of (2,2) supersymmetric GLSMs, relating the (0,2) supersymmetric analogue of the A-twist of the original gauge theory (known[4] as the A/2-twist) to the (0,2) supersymmetric analogue of the B-twist (known as the B/2-twist) of the mirror Landau-Ginzburg model. Our ansatz will apply to both abelian and nonabelian theories obtained as deformations of (2,2) supersymmetric theories, but with a restriction on the allowed deformations, a restriction on the form of the functions $E = \overline{D}_+ \Psi$, which we shall describe in a moment.

Our ansatz will follow the same pattern and have the same basic properties as the other gauge theory mirrors discussed in the previous section. For example, it will have the same symmetries, realizing classically any anomalies of the original theory, as we shall see later in this section. For another example, quantum sheaf cohomology ring relations arising from quantum corrections in the original (0,2) supersymmetric gauge theory are realized classically in the mirror as critical locus relations, just as in the (2,2) supersymmetric models, as we shall see explicitly later in this section. For a third example, integrating out the X and Y fields in the mirror reproduces the twisted one-loop effective superpotential of the original gauge theory, just as in (2,2) supersymmetric theories, as we discuss in section 4.1. More systematically, all (topological) correlation functions of the original gauge theory, including quantum effects, are reproduced from classical contributions to correlation functions in the mirror B/2-twisted Landau-Ginzburg model, just as in (2,2) supersymmetric theories, as we discuss in section 4.2.

For simplicity, in this section we will assume the original gauge theory has no superpotential, and will discuss mirrors to theories with (0,2) superpotentials in section 11. We do not claim a physical proof of this proposal, though in later sections we will provide numerous consistency tests.

We will consider (0,2) theories that are deformations of (2,2) theories. Now, (2,2) supersymmetric multiplets are equivalent to pairs of (0,2) supersymmetric multiplets. For example, a (2,2) supersymmetric chiral superfield Φ is equivalent to a pair of (0,2) supersymmetric multiplets:

- a (0,2) supersymmetric chiral multiplet Φ,

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[3]For introductions to (0,2) GLSMs and (0,2) Landau-Ginzburg models, we recommend [44, 45].

[4]For an introduction to the A/2 and B/2 twists, we refer the reader to e.g. [10, 24, 25].

- a (0,2) supersymmetric Fermi multiplet Ψ, with $\overline{D}_+\Psi$ a holomorphic function of chiral superfields.

On the (2,2) locus, $\overline{D}_+\Psi$ is uniquely specified.

We will consider (0,2) deformations encoded in $\overline{D}_+\Psi$, deforming this function to a more general holomorphic function of the chiral superfields (and breaking (2,2) supersymmetry to (0,2)). Specifically, we consider deformations obeying the following two constraints:

- We assume that $\overline{D}_+\Psi$ is linear in chiral superfields, rather than a more general holomorphic function of chiral superfields. This may sound very restrictive, but in fact, it has been argued that only linear terms contribute to A/2-twisted GLSMs[5] – nonlinear terms are irrelevant. (This was conjectured in [15][section 3.5], [46][section A.3], and rigorously proven in [16, 17] for abelian GLSMs. It also is a consequence of supersymmetric localization [18], and see in addition [47][appendix A].)
- We assume that $\overline{D}_+\Psi$ is also diagonal, meaning, for theories which are deformations of (2,2) theories, that for any Fermi superfield Ψ, $\overline{D}_+\Psi$ is proportional to the chiral superfield with which it is partnered on the (2,2) locus.

On the (2,2) locus, the $\overline{D}_+\Psi$ are both linear and diagonal, and there exist nontrivial (0,2) deformations which are also linear and diagonal. The constraints above, that $\overline{D}_+\Psi$ be both linear and diagonal, imply the form

$$\overline{D}_+\Psi_i = E_i(\sigma)\Phi_i. \tag{3.1}$$

This form is not the most general possible (0,2) deformation, but nevertheless still allows for nontrivial deformations, and in any event is the most general possible deformation for which we have been able to find a mirror construction that obeys all consistency constraints.

Now that we have stated the restrictions, we give the proposal. Let us consider a (0,2) GLSM with connected[6] gauge group G of dimension n and rank r, chiral fields Φ_i and Fermi fields Ψ_i in a (possibly reducible)

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[5]For A/2-twisted nonlinear sigma models, this story is not settled, not least because we know of no simple way to distinguish the UV linear from UV nonlinear deformations in the IR.

[6]It is very straightforward to extend this proposal to $O(k)$ gauge theories in the same fashion as the (2,2) case, discussed in [40], but we shall not discuss any examples of $O(k)$ (0,2) mirrors in this paper.

representation R for $i = 1, \cdots, N = \dim R$. If $\mathcal{W}$ is the Weyl group of G, then the proposed mirror theory is a $\mathcal{W}$-orbifold of a (0,2) Landau-Ginzburg model given by the following matter fields:

- r chiral fields σ_a and r Fermi fields Υ_a, $a = 1, \cdots, r$,
- chiral fields Y_i and Fermi fields F_i where $i = 1, \cdots, N$,
- $n - r$ chiral fields $X_{\tilde{\mu}}$ and $n - r$ Fermi fields $\Lambda_{\tilde{\mu}}$,

following the same pattern as the (2,2) nonabelian mirror proposal [38].

For linear and diagonal $\overline{D}_+ \Psi$ as above, the proposed (0,2) superpotential of the mirror Landau-Ginzburg orbifold is

$$\begin{aligned} W = & \sum_{a=1}^{r} \Upsilon_a \left(\sum_{i=1}^{N} \rho_i^a Y_i \; - \; \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} \; - \; t_a \right) \\ & + \sum_{i=1}^{N} F_i \left(E_i(\sigma) - \exp(-Y_i) \right) \; + \; \sum_{\tilde{\mu}=1}^{n-r} \Lambda_{\tilde{\mu}} \left(1 \; - \; \sum_{a=1}^{r} \sigma_a \alpha_{\tilde{\mu}}^a X_{\tilde{\mu}}^{-1} \right), \end{aligned} \tag{3.2}$$

where ρ_i^a is the a-th component of the weight ρ_i of representation R, and $\alpha_{\tilde{\mu}}$, $\tilde{\mu} = 1, \cdots, n-r$ are the roots of G.

In later sections, we will slightly modify the index structure above, to be more convenient in each case, just as in [38–40]. For example, if the matter representation R consists of multiple fundamentals, we will break i into separate color and flavor indices.

The Weyl orbifold group acts on the superpotential above in essentially the same form as discussed in detail in [38–40], so we will be brief. In broad brushstrokes, the orbifold group acts by a combination of exchanging fields and multiplying by signs. In the present case, such actions happen on pairs (Y_i, F_i), $(X_{\tilde{\mu}}, \Lambda_{\tilde{\mu}})$, (σ_a, Υ_a) simultaneously. For example, if Y_i is swapped with Y_j, then simultaneously F_i is swapped with F_j. If Y_i is multiplied by a sign, then simultaneously F_i is multiplied by a sign. It is then straightforward to show that the superpotential above is invariant under the orbifold group, following the same arguments as in [38–40].

Furthermore, because the $\Lambda_{\tilde{\mu}}$ terms have the same form as on the (2,2) locus, the part of the excluded locus corresponding to $X_{\tilde{\mu}}$ poles is the same as on the (2,2) locus, and so, for mirrors to connected gauge groups, the fixed points of the Weyl orbifold do not intersect non-excluded critical loci. In passing, another part of the excluded locus is defined by the fact that $\exp(-Y)$ is nonzero for finite Y, and that part of the excluded locus will change as the $\exp(-Y)$'s are now determined by the E's.

Most of the superpotential above is simply the (0,2) version of the (2,2) mirrors of [32, 38–40], with the exception of the FE terms in the second line. For a (2,2) supersymmetric mirror, each of those E's would be

$$E_i(\sigma) \;=\; \sum_{a=1}^{r} \rho_i^a \sigma_a. \tag{3.3}$$

Allowing for more general E's encodes the (0,2) deformation. We should also observe that in the original (0,2) gauge theory, those E's are not in the superpotential; the fact that they appear in the mirror (0,2) superpotential is as one expects for mirror symmetry.

Just as in [38], we omit the Kähler potential from our ansatz, partly because it is not pertinent to the tests we will perform. For abelian (0,2) GLSMs, detailed discussions of dualities and corresponding Kähler potentials can be found in [34].

The constraints implied by the Fermi fields imply the operator mirror map

$$\exp(-Y_i) \;=\; E_i(\sigma), \tag{3.4}$$

$$X_{\tilde{\mu}} \;=\; \sum_{a=1}^{r} \alpha_{\tilde{\mu}}^a \sigma_a, \tag{3.5}$$

a precise analogue of the operator mirror map (2.5) in the (2,2) case, as well as the constraints

$$\sum_{i=1}^{N} \rho_i^a Y_i \;-\; \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} \;=\; t_a. \tag{3.6}$$

Exponentiating the constraints and applying the operator mirror map, we get the relations

$$\left[\prod_i E_i(\sigma)^{\rho_i^a} \right] \left[\prod_{\tilde{\mu}} X_{\tilde{\mu}}^{\alpha_{\tilde{\mu}}^a} \right] \;=\; q_a. \tag{3.7}$$

Just as in the (2,2) case [38, section 3.3], and as we will see in more detail in section 4, the factor

$$\left[\prod_{\tilde{\mu}} X_{\tilde{\mu}}^{\alpha_{\tilde{\mu}}^a} \right] \tag{3.8}$$

just contributes a phase, so that these relations reduce to

$$\prod_i E_i(\sigma)^{\rho_i^a} = \tilde{q}_a, \tag{3.9}$$

for suitably phase-shifted $\tilde{q}_a \propto q_a$, which are precisely the quantum sheaf cohomology relations for these theories (see *e.g.* [18]). Thus, as expected, the quantum sheaf cohomology ring relations of the original theory are realized classically in the mirror, just as in (2,2) supersymmetric mirrors.

The right-chiral $U(1)_R$ symmetry of the original gauge theory is realized in the mirror as

$$\begin{aligned} Y_i \; &\mapsto \; Y_i - i\alpha, \quad F_i \; \mapsto \; F_i \;\; (\text{invariant}) \,, \\ X_{\tilde{\mu}} \; &\mapsto \; X_{\tilde{\mu}} \exp\left(+i\alpha\right), \quad \Lambda_{\tilde{\mu}} \; \mapsto \; \Lambda_{\tilde{\mu}} \exp\left(+i\alpha\right), \\ \sigma_a \; &\mapsto \; \sigma_a \exp\left(+i\alpha\right), \quad \Upsilon_a \; \mapsto \; \Upsilon_a \exp\left(+i\alpha\right), \end{aligned} \tag{3.10}$$

and with a corresponding phase rotation of the superspace coordinates, where α parametrizes the symmetry, following exactly the same pattern as equation (2.6) for the (2,2) mirror, and with the same result: the axial anomaly of the original gauge theory is mirror to a classical obstruction that can be cured by a shift of the (mirror to the) theta angle.

The left-chiral $U(1)$ symmetry (an R-symmetry on the (2,2) locus) of the original gauge theory is realized in the mirror as

$$\begin{aligned} Y_i \; &\mapsto \; Y_i - i\alpha, \quad F_i \; \mapsto \; F_i \exp\left(-i\alpha\right), \\ X_{\tilde{\mu}} \; &\mapsto \; X_{\tilde{\mu}} \exp\left(+i\alpha\right), \quad \Lambda_{\tilde{\mu}} \; \mapsto \; \Lambda_{\tilde{\mu}} \;\; (\text{invariant}) \,, \\ \sigma_a \; &\mapsto \; \sigma_a \exp\left(+i\alpha\right), \quad \Upsilon_a \; \mapsto \; \Upsilon_a \;\; (\text{invariant}) \,, \end{aligned} \tag{3.11}$$

where α parametrizes the symmetry, and with no phase rotation of the superspace coordinates. As in the right-chiral case, the anomaly of the original theory is realized in the mirror by a classical obstruction that can be cured by a shift of the (mirror to the) theta angle.

4. Justification

We saw in the previous section that the proposed (0,2) mirror possesses many of the desired properties of a mirror: it realizes classically the quantum sheaf cohomology ring relations of the original theory, and it has the same symmetries, realizing anomalies classically in the mirror.

In this section, we will provide further general tests of the (0,2) mirror proposal of the previous section. Specifically, we will reproduce the one-loop effective (0,2) superpotential of [15] and also argue how correlation functions in these theories reproduce those of the original gauge theories, in cases in which vacua are isolated. Our arguments in this section will be somewhat formal, but in concrete examples in later sections we will verify these properties explicitly.

4.1. Integrate out fields

In this section, we will integrate out fields and recover the one-loop effective superpotential of the original gauge theory, a standard property of (2,2) gauge theory mirrors that also holds in this (0,2) supersymmetric mirror proposal.

First, following [38], to better understand the properties of this theory, we integrate out the fields $X_{\tilde{\mu}}$ and $\Lambda_{\tilde{\mu}}$. This is an option because they have nonzero masses; phrased simply,

$$\frac{\partial^2 W}{\partial \Lambda_{\tilde{\mu}} \partial X_{\tilde{\nu}}} = \sum_a \frac{\sigma_a \alpha^a_{\tilde{\mu}}}{X^2_{\tilde{\mu}}} \delta_{\tilde{\mu}\tilde{\nu}}, \tag{4.1}$$

whose zero locus defines part of the excluded locus, as explained in [38]. The Hessian of $X_{\tilde{\mu}}$ is

$$H_X = \prod_{\tilde{\mu}} \left(\sum_{a=1}^{r} \sigma_a \alpha^a_{\mu} \right)^{-1}, \tag{4.2}$$

which, when integrating out $X_{\tilde{\mu}}$, $\Lambda_{\tilde{\mu}}$, generates a factor in the path integral measure which vanishes along the excluded locus, exactly the same as in (2,2) mirrors [38]. The equations of motion of $X_{\tilde{\mu}}$ are

$$X_{\tilde{\mu}} = \sum_{a=1}^{r} \sigma_a \alpha^a_{\tilde{\mu}}. \tag{4.3}$$

Therefore, integrating out $X_{\tilde{\mu}}$ and $\Lambda_{\tilde{\mu}}$ amounts to eliminating the terms proportional to $\Lambda_{\tilde{\mu}}$ and $\ln X_{\tilde{\mu}}$ in (3.2) and shifting the FI parameters t_a to $\tilde{t}_a$, just as happens in (2,2) mirrors [38], reproducing a phase discussed in [48, section 10]. For example, for each $U(k)$ factor of the gauge group,

$$\alpha^a_{\tilde{\mu}} = \alpha^a_{bc} = \delta^a_c - \delta^a_b \tag{4.4}$$

for $a, b, c = 1, \cdots, k$ and $b \neq c$ and thus

$$\begin{aligned} \sum_{\tilde{\mu}} \alpha^a_{\tilde{\mu}} \ln X_{\tilde{\mu}} &= \sum_{b \neq c} \alpha^a_{bc} \ln (\sigma_c - \sigma_b) \\ &= \sum_{b \neq a} \ln \left(\frac{\sigma_a - \sigma_b}{\sigma_b - \sigma_a} \right) = (k-1)\pi i \end{aligned} \tag{4.5}$$

from the equation of motion.

Therefore, after integrating out $X_{\tilde{\mu}}$ and $\Lambda_{\tilde{\mu}}$, the superpotential (3.2) reduces to

$$\tilde{W} = \sum_{a=1}^{r} \Upsilon_a \left(\sum_{i=1}^{N} \rho_i^a Y_i - \tilde{t}_a \right) + \sum_{i=1}^{N} F_i \left(E_i(\sigma) - \exp(-Y_i) \right) \tag{4.6}$$

through a redefinition $\tilde{t}_a$ of t_a. The equations of motion of σ_a and Y_i derived from (4.6) then gives the mirror map (3.4) and the expected quantum sheaf cohomology relations (3.9).

Let us now also integrate out (Y_i, F_i). We will recover the one-loop effective superpotential of the original gauge theory on the Coulomb branch, just as happens in (2,2) supersymmetric mirrors. As before, it is legitimate to do so because these fields have nonzero mass:

$$\frac{\partial^2 W}{\partial Y_i \partial F_j} = \delta_{ij} \exp(-Y_i), \tag{4.7}$$

whose zero locus defines part of the excluded locus, as explained in [38]. From the superpotential above, we find

$$\exp(-Y_i) = E_i(\sigma), \tag{4.8}$$

or simply

$$Y_i = -\ln E_i(\sigma), \tag{4.9}$$

and plugging back in we recover

$$\tilde{W} = \sum_{a=1}^{r} \Upsilon_a \left(- \sum_{i=1}^{N} \rho_i^a \ln E_i(\sigma) - \tilde{t}_a \right). \tag{4.10}$$

This matches [15, equ'ns (3.22)-(3.23)]. Thus, we see that integrating out fields recovers the one-loop effective superpotential along the Coulomb branch, exactly as happens in (2,2) supersymmetric gauge theory mirrors.

4.2. Correlation functions

In this section, we will compare correlation functions in the B/2-twisted Landau-Ginzburg model just defined (3.2) with corresponding A/2 model correlation functions, in cases with isolated Coulomb branch vacua, and along the way, recover the one-loop effective (0,2) superpotential of [15] along the Coulomb branch.

Now, for a (0,2) superpotential of the form $W = F^i J_i$ with isolated vacua, correlation functions are schematically of the form [49]

$$\langle f \rangle \;=\; \sum_{\text{vacua}} \frac{f}{\det \partial_i J_j}, \tag{4.11}$$

closely related to formulas for correlation functions in (2,2) Landau-Ginzburg models involving determinants of matrices of second derivatives of the superpotential. Thus, we need to compute some analogues of Hessians.

The Hessian of Y_i is

$$H_Y = \prod_i \exp(-Y_i) = \prod_i E_i(\sigma),$$

which is nonzero at generic points on the Coulomb branch. From (3.4), integrating out Y_i and F_i reduces (4.6) to

$$W_{\text{eff}} \;=\; \sum_{a=1}^{r} \Upsilon_a J^a_{\text{eff}} \;=\; \sum_{a=1}^{r} \Upsilon_a \left(-\sum_{i=1}^{N} \rho_i^a \ln E_i(\sigma) - \tilde{t}_a \right), \tag{4.12}$$

which is the same as the effective superpotential on the Coulomb branch of the original GLSM. Consequently, assuming isolated vacua, for any operator $\mathcal{O}(\sigma)$, the B/2 correlation functions of our proposed Landau-Ginzburg mirror are [49]

$$\langle \mathcal{O}(\sigma) \rangle \;=\; \frac{1}{|\mathcal{W}|} \sum_{J^a_{\text{eff}}=0} \frac{\mathcal{O}(\sigma)}{\left(\det_{a,b} \partial_b J^a_{\text{eff}}\right) H_X H_Y}, \tag{4.13}$$

$$\;=\; \frac{1}{|\mathcal{W}|} \sum_{J^a_{\text{eff}}=0} \frac{\mathcal{O}(\sigma) \prod_{\tilde{\mu}} \left(\sum_{a=1}^{r} \sigma_a \alpha^a_{\tilde{\mu}} \right)}{\left(\det_{a,b} \partial_b J^a_{\text{eff}}\right) \left(\prod_i E_i(\sigma)\right)}, \tag{4.14}$$

which is the same as the A/2 correlation function computed from the original GLSM [18, equ'n (3.63)]. (The factor of $1/|\mathcal{W}|$ reflects the Weyl orbifold, which acts freely on the critical locus, as in [38], so that twisted sectors do not enter this computation, at least for mirrors to theories with connected gauge groups.)

5. Specialization to abelian theories

Let's consider a GLSM with gauge group $U(1)^r$. The chiral field Φ_i and Fermi field Ψ_i have charge Q_i^a under the a-th $U(1)$, for $i = 1, \cdots, N$. Assuming linear and diagonal (0,2) deformations, as discussed before, these fields satisfy

$$\overline{D}_+ \Psi_i \;=\; \sum_{a=1}^{r} E_i^a \sigma_a \Phi_i, \tag{5.1}$$

where $E_i^a = Q_i^a$ on the (2,2) locus.

In the abelian case, the fields X_μ and Λ_μ are absent in the mirror theory. The matter content of the mirror Landau-Ginzburg model thus consists of chiral fields σ_a, Y_i and Fermi fields Υ_a, F_i, $a = 1, \cdots, r, i = 1, \cdots, N$. The superpotential is

$$W \;=\; \sum_{a=1}^{r} \Upsilon_a \left(\sum_{i=1}^{N} Q_i^a Y_i - t^a \right) \;+\; \sum_{i=1}^{N} F_i \left(\sum_{a=1}^{r} E_i^a \sigma_a - \exp(-Y_i) \right). \tag{5.2}$$

Specializing equation (3.4), the operator mirror map in this case is

$$\exp(-Y_i) \;=\; \sum_{a=1}^{r} E_i^a \sigma_a \tag{5.3}$$

and the effective superpotential is

$$W_{\text{eff}} \;=\; \sum_{a=1}^{r} \Upsilon_a J^a_{\text{eff}} \;=\; \sum_{a=1}^{r} \Upsilon_a \left(- \sum_{i=1}^{N} Q_i^a \ln \left(\sum_{b=1}^{r} E_i^b \sigma_b \right) - t^a \right), \tag{5.4}$$

which reproduces the expected correlation functions

$$\langle \mathcal{O}(\sigma) \rangle \;=\; \sum_{J^a_{\text{eff}}=0} \frac{\mathcal{O}(\sigma)}{\left(\det_{a,b} \partial_b J^a_{\text{eff}}\right) H_Y}, \tag{5.5}$$

where

$$H_Y \;=\; \prod_{i=1}^{N} \left(\sum_{a=1}^{r} E_i^a \sigma_a \right). \tag{5.6}$$

6. Comparison to previous abelian proposal

A proposal was made for a systematic mirror construction in abelian (0,2) GLSMs in [37]. The proposal of this paper both generalizes and simplifies the proposal given there. In this section, we will explicitly relate our ansatz to that discussed there. (Special cases have already been discussed, in sections 7.5 and 8.)

Briefly, the proposal in [37] considered abelian (0,2) GLSMs with E's that are both linear and diagonal, as here, but with two additional restrictions:

- To compute the mirror, one picked an invertible submatrix S of the charge matrix,
- and the (0,2) deformations vanished for E's corresponding to rows of S.

The physics of the resulting mirror was independent of choices, but nevertheless this was a more restrictive mirror than that given in this paper.

We will outline a derivation of the construction in [37] from the mirror in this paper, but first, with the benefit of hindsight, let us outline in general terms how they are related.

- In the proposal of this paper, to generate a lower-energy Landau-Ginzburg model, we may for example integrate out a subset of the F Fermi fields, and solve for the σ_a. This procedure only works if the corresponding submatrix of the E's is invertible, and so, broadly speaking, corresponds to a choice of invertible submatrix.
- Assuming that the E submatrix chosen above is the same as on the (2,2) locus removes the necessity of keeping track of overall numerical factors multiplying partition functions and correlation functions, the subtlety discussed in *e.g.* subsection 7.3.1.

Next, we shall outline a derivation of the ansatz of [37] from the proposal of this paper. First, they wrote their linear diagonal $\overline{D}_+ \Psi_i$ in terms of deformations B_{ij} off the (2,2) locus, as

$$E_i = \sum_j \sum_a (\delta_{ij} + B_{ij}) \, Q_i^a \sigma_a. \tag{6.1}$$

For these E_i, our ansatz (3.2) can be written as

$$\begin{aligned} W &= \sum_{a=1}^{r} \Upsilon_a \left(\sum_{i=1}^{N} Q_i^a \sigma_a \; - \; t_a \right) \\ &\quad + \sum_{i=1}^{N} F_i \left(\sum_a Q_i^a \sigma_a \; + \; \sum_j B_{ij} Q_j^a \sigma_a \; - \; \exp\left(-Y_i\right) \right). \end{aligned} \tag{6.2}$$

Now, in the ansatz of [37], one picks an invertible submatrix S of the charge matrix, and for i corresponding to a column of S, $B_{ij} = 0$. As a result, for those i, the F_i terms are simply

$$F_i \left(\sum_a Q_i^a \sigma_a \; - \; \exp\left(-Y_i\right) \right), \tag{6.3}$$

and so we have a constraint that relates, for those i,

$$\sum_a Q_i^a \sigma_a \; = \; \exp\left(-Y_i\right), \tag{6.4}$$

or equivalently, in the notation of [37],

$$\sum_a S_{i_S}^a \sigma_a \; = \; \exp\left(-Y_i\right). \tag{6.5}$$

Solving for σ_a, we have

$$\sigma_a \; = \; \sum_{i_S} \left(S^{-1}\right)_{a i_S} \exp\left(-Y_{i_S}\right), \tag{6.6}$$

and plugging back in, our (0,2) superpotential becomes

$$\begin{aligned} W &= \sum_{a=1}^{r} \Upsilon_a \left(\sum_{i=1}^{N} Q_i^a \sigma_a \; - \; t_a \right) \\ &\quad + \sum_{i=1}^{N} F_i \left(\sum_a Q_i^a \sigma_a + \sum_{a,j,i_S} B_{ij} Q_j^a \left(S^{-1}\right)_{a i_S} \exp\left(-Y_{i_S}\right) - \exp\left(-Y_i\right) \right), \end{aligned} \tag{6.7}$$

which is precisely the (0,2) superpotential of [37].

7. Example: $\mathbb{P}^n \times \mathbb{P}^m$

So far we have given general arguments that expected properties of the mirror always hold for this proposal: anomalies of the original theory are realized classically in the mirror, quantum sheaf cohomology ring relations arise from classical critical locus constraints, correlation functions match, and integrating out fields returns the one-loop twisted effective action of the original theory, all as expected for a gauge theory mirror ala [32, 38].

Now, general arguments are well and good, but to make the discussion more concrete, working through examples can also be helpful. To that end, in this section we work through the first of several examples, to see concretely how the mirror works in special cases.

7.1. Setup

In this section we will compare to proposals for (0,2) mirrors to $\mathbb{P}^n \times \mathbb{P}^m$ with a deformation of the tangent bundle, as discussed in [35].

In this case, a general deformation of the tangent bundle is described as the cokernel

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{7.1}$$

where

$$E = \begin{bmatrix} Ax & Bx \\ Cy & Dy \end{bmatrix}, \tag{7.2}$$

where x, y are vectors of homogeneous coordinates on $\mathbb{P}^n$, $\mathbb{P}^m$, respectively, and where A, B are constant $(n+1)\times(n+1)$ matrices, and C, D are constant $(m+1)\times(m+1)$ matrices. In this language, the (2,2) locus corresponds for example to the case that A and D are identity matrices, and $B = 0 = C$.

Physically, in the corresponding (0,2) GLSM, we can write

$$\overline{D}_+ \Lambda_i \;=\; (A_{ij}\sigma + B_{ij}\tilde{\sigma})\, x_j, \quad \overline{D}_+ \tilde{\Lambda}_j \;=\; (C_{jk}\sigma + D_{jk}\tilde{\sigma})\, y_k, \tag{7.3}$$

and so we have

$$E_{ij}(\sigma, \tilde{\sigma}) \;=\; (A\sigma + B\tilde{\sigma})_{ij}, \quad \tilde{E}_{jk}(\sigma, \tilde{\sigma}) \;=\; (C\sigma + D\tilde{\sigma})_{jk}. \tag{7.4}$$

The (0,2) mirror ansatz of this paper is only defined for diagonal E's, so we shall assume the matrices A, B, C, D are diagonal:

$$A = \mathrm{diag}\,(a_0, \cdots, a_n)\,, \tag{7.5}$$

$$B = \mathrm{diag}\,(b_0, \cdots, b_n)\,, \tag{7.6}$$

$$C = \mathrm{diag}\,(c_0, \cdots, c_m)\,, \tag{7.7}$$

$$D = \mathrm{diag}\,(d_0, \cdots, d_m)\,. \tag{7.8}$$

We also define

$$E_i(\sigma, \tilde{\sigma}) \;=\; a_i\sigma + b_i\tilde{\sigma}, \quad \tilde{E}_i(\sigma, \tilde{\sigma}) \;=\; c_i\sigma + d_i\tilde{\sigma}. \tag{7.9}$$

Following the (0,2) mirror ansatz given earlier, we take the (0,2) mirror to be defined by the superpotential

$$\begin{aligned} W = \Upsilon_1 \left(\sum_{i=1}^{n} Y_i \;-\; t_1 \right) \;+\; \Upsilon_2 \left(\sum_{j=0}^{m} \tilde{Y}_j \;-\; t_2 \right) \\ + \sum_{i=0}^{n} F_i \left(E_i(\sigma, \tilde{\sigma}) \;-\; \exp\left(-Y_i\right)\right) \\ + \sum_{j=0}^{m} \tilde{F}_j \left(\tilde{E}_j(\sigma, \tilde{\sigma}) \;-\; \exp\left(-\tilde{Y}_j\right)\right). \end{aligned} \tag{7.10}$$

As a first consistency test, let us verify that this produces the quantum sheaf cohomology ring of $\mathbb{P}^n \times \mathbb{P}^m$. First, we integrate out the Υ_i, which gives the usual constraints

$$\prod_{i=0}^{n} \exp\left(-Y_i\right) \;=\; q_1, \quad \prod_{j=0}^{m} \exp\left(-\tilde{Y}_j\right) \;=\; q_2. \tag{7.11}$$

Integrating out the F_i, $\tilde{F}_j$ gives the operator mirror maps

$$\exp\left(-Y_i\right) \;=\; E_i(\sigma, \tilde{\sigma}), \quad \exp\left(-\tilde{Y}_j\right) \;=\; \tilde{E}_j(\sigma, \tilde{\sigma}), \tag{7.12}$$

and combining these with the constraints (7.11), one immediately has

$$\begin{aligned} \det(A\sigma + B\tilde{\sigma}) \;=\; \prod_i E_i(\sigma, \tilde{\sigma}) \;=\; q_1, \\ \det(C\sigma + D\tilde{\sigma}) \;=\; \prod_j \tilde{E}_j(\sigma, \tilde{\sigma}) \;=\; q_2, \end{aligned} \tag{7.13}$$

which are precisely the quantum sheaf cohomology ring relations for this model [14–18].

7.2. Correlation functions in the UV

Before going on to integrate out some of the fields, let us take a moment to explicitly compute two-point B/2-model correlation functions in the case of the mirror to $\mathbb{P}^1 \times \mathbb{P}^1$. (As we already know the chiral ring matches that of the A/2 model, from the results of the immediately preceding subsection, computing the two-point correlation functions suffices to determine all of the B/2-model correlation functions.)

Correlation functions for the $\mathbb{P}^1 \times \mathbb{P}^1$ model were computed in [18][section 4.2]. We repeat the highlights here for completeness. The two-point correlation functions have the form

$$\langle \sigma\sigma \rangle \;=\; -\frac{\Gamma_1}{\alpha}, \;\; \langle \sigma\tilde{\sigma} \rangle \;=\; +\frac{\Delta}{\alpha}, \;\; \langle \tilde{\sigma}\tilde{\sigma} \rangle \;=\; -\frac{\Gamma_2}{\alpha}, \tag{7.14}$$

where

$$\begin{aligned} \gamma_{AB} &= \det(A+B) - \det A - \det B, \\ \gamma_{CD} &= \det(C+D) - \det C - \det D, \\ \Gamma_1 &= \gamma_{AB} \det D - \gamma_{CD} \det B, \\ \Gamma_2 &= \gamma_{CD} \det A - \gamma_{AB} \det C, \\ \Delta &= (\det A)(\det D) - (\det B)(\det C), \\ \alpha &= \Delta^2 - \Gamma_1\Gamma_2. \end{aligned} \tag{7.15}$$

We can compute correlation functions in the present mirror B/2-twisted Landau-Ginzburg model with superpotential (7.10 using the methods of [49]. Specializing to $n = m = 1$, we have six functions J_i, corresponding to the coefficients of $\Upsilon_{1,2}$, $F_{1,2}$, $\tilde{F}_{1,2}$, and six fields σ, $\tilde{\sigma}$, $Y_{0,1}$, $\tilde{Y}_{0,1}$. The resulting matrix of derivatives $(\partial_i J_j)$ has the form

$$(\partial_i J_j) \;=\; \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ a_0 & b_0 & \exp(-Y_0) & 0 & 0 & 0 \\ a_1 & b_1 & 0 & \exp(-Y_1) & 0 & 0 \\ c_0 & d_0 & 0 & 0 & \exp\left(-\tilde{Y}_0\right) & 0 \\ c_1 & d_1 & 0 & 0 & 0 & \exp\left(-\tilde{Y}_1\right) \end{bmatrix}, \tag{7.16}$$

and then correlation functions have the form

$$\langle f(\sigma, \tilde{\sigma}) \rangle \;=\; \sum_{J=0} \frac{f(\sigma, \tilde{\sigma})}{\det(\partial_i J_j)}, \tag{7.17}$$

where the sum is over the solutions of $\{J_i = 0\}$. It is straightforward to compute that the resulting correlation functions precisely match those listed above from the A/2 model [18][section 4.2].

7.3. More nearly standard expressions

More nearly standard expressions for Landau-Ginzburg mirrors do not involve σ fields, so in this section, we shall integrate out these fields to derive expressions for mirrors of a more nearly standard form. We will encounter some interesting subtleties.

Specifically, some other expressions for possible (0,2) mirrors to $\mathbb{P}^n \times \mathbb{P}^m$ are in [35, 37]. Those expressions have precisely n Y's and m $\tilde{Y}$'s, so we first integrate out the Υ_i, eliminating Y_0, $\tilde{Y}_0$:

$$\exp\left(-Y_0\right) \;=\; q_1 \prod_{i=1}^{n} \exp\left(+Y_i\right), \quad \exp\left(-\tilde{Y}_0\right) \;=\; q_2 \prod_{j=1}^{m} \exp\left(+\tilde{Y}_j\right). \tag{7.18}$$

Next, we can either integrate out some of the Fermi fields F_i, $\tilde{F}_j$, and then integrate out σ's, or we can integrate out σ's first, and then some of the Fermi fields. This order-of-operations ambiguity does not exist in (2,2) theories. The results are independent of choices, as one should expect, but we illustrate both methods next, to illustrate various subtleties in both the analysis and the normalization of the results. In later analyses in this paper, we will be much more brief.

7.3.1. First method.

Having integrating out the Υ_i, our strategy in this approach is to next integrate out some F, $\tilde{F}$ (as many as σ's), and then use the resulting constraints to eliminate σ's.

The expressions in [35, 37] have as many F's as Y's, so we need to integrate out one F and one $\tilde{F}$. This will mean solving for σ and $\tilde{\sigma}$ in terms of other variables. There are a number of ways to proceed, and indeed, one expects that there will be many equivalent but different-looking expresions for σ, $\tilde{\sigma}$ in terms of Y_i and $\tilde{Y}_j$. To pick one, we choose an index i and j such that the expressions we get from integrating out the corresponding F and

$\tilde{F}$, namely

$$\exp(-Y_i) \;=\; E_i(\sigma,\tilde{\sigma}), \quad \exp\left(-\tilde{Y}_j\right) \;=\; \tilde{E}_j(\sigma,\tilde{\sigma}), \tag{7.19}$$

can be inverted to solve for σ, $\tilde{\sigma}$ in terms of Y_i, $\tilde{Y}_j$. Put another way, using an index I to denote either i or j, and writing, schematically,

$$E_I(\sigma,\tilde{\sigma}) \;=\; S_I^{\alpha}\sigma_{\alpha}, \tag{7.20}$$

we pick two indices I such that the resulting 2×2 matrix S is invertible. (Here we are deliberately making contact with the notation used in [37].)

Suppose, for example, that the two equations

$$\exp(-Y_0) \;=\; E_0(\sigma,\tilde{\sigma}), \quad \exp\left(-\tilde{Y}_0\right) \;=\; \tilde{E}_0(\sigma,\tilde{\sigma}), \tag{7.21}$$

can be inverted to solve for σ, $\tilde{\sigma}$. Let us do this explicitly, and examine the result. From our earlier discussion,

$$E_0(\sigma,\tilde{\sigma}) \;=\; a_0\sigma + b_0\tilde{\sigma}, \quad \tilde{E}_0(\sigma,\tilde{\sigma}) \;=\; c_0\sigma + d_0\tilde{\sigma}. \tag{7.22}$$

Assuming that

$$\Delta_0 \;\equiv\; \det\left[\begin{array}{cc} a_0 & b_0 \\ c_0 & d_0 \end{array}\right] \;\neq\; 0, \tag{7.23}$$

we first integrate out F_0, $\tilde{F}_0$ to get the constraints (7.21), and then these equations to find

$$\sigma \;=\; \frac{1}{\Delta_0}\left(d_0\exp(-Y_0) \;-\; b_0\exp\left(-\tilde{Y}_0\right)\right), \tag{7.24}$$

$$\tilde{\sigma} \;=\; \frac{1}{\Delta_0}\left(-c_0\exp(-Y_0) \;+\; a_0\exp\left(-\tilde{Y}_0\right)\right). \tag{7.25}$$

Then, after finally integrating out σ and $\tilde{\sigma}$, the (0,2) superpotential reduces to

$$
\begin{aligned}
W &= \sum_{i=1}^{n} F_i \left(a_i \sigma + b_i \tilde{\sigma} - \exp\left(-Y_i\right) \right) \\
&\quad + \sum_{j=1}^{m} \tilde{F}_j \left(c_j \sigma + d_j \tilde{\sigma} - \exp\left(-\tilde{Y}_j\right) \right), \\
&= \sum_{i=1}^{n} F_i \left[\frac{(a_i d_0 - b_i c_0)}{\Delta_0} q_1 \prod_{i'=1}^{n} \exp\left(+Y_{i'}\right) \right. \\
&\qquad \left. + \frac{(-a_i b_0 + b_i a_0)}{\Delta_0} q_2 \prod_{j'=1}^{m} \exp\left(+\tilde{Y}_{j'}\right) - \exp\left(-Y_i\right) \right] \\
&\quad + \sum_{j=1}^{m} \tilde{F}_j \left[\frac{(c_j d_0 - d_j c_0)}{\Delta_0} q_1 \prod_{i'=1}^{n} \exp\left(+Y_{i'}\right) \right. \\
&\qquad \left. + \frac{(-c_j b_0 + d_j a_0)}{\Delta_0} q_2 \prod_{j'=1}^{m} \exp\left(+\tilde{Y}_{j'}\right) - \exp\left(-\tilde{Y}_j\right) \right].
\end{aligned} \tag{7.26}
$$

Before going on, there is a subtlety we should discuss, that will become important when comparing correlation functions between the UV and lower-energy theories. Specifically, when we integrated out σ and $\tilde{\sigma}$, one effect is to multiply the path integral by a constant. Specifically, after integrating out F_0 and $\tilde{F}_0$, we had constraints which schematically appear in the B/2 model path integral in the form

$$\int d\sigma d\tilde{\sigma}\, \delta\left(a_0\sigma + b_0\tilde{\sigma} - \exp\left(-Y_0\right)\right) \delta\left(c_0\sigma + d_0\tilde{\sigma} - \exp\left(-\tilde{Y}_0\right)\right). \tag{7.27}$$

Then, integrating over σ, $\tilde{\sigma}$ generates a factor of

$$\frac{1}{a_0 d_0 - b_0 c_0} = \frac{1}{\Delta_0} \tag{7.28}$$

from the Jacobian. This will multiply correlation functions in the lower-energy theory, and we will see later in subsection 7.4 that this will be required in order for the lower-energy-theory's correlation functions to match the UV correlation functions.

7.3.2. Second method. As a consistency test, and to illuminate the underlying methods, we will now rederive the same result via a different approach. Having integrated out the Υ_i, our strategy in this approach is to next integrate out the σ_a. This will generate constraints on the F, $\tilde{F}$, which we will use to write some in terms of the others. (This is the opposite order of operations from the previous approach.)

The result of this method will be an expression for the (0,2) mirror that is not of the form described in [35, 37], and also does not respect symmetries of the parametrization.

We restrict to $\mathbb{P}^1 \times \mathbb{P}^1$ for simplicity. Integrating out σ_a, we have the constraints

$$\sum_{i=0}^{n} a_i F_i + \sum_{j=0}^{m} c_j \tilde{F}_j = 0, \tag{7.29}$$

$$\sum_{i=0}^{n} b_i F_i + \sum_{j=0}^{m} d_j \tilde{F}_j = 0. \tag{7.30}$$

Solving for F_0, $\tilde{F}_0$, we find

$$F_0 = -\frac{1}{\Delta_0}\left[\sum_{i=1}^{n}(a_i d_0 - b_i c_0)F_i + \sum_{j=1}^{m}(c_j d_0 - c_0 d_j)\tilde{F}_j\right], \tag{7.31}$$

$$\tilde{F}_0 = -\frac{1}{\Delta_0}\left[\sum_{i=1}^{n}(a_0 b_i - b_0 a_i)F_i + \sum_{j=1}^{m}(d_j a_0 - b_0 c_j)\tilde{F}_j\right] \tag{7.32}$$

where

$$\Delta_0 = a_0 d_0 - b_0 c_0. \tag{7.33}$$

Plugging this back into the (0,2) superpotential, we have

$$W = -\sum_{i=0}^{n} F_i \exp(-Y_i) - \sum_{j=0}^{m} \tilde{F}_j \exp\left(-\tilde{Y}_j\right), \tag{7.34}$$

$$= -\sum_{i=1}^{n} F_i \left[\exp(-Y_i) \; - \; \frac{(a_i d_0 - b_i c_0)}{\Delta_0} q_1 \prod_{k=1}^{n} \exp(+Y_k) \right.$$
$$\left. - \; \frac{(a_0 b_i - b_0 a_i)}{\Delta_0} q_2 \prod_{k=1}^{m} \exp\left(+\tilde{Y}_k\right) \right]$$
$$- \sum_{j=1}^{m} \tilde{F}_j \left[\exp\left(-\tilde{Y}_j\right) \; - \; \frac{(c_j d_0 - c_0 d_j)}{\Delta_0} q_1 \prod_{k=1}^{n} \exp(+Y_k) \right.$$
$$\left. - \; \frac{(d_j a_0 - b_0 c_j}{\Delta_0} q_2 \prod_{k=1}^{m} \exp\left(+\tilde{Y}_k\right) \right].$$

This precisely matches the superpotential (7.26) derived from integrating out fields in a different order, as expected.

As in the first ordering, there is a subtlety we have glossed over, a multiplicative factor arising when integrating out some of the fields. Here, the factor arises when integrating out F_0, $\tilde{F}_0$, for the same reasons as before: schematically, the B/2 model path integral measure contains a factor of the form

$$\int dF_0 d\tilde{F}_0 \, \delta(a_0 F_0 + b_0 \tilde{F}_0 + \cdots) \, \delta(c_0 F_0 + d_0 \tilde{F}_0 + \cdots), \tag{7.35}$$

which again generates a numerical factor[7] of Δ_0^{-1} that multiplies correlation functions, and which will be important in subsection 7.4.

7.4. Correlation functions in the lower-energy theory

Next, we compute correlation functions in the new theory, for the case of $\mathbb{P}^1 \times \mathbb{P}^1$, obtained after integrating out fields, and compare to the results for correlation functions computed in the UV theory, before integrating out fields. We will see an important subtlety.

[7]Tracing through this a bit more carefully, the numerical factor arises from the delta functions, which arose from bosonic fields (σ's), hence the numerical factor is δ_0^{-1} instead of $(\Delta_0^{-1})^{-1} = \Delta_0$ as one might have expected from a fermionic integral.

Using the mirror (0,2) superpotential (7.34), and the operator mirror map

$$\sigma = \frac{1}{\Delta_0}\left(d_0 \exp\left(-Y_0\right) - b_0 \exp\left(-\tilde{Y}_0\right)\right), \tag{7.36}$$

$$= \frac{1}{\Delta_0}\left(d_0 q_1 \exp\left(+Y_1\right) - b_0 q_2 \exp\left(+\tilde{Y}_1\right)\right), \tag{7.37}$$

$$\tilde{\sigma} = \frac{1}{\Delta_0}\left(a_0 \exp\left(-\tilde{Y}_0\right) - c_0 \exp\left(-Y_0\right)\right), \tag{7.38}$$

$$= \frac{1}{\Delta_0}\left(a_0 q_2 \exp\left(+\tilde{Y}_1\right) - c_0 q_1 \exp\left(+Y_1\right)\right), \tag{7.39}$$

where

$$\Delta_0 = a_0 d_0 - b_0 c_0, \tag{7.40}$$

using the methods of [49], we find that the two-point functions computed from the mirror above are all Δ_0 times the A/2 model correlation functions in [18][section 4.2], reviewed in section 7.2, or in other words,

$$\langle \sigma\sigma \rangle_{\text{mirror}} = -\Delta_0 \frac{\Gamma_1}{\alpha}, \quad \langle \sigma\tilde{\sigma} \rangle_{\text{mirror}} = +\Delta_0 \frac{\Delta}{\alpha}, \quad \langle \tilde{\sigma}\tilde{\sigma} \rangle_{\text{mirror}} = -\Delta_0 \frac{\Gamma_2}{\alpha}, \tag{7.41}$$

However, we still need to take into account the subtlety discussed in subsection 7.3. Specifically, when deriving the (0,2) Landau-Ginzburg model above from the UV presentation, we had to perform changes-of-variables when integrating out fields, with the effect that low-energy correlation functions should be multiplied by factors of $1/\Delta_0$. Taking that subtlety into account, and dividing out the extra Δ_0 factors, we find that the correct two-point functions precisely match both those of the A/2 model [18][section 4.2], as well as those of the original (UV) theory described in subsection 7.1.

It is also straightforward to compute four-point functions. Their values in the A/2 model are given in [35][appendix A.1]. When one computes them in the (lower-energy) Landau-Ginzburg model above, not taking into account the subtlety discussed above, one finds that the Landau-Ginzburg correlation functions are Δ_0 times the A/2 model correlation functions. Taking into account the subtlety above, the overall factor of $1/\Delta_0$ multiplying all correlation functions, fixes the four-point functions also. In any event, once one knows that the two-point functions and the quantum sheaf cohomology relations match, all of the higher-point functions are guaranteed to match.

7.5. Comparison to other (0,2) mirrors

Now, let us compare to the (0,2) mirrors in [35, 37], for brevity just for the case of $\mathbb{P}^1 \times \mathbb{P}^1$. As a matter of principle, these mirrors need not necessarily match – there could be multiple different UV theories describing the same IR physics. Nevertheless, in special families, we will see that there is a match.

For example, in [35][section 4.2], it was argued that one (0,2) Landau-Ginzburg model those B/2 correlation functions correctly match those of the corresponding A/2 theory on $\mathbb{P}^1 \times \mathbb{P}^1$ had superpotential

$$W \;=\; F_1 J_1 \;+\; \tilde{F}_1 \tilde{J}_1, \tag{7.42}$$

where

$$J_1 \;=\; a X_1 \;-\; \frac{q_1}{X_1} \;+\; b\frac{\tilde{X}_1^2}{X_1} \;+\; \mu \tilde{X}_1, \tag{7.43}$$

$$\tilde{J}_1 \;=\; d\tilde{X}_1 \;-\; \frac{q_2}{\tilde{X}_1} \;+\; c\frac{X_1^2}{\tilde{X}_1} \;+\; \nu X_1, \tag{7.44}$$

with

$$\mu \;=\; \det(A+B) - \det A - \det B, \quad \nu \;=\; \det(C+D) - \det C - \det D, \tag{7.45}$$

and operator mirror map

$$\sigma \;=\; X_1, \quad \tilde{\sigma} \;=\; \tilde{X}_1. \tag{7.46}$$

These expressions have the good property that they are in terms of determinants of the matrices A, B, C, D, and so respect global symmetries of the original theory. For that matter, the A/2 correlation functions only depend upon those determinants, which is explicit in the mirrors constructed in [35].

For purposes of comparison, for $\mathbb{P}^1 \times \mathbb{P}^1$, the superpotential (7.34) takes the form

$$\begin{aligned} W = -F_1 \Bigg[&\exp(-Y_1) \;-\; q_1 \frac{(a_1 d_0 - b_1 c_0)}{\Delta_0} \exp(+Y_1) \\ &-\; q_2 \frac{(b_1 a_0 - a_1 b_0)}{\Delta_0} \exp\left(+\tilde{Y}_1\right) \Bigg] \end{aligned} \tag{7.47}$$

$$- \tilde{F}_1 \left[\exp\left(-\tilde{Y}_1\right) \; - \; q_1 \frac{(c_1 d_0 - d_1 c_0)}{\Delta_0} \exp\left(+Y_1\right) \right.$$
$$\left. - \; q_2 \frac{(d_1 a_0 - c_1 b_0)}{\Delta_0} \exp\left(+\tilde{Y}_1\right) \right].$$

On the face of it, this clearly does not match the mirror proposal of [35], and in fact, is not even written in terms of global-symmetry-invariant determinants of A, B, C, D. Nevertheless, as we have seen, it does reproduce the same correlation functions.

One could imagine using global symmetry transformations to rotate to $a_0 = d_0 = 1, b_0 = c_0 = 0$, the case considered in [37][section 5.1], in which case the result above reduces to

$$\begin{aligned} W \;\; &= \;\; -F_1 \left[\exp\left(-Y_1\right) \; - \; q_1 a_1 \exp\left(+Y_1\right) \; - \; q_2 b_1 \exp\left(+\tilde{Y}_1\right) \right] \\ &\quad - \tilde{F}_1 \left[\exp\left(-\tilde{Y}_1\right) \; - \; q_1 c_1 \exp\left(+Y_1\right) \; - \; q_2 d_1 \exp\left(+\tilde{Y}_1\right) \right]. \end{aligned} \tag{7.48}$$

In this case,

$$a \; = \; a_1, \quad b \; = \; 0 \; = \; c, \quad d \; = \; d_1, \quad \mu \; = \; b_1, \quad \nu \; = \; c_1, \tag{7.49}$$

with operator mirror map

$$\sigma \; = \; q_1 \exp\left(+Y_1\right), \quad \tilde{\sigma} \; = \; q_2 \exp\left(+\tilde{Y}_1\right). \tag{7.50}$$

If we change variables as

$$\exp\left(-Y_0\right) \; = \; q_1 \exp\left(+Y_1\right), \quad \exp\left(-\tilde{Y}_0\right) \; = \; q_2 \exp\left(+\tilde{Y}_1\right), \tag{7.51}$$

then we can rewrite the superpotential as

$$\begin{aligned} W \;\; &= \;\; -F_1 \left[q_1 \exp\left(+Y_0\right) \; - \; a_1 \exp\left(-Y_0\right) \; - \; b_1 \exp\left(-\tilde{Y}_0\right) \right] \\ &\quad - \tilde{F}_1 \left[q_2 \exp\left(+\tilde{Y}_0\right) \; - \; c_1 \exp\left(-Y_0\right) \; - \; d_1 \exp\left(-\tilde{Y}_0\right) \right], \end{aligned} \tag{7.52}$$

which precisely matches the (0,2) mirror in [35] for the case $a_0 = d_0 = 1$, $b_0 = c_0 = 0$. We will return to this case, which also arose in [37], in a more systematic analysis in section 6.

8. Example: Hirzebruch surfaces

In this section we will compare to proposals for (0,2) mirrors to Hirzebruch surfaces with a deformation of the tangent bunde, as discussed in [36]. Our analysis will follow the same form as that for the mirror to $\mathbb{P}^n \times \mathbb{P}^m$, so we will be comparatively brief.

A Hirzebruch surface $\mathbb{F}_n$ can be described by a GLSM with gauge group $U(1)^2$ and matter fields

	x_0	x_1	w	s
$U(1)_1$	1	1	n	0
$U(1)_2$	0	0	1	1

A deformation $\mathcal{E}$ of the tangent bundle is described mathematically as the cokernel

$$0 \longrightarrow \mathcal{O}^2 \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0, \tag{8.1}$$

where

$$* = \begin{bmatrix} Ax & Bx \\ \gamma_1 w & \beta_1 w \\ \gamma_2 s & \beta_2 s \end{bmatrix}, \tag{8.2}$$

and $x = [x_0, x_1]^T$. In principle, additional nonlinear deformations are also possible, but as they do not contribute to quantum sheaf cohomology rings (see section 3), we omit them here. The (2,2) locus corresponds to the case $A = I$, $B = 0$, $\gamma_1 = n$, $\beta_1 = 1$, $\gamma_2 = 0$, $\beta_2 = 1$.

For a general (0,2) theory (with linear diagonal deformations), the E's take the form

$$\overline{D}_+\Lambda_{x,i} = ((\sigma A + \tilde{\sigma} B)x)_i, \quad \overline{D}_+\Lambda_w = (\gamma_1\sigma + \beta_1\tilde{\sigma})w, \quad \overline{D}_+\Lambda_s \, (\gamma_2\sigma + \beta_2\tilde{\sigma})s, \tag{8.3}$$

where the Λ's are the Fermi superfield partners to the bosonic chiral fields. Our mirror construction applies to diagonal deformations, so we only consider the case that

$$\begin{aligned} \overline{D}_+\Lambda_{x,0} &= (a_0\sigma + b_0\tilde{\sigma})x_0, & \overline{D}_+\Lambda_{x,1} &= (a_1\sigma + b_1\tilde{\sigma})x_1, \\ \overline{D}_+\Lambda_w &= (\gamma_1\sigma + \beta_1\tilde{\sigma})w, & \overline{D}_+\Lambda_s &= (\gamma_2\sigma + \beta_2\tilde{\sigma})s. \end{aligned} \tag{8.4}$$

From our ansatz, the mirror Landau-Ginzburg model has fields

- $\sigma, \tilde{\sigma}$,

- $(Y_{0,1}, F_{0,1})$, corresponding to $(x_{0,1}, \Lambda_{x,0-1})$ of the A/2 model,
- (Y_w, F_w), corresponding to (w, Λ_w) of the A/2 model,
- (Y_s, F_s), corresponding to (s, Λ_s) of the A/2 model,

and superpotential

$$\begin{aligned} W &= \Upsilon_1 \left(Y_0 + Y_1 + nY_w - t_1\right) + \Upsilon_2 \left(Y_w + Y_s - t_2\right) \\ &\quad + F_0 \left(a_0\sigma + b_0\tilde{\sigma} - \exp\left(-Y_0\right)\right) + F_1 \left(a_1\sigma + b_1\tilde{\sigma} - \exp\left(-Y_1\right)\right) \\ &\quad + F_w \left(\gamma_1\sigma + \beta_1\tilde{\sigma} - \exp\left(-Y_w\right)\right) + F_s \left(\gamma_2\sigma + \beta_2\tilde{\sigma} - \exp\left(-Y_s\right)\right). \end{aligned} \tag{8.5}$$

The operator mirror map is defined by the constraints imposed by the F's:

$$\exp\left(-Y_0\right) = a_0\sigma + b_0\tilde{\sigma}, \tag{8.6}$$

$$\exp\left(-Y_1\right) = a_1\sigma + b_1\tilde{\sigma}, \tag{8.7}$$

$$\exp\left(-Y_w\right) = \gamma_1\sigma + \beta_1\tilde{\sigma}, \tag{8.8}$$

$$\exp\left(-Y_s\right) = \gamma_2\sigma + \beta_2\tilde{\sigma}, \tag{8.9}$$

and using the mirror D-term relations imposed by the Υ's, namely

$$\exp\left(-Y_0 - Y_1 - nY_w\right) = q_1, \quad \exp\left(-Y_w - Y_s\right) = q_2, \tag{8.10}$$

we quickly derive the quantum sheaf cohomology (chiral ring) relations

$$\begin{aligned} \left(a_0\sigma + b_0\tilde{\sigma}\right)\left(a_1\sigma + b_1\tilde{\sigma}\right)\left(\gamma_1\sigma + \beta_1\tilde{\sigma}\right)^n &= q_1, \\ \left(\gamma_1\sigma + \beta_1\tilde{\sigma}\right)\left(\gamma_2\sigma + \beta_2\tilde{\sigma}\right) &= q_2, \end{aligned} \tag{8.11}$$

or equivalently

$$\begin{aligned} \det\left(A\sigma + B\tilde{\sigma}\right)\left(\gamma_1\sigma + \beta_1\tilde{\sigma}\right)^n &= q_1, \\ \left(\gamma_1\sigma + \beta_1\tilde{\sigma}\right)\left(\gamma_2\sigma + \beta_2\tilde{\sigma}\right) &= q_2, \end{aligned} \tag{8.12}$$

which precisely match the known quantum sheaf cohomology ring relations for this case [14–17].

Next, we integrate out some of the fields to find a lower-energy effective Landau-Ginzburg description of the same physics. If we integrate out F_0,

F_w, we get the constraints

$$a_0\sigma + b_0\tilde{\sigma} = \exp(-Y_0), \tag{8.13}$$
$$\gamma_1\sigma + \beta_1\tilde{\sigma} = \exp(-Y_w), \tag{8.14}$$

which can be solved to give

$$\sigma = \frac{1}{\Delta_0}\left(\beta_1 \exp(-Y_0) - b_0 \exp(-Y_w)\right), \tag{8.15}$$
$$\tilde{\sigma} = \frac{1}{\Delta_0}\left(a_0 \exp(-Y_w) - \gamma_1 \exp(-Y_0)\right), \tag{8.16}$$

for

$$\Delta_0 = a_0\beta_1 - b_0\gamma_1. \tag{8.17}$$

Using the Υ constraints to eliminate Y_0, Y_w, we have

$$\exp(-Y_w) = q_2 \exp(+Y_s), \tag{8.18}$$
$$\begin{aligned} \exp(-Y_0) &= q_1 \exp(+Y_1) \exp(+nY_w) \\ &= (q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s), \end{aligned} \tag{8.19}$$

and finally plugging in we get the lower-energy effective superpotential

$$\begin{aligned} W = {} & F_1\left(a_1\sigma + b_1\tilde{\sigma} - \exp(-Y_1)\right) \\ & + F_s\left(\gamma_2\sigma + \beta_2\tilde{\sigma} - \exp(-Y_s)\right), \\ = {} & F_1\Bigg(\frac{(a_1\beta_1 - b_1\gamma_1)}{\Delta_0}\exp(-Y_0) \\ & \quad + \frac{(-a_1 b_0 + b_1 a_0)}{\Delta_0}\exp(-Y_w) - \exp(-Y_1)\Bigg) \\ & + F_s\Bigg(\frac{(\gamma_2\beta_1 - \beta_2\gamma_1)}{\Delta_0}\exp(-Y_0) \\ & \quad + \frac{(-\gamma_2 b_0 + \beta_2 a_0)}{\Delta_0}\exp(-Y_w) - \exp(-Y_s)\Bigg), \end{aligned} \tag{8.20}$$

$$
\begin{aligned}
&= F_1 \Bigg[\frac{(a_1\beta_1 - b_1\gamma_1)}{\Delta_0} (q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s) \\
&\quad + \frac{(-a_1 b_0 + b_1 a_0)}{\Delta_0} q_2 \exp(+Y_s) \; - \; \exp(-Y_1) \Bigg] \\
&\quad + F_s \Bigg[\frac{(\gamma_2\beta_1 - \beta_2\gamma_1)}{\Delta_0} (q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s) \\
&\quad + \frac{(-\gamma_2 b_0 + \beta_2 a_0)}{\Delta_0} q_2 \exp(+Y_s) \; - \; \exp(-Y_s) \Bigg].
\end{aligned}
$$

To be clear, because of the change of variables we performed in constraints above, to match A/2 correlation functions, correlation functions in this model must be multiplied by a factor of $1/\Delta_0$, just as in our analysis in subsection 7.3.

As a consistency check, let us quickly verify from the mirror (8.20) above, plus the operator mirror map (8.15), (8.16), that the quantum sheaf cohomology relations are obeyed. Briefly,

$$a_0\sigma + b_0\tilde{\sigma} = (q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s) \quad \text{from the operator mirror map,} \tag{8.21}$$

$$a_1\sigma + b_1\tilde{\sigma} = \exp(-Y_1) \quad \text{from the } F_1 \text{ constraint,} \tag{8.22}$$

$$\gamma_1\sigma + \beta_1\tilde{\sigma} = q_2 \exp(+Y_s) \quad \text{from the operator mirror map,} \tag{8.23}$$

$$\gamma_2\sigma + \beta_2\tilde{\sigma} = \exp(-Y_s) \quad \text{from the } F_s \text{ constraint,} \tag{8.24}$$

hence

$$(a_0\sigma + b_0\tilde{\sigma})(a_1\sigma + b_1\tilde{\sigma})(\gamma_1\sigma + \beta_1\tilde{\sigma})^n = q_1, \tag{8.25}$$

$$(\gamma_1\sigma + \beta_1\tilde{\sigma})(\gamma_2\sigma + \beta_2\tilde{\sigma}) = q_2, \tag{8.26}$$

which are precisely the quantum sheaf cohomology ring relations (8.12) for this case.

Now, consider the mirror in the special case that $a_0 = 1$, $b_0 = 0$, $\beta_1 = 1$, $\gamma_1 = n$, in other words, that they take their values on the (2,2) locus. In this case, $\Delta_0 = 1$, and the mirror above becomes

$$
\begin{aligned}
W \; = \; & F_1 \big[(a_1 - n b_1)(q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s) \\
& \quad + b_1 q_2 \exp(+Y_s) \; - \; \exp(-Y_1) \big] \\
& + F_s \big[(\gamma_2 - n\beta_2)(q_1 q_2^{-n}) \exp(+Y_1) \exp(-nY_s) \\
& \quad + \beta_2 q_2 \exp(+Y_s) \; - \; \exp(-Y_s) \big].
\end{aligned} \tag{8.27}
$$

Using the operator mirror map, we can write this more simply as

$$\begin{aligned} W \;&=\; F_1\left[a_1\sigma \,+\, b_1\tilde{\sigma} \,-\, \exp\left(-Y_1\right)\right] \\ &\quad + F_s\left[\gamma_2\sigma \,+\, \beta_2\tilde{\sigma} \,-\, \exp\left(-Y_s\right)\right]. \end{aligned} \tag{8.28}$$

Now, we can perform a change of variables to relate this to the $\mathbb{F}_n$ mirror described in [36][section 4.2], [37][section 5.2.1]. To relate to their notation, if we define X_1, X_3 by

$$\sigma \;=\; X_1 \;=\; \exp\left(-Y_0\right), \tag{8.29}$$

$$\tilde{\sigma} \;=\; X_3 - nX_1 \;=\; \exp\left(-Y_w\right), \tag{8.30}$$

then the (0,2) superpotential above becomes

$$\begin{aligned} W \;&=\; F_1\left[a_1 X_1 \,+\, b_1\left(X_3 - nX_1\right) \,-\, \frac{q_1}{X_1 X_3^n}\right] \\ &\quad + F_s\left[\gamma_2 X_1 \,+\, \beta_2\left(X_3 - nX_1\right) \,-\, \frac{q_2}{X_3}\right]. \end{aligned} \tag{8.31}$$

For the case we are considering ($a_0 = 1$, $b_0 = 0$, $\gamma_1 = n$, $\beta_1 = 1$),

$$a \;=\; \det A \;=\; a_1, \tag{8.32}$$

$$b \;=\; \det B \;=\; 0, \tag{8.33}$$

$$\mu_{AB} \;=\; b_1, \tag{8.34}$$

the coefficient of F_1 can be identified with the J_1 in [36][section 4.2], [37][section 5.2.1], and their J_2 is nJ_1 plus the coefficient of F_s. After a trivial linear rotation of F_1, F_s, we see that this change of variables identifies, in this case, the (0,2) mirror superpotential to $\mathbb{F}_n$ above, derived from our general ansatz, with that discussed in [36, 37]. This matching was not necessary – there can be different UV representations of the same IR physics – but it is certainly satisfying. We will discuss a more general form of this construction in section 6.

9. Example: Grassmannians

So far all of our examples have involved abelian GLSMs. We next turn to a nonabelian example. The Grassmannian $G(k, N)$ is described by a $U(k)$ GLSM with chirals Φ_i^a and Fermis Ψ_i^a in N copies of the fundamental representation, $a \in \{1, \cdots, k\}$, $i \in \{1, \cdots, N\}$. For linear and diagonal (0,2) deformations off the (2,2) locus [19]

$$\overline{D}_+ \Psi_i^a \;=\; \left(\sigma_b^a + B_i^j \, (\mathrm{Tr}\, \sigma) \right) \Phi_j^b, \tag{9.1}$$

where B is diagonal, $B = \mathrm{diag}(b_1, \cdots, b_N)$. The mirror theory consists of chiral fields $\sigma_a, Y_{ia}, X_{\mu\nu}$ and Fermi fields $\Upsilon_a, F_{ia}, \Lambda_{\mu\nu}$ with $a, \mu, \nu = 1, \cdots, k, i = 1, \cdots, N$ and $\mu \neq \nu$, in the notation of [38]. For the fundamental representation of $U(k)$, the a-th component of the weight associated with Y_{ib} is

$$\rho_{ib}^a = \delta_b^a \tag{9.2}$$

and the roots are given by

$$\alpha_{\mu\nu}^a = \delta_\nu^a - \delta_\mu^a, \tag{9.3}$$

therefore the superpotential reads

$$\begin{aligned} W = & \sum_{a=1}^k \Upsilon_a \left(\sum_{i=1}^N Y_{ia} + \sum_{\mu \neq a} (\ln X_{a\mu} - \ln X_{\mu a}) - t \right) \\ & + \sum_{i=1}^N \sum_{a=1}^k F_{ia} \left(\sigma_a + b_i \left(\sum_b \sigma_b \right) - \exp(-Y_{ia}) \right) \\ & + \sum_{\mu \neq \nu} \Lambda_{\mu\nu} \left(1 + \frac{\sigma_\mu - \sigma_\nu}{X_{\mu\nu}} \right), \end{aligned} \tag{9.4}$$

which gives the operator mirror map

$$\exp(-Y_{ia}) \;=\; \sigma_a + b_i \left(\sum_b \sigma_b \right). \tag{9.5}$$

Next, we compute the excluded locus. From the $X_{\mu\nu}$ poles, since $X_{\mu\nu} = \sigma_\nu - \sigma_\nu$ along the critical locus, we have

$$\sigma_a \;\neq\; \sigma_b \tag{9.6}$$

for $a \neq b$. That part is the same as on the (2,2) locus. From the fact that $\exp(-Y) \neq 0$, the F_{ia} coefficients imply that

$$\sigma_a \;+\; b_i \left(\sum_c \sigma_c \right) \;\neq\; 0, \tag{9.7}$$

for all a and i, which is a deformation of what one gets on the (2,2) locus.

Let us take a moment to examine the second excluded locus condition further. If we sum over σ_a, we get

$$(1 + kb_i)\left(\sum_c \sigma_c\right) \neq 0 \tag{9.8}$$

for all i, hence for example

$$1 + kb_i \neq 0 \tag{9.9}$$

for all i. This condition is closely related to a constraint that arises on the B_i^j in order for the gauge bundle defined by the $\overline{D}_+\Psi$ to be a bundle, and not some more general sheaf. Specifically, it was shown in [20][theorem 3.3] that the B's define a bundle, and not a sheaf, if and only if there do not exist k eigenvalues of B that sum to -1. The excluded locus condition we have just derived on the Coulomb branch implies that none of the B eigenvalues equals $-1/k$, which is closely related.

Next, let us recover the A/2 model. Upon integrating out $X_{\mu\nu}$ and Y_{ia}, we get

$$W_{\rm eff} = \sum_{a=1}^{k} \Upsilon_a \left(-\ln \prod_{i=1}^{N} \left(\sigma_a + b_i \left(\sum_b \sigma_b \right) \right) - t \right) \tag{9.10}$$

and

$$H_X = \prod_{\mu \neq \nu} (\sigma_\mu - \sigma_\nu)^{-1}, \tag{9.11}$$

$$H_Y = \prod_{i=1}^{N} \prod_{a=1}^{k} \left(\sigma_a + b_i \left(\sum_b \sigma_b \right) \right), \tag{9.12}$$

which reproduce the A/2 correlation functions of the $U(k)$ GLSM

$$\langle \mathcal{O}(\sigma) \rangle = \frac{1}{k!} \sum_{J^a_{\rm eff}=0} \frac{\mathcal{O}(\sigma)}{\left(\det_{a,b} \partial_b J^a_{\rm eff}\right) H_X H_Y}. \tag{9.13}$$

Next, we shall integrate out some of the fields to construct a lower-energy Landau-Ginzburg model in the pattern of [38][section 4.1]. Beginning with the (0,2) superpotential (9.4), integrating out the Υ_a gives the constraints

$$\sum_{i=1}^{N} Y_{ia} + \sum_{\mu \neq a} \ln\left(\frac{X_{a\mu}}{X_{\mu a}}\right) = t. \tag{9.14}$$

Using these to eliminate Y_{Na}, we have

$$Y_{Na} = -\sum_{i=1}^{N-1} Y_{ia} - \sum_{\mu \neq a} \ln\left(\frac{X_{a\mu}}{X_{\mu a}}\right) = t, \tag{9.15}$$

and so we define

$$\Pi_a = \exp\left(-Y_{Na}\right), \tag{9.16}$$

$$= q\left[\prod_{i=1}^{N-1} \exp\left(+Y_{ia}\right)\right]\left[\prod_{\mu \neq a} \frac{X_{a\mu}}{X_{\mu a}}\right], \tag{9.17}$$

which happens to match the Π_a defined in the (2,2) mirror of $G(k,N)$ in [38][section 4.1].

Next, we integrate out F_{Na}, which gives constraints

$$\sigma_a + b_N\left(\sum_c \sigma_c\right) = \exp\left(-Y_{Na}\right) = \Pi_a. \tag{9.18}$$

These equations can be solved to give

$$\sigma_a = \frac{1}{1+kb_N}\left[(1+(k-1)b_N)\,\Pi_a - b_N\sum_{c\neq a}\Pi_c\right]. \tag{9.19}$$

Plugging this back in, we get our expression for a mirror Landau-Ginzburg theory:

(9.20)

$$\begin{aligned} W = & \sum_{i=1}^{N-1} \sum_{a=1}^{k} F_{ia} \left(\sigma_a + b_i \left(\sum_c \sigma_c \right) \; - \; \exp\left(-Y_{ia}\right) \right) \\ & + \sum_{\mu \neq \nu} \Lambda_{\mu\nu} \left(1 \; + \; \frac{\sigma_\mu - \sigma_\nu}{X_{\mu\nu}} \right), \\ = & \sum_{i=1}^{N-1} \sum_{a=1}^{k} F_{ia} \left[\frac{1}{1 + k b_N} \left((1 + (k-1) b_N + b_i) \, \Pi_a \; + \; (b_i - b_N) \sum_{c \neq a} \Pi_c \right) \right. \\ & \left. \qquad - \; \exp\left(-Y_{ia}\right) \right] \\ & + \sum_{\mu \neq \nu} \Lambda_{\mu\nu} \left(1 \; + \; \frac{\Pi_\mu - \Pi_\nu}{X_{\mu\nu}} \right). \end{aligned}$$

As in earlier discussions, we have glossed over a subtlety: when integrating out the F_{Na}, we omitted a Jacobian factor of

(9.21)

$$\det(\mathrm{Jac})^{-1} \; = \; \det \begin{bmatrix} 1 + b_N & b_N & \cdots & b_N \\ b_N & 1 + b_N & \cdots & b_N \\ \vdots & & & \vdots \\ b_N & b_N & \cdots & 1 + b_N \end{bmatrix}^{-1} \; = \; \frac{1}{1 + k b_N},$$

which should be multiplied into correlation functions in order to match against A/2 results.

As a consistency check, when all the $b_i = 0$, the (0,2) superpotential above reduces to

$$W = \sum_{i=1}^{N-1} \sum_{a=1}^{k} F_{ia} \left(\Pi_a \; - \; \exp\left(-Y_{ia}\right) \right) \; + \; \sum_{\mu \neq \nu} \Lambda_{\mu\nu} \left(1 \; + \; \frac{\Pi_\mu - \Pi_\nu}{X_{\mu\nu}} \right), \tag{9.22}$$

which is precisely the (0,2) expansion of the (2,2) mirror superpotential

$$W \; = \; \sum_{i=1}^{N-1} \sum_{a=1}^{k} \exp\left(-Y_{ia}\right) \; + \; \sum_{\mu \neq \nu} X_{\mu\nu} \; + \; \sum_{a=1}^{k} \Pi_a \tag{9.23}$$

computed in [38][section 4.1].

Next, we will derive the quantum sheaf cohomology relations from this lower-energy Landau-Ginzburg model. The $\Lambda_{\mu\nu}$ imply the constraints

$$X_{\mu\nu} \;=\; \Pi_\nu - \Pi_\mu \tag{9.24}$$

along the critical locus, and similarly from the F_{ia},

$$\exp\left(-Y_{ia}\right) = \frac{1}{1+kb_N}\left((1+(k-1)b_N + b_i)\,\Pi_a \right. \tag{9.25}$$
$$\left. +\; (b_i - b_N)\sum_{c\neq a}\Pi_c \right),$$
$$= \sigma_a \;+\; b_i\left(\sum_c \sigma_c\right) \tag{9.26}$$

along the critical locus. Plugging into the definition of Π_a, we have

$$\Pi_a \;=\; q\left[\prod_{i=1}^{N-1}\exp\left(+Y_{ia}\right)\right](-)^{k-1}, \tag{9.27}$$

hence

$$\Pi_a \prod_{i=1}^{N-1}\left[\sigma_a \;+\; b_i\left(\sum_c \sigma_c\right)\right] \;=\; (-)^{k-1}q, \tag{9.28}$$

or more simply

$$\det\left(I\sigma_a + B(\mathrm{Tr}\,\sigma)\right) \;=\; \prod_{i=1}^{N}\left[\sigma_a \;+\; b_i\left(\sum_c \sigma_c\right)\right] \;=\; (-)^{k-1}q, \tag{9.29}$$

This is precisely the physical description of the quantum sheaf cohomology ring relation in the A/2 model on $G(k,n)$ with the tangent bundle deformation described above [19], as expected. Thus, we see this mirror correctly duplicates the quantum sheaf cohomology ring.

Now, let us perform some consistency checks by computing correlation functions in the mirror Landau-Ginzburg model above in two simple examples and comparing to known results.

Our first example is the special case of $G(1,3) = \mathbb{P}^2$. This has no mathematically nontrivial tangent bundle deformations, but nontrivial parameters can still enter the GLSM and appear in correlation functions, and so it will give a nontrivial test. In this case, the (0,2) superpotential above reduces to

$$W = \sum_{i=1}^{2} F_i \left[\frac{1}{1+b_3} (1+b_i)\, \Pi \; - \; \exp(-Y_i) \right], \tag{9.30}$$

with

$$\Pi = q \prod_{i=1}^{2} \exp(+Y_i), \quad \sigma = \frac{1}{1+b_3} \Pi. \tag{9.31}$$

The matrix of derivatives of the superpotential terms is

$$(\partial_i J_j) = \frac{1}{1+b_3} \left[\begin{array}{cc} (1+b_1)\Pi + (1+b_3)\exp(-Y_1) & (1+b_1)\Pi \\ (1+b_2)\Pi & (1+b_2)\Pi + (1+b_3)\exp(-Y_2) \end{array} \right], \tag{9.32}$$

and using the methods of [49], we find

$$\langle \sigma^2 \rangle = \frac{1}{(1+b_1)(1+b_2)}, \quad \langle \sigma^5 \rangle = \frac{q}{(1+b_1)^2 (1+b_2)^2 (1+b_3)}. \tag{9.33}$$

These are exactly $(1+b_3)$ times the A/2 correlation functions for this model given in [19][section 4.1], which are

$$\langle \sigma^2 \rangle = \frac{1}{(1+b_1)(1+b_2)(1+b_3)}, \quad \langle \sigma^5 \rangle = \frac{q}{(1+b_1)^2 (1+b_2)^2 (1+b_3)^2}. \tag{9.34}$$

As predicted, we multiply the (lower-energy) Landau-Ginzburg model correlation functions by $1/(1+b_3)$ to get the A/2 model correlation functions.

Next, consider the case of $G(2,3) = \mathbb{P}^2$. This model, mirror to a $U(2)$ gauge theory, again has no mathematically nontrivial tangent bundle deformations, but will also serve as a test of correlation functions, as nontrivial parameters do enter the GLSM and appear in correlation functions. Briefly, one now constructs a matrix of derivatives of the functions multiplying F_{11}, F_{12}, F_{21}, F_{22}, Λ_{12}, Λ_{21}, with respect to Y_{11}, Y_{12}, Y_{21}, Y_{22}, X_{12}, X_{21}, and using the methods of [49], we find

$$\langle \sigma_1^2 \rangle = \frac{1+2b_3}{\Delta} (-1 - 2I_2 - 2I_1), \tag{9.35}$$

$$\langle \sigma_1 \sigma_2 \rangle = \frac{1+2b_3}{\Delta} (2 + 2I_2 + 2I_1), \tag{9.36}$$

$$\langle \sigma_2^2 \rangle = \frac{1+2b_3}{\Delta} (-1 - 2I_2 - 2I_1), \tag{9.37}$$

where, following the notation of [19],

$$I_1 = \sum_i b_i, \tag{9.38}$$

$$I_2 = \sum_{i<j} b_i b_j, \tag{9.39}$$

$$I_3 = b_1 b_2 b_3, \tag{9.40}$$

$$\Delta = 2 \prod_{i<j} (1 + b_i + b_j) \,. \tag{9.41}$$

The correlation functions above are precisely $(1 + 2b_3)$ times the A/2 model correlation functions computed in [19], precisely as expected from the normalization subtlety discussed in section 7.3.

10. Example: Flag manifolds

In this section, we will briefly outline mirrors to flag manifolds. The GLSM describing the flag manifold $F(k_1, k_2, \cdots, k_n, N)$ is a quiver gauge theory with gauge group $U(k_1) \times \cdots \times U(k_n)$ [50]. For each $s = 1, \cdots, n-1$, there is a chiral multiplet $\Phi_{s,s+1}$ and a Fermi multiplet $\Psi_{s,s+1}$ transforming in the fundamental representation of $U(k_s)$ and in the antifundamental representation of $U(k_{s+1})$. There are also chiral multiplets $\Phi^i_{n,n+1}$ and Fermi multiplets $\Psi^i_{n,n+1}$ transforming in the fundamental representation of $U(k_n)$ for $i = 1, \cdots, N$. The E-terms of this theory are given by [21]

(10.1)

$$\overline{D}_+ \Psi_{s,s+1} \;=\; \Phi_{s,s+1}\Sigma^{(s)} \;-\; \Sigma^{(s+1)}\Phi_{s,s+1} \;+\; \sum_{t=1}^{n} u^s_t \left(\mathrm{Tr}\,\Sigma^{(t)}\right) \Phi_{s,s+1},$$

$$s = 1, \cdots, n-1,$$

$$\overline{D}_+ \Psi^i_{n,n+1} \;=\; \Phi_{n,n+1}\Sigma^{(n)} \;+\; \sum_{t=1}^{n} \left(\mathrm{Tr}\,\Sigma^{(t)}\right) A_t{}^i_j \Phi^j_{n,n+1}, \;\; i, j = 1, \cdots, N.$$

The matrices A_t are assumed to be diagonal in this paper, i.e.

$$A_t{}^i_j = A_{ti}\delta^i_j. \tag{10.2}$$

The mirror theory is a Landau-Ginzburg model consisting of chiral fields

$$\sigma^{(s)}_{a_s}, \;\; Y^{(s)}{}^{a_s}_{b_s}, \;\; X^{(s)}_{\mu_s\nu_s} \tag{10.3}$$

and Fermi fields

$$\Upsilon^{(s)}_{a_s},\ F^{(s)}{}^{a_s}_{b_s},\ \Lambda^{(s)}_{\mu_s\nu_s} \tag{10.4}$$

for $s = 1, \cdots, n$, $a_s = 1, \cdots, k_s$, $b_s = 1, \cdots, k_{s+1}$, $\mu_s, \nu_s = 1, \cdots, k_s$ and $\mu_s \neq \nu_s$ where $k_{n+1} = N$.

The superpotential is

$$\begin{aligned} W =& \sum_{s=1}^{n}\sum_{a_s=1}^{k_s} \Upsilon^{(s)}_{a_s} \left(\sum_{b_s=1}^{k_{s+1}} Y^{(s)}{}^{a_s}_{b_s} - \sum_{\alpha_s=1}^{k_{s-1}} Y^{(s-1)}{}^{\alpha_s}_{a_s} \right. \\ & \left. + \sum_{\mu_s\neq a_s} (\ln X^{(s)}_{a_s\mu_s} - \ln X^{(s)}_{\mu_s a_s}) - t_s \right) \\ & + \sum_{s=1}^{n}\sum_{a_s=1}^{k_s}\sum_{b_s=1}^{k_{s+1}} F^{(s)}{}^{a_s}_{b_s} \left(E^{(s)}{}^{a_s}_{b_s}(\sigma) - \exp\left(-Y^{(s)}{}^{a_s}_{b_s}\right)\right) \\ & + \sum_{s=1}^{n}\sum_{\mu_s\neq\nu_s} \Lambda^{(s)}_{\mu_s\nu_s} \left(1 + \frac{\sigma^{(s)}_{\mu_s} - \sigma^{(s)}_{\nu_s}}{X^{(s)}_{\mu_s\nu_s}}\right), \end{aligned} \tag{10.5}$$

where $k_0 = 0$,

$$E^{(s)}{}^{a_s}_{b_s}(\sigma) \ = \ \sigma^{(s)}_{a_s} \ - \ \sigma^{(s+1)}_{b_s} \ + \ \sum_{t=1}^{n} u^s_t \mathrm{Tr}\, \sigma^{(t)} \tag{10.6}$$

for $s = 1, \cdots, n$, $a_s = 1, \cdots, k_s$, $b_s = 1, \cdots, k_{s+1}$ and

$$E^{(n)}{}^{a_n}_{b_n}(\sigma) \ = \ \sigma^{(n)}_{a_n} \ + \ \sum_{t=1}^{n} A_{tb_n} \mathrm{Tr}\, \sigma^{(t)} \tag{10.7}$$

for $a_n = 1, \cdots, k_n$ and $b_n = 1, \cdots, N$. Again, integrating out $X^{(s)}_{\mu_s\nu_s}$ and $\Lambda^{(s)}_{\mu_s\nu_s}$ shifts the FI parameters

$$t_s \to t_s + (k_s - 1)\pi i. \tag{10.8}$$

11. Hypersurfaces

So far, our examples have involved mirrors to GLSMs without a superpotential. One can add a superpotential to the original theory, following the same prescription as [38]; namely, one assigns R-charges to the fields, and then takes the mirrors to fields with nonzero R charges, following the same pattern as in [38]. For example, if a chiral field ϕ of the original theory has R-charge r, then the fundamental field in the mirror is

$$X \equiv \exp\left(-(r/2)Y\right), \tag{11.1}$$

and the theory has a $\mathbb{Z}_{2/r}$ orbifold.

As a result, the mirror (0,2) theory does not depend upon the details of the original superpotential, only upon R-charges. For (2,2) theories, such statements are standard, but in (0,2) theories, they have come to be believed only somewhat more recently [15], and only as statements about GLSM descriptions. In any event, the point is that our mirror construction implicitly reproduces the conjecture of [15] that A/2-twisted GLSMs are independent of precise superpotential terms, and depend only upon R-charges.

12. Conclusions

In this paper we have described an extension of the nonabelian mirror proposal of [38] from two-dimensional (2,2) supersymmetric theories to (0,2) supersymmetric theories. The result is a simple systematic ansatz which both generalizes and simplifies previous approaches to Hori-Vafa-style (0,2) abelian mirrors [34–37], and also applies to nonabelian cases [38–40]. We have demonstrated that this mirror proposal has the desired properties of a gauge theoretic mirror: it reproduces symmetries, correlation functions and quantum sheaf cohomology rings, and demonstrated how one can recover the one-loop effective superpotential of the original theory, in general cases. In addition, we have checked the proposal in specific examples of mirrors in abelian and nonabelian theories.

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