

The duality covariant geometry and DSZ quantization of abelian gauge theory

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We develop the Dirac-Schwinger-Zwanziger (DSZ) quantization of classical abelian gauge theories with general duality structure on oriented and connected Lorentzian four-manifolds (M, g) of arbitrary topology, obtaining as a result the duality-covariant geometric formulation of such theories through connections on principal bundles. We implement the DSZ condition by restricting the field strengths of the theory to those which define classes originating in the degree-two cohomology of a local system valued in the groupoid of integral symplectic spaces. We prove that such field strengths are curvatures of connections \mathcal{A} defined on principal bundles P whose structure group G is the disconnected *non-abelian* group of automorphisms of an integral affine symplectic torus. The connected component of the identity of G is a torus group, while its group of connected components is a modified Siegel modular group which coincides with the group of local duality transformations of the theory. This formulation includes electromagnetic and magnetoelectric gauge potentials on an equal footing and describes the equations of motion through a first-order *polarized self-duality condition* for the curvature of \mathcal{A} . The condition involves a combination of the Hodge operator of (M, g) with a taming of the duality structure determined by P , whose choice encodes the self-couplings of the theory. This description is reminiscent of the theory of four-dimensional euclidean instantons, even though we consider a two-derivative theory in Lorentzian signature. We use this formulation to characterize the hierarchy of duality groups of abelian gauge theory, providing a gauge-theoretic description of the electromagnetic duality group as the discrete remnant of the gauge group of P . We also perform the time-like reduction of the polarized self-duality condition to a Riemannian three-manifold, obtaining a new type of Bogomolny equation which is modified by the given taming and duality structure induced by P . We give explicit examples of such solutions, which we call *polarized dyons*.

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Introduction

Abelian gauge theory on Lorentzian four-manifolds is a natural extension of Maxwell electrodynamics, which locally describes a finite number of abelian gauge fields interacting through couplings which are allowed to vary over space-time. Such theories occur frequently in high energy physics. For instance, the low energy limit of a non-abelian gauge theory coupled to scalar fields which can be maximally higgsed contains an abelian gauge theory sector; this occurs in particular on the Coulomb branch of supersymmetric non-abelian gauge theories. Moreover, the universal bosonic sector of four-dimensional ungauged supergravity involves a fixed number of abelian gauge fields interacting with each other through couplings which can vary over space-time due to their dependence on the scalars of the theory. This sector of four-dimensional supergravity can be described by pulling back an abelian gauge theory defined on the target space of the sigma model of the scalar fields [32].

The *local* behavior of abelian gauge theories (including their supersymmetric extensions) was studied intensively in the physics literature, where the subject has achieved the level of textbook material [16]. Despite intense

activity, a global geometric formulation of such theories on arbitrary spacetimes is still missing. At the level of field strengths, such a description was given in [32] (see [34] for a summary) in the wider context of the geometric description of the universal sector of classical supergravity in four dimensions. As explained there and recalled in Section 1, the global formulation requires the specification of a “duality structure”, defined as a flat symplectic vector bundle $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ on the spacetime manifold M . The even rank $2n$ of \mathcal{S} equals the number of field strengths, where *both* electromagnetic and magnetoelectric fields are included. When the spacetime is not simply connected, such a bundle need not be trivial and it ‘twists’ the local formulation in such a way that the combination of all electromagnetic and magnetoelectric field strengths can be described globally by a $d_{\mathcal{D}}$ -closed two-form \mathcal{V} defined on M and valued in \mathcal{S} , where $d_{\mathcal{D}}$ is the differential induced by the flat connection \mathcal{D} of the duality structure. A classical electromagnetic field strength *configuration* is a $d_{\mathcal{D}}$ -closed two-form \mathcal{V} . The classical equations of motion are encoded by the condition that \mathcal{V} be self-dual with respect to a ‘polarized Hodge operator’ obtained by tensoring the Hodge operator $*_g$ defined by the Lorentzian spacetime metric g with a (generally non-flat) taming \mathcal{J} of the symplectic bundle (\mathcal{S}, ω) , an object which encodes all couplings and theta angles in a fully geometric manner. A classical field strength *solution* is a field strength configuration which obeys the polarized self-duality condition. This provides a global formulation of the theory on oriented Lorentzian four manifolds of arbitrary topology, which is manifestly covariant with respect to electromagnetic duality. In this formulation, classical duality transformations are described by (based) flat symplectic automorphisms of Δ . Such theories admits global solutions which correspond to ‘classical electromagnetic U-folds’ – a notion which had been used previously in the physics literature without being given a mathematically clear definition.

While the treatment found in the physics literature discusses *local* gauge potentials and *local* gauge transformations (which are described using differential forms defined locally on spacetime), a fully general and manifestly duality-covariant geometric formulation of such theories in terms of connections on an appropriate principal bundle has not yet been given. Such a formulation is required by the Aharonov-Bohm effect [1] and by Dirac-Schwinger-Zwanziger (DSZ) “quantization” [23, 42, 47], which force the field strengths to obey an *integrality condition* implied by the requirement of a consistent coupling between classical gauge fields and quantum charge carriers. Imposing this condition restricts the set of allowed field strengths, defining a so-called *prequantum abelian gauge theory*. As pointed out in [32],

the general formulation of this condition involves the choice of a *Dirac system* \mathcal{L} for Δ , defined as a \mathcal{D} -flat fiber sub-bundle of \mathcal{S} whose fibers are full symplectic lattices inside the fibers of \mathcal{S} . Every Dirac system has a *type* $\mathbf{t} = (t_1, \dots, t_n)$, where t_1, \dots, t_n are positive integers such that $t_1 | t_2 | \dots | t_n$. A duality structure is called *semiclassical* if it admits a Dirac system. The choice of a Dirac system \mathcal{L} refines a semiclassical duality structure Δ to an *integral duality structure* $\mathbf{\Delta} = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ and reduces the group of duality transformations to a discrete group, which generalizes the arithmetic duality group known from the local formulation found in the physics literature. In the global setting, the Dirac system replaces the “Dirac lattice” of the local approach and makes the DSZ condition is rather subtle since it requires the use of cohomology with local coefficients (see [28, 43, 44, 48]). In particular, the bundle-valued two-form \mathcal{V} (which describes all electromagnetic and magnetoelectric field strengths simultaneously) *cannot* in general be the curvature of a connection defined on a principal torus bundle. Indeed, the vector bundle \mathcal{S} is generally non-trivial, while the adjoint bundle of any principal torus bundle is trivial.

In the present paper, we ‘solve’ the DSZ integrality condition determined by a Dirac system \mathcal{L} by giving the geometric formulation of prequantum abelian gauge theory in terms of connections defined on an adequate principal bundle P . More precisely, we show that the combined field strength \mathcal{V} is the curvature of a connection defined on a principal bundle with *non-abelian* and *disconnected* structure group G , whose connected component of the identity is a torus group and whose group of connected components is a modified Siegel modular group. This shows that the manifestly duality-covariant formulation of such gauge theories is not truly abelian, since it involves a non-abelian structure group. Instead, the structure group G is *weakly abelian*, in the sense that only its *Lie algebra* is abelian. As a consequence, the gauge group of P has a “discrete remnant” which encodes equivariance of the theory under electromagnetic duality transformations. Using this framework, we show that the electromagnetic duality transformations of the theory are given by (non-infinitesimal) gauge transformations, a fact that provides a geometric interpretation for the former. Principal connections \mathcal{A} on P describe the combined electromagnetic and magnetoelectric *gauge potentials* of the theory. The polarized self-duality condition becomes a first-order differential equation for \mathcal{A} which is reminiscent of the instanton equations, though the signature of our spacetime is Lorentzian and the theory is of second order. These results provide a global geometric formulation of prequantum abelian gauge theory as a theory of principal connections, in a manner that is manifestly covariant under electromagnetic duality.

To extract the principal bundle description, we proceed as follows. We first show that integral duality structures of rank $2n$ and type \mathfrak{t} are associated to *Siegel systems* of rank $2n$, which we define to be local systems Z of free abelian groups of rank $2n$ whose monodromy is contained in the *modified Siegel modular* group $\Gamma = \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ of type \mathfrak{t} . The latter is defined as the group of automorphisms of a full symplectic lattice of rank $2n$ and type \mathfrak{t} and is a subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ which contains the usual Siegel modular group $\mathrm{Sp}(2n, \mathbb{Z})$, to which it reduces when \mathfrak{t} coincides with the *principal type* $\delta = (1, \dots, 1)$. The vector bundle of the duality structure is given by $\mathcal{S} = Z \otimes_{\mathbb{Z}} \mathbb{R}$ while the flat connection \mathcal{D} is induced by the monodromy connection of Z . A classical field strength configuration \mathcal{V} is called *integral* if it satisfies the DSZ quantization condition, which states that the cohomology class $[\mathcal{V}]_{\mathcal{D}} \in H_{\mathcal{D}}^2(M, \mathcal{S})$ of \mathcal{V} with respect to $d_{\mathcal{D}}$ lies in the image of the local coefficient cohomology group $H^2(M, Z)$ through the natural morphism $H^2(M, Z) \rightarrow H_{\mathcal{D}}^2(M, \mathcal{S})$, where $H_{\mathcal{D}}^2(M, \mathcal{S})$ is the second cohomology of the complex $(\Omega^*(M, \mathcal{S}), d_{\mathcal{D}})$. We then show that \mathcal{V} is integral if and only if it coincides with the adjoint curvature $\mathcal{V}_{\mathcal{A}}$ of a connection \mathcal{A} defined on a principal bundle P (called a *Siegel bundle*) whose structure group is the group $G = \mathrm{Aff}_{\mathfrak{t}}$ of automorphisms of an *integral affine symplectic torus* and whose adjoint bundle identifies with \mathcal{S} . The group $\mathrm{Aff}_{\mathfrak{t}}$ is a semidirect product $A \rtimes \Gamma$ of the torus group $A = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the modified Siegel modular group $\Gamma = \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ and hence has a countable group of components which is isomorphic with Γ . The $d_{\mathcal{D}}$ -cohomology class of $\mathcal{V}_{\mathcal{A}}$ coincides with the image in $H_{\mathcal{D}}^2(M, \mathcal{S})$ of the *twisted Chern class* $c(P) \in H^2(M, Z)$ of P . The Siegel system Z (and hence the duality structure Δ) is uniquely determined by the Siegel bundle P and Siegel bundles are determined up to isomorphism by the pair (Z, c) . The classifying space of such principal bundles is a *twisted Eilenberg MacLane space* [27] of type 2, namely a $K(\mathbb{Z}^{2n}, 2)$ -fibration over $K(\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}), 1)$ whose κ -invariant is trivial and whose monodromy is induced by the fundamental action of $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ on \mathbb{Z}^{2n} .

There exists a large group of classical ‘pseudo-duality’ transformations which identifies the spaces of classical field strength configurations and solutions of *different* abelian gauge theories. Locally, such transformations involve matrices $T \in \mathrm{Sp}(2n, \mathbb{R})$ acting on the field strengths and the DSZ integrality condition with respect to a Dirac lattice of type \mathfrak{t} restricts T to lie in the arithmetic group $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. As already pointed out in [32], the global theory of such duality transformations is much richer. We develop this theory from scratch, defining a hierarchy of duality groups and providing short exact sequences to compute them. We exploit this geometric framework to show that the electromagnetic duality transformations of

abelian gauge theory correspond to gauge transformations of Siegel bundles, elucidating the geometric origin of electromagnetic duality. In particular, we emphasize the role played by the *type* \mathfrak{t} of the underlying Dirac system and by the monodromy representation of the Siegel system Z in the correct definition and computation of the discrete duality group of the prequantum theory. The fact that a symplectic lattice need not be of principal type is well-known in the theory of symplectic tori as well as in that of Abelian varieties, where non-principal types correspond to non-principal polarizations. The physical implications of non-principal types have been systematically explored only recently in the context of supersymmetric field theories, see for instance [4–8, 14, 15] and references therein.

The construction of the present paper produces a class of geometric gauge models which is amenable to the methods of mathematical gauge theory [24]. In particular, it allows for the study of moduli spaces of solutions (which, as shown in [32], can be viewed as ‘electromagnetic U-folds’) using techniques borrowed from the theory of instantons. In this spirit, we perform the time-like reduction of the polarized self-duality equations, obtaining a novel system of Bogomolny-like equations. Solutions of these equations define *polarized abelian dyons*, of which we describe a few examples.

The paper is organized as follows. Section 1 recalls the description of classical abelian gauge theories with arbitrary duality structure in terms of combined electromagnetic and magnetoelectric field strengths, following [32]. In the same section, we describe the hierarchy of duality groups of such theories and give a few short exact sequences to characterize them. Section 2 discusses the DSZ integrality condition for general duality structures, relating the notion of Dirac system (which appears in its formulation) to various equivalent objects. Section 3 discusses Siegel bundles and connections showing in particular that a Siegel induces a canonical integral duality structure. In Section 4, we give the formulation of “prequantum” abelian gauge theory (defined as classical abelian gauge theory supplemented by the DSZ integrality condition) as a theory of principal connections on a Siegel bundle. Section 5 discusses the time-like dimensional reduction of the polarized self-duality equations, which leads to the notion of polarized abelian dyon. In the same section, we construct examples of polarized abelian dyons on the punctured affine 3-space. Appendix A recalls the duality-covariant formulation of abelian gauge theory on contractible Lorentzian four-manifolds, starting from the local treatment found in the physics literature. Appendix B discusses integral symplectic spaces and integral symplectic tori, introducing certain notions used in the main text.

0.1. Notations and conventions

All manifolds and fiber bundles considered in the paper are smooth, Hausdorff and paracompact. The manifold denoted by M is assumed to be connected. In our convention, a Lorentzian four-manifold (M, g) has “mostly plus” signature $(3, 1)$. If E is a fiber bundle defined on a manifold M , we denote by $\mathcal{C}^\infty(M, E)$ the set of globally-defined smooth sections of E and by $\mathcal{C}^\infty(E)$ the sheaf of smooth sections of E .

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1. Classical abelian gauge theory

In this section we introduce the configuration space and equations of motion defining *classical abelian gauge theory* on an oriented Lorentzian four-manifold (M, g) and discuss its global dualities and symmetries. Appendix A gives the description of this theory for the special case when M is contractible, which recovers the local treatment found in the physics literature. The global formulation presented in this section was proposed in a wider context in reference [32], to which we refer the reader for certain details. The definition of classical abelian gauge theory on (M, g) is given in terms of field strengths and relies on the choice of a *duality structure* (defined as a flat symplectic vector bundle $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ on M) equipped with a *taming* \mathcal{J} of (\mathcal{S}, ω) , which encodes the gauge-kinetic functions (coupling constants and theta angles) in a globally-correct and frame-independent manner. The equations of motion of the theory are encoded by the \mathcal{J} -polarized *self-duality condition* for the combined electromagnetic and magnetoelectric *field strengths*, which are modeled mathematically by a \mathcal{D} -flat two-form valued in the underlying vector bundle \mathcal{S} of the duality structure.

1.1. Duality structures

Let M be a connected smooth manifold and (\mathcal{S}, ω) be a symplectic vector bundle defined on M with symplectic structure ω . Since $\mathrm{Sp}(2n, \mathbb{R})$ and

$\mathrm{GL}(n, \mathbb{C})$ are homotopy equivalent to their common maximal compact subgroup $\mathrm{U}(n)$, the classification of symplectic, complex and Hermitian vector bundles defined on M are equivalent. In particular, any complex vector bundle admits a Hermitian pairing and any symplectic vector bundle admits a complex structure which is compatible with its symplectic pairing and makes it into a Hermitian vector bundle. Thus a real vector bundle of even rank admits a symplectic pairing if and only if it admits a complex structure. The classifying spaces $\mathrm{BSp}(2n, \mathbb{R})$ and $\mathrm{BU}(n)$ are homotopy equivalent, hence the fundamental characteristic classes of a symplectic vector bundle (\mathcal{S}, ω) are Chern classes, which we denote by $c_k(\mathcal{S}, \omega)$.

Remark 1.1. Suppose that $\dim M = 4$ and let \mathcal{S} be an oriented real vector bundle of rank $2n$ defined on M , thus $w_1(\mathcal{S}) = 0$. If \mathcal{S} admits a complex structure \mathcal{J} inducing its orientation, then its third Stiefel-Whitney class $w_3(\mathcal{S})$ must vanish and its even Stiefel-Whitney classes $w_2(\mathcal{S})$ and $w_4(\mathcal{S})$ must coincide with the mod 2 reduction of the Chern classes $c_1(\mathcal{S}, \mathcal{J})$ and $c_2(\mathcal{S}, \mathcal{J})$ of the complex rank n vector bundle defined by \mathcal{S} and \mathcal{J} . In particular, the third *integral* Stiefel-Whitney class $W_3(\mathcal{S}) \in H^3(M, \mathbb{Z})$ must vanish, i.e. \mathcal{S} must admit a Spin^c structure (notice that $W_5(\mathcal{S})$ vanishes for dimension reasons). These conditions are not always sufficient. To state the necessary and sufficient conditions (see [37]), we distinguish the cases:

- $n = 1$, i.e. $\mathrm{rk}\mathcal{S} = 2$. Then \mathcal{S} always admits a complex structure (equivalently, a symplectic pairing) which induces its orientation (since $\mathrm{SO}(2) = \mathrm{U}(1)$).
- $n = 2$, i.e. $\mathrm{rk}\mathcal{S} = 4$. In this case, \mathcal{S} admits a complex structure (equivalently, a symplectic pairing) which induces its orientation if and only if it satisfies $W_3(\mathcal{S}) = 0$ and $\mathfrak{c}^4(\mathcal{S}) = 0$, where $\mathfrak{c}^4(\mathcal{S}) \in H^4(M, \mathbb{Z})$ is an integral obstruction class described in [37, Theorem II].
- $n \geq 3$, i.e. $\mathrm{rk}\mathcal{S} \geq 6$. Then \mathcal{S} admits a complex structure (equivalently, a symplectic pairing ω) which induces its orientation if and only if $W_3(\mathcal{S}) = 0$.

Notice that an oriented real vector bundle \mathcal{S} of rank four on a four-manifold M is determined up to isomorphism by its first Pontryagin class $p_1(\mathcal{S}) \in H^4(M, \mathbb{Z})$, its second Stiefel-Whitney class $w_2(\mathcal{S}) \in H^2(M, \mathbb{Z}_2)$ and its Euler class $e(\mathcal{S}) \in H^4(M, \mathbb{Z})$.

Definition 1.2. A *duality structure* $\Delta \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$ on M is a flat symplectic vector bundle (\mathcal{S}, ω) over M equipped with a flat connection \mathcal{D} :

$\mathcal{C}^\infty(M, \mathcal{S}) \rightarrow \Omega^1(M, \mathcal{S})$ which preserves ω . The *rank* of Δ is the rank of the vector bundle \mathcal{S} , which is necessarily even.

Notice that the Chern classes $c_1(\mathcal{S}, \omega)$ and $c_2(\mathcal{S}, \omega)$ of the underlying symplectic vector bundle of a duality structure must be torsion classes.

Definition 1.3. A *based isomorphism of duality structures* from $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ to $\Delta' = (\mathcal{S}', \omega', \mathcal{D}')$ is a based isomorphism of vector bundles $f : \mathcal{S} \xrightarrow{\sim} \mathcal{S}'$ which satisfies the conditions $\omega' \circ (f \otimes f) = \omega$ and $\mathcal{D}' \circ f = (\text{id}_{T^*M} \otimes f) \circ \mathcal{D}$.

Here and below, we let based morphisms of vector bundles act on sections in the natural manner. We denote by $\text{Dual}(M)$ be the groupoid of duality structures defined on M and based isomorphisms of such.

The group $\text{Aut}_b(\mathcal{S}, \omega)$ of based automorphisms of a symplectic vector bundle (\mathcal{S}, ω) is called its group of *symplectic gauge transformations*. Such transformations φ act on the set of linear connections \mathcal{D} defined on \mathcal{S} through:

$$\mathcal{D} \rightarrow (\text{id}_{T^*M} \otimes \varphi) \circ \mathcal{D} \circ \varphi^{-1}$$

and preserve the set of flat symplectic connections. The group $\text{Aut}_b(\Delta)$ of based automorphism of a duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ coincides with the stabilizer of \mathcal{D} in $\text{Aut}_b(\mathcal{S}, \omega)$. For any such duality structure, we have:

$$c_1(\mathcal{S}, \omega) = \delta(\hat{c}_1(\mathcal{D})), \quad c_2(\mathcal{S}, \omega) = \delta(\hat{c}_2(\mathcal{D})),$$

where $\hat{c}_1(\mathcal{D}) \in H^1(M, \text{U}(1))$ and $\hat{c}_2(\mathcal{D}) \in H^3(M, \text{U}(1))$ are the Cheeger-Chern-Simons invariants of the flat connection \mathcal{D} and $\delta : H^i(M, \mathbb{C}/\mathbb{Z}) \rightarrow H^{i+1}(M, \mathbb{Z})$ are the Bockstein morphisms in the long exact sequence:

$$\begin{aligned} \dots \rightarrow H^i(M, \mathbb{Z}) \rightarrow H^i(M, \mathbb{R}) \xrightarrow{\text{exp}^*} H^i(M, \text{U}(1)) \\ \xrightarrow{\delta} H^{i+1}(M, \mathbb{Z}) \rightarrow H^{i+1}(M, \mathbb{R}) \rightarrow \dots \end{aligned}$$

induced by the exponential sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \text{U}(1) \rightarrow 0.$$

The Cheeger-Chern-Simons invariants depend only on the gauge equivalence class of \mathcal{D} .

1.2. The twisted de Rham complex of a duality structure

Given a duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ on a connected manifold M , let $d_{\mathcal{D}} : \Omega(M, \mathcal{S}) \rightarrow \Omega(M, \mathcal{S})$ be the exterior covariant derivative twisted by \mathcal{D} (notice that $d_{\mathcal{D}}|_{\Omega^0(M, \mathcal{S})} = \mathcal{D}$). This defines a cochain complex:

$$(1) \quad 0 \rightarrow \mathcal{C}^\infty(M, \mathcal{S}) \xrightarrow{\mathcal{D}} \Omega^1(M, \mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^2(M, \mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^3(M, \mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^4(M, \mathcal{S}) \rightarrow 0,$$

whose total cohomology group (viewed as a \mathbb{Z} -graded abelian group) we denote by $H_{\mathcal{D}}^*(M, \mathcal{S})$.

Definition 1.4. The vector spaces $H_{\mathcal{D}}^k(M, \mathcal{S})$ (where $k = 0, \dots, 4$) are called the *twisted de Rham cohomology spaces* of the duality structure Δ .

Let $\Omega_{\text{flat}}^k(\mathcal{S})$ be the locally-constant sheaf of \mathcal{D} -flat \mathcal{S} -valued k -forms. Then a straightforward modification of the Poincaré lemma shows that the complex of sheaves:

$$0 \rightarrow \mathcal{C}_{\text{flat}}^\infty(\mathcal{S}) \hookrightarrow \Omega^0(\mathcal{S}) \xrightarrow{\mathcal{D}} \Omega^1(\mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^2(\mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^3(\mathcal{S}) \xrightarrow{d_{\mathcal{D}}} \Omega^4(\mathcal{S}) \rightarrow 0$$

is exact and hence provides a resolution of the locally-constant sheaf $\mathcal{C}_{\text{flat}}^\infty(\mathcal{S})$. Since M is paracompact, each of the sheaves $\Omega^k(\mathcal{S})$ is acyclic. Thus the sheaf cohomology of $\Omega_{\text{flat}}^0(\mathcal{S})$ can be computed as the cohomology of the complex (1). This gives a natural isomorphism of graded vector spaces:

$$H_{\mathcal{D}}^*(M, \mathcal{S}) \simeq H^*(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{S})),$$

where the right hand side denotes sheaf cohomology.

1.3. Flat systems of symplectic vector spaces

Let $\Pi_1(M)$ be the fundamental groupoid of a connected manifold M , whose objects are the points of M and whose set of morphisms $\Pi_1(m, m')$ from m to m' is the set of homotopy classes of piecewise-smooth curves starting at m and ending at m' . Let Symp be the groupoid of finite-dimensional symplectic vector spaces (see Appendix B). The functor category $[\Pi_1(M), \text{Symp}]$ is a groupoid since all its morphisms (which are natural transformations) are invertible.

Definition 1.5. A *flat system of symplectic vector spaces* (or *Symp-valued local system*) on M is a functor $F : \Pi_1(M) \rightarrow \text{Symp}$, i.e. an object of the

groupoid $[\Pi_1(M), \text{Symp}]$. An isomorphism of such systems is an isomorphism in this groupoid. A flat system F of symplectic vector spaces has rank¹ $2n$ if $\dim F(m) = 2n$ for all $m \in M$.

Recall that a *symplectic representation* of a group G is a representation $\rho : G \rightarrow \text{Aut}(V, \omega)$ of G through automorphisms of a finite-dimensional symplectic vector space (V, ω) . An *equivalence (or isomorphism) of symplectic representations* from $\rho : G \rightarrow \text{Aut}(V, \omega)$ to $\rho' : G \rightarrow \text{Aut}(V', \omega')$ is an isomorphism of symplectic vector spaces $\varphi : (V, \omega) \xrightarrow{\sim} (V', \omega')$ such that $\varphi \circ \rho(g) = \rho'(g) \circ \varphi$ for all $g \in G$. We denote by $\text{SympRep}(G)$ the groupoid of symplectic representations of G equipped with this notion of isomorphism.

Definition 1.6. Let F be a flat system of symplectic vector spaces on M . The *holonomy representation* of F at a point $m \in M$ is the morphism of groups $\text{hol}_m(F) : \pi_1(M, m) = \Pi_1(m, m) \rightarrow \text{Aut}(F(m)) = \text{Symp}(F(m), F(m))$ defined through:

$$\text{hol}_m(F)(\mathbf{c}) = F(\mathbf{c}), \quad \forall \mathbf{c} \in \pi_1(M, m).$$

The *holonomy group* of F at m is the subgroup of $\text{Aut}(F(m))$ defined through:

$$\text{Hol}_m(F) \stackrel{\text{def.}}{=} \text{im}(\text{hol}_m(F)).$$

Notice that $\text{hol}_m(F)$ is a symplectic representation of $\pi_1(M, m)$ on $F(m)$. Any homotopy class $\gamma \in \Pi_1(m, m')$ of paths from m to m' induces an isomorphism of symplectic vector spaces $F(\gamma) : F(m) \xrightarrow{\sim} F(m')$ which intertwines the holonomy representations of F at the points m and m' :

$$F(\gamma) \circ \text{hol}_m(F)(\mathbf{c}) = \text{hol}_{m'}(F)(\gamma \mathbf{c} \gamma^{-1}) \circ F(\gamma), \quad \forall \mathbf{c} \in \pi_1(M, m).$$

Since M is connected, it follows that the holonomy representation of F at a fixed basepoint $m_0 \in M$ determines its holonomy representation at any other point of M . Moreover, the isomorphism class of the holonomy group of F at m_0 does not depend on the choice of $m_0 \in M$.

Proposition 1.7. *Let $m_0 \in M$ be any point of M . Then the map hol_{m_0} which sends a Symp-valued local system F defined on M to its holonomy representation $\text{hol}_{m_0}(F)$ at m_0 and sends an isomorphism $f : F \xrightarrow{\sim} F'$ of Symp-valued local systems to the equivalence of representations $\text{hol}_{m_0}(f) \stackrel{\text{def.}}{=}$*

¹The rank is constant on M since M is connected.

$f(m_0) : \text{hol}_{m_0}(F) \xrightarrow{\sim} \text{hol}_{m_0}(F')$ defines an equivalence of groupoids from $[\Pi_1(M), \text{Symp}]$ to $\text{SympRep}(\pi_1(M, m_0))$. In particular, isomorphism classes of flat systems of symplectic vector spaces of rank $2n$ defined on M are in bijection with the points of the character variety:

$$(2) \quad \mathfrak{R}(\pi_1(M, m_0), \text{Sp}(2n, \mathbb{R})) \stackrel{\text{def.}}{=} \text{Hom}(\pi_1(M, m_0), \text{Sp}(2n, \mathbb{R})) / \text{Sp}(2n, \mathbb{R}),$$

where $\text{Sp}(2n, \mathbb{R})$ acts on $\text{Hom}(\pi_1(M), \text{Sp}(2n, \mathbb{R}))$ through its adjoint representation.

Proof. An isomorphism $f : F \xrightarrow{\sim} F'$ of Symp -valued local systems is a collection of isomorphisms of symplectic vector spaces $f(m) : F(m) \rightarrow F'(m)$ for all $m \in M$ which satisfies:

$$f(m) \circ F(\gamma) = F'(\gamma) \circ f(m), \quad \forall \gamma \in \Pi_1(m, m'), \quad \forall m, m' \in M.$$

Taking $m' = m = m_0$ in this relation shows that $f(m)$ is an equivalence between the holonomy representations of F and F' at m :

$$\text{hol}_{m_0}(F')(\mathbf{c}) = f(m) \circ \text{hol}_{m_0}(F)(\mathbf{c}) \circ f(m_0)^{-1}, \quad \forall \mathbf{c} \in \pi_1(M, m_0).$$

Conversely, it is easy to see that any such equivalence of representations extends to an isomorphism from F to F' . □

1.4. The flat system of symplectic vector spaces defined by a duality structure

Definition 1.8. The *parallel transport functor* $\mathcal{T}_\Delta \in [\Pi_1(M), \text{Symp}]$ of a duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is the functor which associates to each point $m \in M$ the symplectic vector space $\mathcal{T}_\Delta(m) \stackrel{\text{def.}}{=} (\mathcal{S}_m, \omega_m)$ and to the homotopy class (with fixed endpoints) $\mathbf{c} \in \Pi_1(M)$ of any piecewise-smooth curve $c : [0, 1] \rightarrow M$ the isomorphism of symplectic vector spaces:

$$\mathcal{T}_\Delta(\mathbf{c}) : (\mathcal{S}_{c(0)}, \omega_{c(0)}) \xrightarrow{\sim} (\mathcal{S}_{c(1)}, \omega_{c(1)})$$

given by the parallel transport of \mathcal{D} .

Notice that $\mathcal{T}_\Delta : \Pi_1(M) \rightarrow \text{Symp}$ is a flat system of symplectic vector spaces. The map which sends Δ to \mathcal{T}_Δ extends in an obvious manner to an equivalence of groupoids:

$$\mathcal{T} : \text{Dual}(M) \xrightarrow{\sim} [\Pi_1(M), \text{Symp}],$$

which sends a based isomorphism $f : \Delta = (\mathcal{S}, \omega, \mathcal{D}) \xrightarrow{\sim} \Delta' = (\mathcal{S}', \omega', \mathcal{D}')$ of duality structures to the invertible natural transformation $\mathcal{T}(f) : \mathcal{T}_\Delta \xrightarrow{\sim} \mathcal{T}_{\Delta'}$ given by the isomorphisms of symplectic vector spaces:

$$\mathcal{T}(f)(m) \stackrel{\text{def.}}{=} f_m : (\mathcal{S}_m, \omega_m) \xrightarrow{\sim} (\mathcal{S}'_m, \omega_m), \quad \forall m \in M.$$

These isomorphisms intertwine $\mathcal{T}^\Delta(\mathbf{c})$ and $\mathcal{T}^{\Delta'}(\mathbf{c})$ for any $\mathbf{c} \in \Pi_1(M)(m, m')$ since f satisfies $\mathcal{D}' \circ f = (\text{id}_{T^*M} \otimes f) \circ \mathcal{D}$. Hence one can identify duality structures and systems of flat symplectic vector spaces defined on M . For any duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ on M , the holonomy representation $\text{hol}_m(\mathcal{D})$ of the flat connection \mathcal{D} at $m \in M$ coincides with the holonomy representation of the flat system of symplectic vector spaces defined by Δ :

$$\text{hol}_m(\mathcal{D}) = \text{hol}_m(\mathcal{T}_\Delta).$$

In particular, the holonomy groups of \mathcal{D} and \mathcal{T}_Δ at $m \in M$ coincide:

$$\text{Hol}_m(\mathcal{D}) = \text{Hol}_m(\mathcal{T}_\Delta).$$

Since M is connected, Proposition 1.7 implies that isomorphism classes of duality structures defined on M are in bijection with the character variety (2).

1.5. Trivial duality structures

Definition 1.9. Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be a duality structure defined on M . We say that Δ is:

- 1) *topologically trivial*, if the vector bundle \mathcal{S} is trivializable, i.e. if it admits a global frame.
- 2) *symplectically trivial* if the symplectic vector bundle (\mathcal{S}, ω) is isomorphic to the trivial symplectic vector bundle, i.e. if it admits a global symplectic frame (a global frame in which ω has the standard form).
- 3) *trivial* (or *holonomy trivial*), if Δ admits a global *flat* symplectic frame, i.e. if the holonomy group of \mathcal{D} is the trivial group.

A symplectically trivial duality structure is automatically topologically trivial, while a holonomy trivial duality structure is symplectically trivial. If M is simply-connected then every duality structure is holonomy trivial. A global flat symplectic frame of a holonomy-trivial duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ of rank $2n$ has the form:

$$\mathcal{E} = (e_1, \dots, e_n, f_1, \dots, f_n),$$

where e_i, f_j are \mathcal{D} -flat sections of \mathcal{S} such that:

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \omega(e_i, f_j) = -\omega(f_i, e_j) = \delta_{ij}, \quad \forall i, j = 1, \dots, n.$$

Any choice of such a frame induces a trivialization isomorphism $\tau_{\mathcal{E}} : \Delta \xrightarrow{\sim} \Delta_n$ between Δ and the *canonical trivial duality structure*:

$$(3) \quad \Delta_n \stackrel{\text{def.}}{=} (\mathbb{R}^{2n}, \underline{\omega}_{2n}, d),$$

of rank $2n$, where $\underline{\omega}_{2n}$ is the constant symplectic pairing induced on the trivial vector bundle $\mathbb{R}^{2n} = M \times \mathbb{R}^{2n}$ by the canonical symplectic pairing ω_{2n} of \mathbb{R}^{2n} and $d : \mathcal{C}^\infty(M, \mathbb{R}^{2n}) \rightarrow \Omega^1(M, \mathbb{R}^{2n})$ is the ordinary differential. Any two flat symplectic frames \mathcal{E} and \mathcal{E}' of Δ are related by a based automorphism $T \in \text{Aut}_b(\Delta)$:

$$\mathcal{E}' = T\mathcal{E},$$

which corresponds to the constant automorphism $\tau_{\mathcal{E}'} \circ \tau_{\mathcal{E}}^{-1} \in \text{Aut}_b(\Delta_n)$ of Δ_n induced by an element $\hat{T} \in \text{Sp}(2n, \mathbb{R})$.

1.6. Electromagnetic structures

As before, let M be a connected manifold.

Definition 1.10. An *electromagnetic structure* defined on M is a pair $\Xi = (\Delta, \mathcal{J})$, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is a duality structure on M and $\mathcal{J} \in \text{End}(\mathcal{S})$ is a *taming* of the symplectic vector bundle (\mathcal{S}, ω) , i.e. a complex structure on \mathcal{S} which:

- is compatible with ω , i.e. it satisfies:

$$\omega(\mathcal{J}x, \mathcal{J}y) = \omega(x, y) \quad \forall (x, y) \in \mathcal{S} \times_M \mathcal{S}$$

- has the property that the symmetric bilinear pairing $Q_{\mathcal{J},\omega}$ defined on \mathcal{S} through:

$$(4) \quad Q_{\mathcal{J},\omega}(x, y) \stackrel{\text{def.}}{=} \omega(\mathcal{J}x, y) \quad \forall (x, y) \in \mathcal{S} \times_M \mathcal{S}$$

is positive-definite.

The *rank* of Ξ is the rank of Δ .

Notice that the taming \mathcal{J} is *not* required to be flat with respect to \mathcal{D} . The bundle-valued one-form:

$$\Psi_{\Xi} \stackrel{\text{def.}}{=} \mathcal{D}^{\text{ad}}(\mathcal{J}) = \mathcal{D} \circ \mathcal{J} - (\text{id}_{T^*M} \otimes \mathcal{J}) \circ \mathcal{D} \in \Omega^1(M, \text{End}(\mathcal{S}))$$

is called the *fundamental form* of Ξ and measures the failure of \mathcal{D} to preserve \mathcal{J} . The electromagnetic structure Ξ is called *unitary* if $\Psi_{\Xi} = 0$. We refer the reader to [32] for more detail on the fundamental form.

Definition 1.11. Let $\Xi = (\Delta, \mathcal{J})$ and $\Xi' = (\Delta', \mathcal{J}')$ be two electromagnetic structures defined on M . A *based isomorphism of electromagnetic structures* from Ξ to Ξ' is a based isomorphism of duality structures $f : \Delta \xrightarrow{\sim} \Delta'$ such that $\mathcal{J}' = f \circ \mathcal{J} \circ f^{-1}$.

Remark 1.12. A taming \mathcal{J} of a duality structure Δ can also be described using a *positive complex polarization*. Let $\Delta_{\mathbb{C}} = (\mathcal{S}_{\mathbb{C}}, \omega_{\mathbb{C}}, \mathcal{D}_{\mathbb{C}})$ be the complexification of Δ . Then \mathcal{J} is equivalent to a complex Lagrangian sub-bundle $L \subset \mathcal{S}_{\mathbb{C}}$ such that:

$$\mathbf{i}\omega(x, \bar{x}) > 0, \quad \forall x \in \dot{\mathcal{S}}_{\mathbb{C}},$$

where $\dot{\mathcal{S}}_{\mathbb{C}}$ is the complement of the zero section in $\mathcal{S}_{\mathbb{C}}$. By definition, such a Lagrangian sub-bundle is a positive complex polarization of the symplectic vector bundle (\mathcal{S}, ω) . A detailed description of this correspondence can be found in [32, Appendix A]. Note that the physics literature sometimes uses a local version of positive polarizations when discussing Abelian gauge theory. In our global framework this requires the supplementary step of complexifying the vector bundle \mathcal{S} .

Definition 1.13. A *taming map* of size $2n$ defined on M is a smooth map $\mathcal{J} : M \rightarrow \text{Mat}(2n, \mathbb{R})$ such that $\mathcal{J}(m)$ is a taming of the standard symplectic form ω_{2n} of \mathbb{R}^{2n} for all $m \in M$ (in particular, we have $\mathcal{J}(m) \in \text{Sp}(2n, \mathbb{R})$ for all m).

We denote by $\mathfrak{J}_n(M)$ the set of all taming maps of size $2n$. Let $\Xi = (\Delta, \mathcal{J})$ be an electromagnetic structure of rank $2n$ defined on M whose underlying duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is holonomy trivial. Choosing a flat global symplectic frame \mathcal{E} , we can identify Δ with the canonical trivial duality structure (3) through an isomorphism $\tau_{\mathcal{E}} : \Delta \xrightarrow{\sim} \Delta_n$. This identifies \mathcal{J} with the section $\tau_{\mathcal{E}} \circ \mathcal{J} \circ \tau_{\mathcal{E}}^{-1}$ of the trivial vector bundle $\underline{\mathbb{R}}^{2n}$, which in turn can be viewed as a smooth map from M to \mathbb{R}^{2n} . Since $\tau_{\mathcal{E}}$ transports ω to the constant symplectic form induced on $\underline{\mathbb{R}}^{2n}$ by ω_{2n} , it is easy to see that this map is a taming map of size $2n$ defined on M . Hence any choice of global flat symplectic frame identifies the set of tamings of Δ with $\mathfrak{J}_n(M)$.

1.7. The polarized Hodge operator

Let (M, g) be an oriented Lorentzian four-manifold. Let $\Xi = (\Delta, \mathcal{J})$ be an electromagnetic structure of rank $2n$ defined on M and let $Q := Q_{\mathcal{J}, \omega}$ be the Euclidean scalar product induced by \mathcal{J} and ω on \mathcal{S} as in equation (4). The Hodge operator $*_g$ of (M, g) extends trivially to an automorphism of the vector bundle $\Lambda^*(M, \mathcal{S}) \stackrel{\text{def.}}{=} \Lambda^*T^*M \otimes \mathcal{S}$, which we denote by the same symbol.

Definition 1.14. The \mathcal{J} -polarized Hodge operator of (M, g, Ξ) is the automorphism $\star_{g, \mathcal{J}}$ of the vector bundle $\Lambda^*(M, \mathcal{S})$ defined through:

$$\star_{g, \mathcal{J}} \stackrel{\text{def.}}{=} *_g \otimes \mathcal{J} = \mathcal{J} \otimes *_g.$$

Let $(\cdot, \cdot)_g$ be the pseudo-Euclidean scalar product induced by g on the total exterior bundle $\Lambda^*(M) \stackrel{\text{def.}}{=} \Lambda^*T^*M$. Together with Q , this pairing induces a pseudo-Euclidean scalar product $(\cdot, \cdot)_{g, Q}$ on the vector bundle $\Lambda^*(M, \mathcal{S})$, which is uniquely determined by the condition:

$$(\rho_1 \otimes \xi_1, \rho_2 \otimes \xi_2)_{g, Q} = \delta_{k_1, k_2} (-1)^{k_1} Q(\xi_1, \xi_2) (\rho_1, \rho_2)_g$$

for all $\rho_1 \in \Omega^{k_1}(M)$, $\rho_2 \in \Omega^{k_2}(M)$ and $\xi_1, \xi_2 \in \mathcal{C}^\infty(M, \mathcal{S})$. The polarized Hodge operator $\star_{g, \mathcal{J}}$ induces an involutive automorphism of the space $\Omega^2(M, \mathcal{S})$. We have a $(\cdot, \cdot)_{g, Q}$ -orthogonal splitting:

$$\Lambda^2(M, \mathcal{S}) = \Lambda^2_+(M, \mathcal{S}) \oplus \Lambda^2_-(M, \mathcal{S})$$

into eigenbundles of $\star_{g, \mathcal{J}}$ with eigenvalues $+1$ and -1 for the polarized Hodge operator. The sections of $\Lambda^2_+(M, \mathcal{S})$ and $\Lambda^2_-(M, \mathcal{S})$ are called respectively

polarized self-dual and *polarized anti-self-dual* \mathcal{S} -valued two-forms. Notice that the definition of these notions does *not* require complexification of \mathcal{S} .

1.8. Classical abelian gauge theory

Let (M, g) be an oriented Lorentzian four-manifold.

Definition 1.15. The *classical configuration space* defined by the duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ on M is the vector space of two-forms valued in \mathcal{S} which are closed with respect to $d_{\mathcal{D}}$:

$$\text{Conf}(M, \Delta) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \Omega^2(M, \mathcal{S}) \mid d_{\mathcal{D}}\mathcal{V} = 0 \} .$$

Definition 1.16. The *classical abelian gauge theory* determined by an electromagnetic structure $\Xi = (\Delta, \mathcal{J})$ on (M, g) is defined by the polarized self-duality condition:

$$\star_{g, \mathcal{J}} \mathcal{V} = \mathcal{V}$$

for $\mathcal{V} \in \text{Conf}(M, \Delta)$. The solutions of this equation are called *classical electromagnetic field strengths* and form the vector space:

$$\text{Sol}(M, g, \Xi) = \text{Sol}(M, g, \Delta, \mathcal{J}) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \text{Conf}(M, \Delta) \mid \star_{g, \mathcal{J}} \mathcal{V} = \mathcal{V} \} .$$

Notice that classical abelian gauge theory is formulated in terms of field strengths. In later sections, we will formulate the pre-quantum version of this theory (which is obtained by imposing an appropriate DSZ quantization condition) in terms of connections defined on certain principal bundles called *Siegel bundles* (see Definition 3.2). The classical theory simplifies when M is contractible, since in this case any duality structure defined on M is holonomy-trivial. This case corresponds to the local abelian gauge theory discussed in Appendix A, which makes contact with the formulation used in the physics literature. Notice that the traditional physics formulation (which is valid only locally or for holonomy-trivial duality structures) relies on gauge-kinetic functions, leading to complicated formulas which obscure the geometric structure displayed by Definition 1.16. It was shown in [32] that elements of $\text{Sol}(M, g, \Xi)$ correspond to classical electromagnetic U-folds when the duality structure Δ is not holonomy trivial.

1.9. Classical duality groups

Let (M, g) be an oriented Lorentzian four-manifold and \mathcal{S} be a vector bundle defined on M . Let $\text{Aut}(\mathcal{S})$ be the group of those *unbased* vector bundle automorphisms of \mathcal{S} which cover orientation-preserving diffeomorphisms of M and let $\text{Aut}_b(\mathcal{S}) \subset \text{Aut}(\mathcal{S})$ the subgroup of based automorphisms, i.e. those automorphisms which cover the identity of M . Let $\text{Diff}(M)$ be the group of *orientation-preserving* diffeomorphisms of M . Given $u \in \text{Aut}(\mathcal{S})$, let $f_u \in \text{Diff}(M)$ be the orientation-preserving diffeomorphism of M covered by u . This defines a morphism of groups:

$$\text{Aut}(\mathcal{S}) \ni u \rightarrow f_u \in \text{Diff}(M).$$

The group $\text{Aut}(\mathcal{S})$ admits an \mathbb{R} -linear representation:

$$\text{Aut}(\mathcal{S}) \ni u \rightarrow \mathbb{A}_u \in \text{Aut}(\Omega^*(M, \mathcal{S}))$$

given by push-forward; we will occasionally also use the notation: $u \cdot (-) \stackrel{\text{def.}}{=} \mathbb{A}_u(-)$. When f_u is not the identity the push-forward action of $u \in \text{Aut}(\mathcal{S})$ must be handled with care; we refer the reader to [32, Appendix D] for a detailed description of this operation and its properties. On decomposable elements of $\Omega^*(M, \mathcal{S}) = \Omega^*(M) \otimes C^\infty(M, \mathcal{S})$, it is given by:

$$\mathbb{A}_u(\alpha \otimes \xi) = u \cdot (\alpha \otimes \xi) = (f_{u*}\alpha) \otimes (u \cdot \xi) = (f_{u*}\alpha) \otimes (u \circ \xi \circ f_u^{-1}),$$

where $f_{u*}\alpha$ is the push-forward of α on M as defined in [32, Appendix D]. For instance, if $\alpha \in \Omega^1(M)$ we have:

$$\begin{aligned} (f_{u*}\alpha)(v) &= \alpha(f_{u*}^{-1}(v)) \circ f_u^{-1} \\ &= \alpha(df_u^{-1}(v) \circ f_u) \circ f_u^{-1} \in C^\infty(M), \quad \forall v \in \mathfrak{X}(M). \end{aligned}$$

In particular, restricting to a given point $m \in M$ we obtain the familiar formula:

$$(f_{u*}\alpha)(v)_m = \alpha(f_{u*}^{-1}(v))_{f_u^{-1}(m)} = \alpha_{f_u^{-1}(m)}(df_u^{-1}|_m(v_m)).$$

Recall that $f_{u*}^{-1}(v) \in \mathfrak{X}(M)$ is again a vector field on M whereas $df_u^{-1}(v): M \rightarrow TM$ is a vector field *along* f_u . For any duality structure

$\Delta = (\mathcal{S}, \omega, \mathcal{D})$ defined on M , let:

$$\text{Aut}(\Delta) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(\mathcal{S}) \mid \omega_u = \omega, \mathcal{D}_u = \mathcal{D}\},$$

be the group of *unbased* automorphisms of Δ , defined as the stabilizer of the pair (ω, \mathcal{D}) in $\text{Aut}(\mathcal{S})$. Here \mathcal{D}_u is the push-forward of the connection \mathcal{D} by u , which is defined through:

$$\begin{aligned} (\mathcal{D}_u)_v(s) &= u \cdot \mathcal{D}_{f_{u*}^{-1}(v)}(u^{-1} \cdot s) \\ &= u \circ (\mathcal{D}_{f_{u*}^{-1}(v)}(u^{-1} \circ s \circ f_u)) \circ f_u^{-1}, \quad \forall s \in \Gamma(\mathcal{S}) \quad \forall v \in \mathfrak{X}(M), \end{aligned}$$

where $f_{u*}^{-1} \cdot v = df_u^{-1}(v) \circ f_u$. Note that if $s \in \Gamma(\mathcal{S})$ is a parallel section of \mathcal{S} with respect to \mathcal{D} , then $u \cdot s \in \Gamma(\mathcal{S})$ is parallel with respecto \mathcal{D}_u for all $u \in \text{Aut}(\Delta)$. We have a short exact sequence:

$$1 \rightarrow \text{Aut}_b(\Delta) \rightarrow \text{Aut}(\Delta) \rightarrow \text{Diff}_\Delta(M) \rightarrow 1,$$

where $\text{Diff}_\Delta(M) \subset \text{Diff}(M)$ is the subgroup formed by those orientation-preserving diffeomorphisms of M that can be covered by elements of $\text{Aut}(\Delta)$. Given a taming $\mathcal{J} \in \text{End}(\mathcal{S})$ of (\mathcal{S}, ω) , we define:

$$\mathcal{J}_u \stackrel{\text{def.}}{=} u \circ \mathcal{J} \circ u^{-1} \in \text{End}(\mathcal{S}),$$

which is a taming of the duality structure $\Delta_u \stackrel{\text{def.}}{=} (\mathcal{S}, \omega_u, \mathcal{D}_u)$. On the other hand, if g is a Lorentzian metric on M , we set:

$$g_u \stackrel{\text{def.}}{=} (f_u)_*(g).$$

Finally, we define $\Xi_u \stackrel{\text{def.}}{=} (\Delta_u, \mathcal{J}_u)$ and we denote by $\text{Iso}(M, g)$ the group of orientation-preserving isometries of (M, g) .

Definition 1.17. Let $\Xi = (\Delta, \mathcal{J})$ be an electromagnetic structure defined on M .

- The group $\text{Aut}(\Delta)$ of unbased automorphisms of Δ is called the *unbased pseudo-duality group* defined by Δ . The *unbased pseudo-duality transformation* defined by $u \in \text{Aut}(\Delta)$ is the linear isomorphism:

$$(5) \quad \mathbb{A}_u : \text{Conf}(M, \Delta) \xrightarrow{\sim} \text{Conf}(M, \Delta),$$

which restricts to an isomorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Delta, \mathcal{J}) \xrightarrow{\sim} \text{Sol}(M, g_u, \Delta, \mathcal{J}_u).$$

- The *unbased duality group* $\text{Aut}(g, \Delta)$ of the pair (g, Δ) is the stabilizer of g in $\text{Aut}(\Delta)$:

$$\text{Aut}(g, \Delta) \stackrel{\text{def.}}{=} \text{Stab}_{\text{Aut}(\Delta)}(g) = \{u \in \text{Aut}(\Delta) \mid f_u \in \text{Iso}(M, g)\} .$$

The *unbased duality transformation* defined by $u \in \text{Aut}(g, \Delta)$ is the linear isomorphism (5), which restricts to an isomorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Delta, \mathcal{J}) \xrightarrow{\sim} \text{Sol}(M, g, \Delta, \mathcal{J}_u) .$$

- The group $\text{Aut}_b(\Delta)$ is called the (*electromagnetic*) *duality group* defined by Δ . The *duality transformation* defined by $u \in \text{Aut}_b(\Delta)$ is the linear isomorphism (5), which restricts to an isomorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Delta, \mathcal{J}) \rightarrow \text{Sol}(M, g, \Delta, \mathcal{J}_u) .$$

Remark 1.18. The fact that \mathbb{A}_u restricts as stated above follows from a direct yet subtle computation. Since the conditions involved are linear, it is enough to verify it on an homogeneous element. If $\mathcal{V} = \alpha \otimes \xi \in \text{Sol}(M, \Delta)$, $u \in \text{Aut}(\Delta)$ and $v \in \mathfrak{X}(M)$ we compute:

$$\begin{aligned} d_{(\mathcal{D}_u)_v} \mathbb{A}_u(\mathcal{V}) &= d_{(\mathcal{D}_u)_v} ((f_{u*}\alpha) \otimes (u \cdot \xi)) \\ &= (\iota_v f_{u*} d\alpha) \otimes (u \cdot \xi) + (f_{u*}\alpha) \otimes ((\mathcal{D}_u)_v u \cdot \xi) \\ &= (f_{u*} \iota_{f_u^{-1}(v)} d\alpha) \otimes (u \cdot \xi) + (f_{u*}\alpha) \otimes (u \cdot \mathcal{D}_{f_u^{-1}(v)} \xi) \\ &= \mathbb{A}_u ((\iota_{f_u^{-1}(v)} d\alpha) \otimes (\xi) + \alpha \otimes (\mathcal{D}_{f_u^{-1}(v)} \xi)) \\ &= \mathbb{A}_u (d_{\mathcal{D}_{f_u^{-1}(v)}} \mathcal{V}) = 0 . \end{aligned}$$

On the other hand, if $\mathcal{V} = \alpha \otimes \xi \in \text{Sol}(M, g, \Xi)$ we compute:

$$\begin{aligned} \star_{g_u, \mathcal{J}_u} \mathbb{A}_u(\mathcal{V}) &= (\star_{g_u} f_{u*}\alpha) \otimes (\mathcal{J}_u \circ u \cdot \xi) \\ &= f_{u*} (\star_g \alpha) \otimes u \circ \mathcal{J}(\xi) \circ f_u^{-1} = \mathbb{A}_u(\star_{g, \mathcal{J}} \mathcal{V}) = \mathbb{A}_u(\mathcal{V}) , \end{aligned}$$

implying that the restrictions of \mathbb{A}_u to $\text{Sol}(M, \Delta)$ and $\text{Conf}(M, g, \Xi)$ are well-defined as stated above.

We have obvious inclusions:

$$(6) \quad \text{Aut}_b(\Delta) \subset \text{Aut}(g, \Delta) \subset \text{Aut}(\Delta)$$

and a short exact sequence:

$$(7) \quad 1 \rightarrow \text{Aut}_b(\Delta) \rightarrow \text{Aut}(g, \Delta) \rightarrow \text{Iso}_\Delta(M, g) \rightarrow 1 ,$$

where $\text{Iso}_\Delta(M, g) \subset \text{Iso}(M, g)$ denotes the group formed by those orientation-preserving isometries of (M, g) that can be covered by elements of $\text{Aut}(g, \Delta)$.

The classical duality group $\text{Aut}_b(\Delta)$ consists of all based automorphisms of \mathcal{S} which preserve both ω and \mathcal{D} . Therefore (see [24, Lemma 4.2.8]) fixing a point $m_0 \in M$ it can be realized as the centralizer $C_{m_0}(\Delta)$ of the holonomy group $\text{Hol}_{m_0}(\mathcal{D})$ of \mathcal{D} at m_0 inside the group $\text{Aut}(\mathcal{S}_{m_0}, \omega_{m_0}) \simeq \text{Sp}(2n, \mathbb{R})$:

$$\text{Aut}_b(\Delta) \simeq C_{m_0}(\Delta).$$

In particular, $\text{Aut}_b(\Delta)$ is a closed subgroup of $\text{Sp}(2n, \mathbb{R})$ and hence it is a finite-dimensional Lie group. The same holds for $\text{Iso}_\Delta(M, g)$, which is a closed subgroup of the finite-dimensional² Lie group $\text{Iso}(M, g)$. The sequence (7) implies that $\text{Aut}(g, \Delta)$ is also a finite-dimensional Lie group. In general, the groups defined above differ markedly from their local counterparts described in Appendix A.2. We stress that the latter are not the adequate groups to consider when dealing with electromagnetic U-folds (since in that case the duality structure is not holonomy trivial).

Definition 1.19. Let $\Xi = (\Delta, \mathcal{J})$ be an electromagnetic structure defined on M .

- The group:

$$\text{Aut}(\Xi) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(\Delta) \mid \mathcal{J}_u = \mathcal{J}\}$$

of unbased automorphisms of Ξ is called the *unbased unitary pseudo-duality group* defined by Ξ . The *unbased unitary pseudo-duality transformation* defined by $u \in \text{Aut}(\Xi)$ is the linear isomorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Xi) \xrightarrow{\sim} \text{Sol}(M, g_u, \Xi).$$

- The *unbased unitary duality group* of the pair (g, Ξ) is the stabilizer of g in $\text{Aut}(\Xi)$:

$$\begin{aligned} (8) \quad \text{Aut}(g, \Xi) &\stackrel{\text{def.}}{=} \text{Stab}_{\text{Aut}(\Xi)}(g) \\ &= \text{Stab}_{\text{Aut}(g, \Delta)}(\mathcal{J}) \\ &= \{u \in \text{Aut}(\Delta) \mid \mathcal{J}_u = \mathcal{J} \ \& \ f_u \in \text{Iso}(M, g)\}. \end{aligned}$$

²It is well-known that the isometry group of any pseudo-Riemannian manifold is a finite-dimensional Lie group. See for example [31, Theorem 5.1, p. 22].

The *unbased unitary duality transformation* defined by $u \in \text{Aut}(g, \Xi)$ is the linear automorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Xi) \xrightarrow{\sim} \text{Sol}(M, g, \Xi).$$

- The group $\text{Aut}_b(\Xi)$ of based automorphisms of Ξ is called the *classical unitary duality group* defined by Ξ :

$$\text{Aut}_b(\Xi) = \text{Stab}_{\text{Aut}_b(\Delta)}(\mathcal{J}) = \{u \in \text{Aut}_b(\Delta) \mid \mathcal{J}_u = \mathcal{J}\}.$$

The *classical unitary duality transformation* defined by $u \in \text{Aut}_b(\Xi)$ is the linear automorphism:

$$\mathbb{A}_u : \text{Sol}(M, g, \Xi) \xrightarrow{\sim} \text{Sol}(M, g, \Xi)$$

We have obvious inclusions:

$$\text{Aut}_b(\Xi) \subset \text{Aut}(g, \Xi) \subset \text{Aut}(\Xi)$$

and a short exact sequence:

$$(9) \quad 1 \rightarrow \text{Aut}_b(\Xi) \rightarrow \text{Aut}(g, \Xi) \rightarrow \text{Iso}_\Xi(M, g) \rightarrow 1,$$

where $\text{Iso}_\Xi(M, g)$ is the group formed by those orientation-preserving isometries of (M, g) which are covered by elements of $\text{Aut}(g, \Xi)$. Arguments similar to those above show that $\text{Aut}_b(\Xi)$, $\text{Aut}(g, \Xi)$ and $\text{Iso}_\Xi(M, g)$ are finite-dimensional Lie groups. The previous definitions give global mathematically rigorous descriptions of several types of *duality groups* associated to abelian gauge theory on Lorentzian four-manifolds. In general, these can differ markedly from their “local” counterparts described in Appendix A.2, which are considered traditionally in the physics literature.

1.10. The case of trivial duality structure

Let (M, g) be an oriented Lorentzian four-manifold. For any $n \geq 0$, the set $\mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R}))$ of smooth $\text{Sp}(2n, \mathbb{R})$ -valued functions defined on M is a group under pointwise multiplication, whose group of automorphisms we denote by $\text{Aut}(\mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})))$.

Lemma 1.20. *Let (\mathcal{S}, ω) be a symplectic vector bundle of rank $2n$ defined on M which is symplectically trivializable. Then any symplectic trivialization*

of (\mathcal{S}, ω) induces an isomorphism of groups:

$$\text{Aut}(\mathcal{S}, \omega) \simeq \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \rtimes_\alpha \text{Diff}(M),$$

where $\alpha : \text{Diff}(M) \rightarrow \text{Aut}(\mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})))$ is the morphism of groups defined through:

$$\alpha(\varphi)(f) \stackrel{\text{def.}}{=} f \circ \varphi^{-1}, \quad \forall \varphi \in \text{Diff}(M), \quad \forall f \in \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})).$$

In particular, we have a short exact sequence of groups:

$$1 \rightarrow \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \rightarrow \text{Aut}(\mathcal{S}, \omega) \rightarrow \text{Diff}(M) \rightarrow 1$$

which splits from the right.

Proof. Let $\tau : \mathcal{S} \xrightarrow{\sim} M \times \mathbb{R}^{2n}$ be a symplectic trivialization of (\mathcal{S}, ω) . Then the map $\text{Ad}(\tau) : \text{Aut}(\mathcal{S}, \omega) \rightarrow \text{Aut}(M \times \mathbb{R}^{2n}, \omega_{2n})$ defined through:

$$\text{Ad}(\tau)(f) \stackrel{\text{def.}}{=} \tau \circ f \circ \tau^{-1}, \quad \forall f \in \text{Aut}(\mathcal{S}, \omega),$$

is an isomorphism of groups. Let $f \in \text{Aut}(\mathcal{S}, \omega)$ be an unbased automorphism of (\mathcal{S}, ω) which covers the diffeomorphism $\varphi \in \text{Diff}(M)$. Then $\text{Ad}(\tau)(f)$ is an unbased automorphism of $M \times \mathbb{R}^{2n}$ which covers φ and hence we have:

$$\text{Ad}(\tau)(f)(m, x) = (\varphi(m), \tilde{f}(m)(x)), \quad \forall (m, x) \in M \times \mathbb{R}^{2n},$$

where $\tilde{f} : M \rightarrow \text{Sp}(2n, \mathbb{R})$ is a smooth map. Setting $h \stackrel{\text{def.}}{=} \tilde{f} \circ \varphi^{-1} \in \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R}))$, we have:

$$(10) \quad \text{Ad}(\tau)(f)(m, x) = (\varphi(m), h(\varphi(m))(x)), \quad \forall (m, x) \in M \times \mathbb{R}^{2n}$$

and the correspondence $f \rightarrow (h, \varphi)$ gives a bijection between $\text{Aut}(\mathcal{S}, \omega)$ and the set $\mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \times \text{Diff}(M)$. If $f_1, f_2 \in \text{Aut}(\mathcal{S}, \omega)$ correspond through this map to the pairs:

$$(h_1, \varphi_1), (h_2, \varphi_2) \in \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \times \text{Diff}(M),$$

then direct computation using (10) gives:

$$\text{Ad}(\tau)(f_1 \circ f_2)(m, x) = ((\varphi_1 \circ \varphi_2)(m), h_1(m)(h_2 \circ \varphi_1^{-1})(m)(x)),$$

showing that $f_1 \circ f_2$ corresponds to the pair $(h_1 \cdot \alpha(\varphi_1)(h_2), \varphi_1 \circ \varphi_2)$. □

Corollary 1.21. *Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be a holonomy trivial duality structure defined on M . Then any trivialization of Δ induces an isomorphism of groups:*

$$\text{Aut}(\Delta) \simeq \text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M).$$

Proof. Follows from Lemma 1.20 by noticing that the action α of $\text{Diff}(M)$ on $\mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R}))$ restricts to the trivial action on the subgroup:

$$\{f \in \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \mid df = 0\} \simeq \text{Sp}(2n, \mathbb{R})$$

of constant $\text{Sp}(2n, \mathbb{R})$ -valued functions defined on M . □

Fix an electromagnetic structure $\Xi = (\Delta, \mathcal{J})$ of rank $2n$ defined on M with holonomy-trivial underlying duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. Choosing a global flat symplectic frame \mathcal{E} of \mathcal{S} , we identify Δ with the canonical trivial duality structure (3) and \mathcal{J} with a taming map of size $2n$. Then \mathcal{D} identifies with the trivial connection and $d_{\mathcal{D}}$ identifies with the exterior derivative $d: \Omega(M, \mathbb{R}^{2n}) \rightarrow \Omega(M, \mathbb{R}^{2n})$ extended trivially to vector-valued forms. Using Lemma 1.20 and its obvious adaptation, we obtain:

$$\begin{aligned} \text{Aut}_b(\mathcal{S}) &\equiv \mathcal{C}^\infty(M, \text{GL}(2n, \mathbb{R})), \\ \text{Aut}(\mathcal{S}) &\equiv \mathcal{C}^\infty(M, \text{GL}(2n, \mathbb{R})) \rtimes_{\alpha} \text{Diff}(M), \\ \text{Aut}_b(\mathcal{S}, \omega) &\equiv \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})), \\ \text{Aut}(\mathcal{S}, \omega) &\equiv \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R})) \rtimes_{\alpha} \text{Diff}(M). \end{aligned}$$

On the other hand, Corollary 1.21 gives:

$$\begin{aligned} \text{Aut}_b(\Delta) &\equiv \text{Sp}(2n, \mathbb{R}), \quad \text{Aut}(\Delta) \equiv \text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M), \\ \text{Aut}(g, \Delta) &= \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M, g). \end{aligned}$$

Moreover, we have:

$$\begin{aligned} \text{Aut}_b(\Xi) &\equiv \text{U}_{\mathcal{J}}(n) \stackrel{\text{def.}}{=} \{\gamma \in \text{Sp}(2n, \mathbb{R}) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J}\}, \\ \text{Aut}(\Xi) &\equiv \{(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \circ f\}, \end{aligned}$$

as well as:

$$\text{Aut}(g, \Xi) \equiv \{(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \circ f\}.$$

Hence we recover the local formulas obtained in Appendix A.2 for the duality groups of local abelian gauge theory. Notice that $\text{Aut}_b(\Xi)$ is isomorphic with

the unitary group $U(n)$ when the electromagnetic structure Ξ is unitary, which amounts to the map \mathcal{J} being constant.

2. The Dirac-Schwinger-Zwanziger condition

The previous section introduced classical abelian gauge theory as a theory of *field strengths*, i.e. a theory of $d_{\mathcal{D}}$ -closed two-forms valued in the flat symplectic vector bundle \mathcal{S} with equation of motion given by the polarized self-duality condition. Well-known arguments originally due to Dirac as well as the Aharonov-Bohm effect [1] imply that a consistent coupling of the theory to quantum charge carriers imposes an integrality condition on field strength configurations. This is traditionally called the DSZ “quantization” condition, even though it constrains *classical* field strength configurations – in fact, only particles which carry the corresponding charges are quantized in such arguments, but *not* the gauge fields themselves. To avoid confusion, we prefer to call it the *DSZ integrality condition*. For local abelian gauge theories of rank $2n$ (which are discussed in Appendix A), this condition can be implemented using a full symplectic lattice in the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_{2n})$, as usually done in the physics literature [42, 47]. For abelian gauge theories with non-trivial electromagnetic structure defined on an arbitrary Lorentzian four-manifold, we shall implement this condition using a *Dirac system*, as originally proposed in [32]. We begin with some preliminaries.

2.1. Principal bundles with discrete structure group

Let Γ be a discrete group and Q be a principal bundle with structure group Γ and projection $p : Q \rightarrow M$. Then the total space of Q is a (generally disconnected) covering space of M . Let $U_Q : \Pi_1(M) \rightarrow \Phi_0(Q)$ be the monodromy transport of the covering map $p : Q \rightarrow M$, where $\Phi_0(Q)$ is the *bare fiber groupoid* of Q . By definition, the objects of $\Phi_0(Q)$ are the fibers of Q while its morphisms are arbitrary bijections between the latter. By definition, the functor U_Q associates to the homotopy class $\mathbf{c} \in \Pi_1(M)(m, m')$ of any curve $c : [0, 1] \rightarrow M$ with $c(0) = m$ and $c(1) = m'$ the bijection $U(\mathbf{c}) : Q_m \xrightarrow{\sim} Q_{m'}$ given by $U_Q(\mathbf{c})(x) \stackrel{\text{def.}}{=} \tilde{c}_x(1) \in Q_{m'}$, where \tilde{c}_x is the the unique lift of c to Q through the point $x \in Q$ (thus $\tilde{c}_x(0) = x$). Notice that x and $U(\mathbf{c})(x)$ lie on the same connected component of Q and hence the diffeomorphism $U_Q(\mathbf{c}) : Q \xrightarrow{\sim} Q$ induces the trivial permutation of $\pi_0(Q)$. For any $\gamma \in \Gamma$, the curve \tilde{c}_x^γ defined through $\tilde{c}_x^\gamma(t) \stackrel{\text{def.}}{=} \tilde{c}(t)\gamma$ for all $t \in [0, 1]$ is a lift of c through

the point $x\gamma$. The homotopy lifting property of p implies that \tilde{c}_x^γ and $\tilde{c}_{x\gamma}$ are homotopic and hence $U_Q(\mathbf{c})(x\gamma) = U(\mathbf{c})(x)\gamma$. This shows that U_Q acts through isomorphisms of Γ -spaces and hence it is in fact a functor:

$$U_Q : \Pi_1(M) \rightarrow \Phi(Q),$$

where $\Phi(Q)$ is the *principal fiber groupoid* of Q (whose objects coincide with those of $\Phi_0(Q)$ but whose morphisms are isomorphisms of Γ -spaces). This implies that U_Q is the parallel transport of a flat principal connection defined on Q , which we shall call the *monodromy connection* of Q . The holonomy morphism:

$$\alpha_m(Q) : \pi_1(M, m) \rightarrow \text{Aut}_\Gamma(Q_m)$$

of this connection at a point $m \in M$ will be called the *monodromy morphism* of Q at m , while its image:

$$\text{Hol}_m(Q) \stackrel{\text{def.}}{=} \text{im}(\alpha_m(Q)) \subset \text{Aut}_\Gamma(Q_m)$$

will be called the *monodromy group* of Q at m . The monodromy morphism at a fixed point $m_0 \in M$ induces a bijection between the set of isomorphism classes of principal Γ -bundles and the character variety:

$$\mathfrak{X}(\pi_1(M, m_0), \Gamma) \stackrel{\text{def.}}{=} \text{Hom}(\pi_1(M, m), \Gamma) / \Gamma.$$

Remark 2.1. For the purposes of this work, the most important class of principal bundles with discrete structure group are principal $\text{Sp}_t(2n, \mathbb{Z})$ bundles, which are naturally associated to Siegel bundles (see Section 3).

2.2. Bundles of finitely-generated free abelian groups

Let \mathcal{F} be a bundle of free abelian groups of rank r defined on M . Then \mathcal{F} is isomorphic with the bundle of groups with fiber \mathbb{Z}^r associated to a principal $\text{GL}(r, \mathbb{Z})$ -bundle Q through the left action $\ell : \text{GL}(r, \mathbb{Z}) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathbb{Z}^r)$:

$$\mathcal{F} \simeq \mathcal{F}_r(Q) \stackrel{\text{def.}}{=} Q \times_\ell \mathbb{Z}^r \simeq \hat{M} \times_{\ell \circ \alpha_m(Q)} \mathbb{Z}^r,$$

where $\alpha_m(Q) : \pi_1(M, m) \rightarrow \text{GL}(r, \mathbb{Z})$ is the monodromy morphism of Q at m . The monodromy connection of Q induces a flat Ehresmann connection

which we shall call the *monodromy connection of \mathcal{F}* and whose parallel transport:

$$U_{\mathcal{F}} : \Pi_1(M) \rightarrow \Phi(\mathcal{F})$$

acts by isomorphisms of groups between the fibers of \mathcal{F} . The holonomy morphism:

$$\sigma_m(\mathcal{F}) : \pi_1(M, m) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathcal{F}_m)$$

at $m \in M$ can be identified with the morphism $\ell \circ \alpha_m(P) : \pi_1(M, m) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathbb{Z}^r)$ upon choosing a basis of \mathcal{F}_m . The holonomy group:

$$\text{Hol}_m(\mathcal{F}) \stackrel{\text{def.}}{=} \text{im}(\sigma_m(\mathcal{F})) \subset \text{Aut}_{\mathbb{Z}}(\mathcal{F}_m) \simeq \text{GL}(r, \mathbb{Z})$$

is called the *monodromy group of \mathcal{F}* at m and identifies with a subgroup of $\text{GL}(r, \mathbb{Z})$ upon choosing an appropriate basis of \mathcal{F}_m .

Conversely, let $\text{Fr}(\mathbb{Z}^r)$ be the set of all bases of the free \mathbb{Z} -module \mathbb{Z}^r . Then $\text{GL}(r, \mathbb{Z})$ has a natural free and transitive left action μ on this set. Taking the set of bases of each of fiber gives the *bundle of frames $\text{Fr}(\mathcal{F})$* of \mathcal{F} . This is a principal $\text{GL}(r, \mathbb{Z})$ -bundle whose monodromy morphism coincides with that of \mathcal{F} . This gives the following result.

Proposition 2.2. *The correspondences $Q \mapsto \mathcal{F}_r(Q)$ and $\mathcal{F} \mapsto \text{Fr}(\mathcal{F})$ extend to mutually quasi-inverse equivalences between the groupoid of bundles of free abelian groups of rank r defined on M and the groupoid of principal $\text{GL}(r, \mathbb{Z})$ -bundles defined on M .*

2.3. Dirac systems and integral duality structures

Definition 2.3. Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be a duality structure defined on M . A *Dirac system* for Δ is a smooth fiber sub-bundle $\mathcal{L} \subset \mathcal{S}$ of full symplectic lattices in (\mathcal{S}, ω) which is preserved by the parallel transport \mathcal{T}_{Δ} of \mathcal{D} . That is, for any piece-wise smooth path $\gamma : [0, 1] \rightarrow M$ we have:

$$\mathcal{T}_{\Delta}(\gamma)(\mathcal{L}_{\gamma(0)}) = \mathcal{L}_{\gamma(1)}.$$

A pair:

$$\Delta \stackrel{\text{def.}}{=} (\Delta, \mathcal{L})$$

consisting of a duality structure Δ and a choice of Dirac system \mathcal{L} for Δ is called an *integral duality structure*.

Let $\text{Dual}_{\mathbb{Z}}(M)$ be the groupoid of integral duality structures defined on M , with the obvious notion of isomorphism. For every $m \in M$, the fiber $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ of an integral duality structure $\Delta = (\Delta, \mathcal{L})$ of rank $2n$ is an *integral symplectic space* of dimension $2n$ (see Appendix B for details). Each such space defines an integral vector (called its *type*) belonging to a certain subset Div^n of $\mathbb{Z}_{>0}^n$ endowed with a partial order relation \leq which makes it into a complete meet semi-lattice. The type of an integral symplectic space depends only on its isomorphism class (which it determines uniquely) and every element of Div^n is realized as a type. Moreover, the group of automorphisms of an integral symplectic space of type $\mathfrak{t} \in \text{Div}^n$ is isomorphic with the *modified Siegel modular group* $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ of type \mathfrak{t} (see Definition B.6). This is a discrete subgroup of $\text{Sp}(2n, \mathbb{R})$ which contains the Siegel modular group $\text{Sp}(2n, \mathbb{Z})$, to which it reduces when \mathfrak{t} equals the *principal type* $\delta = (1, \dots, 1)$. If \mathfrak{t} and \mathfrak{t}' are elements of Div^n such that $\mathfrak{t} \leq \mathfrak{t}'$, then the lattice Λ of any integral symplectic space (V, ω, Λ) of type \mathfrak{t} admits a full rank sublattice Λ' such that (V, ω, Λ') is an integral symplectic space of type \mathfrak{t}' . In this case, we have $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subset \text{Sp}_{\mathfrak{t}'}(2n, \mathbb{Z})$. Since we assume that M is connected and that \mathcal{D} preserves \mathcal{L} , the integral symplectic spaces $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ are isomorphic to each other through the parallel transport of \mathcal{D} , hence their type does not depend on the base-point $m \in M$.

Definition 2.4. The type $\mathfrak{t}_{\Delta} \in \text{Div}^n$ of an integral duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ (and of the corresponding Dirac system \mathcal{L}) is the common type of the integral symplectic spaces $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$, where $m \in M$.

Let $\text{Dual}_{\mathbb{Z}}^{\mathfrak{t}}(M)$ be the full subgroupoid of $\text{Dual}_{\mathbb{Z}}(M)$ consisting of integral duality structures of type \mathfrak{t} .

Definition 2.5. A duality structure Δ is called *semiclassical* if it admits a Dirac system.

Not every duality structure is semiclassical, as the following proposition shows.

Proposition 2.6. *A duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ admits a Dirac system of type $\mathfrak{t} \in \text{Div}^n$ if and only if the holonomy representation of \mathcal{D} at some point (equivalently, at any point) $m \in M$:*

$$\mathcal{T}_{\Delta}|_{\pi_1(M, m)}: \pi_1(M, m) \rightarrow \text{Sp}(\mathcal{S}_m, \omega_m) \simeq \text{Sp}(2n, \mathbb{R})$$

can be conjugated so that its image lies inside the modified Siegel modular group:

$$\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subset \mathrm{Sp}(2n, \mathbb{R})$$

of type \mathfrak{t} . In this case, Δ is semiclassical and the greatest lower bound of those $\mathfrak{t} \in \mathrm{Div}^n$ with this property is called the type of Δ and denoted by \mathfrak{t}_Δ .

Proof. Assume $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ admits a Dirac system \mathcal{L} of type \mathfrak{t} . Then (as explained in Appendix B) the automorphism group of every fiber $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ is isomorphic to $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, which is the automorphism group of the standard integral symplectic space $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ of type \mathfrak{t} . Since \mathcal{L} is preserved by the parallel transport of \mathcal{D} , it follows that we have $\mathcal{T}_\Delta(\pi_1(M, m)) \subset \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ after identifying $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ with $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ and hence $\mathrm{Sp}(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ with $\mathrm{Sp}(2n, \mathbb{R})$. The converse follows immediately from the associated bundle construction. \square

Remark 2.7. A duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ of rank $2n$ admits a Dirac system \mathcal{L} of type $\mathfrak{t} = (t_1, \dots, t_n) \in \mathrm{Div}^n$ if and only if M admits an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ such that for each $\alpha \in I$ there exists a \mathcal{D} -flat frame $(e_1^{(\alpha)}, \dots, e_{2n}^{(\alpha)})$ of $\mathcal{S}|_{U_\alpha}$ with the property:

$$(11) \quad \begin{aligned} \omega(e_i, e_j) &= \omega(e_{n+i}, e_{n+j}) = 0, & \omega(e_i, e_{n+j}) &= t_i \delta_{ij}, \\ \omega(e_{n+i}, e_j) &= -t_i \delta_{ij}, & \forall i, j &= 1, \dots, n. \end{aligned}$$

For each $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ we have:

$$e_k^{(\beta)} = \sum_{l=1}^{2n} T_{lk}^{(\alpha\beta)} e_l^{(\alpha)}, \quad \forall k = 1, \dots, 2n \text{ on } U_\alpha \cap U_\beta,$$

where $T_{lk}^{(\alpha\beta)} \in \mathbb{Z}$ for all $k, l = 1, \dots, 2n$. Furthermore:

$$\oplus_{k=1}^{2n} \mathbb{Z} e_k^{(\alpha)}(m) = \mathcal{L}_m, \quad \forall \alpha \in I, \quad \forall m \in U_\alpha,$$

and the matrices $T^{(\alpha\beta)} \stackrel{\text{def.}}{=} (T_{kl}^{(\alpha\beta)})_{k,l=1,\dots,2n}$ belong to $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$.

Every integral duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ on M defines a parallel transport functor:

$$\mathcal{T}_\Delta : \Pi_1(M) \rightarrow \mathrm{Symp}_{\mathbb{Z}},$$

where $\mathrm{Symp}_{\mathbb{Z}}$ is the groupoid of integral symplectic spaces defined in Appendix B. This functor associates the integral symplectic vector space

$(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ to every point $m \in M$ and the isomorphism of symplectic vector spaces $\mathcal{T}_\Delta(\mathbf{c}) \stackrel{\text{def.}}{=} \mathcal{T}_\Delta(\mathbf{c})$ to every homotopy class $\mathbf{c} \in \Pi_1(M)(m, m')$ of curves from $m \in M$ to $m' \in M$. The functor \mathcal{T}_Δ defines a flat system of integral symplectic vector spaces (that is, a $\text{Symp}_{\mathbb{Z}}$ -valued local system) on M . As in Section 1, the correspondence $\Delta \mapsto \mathcal{T}_\Delta$ extends to an equivalence of groupoids:

$$\mathcal{T} : \text{Dual}_{\mathbb{Z}}(M) \xrightarrow{\sim} [\Pi_1(M), \text{Symp}_{\mathbb{Z}}]$$

between $\text{Dual}_{\mathbb{Z}}(M)$ and the functor groupoid $[\Pi_1(M), \text{Symp}_{\mathbb{Z}}]$. Thus one can identify integral duality structures with $\text{Symp}_{\mathbb{Z}}$ -valued local systems defined on M . This implies the following result, whose proof is similar to that of Proposition 1.7.

Proposition 2.8. *For any $m_0 \in M$, the set of isomorphism classes of integral duality structures of type \mathfrak{t} defined on M is in bijection with the character variety:*

$$(12) \quad \mathfrak{X}(\pi_1(M, m_0), \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})) \stackrel{\text{def.}}{=} \text{Hom}(\pi_1(M, m_0), \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})) / \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}).$$

For later reference, we introduce the following:

Definition 2.9. An *integral electromagnetic structure* defined on M is a pair:

$$\Xi \stackrel{\text{def.}}{=} (\Xi, \mathcal{L}),$$

where $\Xi = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$ is an electromagnetic structure on M and \mathcal{L} is a Dirac system for the duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$.

2.4. Siegel systems

Integral duality structures are associated to certain local systems of free abelian groups of even rank defined on M .

Definition 2.10. Let $n \in \mathbb{Z}_{>0}$. A *Siegel system* of rank $2n$ on M is a bundle Z of free abelian groups of rank $2n$ defined on M equipped with a reduction of its structure group from $\text{GL}(2n, \mathbb{Z})$ to a subgroup of some modified Siegel modular group $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, where $\mathfrak{t} \in \text{Div}^n$. The greatest lower bound of the set of those $\mathfrak{t} \in \text{Div}^n$ with this property is called the *type* of Z and is denoted by \mathfrak{t}_Z .

Let $\text{Sg}(M)$ be the groupoid of Siegel systems on M and $\text{Sg}_{\mathfrak{t}}(M)$ be the full sub-groupoid of Siegel systems of type \mathfrak{t} .

Remark 2.11. Let \mathbb{R} be the trivial real line bundle on M and $U_0 : \Pi_1(M) \rightarrow \Phi(\mathbb{R})$ be the transport functor induced by its trivial flat connection. The following statements are equivalent for a bundle Z of free abelian groups of rank $2n$ defined on M :

- (a) Z is a Siegel system of type \mathfrak{t} defined on M .
- (b) The vector bundle $\mathcal{S} \stackrel{\text{def.}}{=} Z \otimes_{\mathbb{Z}} \mathbb{R}$ carries a symplectic pairing ω which is invariant under the parallel transport $U_Z \otimes_{\mathbb{Z}} U_0$ of the flat connection induced from Z and which makes the triplet $(\mathcal{S}_m, \omega_m, Z_m)$ into an integral symplectic space of type \mathfrak{t} for any $m \in M$.
- (c) For any $m \in M$, the $2n$ -dimensional vector space $\mathcal{S}_m \stackrel{\text{def.}}{=} Z_m \otimes_{\mathbb{Z}} \mathbb{R}$ carries a symplectic form ω_m which makes the triplet $(\mathcal{S}_m, \omega_m, Z_m)$ into an integral symplectic space of type \mathfrak{t} and we have $\text{Hol}_m(Z) = \text{Aut}(\mathcal{S}_m, \omega_m, Z_m)$.

By definition, any Siegel system Z of type \mathfrak{t} is isomorphic with the bundle of groups with fiber \mathbb{Z}^{2n} associated to a principal $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ -bundle Q through the left action $\ell : \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathbb{Z}^{2n})$ of $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ on \mathbb{Z}^{2n} :

$$Z \simeq Z(Q) \stackrel{\text{def.}}{=} Q \times_{\ell} \mathbb{Z}^{2n} \simeq \hat{M} \times_{\ell_{\mathfrak{t} \circ \alpha_m(Q)}} \mathbb{Z}^{2n},$$

where $\alpha_m(Q) : \pi_1(M, m) \rightarrow \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ is the monodromy morphism of Q at m and \hat{M} is the universal cover of M . The monodromy morphism $\sigma_m(Z)$ of Z at m identifies with $\ell \circ \alpha_m(Q)$ upon choosing an integral symplectic basis of the integral symplectic space $(\mathcal{S}_m, \omega_m, Z_m)$. This also identifies the monodromy group $\text{Hol}_m(Z) \subset \text{Aut}_{\mathbb{Z}}(Z_m)$ with $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. Conversely, let $\text{Fr}_{\mathfrak{t}}(\mathbb{Z}^{2n})$ be the set of those bases of the free \mathbb{Z} -module \mathbb{Z}^{2n} in which the standard symplectic form ω_{2n} of $\mathbb{R}^{2n} = \mathbb{Z}^{2n} \otimes_{\mathbb{Z}} \mathbb{R}$ takes the form $\omega_{\mathfrak{t}}$ (see Appendix B). Then $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ has a natural free and transitive left action $\mu_{\mathfrak{t}}$ on this set. Taking the set of bases of each of fiber gives the *bundle of frames* $\text{Fr}(Z)$ of the Siegel system Z , which is a principal $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ -bundle. The previous discussion implies the following result.

Proposition 2.12. *The correspondences $Q \mapsto Z(Q)$ and $Z \mapsto \text{Fr}(Z)$ extend to mutually quasi-inverse equivalence of groupoids between $\text{Prin}_{\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})}(M)$ and $\text{Sg}_{\mathfrak{t}}(M)$.*

In particular, the set of isomorphism classes of Siegel systems of type \mathfrak{t} defined on M is in bijection with the character variety

$\mathfrak{R}(\pi_1(M, m_0), \mathrm{Sp}_t(2n, \mathbb{Z}))$ of equation (12). Let $\underline{1}$ be the unit section of the trivial real line bundle $\underline{\mathbb{R}} = M \times \mathbb{R}$.

Proposition 2.13. *Let Z be a Siegel system defined on M . Then there exists a unique integral duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ such that $\mathcal{S} = Z \otimes_{\mathbb{Z}} \underline{\mathbb{R}}$ and $\mathcal{L} = Z \otimes_{\mathbb{Z}} \mathbb{Z}$ and this duality structure has the same type as Z . Moreover, the parallel transport $U_{\mathcal{D}} : \Pi_1(M) \rightarrow \Phi(\mathcal{S})$ of \mathcal{D} is given by:*

$$(13) \quad U_{\mathcal{D}}(\mathbf{c}) = U_Z(\mathbf{c}) \otimes_{\mathbb{Z}} U_0(m, m'), \quad \forall \mathbf{c} \in \Pi_1(M)(m, m'), \quad \forall m, m' \in M,$$

where $U_Z : \Pi_1(M) \rightarrow \Phi(Z)$ is the monodromy transport of Z and $U_0 : \Pi_1(M) \rightarrow \Phi(\underline{\mathbb{R}})$ is the trivial transport of $\underline{\mathbb{R}}$.

Remark 2.14. The fiber of \mathcal{L} at $m \in M$ is given by:

$$(14) \quad \mathcal{L}_m \stackrel{\text{def.}}{=} \{z \otimes_{\mathbb{Z}} 1 \mid z \in Z_m\} \equiv Z_m,$$

where 1 is the unit element of the field \mathbb{R} . It is clear that the transport $U_{\mathcal{D}}$ defined by (13) gives bijections from \mathcal{L}_m to $\mathcal{L}_{m'}$ and hence preserves \mathcal{L} . Any locally-constant frame (s_1, \dots, s_{2n}) of Z defined above a non-empty open set $V \subset M$ determines a local flat symplectic frame (e_1, \dots, e_{2n}) of Δ defined above V given by:

$$(15) \quad e_i \stackrel{\text{def.}}{=} s_i \otimes_{\mathbb{Z}} 1, \quad \forall i = 1, \dots, 2n$$

and the matrix of ω with respect to this frame is integer-valued.

Proof. The restriction of $U_{\mathcal{D}}$ to \mathcal{L} gives isomorphisms of groups between the fibers (14) of \mathcal{L} and hence it must agree with the monodromy transport U_Z of Z in the sense that:

$$U_{\mathcal{D}}(\mathbf{c})(z \otimes_{\mathbb{Z}} 1) = U_Z(\mathbf{c})(z) \otimes_{\mathbb{Z}} 1, \quad \forall \mathbf{c} \in \Pi_1(M)(m, m'), \quad \forall m, m' \in M.$$

This implies (13) since $U_{\mathcal{D}}$ is \mathbb{R} -linear and \mathcal{L}_m are full lattices in $\mathcal{S}_m = Z_m \otimes_{\mathbb{Z}} \mathbb{R}$. Remark 2.14 gives a \mathcal{D} -flat symplectic pairing ω_Z on \mathcal{S}_Z such that the integral symplectic spaces $(\mathcal{S}_m, \omega, \mathcal{L}_m)$ have type \mathfrak{t}_Z and such that \mathcal{L} is preserved by the parallel transport of \mathcal{D} . □

Remark 2.15. Notice that \mathcal{S} identifies with the vector bundle associated to the frame bundle $\mathrm{Fr}(Z)$ of Z through the linear representation $q = \varphi \circ \iota$:

$\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \rightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{2n})$ defined by the inclusion morphism:

$$\iota : \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R}),$$

where $\mathrm{Sp}(2n, \mathbb{R})$ acts on \mathbb{R}^{2n} through the fundamental representation $\varphi : \mathrm{Sp}(2n, \mathbb{R}) \rightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{2n})$. The representation q preserves the canonical symplectic form $\omega_{\mathfrak{t}}$ of \mathbb{R}^{2n} and the latter induces the symplectic pairing ω of \mathcal{S} .

We denote by $\Delta(Z)$ the integral duality structure defined by Z as in Proposition 2.13. Conversely, any integral duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{L}, \mathcal{D})$ defines a Siegel system $Z(\Delta)$ upon setting:

$$Z(\Delta) \stackrel{\text{def.}}{=} \mathcal{L}$$

and it is easy to see that Δ is isomorphic with $\Delta(\mathcal{L})$. Therefore, we obtain the following result:

Proposition 2.16. *The correspondences $Z \mapsto \Delta(Z)$ and $\Delta \rightarrow Z(\Delta)$ extend to mutually quasi-inverse equivalences of groupoids between $\mathrm{Sg}_{\mathfrak{t}}(M)$ and $\mathrm{Dual}_{\mathbb{Z}}^{\mathfrak{t}}(M)$.*

2.5. Bundles of integral symplectic torus groups

In the following we use the notions of *integral symplectic torus group* discussed in Appendix B.

Definition 2.17. *A bundle of integral symplectic torus groups of rank $2n$ is a bundle \mathcal{A} of $2n$ -dimensional torus groups defined on M whose structure group reduces from $\mathrm{GL}(2n, \mathbb{Z})$ to a subgroup of some modified Siegel modular group $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, where $\mathfrak{t} \in \mathrm{Div}^n$. The greatest lower bound $\mathfrak{t}_{\mathcal{A}}$ of the set of elements $\mathfrak{t} \in \mathrm{Div}^n$ with this property is called the *type* of \mathcal{A} .*

Let \mathcal{A} be a bundle of integral symplectic torus groups of type \mathfrak{t} . Then the zero elements of the fibers determine a section $s_0 \in \mathcal{C}^{\infty}(M, \mathcal{A})$. The structure group $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ acts on \mathcal{A}_m preserving the distinguished point $s_0(m)$ and the abelian group structure of each fiber. Since such a bundle is associated to a principal $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ -bundle, it carries an induced flat Ehresmann connection whose holonomy group is a subgroup of $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ and whose

holonomy representation at $m \in M$ we denote by:

$$(16) \quad \rho_m(\mathcal{A}) : \pi_1(M, m) \rightarrow \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subset \mathrm{GL}(2n, \mathbb{Z}).$$

The parallel transport of this connection preserves the image of the section s_0 as well as a fiberwise symplectic structure which makes each fiber into an integral symplectic torus group of type \mathfrak{t} in the sense of Appendix B. We have:

$$\mathcal{A} \simeq \hat{M} \times_{\rho} (\mathbb{R}^{2n} / \mathbb{Z}^{2n}).$$

Remark 2.18. When M is compact, the fiber bundle \mathcal{A} is virtually trivial by [17, Theorem 1.1.], i.e. the pull-back of \mathcal{A} to some *finite* covering space of M is topologically trivial.

It follows from the above that bundles of integral symplectic torus groups of type \mathfrak{t} defined on M are classified by group morphisms (16), i.e. the set of isomorphism classes of such bundles is in bijection with the character variety $\mathfrak{X}(\pi_1(M, m_0), \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}))$, which also classifies integral duality structures. This also follows from the results below.

Proposition 2.19. *Let $\Delta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ be an integral duality structure of type \mathfrak{t} defined on M . Then the fiberwise quotient:*

$$\mathcal{A}(\Delta) \stackrel{\mathrm{def.}}{=} \mathcal{S} / \mathcal{L},$$

is a bundle of integral symplectic torus groups of type \mathfrak{t} defined on M .

Proof. It is clear that $\mathcal{A}(\Delta)$ is a fiber bundle of even-dimensional torus groups, whose zero section s_0 is inherited from the zero section of \mathcal{S} . The fiberwise symplectic pairing ω of \mathcal{S} descends to a translation-invariant collection of symplectic forms on the fibers of $\mathcal{A}(\Delta)$, making the latter into integral symplectic torus groups of type \mathfrak{t} . Since the parallel transport of \mathcal{D} preserves both \mathcal{L} and ω , this bundle of torus groups inherits a flat Ehresmann connection which preserves both its symplectic structure and the image of the section s_0 and whose holonomy group reduces to $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. In particular, the structure group of $\mathcal{A}(\Delta)$ reduces to $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. □

As explained in Appendix B, any integral symplectic torus group $\mathbf{A} = (A, \Omega)$ of type \mathfrak{t} determines an integral symplectic space $(H_1(A, \mathbb{R}), \omega, H_1(A, \mathbb{Z}))$, where ω is the cohomology class of Ω , viewed as a symplectic pairing on the vector space $H_1(A, \mathbb{R})$.

Proposition 2.20. *Given a bundle \mathcal{A} of integral symplectic torus groups of type \mathfrak{t} defined on M , let $\mathcal{S}_{\mathcal{A}}$ be the vector bundle with fiber at $m \in M$ given by $H_1(\mathcal{A}_m, \mathbb{R})$ and $\mathcal{L}_{\mathcal{A}}$ be the bundle of discrete Abelian groups with fiber at $m \in M$ given by $H_1(\mathcal{A}_m, \mathbb{Z})$. Moreover, let $\omega_{\mathcal{A}}$ be the fiberwise symplectic pairing defined on $\mathcal{S}_{\mathcal{A}}$ through:*

$$\omega_{\mathcal{A},m} = \omega_m, \quad \forall m \in M,$$

where ω_m is the cohomology class of the symplectic form Ω_m of the fiber \mathcal{A}_m . Then the flat Ehresmann connection of \mathcal{A} induces a flat linear connection $\mathcal{D}_{\mathcal{A}}$ on $\mathcal{S}_{\mathcal{A}}$ which makes the quadruplet:

$$\Delta(\mathcal{A}) \stackrel{\text{def.}}{=} (\mathcal{S}_{\mathcal{A}}, \omega_{\mathcal{A}}, \mathcal{D}_{\mathcal{A}}, \mathcal{L}_{\mathcal{A}}),$$

into an integral duality structure of type \mathfrak{t} defined on M .

Proof. $\mathcal{D}_{\mathcal{A}}$ is the connection induced by the flat Ehresmann connection of \mathcal{D} on the bundle of first homology groups of the fibers, which preserves the bundle $\mathcal{L}_{\mathcal{A}}$ of integral first homology groups of these fibers. The remaining statements are immediate. \square

The two propositions above imply the following statement.

Proposition 2.21. *The correspondences $\Delta \mapsto \mathcal{A}(\Delta)$ and $\mathcal{A} \mapsto \Delta(\mathcal{A})$ extend to mutually quasi-inverse equivalences between the groupoid $\text{Dual}_{\mathbb{Z}}^{\mathfrak{t}}(M)$ of integral duality structures of type \mathfrak{t} defined on M and the groupoid $\mathfrak{T}_{\mathfrak{t}}(M)$ of bundles of integral symplectic torus groups of type \mathfrak{t} defined on M .*

Combining everything, we have a chain of equivalences of groupoids:

$$\text{Prin}_{\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})}(M) \simeq \text{Sg}_{\mathfrak{t}}(M) \simeq \mathfrak{T}_{\mathfrak{t}}(M) \simeq \text{Dual}_{\mathbb{Z}}^{\mathfrak{t}}(M).$$

2.6. The exponential sheaf sequence defined by a Siegel system

Let Z be a Siegel system of type $\mathfrak{t} \in \text{Div}^n$ on M . Let $\mathcal{S}_Z \stackrel{\text{def.}}{=} Z \otimes_{\mathbb{Z}} \mathbb{R}$ be the underlying vector bundle of the integral duality structure $\Delta(Z)$ defined by Z and $\mathcal{A}_Z \stackrel{\text{def.}}{=} \mathcal{S}_Z/Z$ be the associated bundle of integral symplectic torus groups. The exponential sequence the torus group $\mathbb{R}^{2n}/\Lambda_{\mathfrak{t}} \simeq \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ (where the canonical symplectic lattice $\Lambda_{\mathfrak{t}}$ of type \mathfrak{t} is defined in Appendix B) induces a short exact sequence of bundles of abelian groups (where j is the

inclusion):

$$0 \rightarrow Z \xrightarrow{j} \mathcal{S}_Z \xrightarrow{\text{exp}} \mathcal{A}_Z \rightarrow 0.$$

In turn, this induces an exact sequence of sheaves of abelian groups:

$$0 \rightarrow \mathcal{C}(Z) \xrightarrow{j} \mathcal{C}^\infty(\mathcal{S}_Z) \xrightarrow{\text{exp}} \mathcal{C}^\infty(\mathcal{A}_Z) \rightarrow 0,$$

where $\mathcal{C}(Z)$ is the sheaf of continuous (and hence locally constant) sections of Z . This induces a long exact sequence in sheaf cohomology, of which we are interested in the following piece:

$$(17) \quad H^1(M, \mathcal{C}(Z)) \xrightarrow{j_*} H^1(M, \mathcal{C}^\infty(\mathcal{S}_Z)) \xrightarrow{\text{exp}_*} H^1(M, \mathcal{C}^\infty(\mathcal{A}_Z)) \\ \xrightarrow{\delta} H^2(M, \mathcal{C}(Z)) \rightarrow H^2(M, \mathcal{C}^\infty(\mathcal{S}_Z)),$$

where δ is the Bockstein morphism. The sheaf $\mathcal{C}^\infty(\mathcal{S}_Z)$ is fine and hence acyclic since M is paracompact and \mathcal{S}_Z is a vector bundle on M . Thus:

$$H^j(M, \mathcal{S}_Z) = 0, \quad \forall j > 0,$$

which by the long sequence above implies that δ is an isomorphism of groups. We also have $H^*(M, \mathcal{C}(Z)) = H^*(M, Z)$, where in the right hand side Z is viewed as a local system of \mathbb{Z}^{2n} coefficients. Hence we can view δ as an isomorphism of abelian groups:

$$(18) \quad \delta : H^1(M, \mathcal{C}^\infty(\mathcal{A}_Z)) \xrightarrow{\sim} H^2(M, Z).$$

2.7. The lattice of charges of an integral duality structure

For every integral duality structure $\Delta = (\Delta, \mathcal{L})$ on M , the sheaf $\mathcal{C}_{\text{flat}}^\infty(\mathcal{A})$ of flat smooth local sections of the bundle $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}(\Delta) = \mathcal{S}/\mathcal{L}$ of integral symplectic torus groups defined by Δ fits into the short exact sequence of sheaves of abelian groups:

$$1 \rightarrow \mathcal{C}(\mathcal{L}) \xrightarrow{j_0} \mathcal{C}_{\text{flat}}^\infty(\mathcal{S}) \xrightarrow{\text{exp}} \mathcal{C}_{\text{flat}}^\infty(\mathcal{A}) \rightarrow 1,$$

where j is the inclusion. This induces a long exact sequence in sheaf cohomology, of which we are interested in the following terms:

$$\dots \rightarrow H^1(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{A})) \xrightarrow{\delta_0} H^2(M, \mathcal{C}(\mathcal{L})) \\ \xrightarrow{j_{0*}} H^2(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{S})) \xrightarrow{\text{exp}_*} H^2(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{A})) \rightarrow \dots,$$

where δ_0 is the Bockstein morphism. Notice that $H^*(M, \mathcal{C}(\mathcal{L})) \simeq H^*(M, Z)$, where $Z = \mathcal{L}$ is the Siegel system defined by \mathcal{L} , which we view of a local system of \mathbb{Z}^{2n} coefficients. Moreover, we have $H^2(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{A})) = H^2(M, \mathcal{A}_{\text{disc}})$, the right hand side being the cohomology with coefficients in the local system defined by \mathcal{A} when the fibers of the latter are endowed with the *discrete* topology. Since $H^*(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{S})) = H_{\mathcal{D}}^*(M, \mathcal{S})$, the sequence above can be written as:

$$(19) \quad \dots \rightarrow H^1(M, \mathcal{A}_{\text{disc}}) \xrightarrow{\delta_0} H^2(M, Z) \xrightarrow{j_{0*}} H_{\mathcal{D}}^2(M, \mathcal{S}) \xrightarrow{\text{exp}_*} H^2(M, \mathcal{A}_{\text{disc}}) \rightarrow \dots$$

Denote by $H^2(M, Z)^{\text{tf}} \subset H^2(M, Z)$ the torsion free part of $H^2(M, Z)$.

Definition 2.22. The lattice:

$$L_{\Delta} \stackrel{\text{def.}}{=} j_{0*}(H^2(M, Z)) = j_{0*}(H^2(M, Z)^{\text{tf}}) \subset H_{\mathcal{D}}^2(M, \mathcal{S})$$

is called the *lattice of charges* defined the integral duality structure Δ . Elements of this lattice are called *integral cohomology classes* or *charges*.

Proposition 2.23. *There exists a natural isomorphism:*

$$(20) \quad H_{\mathcal{D}}^k(M, \mathcal{S}) \simeq H^k(M, Z) \otimes_{\mathbb{Z}[\pi]} \mathbb{R} \simeq H^k(M, Z)^{\text{tf}} \otimes_{\mathbb{Z}[\pi]} \mathbb{R},$$

for all k . In particular, the kernel of j_{0*} coincides with the torsion part of $H^2(M, Z)$ and $j_*(H^2(M, Z))$ is a full lattice in $H_{\mathcal{D}}^2(M, \mathcal{S})$.

Proof. Let $\pi \stackrel{\text{def.}}{=} \pi_1(M, m)$ and $\mathbb{Z}[\pi]$ be the group ring of π . The universal coefficient theorem for cohomology with local coefficients of [29] gives a short exact sequence:

$$0 \rightarrow H^k(M, Z) \otimes_{\mathbb{Z}[\pi]} \mathbb{R} \rightarrow H_{\mathcal{D}}^k(M, \mathcal{S}) \rightarrow \text{Tor}_{\mathbb{Z}[\pi]}(H^{k+1}(M, Z), \mathbb{R}) \rightarrow 0,$$

where \mathbb{R} is the $\mathbb{Z}[\pi]$ -module corresponding to the trivial representation of π in \mathbb{R} . Since the latter module is free, we have $\text{Tor}_{\mathbb{Z}[\pi]}(H^{k+1}(M, Z), \mathbb{R}) = 0$ and the sequence above gives the natural isomorphism (20). \square

Remark 2.24. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{C}(Z) & \xrightarrow{j_0} & \mathcal{C}_{\text{flat}}^\infty(\mathcal{S}_Z) & \xrightarrow{\text{exp}} & \mathcal{C}_{\text{flat}}^\infty(\mathcal{A}_Z) \longrightarrow 0 \\
 & & \downarrow \text{id} & & \downarrow \tau & & \downarrow \iota \\
 0 & \longrightarrow & \mathcal{C}(Z) & \xrightarrow{j} & \mathcal{C}^\infty(\mathcal{S}_Z) & \xrightarrow{\text{exp}} & \mathcal{C}^\infty(\mathcal{A}_Z) \longrightarrow 0
 \end{array}$$

where j_0, j and τ, ι are inclusions. In turn, this induces a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
 H^1(M, Z) & \xrightarrow{j_{0*}} & H_{\mathcal{D}_p}^1(M, \mathcal{S}_Z) & \xrightarrow{\text{exp}_*} & H^1(M, \mathcal{A}_{Z, \text{disc}}) & \xrightarrow{\delta_0} & H^2(M, Z) & \xrightarrow{j_{0*}} & H_{\mathcal{D}_p}^2(M, \mathcal{S}_Z) \\
 \downarrow \text{id} & & \downarrow \tau_* & & \downarrow \iota_* & & \downarrow \text{id} & & \downarrow \tau_* \\
 H^1(M, Z) & \xrightarrow{j_*} & 0 & \xrightarrow{\text{exp}_*} & H^1(M, \mathcal{C}^\infty(\mathcal{A}_Z)) & \xrightarrow{\delta} & H^2(M, Z) & \xrightarrow{j_*} & 0
 \end{array}$$

In particular, we have $\delta_0 = \delta \circ \iota_*$.

2.8. The DSZ integrality condition

Let (M, g) be an oriented and connected Lorentzian four-manifold. Given an integral duality structure $\mathbf{\Delta} = (\Delta, \mathcal{L})$, we implement the DSZ condition by restricting the configuration space $\text{Conf}(M, \Delta)$ to a subsets determined by the charge lattice $L_{\mathbf{\Delta}}$. We will show in later sections that integral field strengths are adjoint curvatures of connections defined on a certain principal bundle.

Definition 2.25. Let $\mathbf{\Delta} = (\Delta, \mathcal{L})$ be an integral duality structure on (M, g) with underlying duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. The set of integral electromagnetic field strength configurations defined by $\mathbf{\Delta}$ on M is the following subset of $\text{Conf}(M, \Delta)$:

$$\text{Conf}(M, \mathbf{\Delta}) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \text{Conf}(M, \Delta) \mid 2\pi[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}} \} ,$$

where $[\mathcal{V}]_{\mathcal{D}} \in H_{\mathcal{D}}^2(M, \mathcal{S})$ is the $d_{\mathcal{D}}$ -cohomology class of the \mathcal{S} -valued two-form $\mathcal{V} \in \text{Conf}(M, \Delta)$ and $L_{\mathbf{\Delta}} \subset H_{\mathcal{D}}^2(M, \mathcal{S})$ is the lattice of charges defined by $\mathbf{\Delta}$.

Definition 2.26. Let $\mathbf{\Xi} = (\Delta, \mathcal{J})$ be an integral electromagnetic structure defined on (M, g) with electromagnetic structure $\Xi = (\Delta, \mathcal{J})$ and integral duality structure $\mathbf{\Delta} = (\Delta, \mathcal{L})$. The set of integral field strength solutions

defined by Ξ on (M, g) is the subset of $\text{Sol}(M, g, \Xi)$ defined through:

$$\text{Sol}(M, g, \Xi) \stackrel{\text{def.}}{=} \text{Sol}(M, g, \Xi) \cap \text{Conf}(M, \Delta)$$

and hence consists of those elements of $\text{Conf}(M, \Delta)$ which satisfy the equations of motion (i.e. the polarized self-duality condition) of the classical abelian gauge theory defined by Ξ .

2.9. Integral duality groups

The DSZ integrality condition restricts the classical duality groups of Subsection 1.9 to certain subgroups.

Definition 2.27. Fix an integral duality structure Δ on (M, g) .

- The *integral unbased pseudo-duality group* defined by Δ is the group $\text{Aut}(\Delta) \subset \text{Aut}(\Delta)$ formed by those elements $u \in \text{Aut}(\Delta)$ which satisfy $u(\mathcal{L}) = \mathcal{L}$.
- The *integral unbased duality group* defined by Δ is the subgroup $\text{Aut}(g, \Delta)$ of $\text{Aut}(\Delta)$ which covers $\text{Iso}(M, g)$.
- The *integral duality group* defined by Δ is the subgroup $\text{Aut}_b(\Delta)$ of $\text{Aut}(\Delta)$ consisting of those elements which cover the identity of M .

Definition 2.28. Fix an integral electromagnetic structure $\Xi = (\Delta, \mathcal{J})$ on (M, g) .

- The *integral unbased unitary pseudo-duality group* defined by Ξ is the group:

$$\text{Aut}(\Xi) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(\Delta) \mid \mathcal{J}_u = \mathcal{J}\}$$

- The *integral unbased unitary duality group* defined by Ξ is:

$$\text{Aut}(g, \Xi) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(\Xi) \mid g_u = g\} .$$

- The *integral unitary duality group* defined by Ξ is the subgroup $\text{Aut}_b(\Xi)$ of $\text{Aut}(\Xi)$ consisting of those elements which cover the identity of M .

It is easy to check that \mathbb{A}_u with u belonging to the groups defined above restrict to transformations similar to those of Subsection 1.9 between the sets of *integral* configurations and solutions. The discrete duality groups introduced above are the global counterparts of the discrete duality group

considered in the physics literature on local abelian gauge theory. The latter is usually taken to be $\mathrm{Sp}(2n, \mathbb{Z})$ due to the fact that the symplectic lattice of charges appearing in the local treatment of abelian gauge theory is traditionally assumed to have principal type $\mathfrak{t} = \delta = (1, \dots, 1)$. As explained in [32] and recalled in Appendix A, $\mathrm{Sp}(2n, \mathbb{Z})$ is not always the correct duality group even in the local case, since the local lattice of charges need not be principal. In Section 4.1, we consider a natural gauge-theoretic extension of the discrete duality groups defined above, which clarifies the geometric origin of electromagnetic duality.

2.10. Trivial integral duality structures

Let Z be a trivialisable Siegel system of type $\mathfrak{t} \in \mathrm{Div}^n$ and $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be the associated duality structure, where $\mathcal{S} = Z \otimes_{\mathbb{Z}} \mathbb{R}$. Pick a flat trivialization $\tau : \mathcal{S} \xrightarrow{\sim} M \times S$ of \mathcal{S} , where $S \simeq \mathbb{R}^{2n}$. This takes ω into a symplectic pairing ω_S on the vector space S and restricts to an isomorphism $\tau_0 : Z \xrightarrow{\sim} M \times \Lambda$ between Z and $M \times \Lambda$, where Λ is a full symplectic lattice in (S, ω_S) . Let $A \stackrel{\mathrm{def.}}{=} S/\Lambda$ be the torus group defined by (S, Λ) and $\mathcal{A} \stackrel{\mathrm{def.}}{=} \mathcal{S}/Z$ be the bundle of torus groups defined by (\mathcal{S}, Z) . Then τ induces a trivialization $\bar{\tau} : \mathcal{A} \xrightarrow{\sim} M \times A$ of \mathcal{A} , which fits into a commutative diagram of fiber bundles:

$$\begin{array}{ccccc}
 Z & \xrightarrow{j} & \mathcal{S} & \longrightarrow & \mathcal{A} \\
 \downarrow \tau_0 & & \downarrow \tau & & \downarrow \bar{\tau} \\
 M \times \Lambda & \xrightarrow{i} & M \times S & \longrightarrow & M \times A
 \end{array}$$

where i and j are inclusions. Since τ identifies \mathcal{D} with the trivial connection on $M \times S$, it induces an isomorphism of graded vector spaces:

$$\tau_* : H_{\mathcal{D}}^*(M, \mathcal{S}) \xrightarrow{\sim} H^*(M, S)$$

whose restriction coincides with the isomorphism of graded abelian groups:

$$\tau_{0*} : H^*(M, Z) \xrightarrow{\sim} H^*(M, \Lambda)$$

induced by τ_0 . Moreover, $\bar{\tau}$ induces an isomorphism of graded abelian groups $\bar{\tau}_* : H^*(M, \mathcal{A}) \xrightarrow{\sim} H^*(M, A)$. Hence the diagram above induces an isomorphism of long exact sequences of abelian groups:

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & H^1(M, \mathcal{A}) & \longrightarrow & H^2(M, \mathbb{Z}) & \xrightarrow{j_*} & H^2(M, S) & \longrightarrow & H^2(M, A) & \longrightarrow & \dots \\
 & & \downarrow \bar{\tau}_* & & \downarrow \tau_{0*} & & \downarrow \tau_* & & \downarrow \bar{\tau}_* & & \\
 \dots & \longrightarrow & H^1(M, A) & \longrightarrow & H^2(M, \Lambda) & \xrightarrow{i_*} & H^2(M, S) & \longrightarrow & H^2(M, A) & \longrightarrow & \dots
 \end{array}$$

Since Λ is free while S is a vector space, we have isomorphisms of abelian groups:

$$H^*(M, S) \simeq H^*(M, \mathbb{R}) \otimes_{\mathbb{R}} S, \quad H^*(M, \Lambda) \simeq_{\mathbb{Z}} H^*(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda.$$

and:

$$H^*(M, S) \simeq H^*(M, \Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^*(M, \Lambda)^{\text{tf}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The latter agrees with the isomorphism (20) through the maps τ_{0*} and τ_* . The map $i_* : H^k(M, \Lambda) \rightarrow H^k(M, S)$ is obtained by tensoring the map $H^k(M, \mathbb{Z}) \rightarrow H^k(M, \mathbb{R})$ with the inclusion $\Lambda \subset S$, while its restriction $i_*^{\text{tf}} : H^k(M, \Lambda)^{\text{tf}} \rightarrow H^k(M, S)$ is obtained by tensoring the inclusion $\Lambda \subset S$ with the map $H^l(M, \mathbb{Z})^{\text{tf}} \rightarrow H^k(M, \mathbb{R})$. Since the latter is injective, it follows that i_*^{tf} is injective and hence $H^l(M, \Lambda)^{\text{tf}}$ identifies with a full lattice in $H^k(M, S)$. Since A and $(S, +)$ are divisible groups while $H_0(M, \mathbb{Z}) = \mathbb{Z}$ and Λ are free, the universal coefficient sequence for cohomology gives isomorphisms:

$$\begin{aligned}
 H^k(M, S) &\simeq \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z}), S) = \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z})^{\text{tf}}, S), \\
 (21) \quad H^k(M, \Lambda) &\simeq \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z}), \Lambda) = \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z})^{\text{tf}}, \Lambda)
 \end{aligned}$$

and:

$$H^k(M, A) \simeq \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z}), A) \simeq \text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z})^{\text{tf}}, A)$$

for all k . The first of these is the period isomorphism:

$$\text{per}(\omega)(c) := \text{per}_c(\omega) = c \cap \omega = \int_c \omega, \quad \forall \omega \in H^k(M, S), \quad \forall c \in H_k(M, \mathbb{Z}).$$

The map $i_* : H^k(M, \Lambda) \rightarrow H^k(M, S)$ agrees with the injective map induced by the inclusion $\Lambda \hookrightarrow S$ through the isomorphisms (21). Hence:

$$\begin{aligned}
 H^k(M, \Lambda) &\simeq \text{per}^{-1}(\text{Hom}_{\mathbb{Z}}(H_k(M, \mathbb{Z}), \Lambda)) = i_*(H^k(M, \Lambda)) \\
 &= \{\omega \in H^k(M, S) \mid \text{per}_c(\omega) \in \Lambda \ \& \ c \in H_k(M, \mathbb{Z})\}.
 \end{aligned}$$

3. Siegel bundles and connections

In this section we use the notion of *integral affine symplectic torus*, for which we refer the reader to Appendix B.

3.1. Automorphisms of integral affine symplectic tori

We denote by $\text{Aff}_\mathfrak{t}$ the group of affine symplectomorphisms of the integral affine symplectic torus $\mathbb{A}_\mathfrak{t} = (\mathbb{A}, \Omega_\mathfrak{t})$ of type $\mathfrak{t} \in \text{Div}^n$. Here \mathbb{A} is the underlying $2n$ -dimensional affine torus (which is a principal homogeneous space for the torus group $U(1)^{2n} \simeq \mathbb{R}^{2n}/\mathbb{Z}^{2n}$), while $\Omega_\mathfrak{t}$ is the integral symplectic form of type \mathfrak{t} on \mathbb{A} , which is translationally-invariant. As explained in Appendix B, $\text{Aff}_\mathfrak{t}$ is a non-compact disconnected Lie group whose connected component of the identity is the $2n$ -dimensional torus group $U(1)^{2n}$. We have $\pi_0(\text{Aff}_\mathfrak{t}) \simeq \text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$ and:

$$(22) \quad \text{Aff}_\mathfrak{t} \simeq U(1)^{2n} \times \text{Sp}_\mathfrak{t}(2n, \mathbb{Z}),$$

where $\text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$ acts on $U(1)^{2n}$ through the restriction of the group morphism defined in equation (B.17) of Appendix B, an action which we denote by juxtaposition. Thus $\text{Aff}_\mathfrak{t}$ identifies with the set $U(1)^{2n} \times \text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$, endowed with the composition rule:

$$\begin{aligned} (a_1, \gamma_1) (a_2, \gamma_2) &= (a_1 + \gamma_1 a_2, \gamma_1 \gamma_2), \\ \forall a_1, a_2 \in U(1)^{2n}, \quad \forall \gamma_1, \gamma_2 \in \text{Sp}_\mathfrak{t}(2n, \mathbb{Z}). \end{aligned}$$

Let $\ell : \text{Aff}_\mathfrak{t} \rightarrow \text{Diff}(\text{Aff}_\mathfrak{t})$ be the left action of $\text{Aff}_\mathfrak{t}$ on itself:

$$\ell(g)(g') \stackrel{\text{def.}}{=} gg', \quad \forall g, g' \in \text{Aff}_\mathfrak{t},$$

and let $\text{pr}_1 : \text{Aff}_\mathfrak{t} \rightarrow U(1)^{2n}$ and $\text{pr}_2 : \text{Aff}_\mathfrak{t} \rightarrow \text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$ be the projections of the set-theoretic decomposition $\text{Aff}_\mathfrak{t} = U(1)^{2n} \times \text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$. Notice that pr_2 is a morphism of groups. Define left actions ℓ_1 and ℓ_2 of $\text{Aff}_\mathfrak{t}$ on $U(1)^{2n}$ and $\text{Sp}_\mathfrak{t}(2n, \mathbb{Z})$ through:

$$\begin{aligned} \ell_1(g)(a) &\stackrel{\text{def.}}{=} \text{pr}_1(\ell(g)(a, 1)), \quad \ell_2(g)(\gamma) \stackrel{\text{def.}}{=} \text{pr}_2(\ell(g)(0, \gamma)), \\ \forall g \in \text{Aff}_\mathfrak{t}, \quad \forall a \in U(1)^{2n}, \quad \forall \gamma \in \text{Sp}_\mathfrak{t}(2n, \mathbb{Z}). \end{aligned}$$

Then ℓ_1 is given by:

$$(23) \quad \ell_1(a, \gamma)(a') = a + \gamma a', \quad \forall \gamma \in \text{Sp}_\mathfrak{t}(2n, \mathbb{Z}), \quad \forall a, a' \in U(1)^{2n}.$$

This action is transitive with stabilizer isomorphic with $\mathrm{Sp}_t(2n, \mathbb{Z})$. On the other hand, ℓ_2 is given by:

$$(24) \quad \ell_2(a, \gamma)(\gamma') = \gamma\gamma' = p_2(a, \gamma)\gamma', \quad \forall \gamma, \gamma' \in \mathrm{Sp}_t(2n, \mathbb{Z}), \quad \forall a \in \mathrm{U}(1)^{2n}$$

and is transitive with stabilizer isomorphic to $\mathrm{U}(1)^{2n}$. This gives the right-split short exact sequence:

$$0 \rightarrow \mathrm{U}(1)^{2n} \rightarrow \mathrm{Aff}_t \rightarrow \mathrm{Sp}_t(2n, \mathbb{Z}) \rightarrow 1.$$

Notice that $\ell = \ell_1 \times \ell_2$. The Lie algebra \mathfrak{aff}_t of Aff_t is abelian and coincides with the Lie algebra of $\mathrm{U}(1)^{2n}$:

$$\mathfrak{aff}_t = \mathbb{R}^{2n} \simeq H_1(\mathbb{A}_t, \mathbb{R}).$$

The exponential map $\exp : \mathfrak{aff}_t \rightarrow \mathrm{Aff}_t$ has kernel $\Lambda_t \simeq H_1(\mathbb{A}_t, \mathbb{Z})$ and image A , giving the exponential sequence:

$$(25) \quad 0 \rightarrow \Lambda_t \rightarrow \mathfrak{aff}_t \xrightarrow{\exp} \mathrm{U}(1)^{2n} \rightarrow 0.$$

Lemma 3.1. *The adjoint representation $\mathrm{Ad} : \mathrm{Aff}_t \rightarrow \mathrm{GL}(2n, \mathbb{R})$ of Aff_t coincides with its fundamental linear representation, that is:*

$$(26) \quad \mathrm{Ad}(a, \gamma)(v) = \gamma(v), \quad \forall (a, \gamma) \in \mathrm{Aff}_t, \quad \forall v \in \mathbb{R}^{2n}.$$

In particular, we have $\mathrm{Ad} = j \circ \mathrm{pr}_2$, where $j : \mathrm{Sp}_t(2n, \mathbb{Z}) \rightarrow \mathrm{GL}(2n, \mathbb{R})$ is the fundamental representation of $\mathrm{Sp}_t(2n, \mathbb{Z})$.

Proof. Let $\alpha = (\gamma, 1) : I \rightarrow \mathrm{Aff}_t = \mathrm{U}(1)^{2n} \times \mathrm{Sp}_t(2n, \mathbb{Z})$ be a smooth path in Aff_t such that $\alpha(0) = \mathrm{Id}$. Set:

$$\frac{d}{dt}\alpha(t)|_{t=0} = v \in \mathbb{R}^{2n}.$$

For every $x = (x_1, x_2) \in \mathrm{Aff}_t = \mathrm{U}(1)^{2n} \times \mathrm{Sp}_t(2n, \mathbb{Z})$ we have:

$$x \alpha(t) x^{-1} = (x_1, x_2) (\gamma, 1) (-x_2^{-1}x_1, x_2^{-1}) = (x_2\gamma, 1).$$

Hence:

$$\frac{d}{dt}(x \alpha(t) x^{-1})|_{t=0} = x_2(v),$$

which immediately implies:

$$\mathrm{Ad}(x) = j \circ \mathrm{pr}_2(x),$$

for every $x \in \text{Aff}_{\mathfrak{t}}$ and hence we conclude. □

3.2. Siegel bundles

Let M be a connected manifold.

Definition 3.2. A Siegel bundle P of rank n and type $\mathfrak{t} \in \text{Div}^n$ is a principal bundle on M with structure group $\text{Aff}_{\mathfrak{t}}$. An *isomorphism of Siegel bundles* is a based isomorphism of principal bundles.

Let $\text{Sieg}(M)$ be the groupoid of Siegel bundles defined on M and $\text{Sieg}_{\mathfrak{t}}(M)$ be the full subgroupoid of Siegel bundles of type \mathfrak{t} . Fix a Siegel bundle P of type $\mathfrak{t} \in \text{Div}^n$, whose projection we denote by π . We introduce several fiber bundles associated to P .

3.2.1. The bundle of integral affine symplectic tori defined by P .

Definition 3.3. A fiber bundle \mathfrak{A} defined on M is called a *bundle of integral affine symplectic tori of rank n* if its fibers are $2n$ -dimensional tori and the structure group of \mathfrak{A} reduces to a subgroup of $\text{Aff}_{\mathfrak{t}}$ for some $\mathfrak{t} \in \text{Div}^n$. The smallest element $\mathfrak{t}_{\mathfrak{A}} \in \text{Div}^n$ with this property is called the *type* of \mathfrak{A} .

Notice that bundles of integral symplectic torus groups coincide with those bundles of integral affine symplectic tori which admit a smooth global section. Indeed, such a section gives a further reduction of structure group from $\text{Aff}_{\mathfrak{t}}$ to $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. Given a Siegel bundle P of type $\mathfrak{t} \in \text{Div}^n$ defined on M , the fiber bundle:

$$\mathfrak{A}(P) \stackrel{\text{def.}}{=} P \times_{\ell_1} \text{U}(1)^{2n}$$

associated to P through the action (23) is a bundle of integral affine symplectic tori of type \mathfrak{t} . The fibers of the latter admit integral symplectic forms of type \mathfrak{t} which vary smoothly over M . The group $\text{Sp}(V, \omega)$ acts freely and transitively on the set $\text{Fr}(V, \omega, \Lambda)$ of integral symplectic bases of any integral symplectic space (V, ω, Λ) . Any bundle \mathfrak{A} of integral affine symplectic tori of type \mathfrak{t} is associated through the action ℓ_1 to its Siegel bundle $P(\mathfrak{A})$ of *unpointed torus symplectic frames*, which has type \mathfrak{t} and whose fiber at $m \in M$ is defined through:

$$P(\mathfrak{A})_m \stackrel{\text{def.}}{=} \text{Fr}(H_1(\mathfrak{A}_m, \mathbb{R}), H_1(\mathfrak{A}_m, \mathbb{Z}), \omega_m) \times \mathfrak{A}_m .$$

Here $\omega_m \stackrel{\text{def.}}{=} [\Omega_m] \in H^2(\mathfrak{A}_m, \mathbb{R}) \simeq \wedge^2 H_1(\mathfrak{A}_m, \mathbb{R})^\vee$ is the cohomology class of the symplectic form Ω_m of \mathfrak{A}_m , viewed as a symplectic pairing defined on $H_1(\mathfrak{A}_m, \mathbb{R})$. More precisely, we have:

Proposition 3.4. *The correspondences $P \rightarrow \mathfrak{A}(P)$ and $\mathfrak{A} \rightarrow P(\mathfrak{A})$ extend to mutually quasi-inverse equivalences of groupoids between $\text{Sieg}(M)$ and the groupoid of bundles of integral affine symplectic tori and these equivalences preserve type.*

This statement parallels a similar correspondence which holds for affine torus bundles (see [11] as well as Theorem 2.2. and Remark 2.3 in [12]).

3.2.2. The Siegel system defined by P . Given a Siegel bundle of type $\mathfrak{t} \in \text{Div}^n$, consider the bundle of discrete abelian groups defined through:

$$Z(P)_m \stackrel{\text{def.}}{=} H_1(\mathfrak{A}(P)_m, \mathbb{Z}), \quad \forall m \in M.$$

Since torus translations act trivially on $H_1(\mathfrak{A}(P)_m, \mathbb{Z})$, the structure group of $Z(P)$ reduces to $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. Thus $Z(P)$ is a Siegel system on M . Moreover, $Z(P)$ is isomorphic with the bundle of discrete abelian groups associated to P through the projection morphism $p_2 : \text{Aff}_{\mathfrak{t}} \rightarrow \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, when the latter is viewed as a left action of $\text{Aff}_{\mathfrak{t}}$ through automorphisms of the group $(\mathbb{Z}^{2n}, +)$.

Definition 3.5. $Z(P)$ is called the *Siegel system defined by P* .

Notice that the the monodromy of P at a point $m \in M$ acts through automorphisms of the integral symplectic space $(H_1(\mathfrak{A}(P)_m, \mathbb{R}), H_1(\mathfrak{A}(P)_m, \mathbb{Z}), [\Omega_m])$.

3.2.3. The adjoint bundle and integral duality structure of P . The adjoint bundle $\text{ad}(P)$ of P can be identified with the tensor product $Z(P) \otimes_{\mathbb{Z}} \mathbb{R}$, whose fiber at $m \in M$ is given by:

$$\text{ad}(P)_m = Z(P)_m \otimes_{\mathbb{Z}} \mathbb{R} \simeq H_1(\mathfrak{A}(P)_m, \mathbb{R}).$$

Notice that $\text{ad}(P)$ carries the fiberwise symplectic pairing ω_P given by $(\omega_P)_m \stackrel{\text{def.}}{=} [\Omega_m]$ for all $m \in M$ (see Lemma 3.1). Since the Lie algebra of $\text{Aff}_{\mathfrak{t}}$ is abelian, the structure group of $\text{ad}(P)$ reduces to $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. The Siegel system $Z(P)$ is naturally a sub-bundle of $\text{ad}(P)$ whose fibers $Z(P)_m = H_1(\mathfrak{A}(P)_m, \mathbb{Z})$ are full symplectic lattices with respect to $[\Omega_m]$.

The monodromy of $Z(P)$ induces a unique flat connection \mathcal{D}_P on $\text{ad}(P)$ whose parallel transport preserves $Z(P)$. Setting $\mathcal{S}_P \stackrel{\text{def.}}{=} \text{ad}(P)$, it follows that the system:

$$\mathbf{\Delta}(P) \stackrel{\text{def.}}{=} (\Delta(P), Z(P)), \text{ where } \Delta(P) \stackrel{\text{def.}}{=} (\mathcal{S}_P, \omega_P, \mathcal{D}_P),$$

is an integral duality structure of type \mathfrak{t} , whose underlying duality structure is $\Delta(P)$.

Proposition 3.6. *There correspondence defined above extends to an essentially surjective functor:*

$$\mathbf{\Delta}: \text{Sieg}(M) \rightarrow \text{Dual}_{\mathbb{Z}}(M),$$

which associates to every Siegel bundle P of type $\mathfrak{t} \in \text{Div}^n$ defined on M the integral duality structure $\mathbf{\Delta}(P)$, which has type \mathfrak{t} .

Proof. It is clear that the correspondence extends to a functor. Given $\mathbf{\Delta} = (\Delta, \mathcal{L}) \in \text{Dual}_{\mathbb{Z}}(M)$, denote by Q the frame bundle of the Siegel system defined by \mathcal{L} , which is a principal $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ -bundle (see Proposition 2.12). Let P be the Siegel bundle associated to Q through the natural left action l of $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ on $\text{Aff}_{\mathfrak{t}}$:

$$P = Q \times_l \text{Aff}_{\mathfrak{t}}.$$

Then $\mathbf{\Delta}(P) = \mathbf{\Delta}$, showing that the functor is essentially surjective. □

3.2.4. The bundle of integral symplectic torus groups defined by P . Consider a Siegel bundle P defined on M .

Definition 3.7. The bundle of integral symplectic torus groups defined by P is the bundle:

$$\mathcal{A}(P) = \mathcal{A}(\mathbf{\Delta}(P)) = \text{ad}(P)/Z(P)$$

of integral symplectic torus groups defined by the integral duality structure $\mathbf{\Delta}(P)$.

3.2.5. Siegel bundles with trivial monodromy.

Proposition 3.8. *Let P be a Siegel bundle of rank n and type $\mathfrak{t} \in \text{Div}^n$ defined on M . Then the following statements are equivalent:*

- (a) *The Siegel system $Z(P)$ has trivial monodromy.*

- (b) $Z(P)$ is trivial as a bundle of discrete Abelian groups.
- (c) The structure group of P reduces to the torus group $U(1)^{2n}$.
- (d) The structure group of the bundle of integral symplectic affine tori $\mathfrak{A}(P)$ reduces to $U(1)^{2n}$.
- (e) The structure group of the bundle of integral symplectic torus groups $\mathcal{A}(P)$ is trivial.
- (f) The duality structure $\Delta(P)$ is holonomy-trivial.

In this case, $\mathfrak{A}(P)$ identifies with a principal torus bundle and $\mathcal{A}(P)$ is a trivial bundle of integral symplectic torus groups. Moreover, P is isomorphic with the fiber product $\mathfrak{A}(P) \times_M \underline{\Gamma}$, where $\underline{\Gamma}$ is the trivial $Sp_{\mathfrak{t}}(2n, \mathbb{Z})$ -bundle defined on M . Thus P identifies with a countable collection of copies of the principal torus bundle $\mathfrak{A}(P)$, indexed by elements of $Sp_{\mathfrak{t}}(2n, \mathbb{Z})$.

Proof. The fact that (a) implies (b) follows from the standard characterization of flat bundles in terms of holonomy representations of the fundamental group of the underlying manifold. If $Z(P)$ is trivial as a bundle of discrete groups then the holonomy representation preserves a global frame of $Z(P)$, which in turn implies, using the explicit form of the adjoint representation of $Aff_{\mathfrak{t}}$, that the holonomy representation takes values in $U(1)^{2n} \subset Aff_{\mathfrak{t}}$. The associated holonomy bundle defines a reduction of P to a principal torus bundle with structure group $U(1)^{2n}$. This immediately implies (d), (e) and (f). Since the flat connection on $\Delta(P)$ is by definition the real linear extension of the flat connection of $Z(P)$ the latter has trivial monodromy if and only if $\Delta(P)$ has trivial monodromy, that is, if and only if $\Delta(P)$ is holonomy-trivial. This proves (f) \Rightarrow (a). □

3.3. Classification of Siegel bundles

Let P be a Siegel bundle of type $\mathfrak{t} \in \text{Div}^n$ defined on M and $\Delta := \Delta(P) = (\mathcal{S}_P = \text{ad}(P), \omega_P, \mathcal{D}_P, Z(P))$ be the integral duality structure defined by P . The Bockstein isomorphism (18) reads:

$$\delta : H^1(M, \mathcal{C}^\infty(\mathcal{A}(P))) \xrightarrow{\sim} H^2(M, Z(P)).$$

It was shown in [11] that P determines a primary characteristic class $c^l(P) \in H^1(M, \mathcal{C}^\infty(\mathcal{A}(P)))$.

Definition 3.9. The *twisted Chen class* of P is:

$$c(P) \stackrel{\text{def.}}{=} \delta(c'(P)) \in H^2(M, Z(P)).$$

Recall from Proposition 3.4 that isomorphism classes of Siegel bundles defined on M are in bijection with isomorphism classes of bundles of integral affine symplectic tori. This allows one to classify Siegel bundles by adapting the classification of affine torus bundles given in [11, Section 2] (see also [12, Theorem 2.2]). Since the modifications of the argument of loc. cit. are straightforward, we simply describe the result. Adapting the argument of [11] we obtain:

Theorem 3.10. *Consider the set:*

$$\Sigma(M) \stackrel{\text{def.}}{=} \{(Z, c) \mid Z \in \text{Ob}[\text{Sg}(M)] \ \& \ c \in H^2(M, Z)\} / \sim,$$

where $(Z, c) \sim (Z', c')$ if and only if there exists an isomorphism of Siegel systems $\varphi : Z \rightarrow Z'$ such that $\varphi_*(c) = c'$. Then the map:

$$P \mapsto (Z(P), c(P))$$

induces a bijection between the set of isomorphism classes of Siegel bundles defined on M and the set $\Sigma(M)$.

A more conceptual explanation of this result is given in [35]. Let $\rho : \text{Sp}_t(2n, \mathbb{Z}) \rightarrow \text{Aut}(\text{U}(1)^{2n})$ denote the action of $\text{Sp}_t(2n, \mathbb{Z})$ on $\text{U}(1)^{2n}$ and $\rho_0 : \text{Sp}_t(2n, \mathbb{Z}) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathbb{Z}^{2n})$ be the corresponding action on \mathbb{Z}^{2n} . Then the classifying space of the Aff_t is a *twisted Eilenberg-McLane space* $L := L_{\text{Sp}_t(2n, \mathbb{Z})}(\mathbb{Z}^{2n}, 2)$ in the sense of [27], which is a sectioned fibration over $\text{B}\Gamma \simeq K(\Gamma, 1)$ whose fibers are homotopy equivalent with $\text{BU}(1)^{2n} \simeq K(\mathbb{Z}^{2n}, 2) \simeq (\mathbb{C}\mathbb{P}^\infty)^{\times 2n}$. This space is a homotopy two-type with:

$$\pi_1(L) = \text{Sp}_t(2n, \mathbb{Z}), \ \pi_2(L) = \mathbb{Z}^{2n}.$$

The results of [27] are used in [35] to show that isomorphism classes of principal Aff_t -bundles P defined on a pointed space X are in bijection with isomorphism classes of pairs (α, c) , where $\alpha : \pi_1(X) \rightarrow \text{Sp}_t(2n, \mathbb{Z})$ is a morphism of groups and $c \in H^2(X, Z_\alpha)$, where Z_α is the local system with fiber \mathbb{Z}^{2n} and monodromy action at the basepoint of X given by $\rho_0 \circ \alpha : \pi_1(X) \rightarrow \text{Aut}_{\mathbb{Z}}(\mathbb{Z}^{2n})$. When $X = M$ is a manifold, this local system coincides with $Z(P)$, while the cohomology class c coincides with $c(P)$.

3.4. Principal connections on Siegel bundles

Let P be a Siegel bundle of type $\mathfrak{t} \in \text{Div}^n$ defined on a connected manifold M , whose projection we denote by $\pi : P \rightarrow M$. For ease of notation, we set $G \stackrel{\text{def.}}{=} \text{Aff}_{\mathfrak{t}}$, $\Gamma \stackrel{\text{def.}}{=} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ and $A \stackrel{\text{def.}}{=} \text{U}(1)^n$. We denote the abelian Lie algebra of A by $\mathfrak{a} \simeq \mathbb{R}^{2n}$. Let $\mathbf{\Delta} = (\Delta, Z)$ be the integral duality structure $\mathbf{\Delta}(P)$ determined by P , where $\Delta = \Delta(P) = (\mathcal{S}, \omega, \mathcal{D})$ with $\mathcal{S} = \mathcal{S}_P = \text{ad}(P)$, $\omega = \omega_P$ and $\mathcal{D} = \mathcal{D}_P$ and $Z = Z(P)$ is the Siegel system determined by P . Let:

$$\text{Conn}(P) = \{ \mathcal{A} \in \Omega^1(P, \mathfrak{a}) \mid r_{a,\gamma}^*(\mathcal{A}) = \gamma^{-1}\mathcal{A} \ \& \ \mathcal{A}_y(X_y^a) = a \ \forall y \in P \ \forall (a, \gamma) \in G \},$$

be the set of principal connections on P , where r_g denotes the right action of $g \in G$ on P and we used the fact that the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{a})$ is given by (26). Let:

$$\Omega_{\text{Ad}}(P, \mathfrak{a}) \stackrel{\text{def.}}{=} \{ \eta \in \Omega(P, \mathfrak{a}) \mid r_{a,\gamma}^*(\eta) = \gamma^{-1}\eta \ \& \ \iota_X \eta = 0 \ \forall (a, \gamma) \in G \ \forall X \in V(P) \},$$

be the set of equivariant horizontal forms on P , where $V(P)$ is the space of vertical vector fields defined on P . Then $\Omega_{\text{Ad}}(P, \mathfrak{a})$ is naturally isomorphic with $\Omega(M, \mathcal{S})$. In particular, the curvature $\Omega_{\mathcal{A}} = d_{\mathcal{A}}\mathcal{A} \in \Omega_{\text{Ad}}^2(P, \mathfrak{a})$ of any principal connection $\mathcal{A} \in \text{Conn}(P)$ identifies with a \mathcal{S} -valued two-form $\mathcal{V}_{\mathcal{A}} \in \Omega^2(M, \mathcal{S})$, which is the adjoint curvature of \mathcal{A} . Since \mathfrak{a} is an abelian Lie algebra, we are in the situation considered in [35]. Hence the covariant exterior derivative defined by \mathcal{A} restricts to the ordinary exterior derivative on the space $\Omega_{\text{Ad}}(P, \mathfrak{a})$:

$$(27) \quad d_{\mathcal{A}}|_{\Omega_{\text{Ad}}(P, \mathfrak{a})} = d : \Omega_{\text{Ad}}(P, \mathfrak{a}) \rightarrow \Omega_{\text{Ad}}(P, \mathfrak{a}).$$

Moreover, the principal curvature of \mathcal{A} is given by:

$$\Omega_{\mathcal{A}} = d_{\mathcal{A}}\mathcal{A}$$

and the Bianchi identity $d_{\mathcal{A}}\Omega_{\mathcal{A}} = 0$ reduces to:

$$(28) \quad d\Omega_{\mathcal{A}} = 0.$$

As explained in [35], relation (27) implies that all principal connections on P induce the same linear connection on the adjoint bundle $\text{ad}(P) = \mathcal{S}$, which

coincides with the flat connection \mathcal{D} of the duality structure Δ defined by P . Moreover, the adjoint curvature $\mathcal{V}_A \in \Omega^2(M, \mathcal{S})$ satisfies:

$$d_{\mathcal{D}}\mathcal{V}_A = 0.$$

Let $\text{Sieg}^c(M)$ be the groupoid of Siegel bundles with connection, whose objects are pairs (P, \mathcal{A}) where P is a Siegel bundle and $\mathcal{A} \in \text{Conn}(P)$ and whose morphisms are connection-preserving based isomorphisms of principal bundles. Let $\text{Dual}_{\mathbb{Z}}^c(M)$ be the groupoid of pairs (Δ, \mathcal{V}) , where Δ is an integral duality structure on M and $\mathcal{V} \in \text{Conf}(M, \Delta)$ is a $d_{\mathcal{D}}$ -closed \mathcal{S} -valued 2-form whose $d_{\mathcal{D}}$ -cohomology class belongs to the charge lattice of Δ . There exists a natural functor:

$$\Delta^c : \text{Sieg}^c(M) \rightarrow \text{Dual}_{\mathbb{Z}}^c(M)$$

which sends (P, \mathcal{A}) to the pair $(\Delta(P), \mathcal{V}_A)$. Let $\text{Sieg}^c(M)_0 \subset \text{Sieg}^c(M)$ be the full subgroupoid consisting of Siegel bundles with flat connection.

Theorem 3.11. *There exists a short exact sequence of groupoids:*

$$1 \rightarrow \text{Sieg}^c(M)_0 \xrightarrow{\kappa} \text{Sieg}^c(M) \xrightarrow{\text{curv}} \text{Dual}_{\mathbb{Z}}^c(M) \rightarrow 1,$$

where κ is the inclusion and curv is the curvature map. In particular, for every integral duality structure Δ on M and every $\mathcal{V} \in \text{Conf}(M, \Delta)$, there exists a Siegel bundle with connection (P, \mathcal{A}) such that:

$$\mathcal{V}_A = \mathcal{V},$$

and the set of pairs $(\mathcal{A}, \mathcal{V}_A)$ with this property is a torsor for $\text{Sieg}^c(M)_0$.

Proof. It is clear that an object in $\text{Sieg}^c(M)$ defines an integral duality structure and a cohomology class in $H_{\mathcal{D}}^2(M, \mathcal{S})$, whence it defines an object in $\text{Sieg}^c(M)$. Functoriality of this assignment follows from invariance of the aforementioned cohomology class under gauge transformations. This is proved in Lemma 4.5. The key ingredient of the proof is now to show that this cohomology class is in fact *integral*, that is, belongs to $j_*(H^2(M, Z))$, where Z is the Siegel system defined by P . This is a technical point which is proved in detail in [35], to which we refer the reader for more details. Once this is proven, it follows from Theorem 3.10 that the curvature map is surjective onto $\text{Dual}_{\mathbb{Z}}^c(M)$ and that its kernel is precisely the pairs of integral duality structures and flat connections. □

The previous theorem shows that *integral* electromagnetic field strengths can always be realized as curvatures of principal connections defined on Siegel bundles, which therefore provide the geometric realization of integral configurations of abelian gauge theory.

Remark 3.12. Theorem 3.11 can be elaborated to obtain the Dirac quantization of abelian gauge theory in terms of a certain *twisted* differential cohomology theory, though we do not pursue this here. Recall that the DSZ quantization of various gauge theories using the framework of algebraic quantum field theory and differentiable cohomology has been considered before in the literature, see [13, 45] and references therein.

3.5. Polarized Siegel bundles and polarized self-dual connections

Let M be a connected manifold.

Definition 3.13. A *polarized Siegel bundle* is a pair $\mathbf{P} = (P, \mathcal{J})$, where P is a Siegel bundle and \mathcal{J} is a taming of the duality structure $\Delta := \Delta(P)$ defined by P .

A polarized Siegel bundle $\mathbf{P} = (P, \mathcal{J})$ determines an integral electromagnetic structure $\Xi_{\mathbf{P}} \stackrel{\text{def.}}{=} (\Delta(P), \mathcal{J})$, where $\Delta(P) = (\Delta(P), Z(P))$ is the integral duality structure defined by P .

Definition 3.14. Let $\mathbf{P} = (P, \mathcal{J})$ be a polarized Siegel bundle. A principal connection $\mathcal{A} \in \text{Conn}(P)$ is called *polarized selfdual*, respectively *polarized anti-selfdual* if its adjoint curvature satisfies:

$$\star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \text{ , respectively } \star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = -\mathcal{V}_{\mathcal{A}} \text{ .}$$

Recall the definitions:

$$\Omega_{\pm, \mathcal{J}}^2(M, \mathcal{S}) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \Omega^2(M, \mathcal{S}) \mid \star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = \pm \mathcal{V}_{\mathcal{A}} \} \text{ ,}$$

where $\Delta(P) = (\mathcal{S}, \omega, \mathcal{D})$.

Remark 3.15. The polarized selfduality condition is a first-order partial differential equation for a connection on a Siegel bundle which, to the best of our knowledge, has not been studied in the literature on mathematical gauge theory.

4. Prequantum abelian gauge theory

Let (M, g) be an oriented and connected Lorentzian four-manifold. As explained in the previous section, imposing the DSZ integrality condition on an abelian gauge theory allows us to identify its prequantum gauge degrees of freedom with principal connections on a Siegel bundle. More precisely, let $\mathbf{P} = (P, \mathcal{J})$ be a polarized Siegel bundle on (M, g) and $\mathbf{\Delta} := \mathbf{\Delta}(P) = (\Delta, Z)$ be the integral duality structure defined by P , where $\Delta := \Delta(P) = (\mathcal{S}, \omega, \mathcal{D})$ (with $\mathcal{S} = \text{ad}(P)$) and $Z := Z(P)$ are the duality structure and Siegel system defined by P . Let $\mathbf{\Xi} := \mathbf{\Xi}(P) = (\mathbf{\Delta}, \mathcal{J})$ be the integral electromagnetic structure defined by \mathbf{P} and $\Xi := \Xi(P) = (\Delta, \mathcal{J})$ be its underlying electromagnetic structure. By Theorem 3.11, the set of integral field strength configurations determined by the integral duality structure $\mathbf{\Delta} = \mathbf{\Delta}(P)$ coincides with the set of adjoint curvatures of principal connections defined on P :

$$\text{Conf}(M, \mathbf{\Delta}) = \{ \mathcal{V}_{\mathcal{A}} \mid \mathcal{A} \in \text{Conn}(P) \},$$

while the set of integral field strength solutions determined by $\mathbf{\Xi} := \mathbf{\Xi}(P) = (\mathbf{\Delta}, \mathcal{J})$ is:

$$\text{Sol}(M, g, \mathbf{\Xi}) = \{ \mathcal{V}_{\mathcal{A}} \mid \mathcal{A} \in \text{Conn}(P) \ \& \ \star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \}.$$

This motivates the following.

Definition 4.1. The set of prequantum gauge configurations determined by P is the affine set:

$$\mathfrak{Conf}(M, P) \stackrel{\text{def.}}{=} \text{Conn}(P)$$

of principal connections defined on P . The adjoint curvature $\mathcal{V}_{\mathcal{A}} \in \text{Conf}(M, \mathbf{\Delta})$ of a principal connection $\mathcal{A} \in \text{Conn}(P)$ is called the *integral field strength configuration* defined \mathcal{A} . The *prequantum abelian gauge theory* defined by \mathbf{P} on (M, g) is described by the condition:

$$(29) \quad \star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \text{ for } \mathcal{A} \in \text{Conn}(P).$$

The solutions \mathcal{A} of this equation are called *gauge potentials* or *polarized self-dual connections* and form the set:

$$\begin{aligned} \mathfrak{Sol}(M, g, \mathbf{P}) &= \mathfrak{Sol}(M, g, P, \mathcal{J}) \\ &\stackrel{\text{def.}}{=} \{ \mathcal{A} \in \text{Conn}(P) \mid \star_{g, \mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \} \subset \mathfrak{Conf}(M, P). \end{aligned}$$

We have:

$$\text{Sol}(M, g, \Xi) = \{\mathcal{V}_A \mid \mathcal{A} \in \mathfrak{Sol}(M, g, \mathbf{P})\}$$

and the adjoint curvature map of P gives surjections:

$$\mathfrak{Conf}(M, P) \rightarrow \text{Conf}(M, \Delta) \text{ and } \mathfrak{Sol}(M, g, \mathbf{P}) \rightarrow \text{Sol}_{\mathbb{Z}}(M, g, \Xi).$$

Remark 4.2. Condition (29) reduces locally to a system of first-order differential equations for $2n$ real-valued one-forms, which describe the local electromagnetic and magnetoelectric potentials of the theory (see Appendix A). Also notice that any $\mathcal{A} \in \mathfrak{Sol}(M, g, \mathbf{P})$ satisfies the following second order equation of Abelian Yang-Mills type:

$$d_{\mathcal{D}} \star_{g, \mathcal{J}} \mathcal{V}_A = 0.$$

4.1. The duality hierarchy of prequantum abelian gauge theory

In this subsection we discuss the duality groups of prequantum abelian gauge theory. Let P be a Siegel bundle of type $\mathfrak{t} \in \text{Div}^n$ defined on M . For simplicity, we use the notations:

$$A \stackrel{\text{def.}}{=} \text{U}(1)^{2n}, \Gamma \stackrel{\text{def.}}{=} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}), G \stackrel{\text{def.}}{=} \text{Aff}_{\mathfrak{t}} = A \rtimes \Gamma$$

and denote the Abelian Lie algebra of G by $\mathfrak{g} = \mathfrak{aff}_{\mathfrak{t}} \simeq \mathbb{R}^{2n}$. Let $q : G \rightarrow \Gamma$ be the epimorphism entering the short exact sequence of groups:

$$1 \rightarrow A \rightarrow G \xrightarrow{q} \Gamma \rightarrow 1,$$

which splits from the right.

Definition 4.3. The *discrete remnant bundle* of P is the principal Γ -bundle $\Gamma(P) \stackrel{\text{def.}}{=} P \times_q \Gamma$.

We denote the adjoint representation of G by $\text{Ad} : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathfrak{g})$ and the adjoint action of G (i.e the action of G on itself by conjugation) by $\text{Ad}_G : G \rightarrow \text{Aut}(G)$. The restriction of the latter to the normal subgroup $A \subset G$ is denoted by $\text{Ad}_G^A : G \rightarrow \text{Aut}(A)$. Since A is Abelian, this factors through q

to the *characteristic morphism* $\rho : \Gamma \rightarrow \text{Aut}(A)$:

$$\text{Ad}_G^A = \rho \circ q,$$

while the adjoint representation factors through q to the *reduced adjoint representation* $\bar{\rho} : \Gamma \rightarrow \text{Aut}_{\mathbb{R}}(\mathfrak{g})$:

$$(30) \quad \text{Ad} = \bar{\rho} \circ q.$$

This representation of Γ on \mathfrak{g} preserves the canonical symplectic form on $\mathbb{R}^{2n} \simeq \mathfrak{g}$. The exponential map of G gives a surjective morphism of Abelian groups $\exp_G : \mathfrak{g} \rightarrow A$ whose kernel is a full symplectic lattice Λ of \mathfrak{g} which identifies with $\Lambda_{\mathfrak{t}}$. This lattice is preserved by the reduced adjoint representation, which therefore induces a morphism of groups:

$$\rho_0 : \Gamma \rightarrow \text{Aut}_{\mathbb{Z}}(\Lambda).$$

Accordingly, $\text{Ad} = \bar{\rho} \circ q$ also preserves Λ and hence induces a morphism of groups:

$$\text{Ad}_0 = \rho_0 \circ q : G \rightarrow \text{Aut}_{\mathbb{Z}}(\Lambda).$$

Let $\mathbf{\Delta} = (\Delta, Z)$ be the duality structure defined by P , where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ (with $\mathcal{S} = \text{ad}(P)$) and $Z = Z(P)$ are the duality structure and Siegel system defined by P . We have:

$$\begin{aligned} \text{ad}(P) &= P \times_{\text{Ad}} \mathfrak{g} = \Gamma(P) \times_{\rho} \mathfrak{g}, \\ Z(P) &= P \times_{\text{Ad}_0} \Lambda = \Gamma(P) \times_{\rho_0} \Lambda, \quad \mathcal{A}(P) = P \times_{\text{Ad}_G^A} A, \end{aligned}$$

where $\mathcal{A}(P) = \mathcal{A}(\mathbf{\Delta}(P)) = \text{ad}(P)/Z(P) = \mathcal{S}/Z$ is the bundle of integral symplectic torus groups defined by P . As shown in [35], the connection \mathcal{D} coincides with the flat connection induced on \mathcal{S} by the monodromy connection of $\Gamma(P)$ and the symplectic pairing ω of $\mathcal{S} = \text{ad}(P) = P \times_{\rho_0} \mathfrak{g}$ coincides with that induced by the canonical symplectic pairing of $\mathbb{R}^{2n} \simeq \mathfrak{g}$.

Let $\text{Aut}(P)$ be the group of those unbased automorphisms of P which cover *orientation-preserving* diffeomorphisms of M . We have a short exact sequence:

$$1 \rightarrow \text{Aut}_b(P) \rightarrow \text{Aut}(P) \rightarrow \text{Diff}_P(M) \rightarrow 1,$$

where $\text{Diff}_P(M) \subset \text{Diff}(M)$ is the group formed by those orientation-preserving diffeomorphisms of M that can be covered by elements of $\text{Aut}(P)$. Here $\text{Aut}_b(P)$ is the group of based automorphisms of P . For any

$u \in \text{Aut}(P)$, denote by $f_u \in \text{Diff}(M)$ the orientation-preserving diffeomorphism of M covered by u . Every $u \in \text{Aut}(P)$ induces an unbased automorphism $\text{ad}_u \in \text{Aut}(\mathcal{S})$ of the adjoint bundle $\mathcal{S} = \text{ad}(P)$ defined through:

$$\text{ad}_u([y, v]) \stackrel{\text{def.}}{=} [u(y), v], \quad \forall [y, v] \in \mathcal{S} = \text{ad}(P) = P \times_{\text{ad}} \mathfrak{g}.$$

Notice that ad_u covers f_u .

Proposition 4.4. *For every $u \in \text{Aut}(P)$, the map $\text{ad}_u : \mathcal{S} \rightarrow \mathcal{S}$ is an unbased automorphism of the integral duality structure Δ defined by P . Moreover, the map $\text{ad}_P : \text{Aut}(P) \rightarrow \text{Aut}(\Delta)$ defined through:*

$$\text{ad}_P(u) = \text{ad}_u \quad \forall u \in \text{Aut}(P)$$

is a morphism of groups.

Proof. It is clear from its definition that ad_u preserves ω and Z . It also preserves \mathcal{D} , since the latter is induced by the monodromy connection of $\Gamma(P)$, which is unique. The fact that ad_P is a morphism of groups is immediate. \square

Notice that ad_P restricts to a morphism $\text{ad}_P : \text{Aut}_b(P) \rightarrow \text{Aut}_b(\Delta)$. We set:

$$g_u \stackrel{\text{def.}}{=} (f_u)_*(g) \quad \forall u \in \text{Aut}(P).$$

Let $\mathbb{A} : \text{Aut}(P) \times \text{Conn}(P) \rightarrow \text{Conn}(P)$ be the affine left action of $\text{Aut}(P)$ on $\text{Conn}(P)$ defined through:

$$\mathbb{A}_u(\mathcal{A}) \stackrel{\text{def.}}{=} u_*(\mathcal{A}) \quad \forall \mathcal{A} \in \text{Conn}(P) \quad \forall u \in \text{Aut}(P),$$

where $u_* : \mathcal{C}^\infty(P, T^*P \otimes \mathfrak{g}) \rightarrow \mathcal{C}^\infty(P, T^*P \otimes \mathfrak{g})$ denotes the push-forward of u extended trivially to \mathfrak{g} -valued forms defined on P .

Lemma 4.5. *For every $u \in \text{Aut}(P)$, we have a commutative diagram of affine spaces and affine maps:*

$$\begin{array}{ccc} \text{Conn}(P) & \xrightarrow{\mathbb{A}_u} & \text{Conn}(P) \\ \downarrow \mathcal{V} & & \downarrow \mathcal{V} \\ 2\pi j_{0*}(c(P)) & \xrightarrow{\text{ad}_u} & 2\pi j_{0*}(c(P)) \end{array}$$

where $\mathcal{V} : \text{Conn}(P) \rightarrow \Omega_{\text{d}\mathcal{D}\text{-cl}}(M, \mathcal{S})$ is the adjoint curvature map of P , $c(P) \in H^2(M, Z)$ is the twisted Chern class of P and the map $j_{0*} :$

$H^2(M, Z) \rightarrow H^2(M, \mathcal{C}_{\text{flat}}^\infty(\mathcal{S})) = H_{\mathcal{D}}^2(M, \mathcal{S})$ is induced by the sheaf inclusion $\mathcal{C}(Z) \hookrightarrow \mathcal{C}_{\text{flat}}^\infty(\mathcal{S})$ (see the exact sequence (19)). Here $2\pi j_{0*}(c(P))$ is viewed as an affine subspace of $\Omega_{\text{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S})$ consisting of $\text{d}_{\mathcal{D}}$ -closed \mathcal{S} -valued forms which differ by $\text{d}_{\mathcal{D}}$ -exact \mathcal{S} -valued forms and hence as an affine space modeled on the vector space $\Omega_{\text{d}_{\mathcal{D}}\text{-ex}}^2(M, \mathcal{S})$.

Proof. It is shown in [35] that the curvature map \mathcal{V} , which is clearly affine, takes values in $2\pi j_*(c(P))$. Therefore, it only remains to prove that if:

$$[\mathcal{V}_{\mathcal{A}}] = 2\pi j_*(c(P)),$$

then:

$$[\mathcal{V}_{\mathbb{A}_u(\mathcal{A})}] = 2\pi j_*(c(P)),$$

or, equivalently, that $[\mathcal{V}_{\mathbb{A}_u(\mathcal{A})}] = [\mathcal{V}_{\mathcal{A}}]$ for every $u \in \text{Aut}_b(P)$. Since \mathcal{A} is a connection, $\mathbb{A}_u(\mathcal{A})$ is also a connection on P , whence there exists an equivariant and horizontal one-form $\hat{\tau} \in \hat{\Omega}^1(P, \mathfrak{a})$ such that:

$$\mathbb{A}_u(\mathcal{A}) = \mathcal{A} + \hat{\tau}.$$

Hence, $\text{d}\mathbb{A}_u(\mathcal{A}) = \text{d}\mathcal{A} + \text{d}\hat{\tau}$, which descends to M as follows:

$$\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} = \mathcal{V}_{\mathcal{A}} + \text{d}_{\mathcal{D}}\tau,$$

where $\tau \in \Omega^1(M, \mathcal{S}_P)$ denotes the one-form with values in \mathcal{S}_P defined by τ . On the other hand, considering $\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} \in \Omega^2(P, \mathfrak{aff}_{\mathfrak{t}})$ as a two-form on P taking values in $\mathfrak{aff}_{\mathfrak{t}}$ a direct computation shows that:

$$\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} = u_*(\mathcal{V}_{\mathcal{A}}) \in \Omega^2(P, \mathfrak{aff}_{\mathfrak{t}}),$$

which, by the equivariance properties of the latter, immediately implies:

$$\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} = \text{ad}_u \cdot \mathcal{V}_{\mathcal{A}} \in \Omega^2(M, \Delta),$$

where the *dot* action of an automorphism of Δ on two-forms taking values in Δ was defined in subsection 1.9. Hence we conclude. \square

Proposition 4.6. *For any $u \in \text{Aut}(P)$, the map $\mathbb{A}_u : \text{Conn}(P) \rightarrow \text{Conn}(P)$ restricts to a bijection:*

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, P, \mathcal{J}) \rightarrow \mathfrak{Sol}(M, g_u, P, \mathcal{J}_u).$$

Proof. Since $\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} = \text{ad}_u \cdot \mathcal{V}_{\mathcal{A}}$, it suffices to prove the relation:

$$(31) \quad \star_{g_u, \mathcal{J}_u} \circ \text{ad}_u = \text{ad}_u \circ \star_{g, \mathcal{J}}.$$

For $\alpha \in \Omega^k(M)$ and $\xi \in \mathcal{C}^\infty(M, \mathcal{S})$, we compute:

$$\begin{aligned} \star_{g_u, \mathcal{J}_u} \text{ad}_u \cdot (\alpha \otimes \xi) &= (*_{g_u} f_{u*} \alpha) \otimes (\mathcal{J}_u \circ \text{ad}_u(\xi) \circ f_u^{-1}) \\ &= f_{u*}(*_g \alpha) \otimes \text{ad}_u \circ \mathcal{J}(\xi) \circ f_u^{-1} = \text{ad}_u \cdot (\star_{g, \mathcal{J}}(\alpha \otimes \xi)), \end{aligned}$$

which implies (31) □

Definition 4.7. Let $\mathbf{P} = (P, \mathcal{J})$ be a polarized Siegel bundle defined on M .

- The group $\text{Aut}(P)$ is the *unbased gauge group* of P . For any $u \in \text{Aut}(P)$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, P, \mathcal{J}) \rightarrow \mathfrak{Sol}(M, g_u, P, \mathcal{J}_u)$$

is the *unbased gauge transformation* induced by u .

- The group:

$$\text{Aut}(g, P) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(P) \mid f_u \in \text{Iso}(M, g)\}$$

is the *unbased gauge duality group* defined by P and g . For any $u \in \text{Aut}(g, P)$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, P, \mathcal{J}) \rightarrow \mathfrak{Sol}(M, g, P, \mathcal{J}_u)$$

is the *unbased gauge duality transformation* induced by u .

- The gauge group $\text{Aut}_b(P)$ of P is the *gauge (electromagnetic) duality group* of the abelian gauge theories with underlying Siegel bundle P . For any $u \in \text{Aut}_b(P)$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, P, \mathcal{J}) \rightarrow \mathfrak{Sol}(M, g, P, \mathcal{J}_u)$$

is called the *gauge duality transformation* induced by u .

Lemma 4.5 implies that for any $u \in \text{Aut}(P)$ we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{Sol}(M, g, P, \mathcal{J}) & \xrightarrow{\mathbb{A}_u} & \mathfrak{Sol}(M, g_u, P, \mathcal{J}_u) \\ \downarrow \nu & & \downarrow \nu \\ \text{Sol}(M, g, \Delta, \mathcal{J}) & \xrightarrow{\text{ad}_u} & \text{Sol}(M, g_u, \Delta, \mathcal{J}_u) \end{array}$$

and similar diagrams for the other groups in the previous definition. Hence gauge transformations of P induce integral pseudo-duality and duality transformations of the abelian gauge theory defined by (P, \mathcal{J}) .

Definition 4.8. Let $\mathbf{P} = (P, \mathcal{J})$ be a polarized Siegel bundle on M .

- The group:

$$(32) \quad \text{Aut}(\mathbf{P}) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(P) \mid \mathcal{J}_u = \mathcal{J}\}$$

is the *unbased unitary gauge group* defined by \mathbf{P} on (M, g) . For any $u \in \text{Aut}(\mathbf{P})$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, \mathbf{P}) \rightarrow \mathfrak{Sol}(M, g_u, \mathbf{P})$$

is the *unbased unitary gauge transformation* induced by u .

- The group:

$$(33) \quad \text{Aut}(g, \mathbf{P}) \stackrel{\text{def.}}{=} \{u \in \text{Aut}(P) \mid g_u = g \text{ and } \mathcal{J}_u = \mathcal{J}\}$$

is the *unbased unitary gauge duality group* defined by \mathbf{P} on (M, g) . For any $u \in \text{Aut}(g, \mathbf{P})$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, \mathbf{P}) \rightarrow \mathfrak{Sol}(M, g, \mathbf{P})$$

is the *unbased unitary gauge duality transformation* induced by u .

- The group:

$$\text{Aut}_b(\mathbf{P}) \stackrel{\text{def.}}{=} \{u \in \text{Aut}_b(P) \mid \mathcal{J}_u = \mathcal{J}\} .$$

is the *unitary gauge group* defined by \mathbf{P} on M . For any $u \in \text{Aut}_b(\mathbf{P})$, the map:

$$\mathbb{A}_u : \mathfrak{Sol}(M, g, \mathbf{P}) \rightarrow \mathfrak{Sol}(M, g, \mathbf{P}) .$$

is the *unitary gauge transformation* induced by u .

Let $\Xi \stackrel{\text{def.}}{=} (\Delta, \mathcal{J})$. For any $u \in \text{Aut}(\mathbf{P})$, we have a commutative diagram:

$$\begin{CD} \mathfrak{Sol}(M, g, \mathbf{P}) @>A_u>> \mathfrak{Sol}(M, g_u, \mathbf{P}) \\ @VV\mathcal{V}V @VV\mathcal{V}V \\ \text{Sol}(M, g, \Xi) @>ad_u>> \text{Sol}(M, g_u, \Xi) \end{CD}$$

and similar diagrams for the other groups in the previous definition. We have a short exact sequence:

$$(34) \quad 1 \rightarrow \text{Aut}_b(P) \rightarrow \text{Aut}(g, P) \rightarrow \text{Iso}_P(M, g) \rightarrow 1,$$

where $\text{Iso}_P(M, g) \subset \text{Iso}(M, g)$ is the group formed by those orientation-preserving isometries that can be covered by elements of $\text{Aut}(g, P)$. Similarly, we have an exact sequence:

$$1 \rightarrow \text{Aut}_b(\mathbf{P}) \rightarrow \text{Aut}(g, \mathbf{P}) \rightarrow \text{Iso}_{\mathbf{P}}(M, g) \rightarrow 1,$$

where $\text{Iso}_{\mathbf{P}}(M, g)$ is the group formed by those orientation-preserving isometries of M which are covered by elements of $\text{Aut}(g, \mathbf{P})$.

Definition 4.9. The *standard subgroup* of the unbased gauge group of P is defined through:

$$C(P) \stackrel{\text{def.}}{=} \ker(\text{ad}_P) \subset \text{Aut}(P).$$

When $\dim M > 0$ and $\dim A > 0$, the group $C(P)$ is infinite-dimensional. The classical duality group of a duality structure was shown to be a finite dimensional Lie group in Section 1. This is no longer true of the gauge groups introduced above. Instead, they are infinite-dimensional extensions of the integral duality groups introduced in Section 2.

Proposition 4.10. *The gauge group of P fits into the short exact sequence of groups:*

$$(35) \quad 1 \rightarrow C(P) \hookrightarrow \text{Aut}_b(P) \xrightarrow{\text{ad}_P} \text{Aut}_b(\Delta) \rightarrow 1.$$

Remark 4.11. There exist similar short exact sequences for the remaining groups introduced in Definition 4.7.

Proof. It suffices to prove that $\text{ad}_P(\text{Aut}_b(P)) = \text{Aut}_b(\Delta)$. Recall that $\mathcal{L} = P \times_{\text{Ad}_0} \Lambda = \Gamma(P) \times_{\rho_0} \Lambda$ and $\mathcal{S} = P \times_{\text{Ad}} \mathfrak{g} = \Gamma(P) \times_{\bar{\rho}} \mathfrak{g}$, where $\Lambda \equiv \Lambda_t$ and $\mathfrak{g} \equiv \mathbb{R}^{2n}$. Also recall that the gauge group $\text{Aut}_b(P)$ is naturally isomorphic with the group $\mathcal{C}^\infty(P, G)^G$ of G -equivariant maps from P to G , where $G = A \times \Gamma = \text{U}(1)^{2n} \rtimes \text{Sp}_t(2n, \mathbb{Z})$ acts on itself through conjugation. This isomorphism takes $u \in \text{Aut}_b(P)$ to the equivariant map $f \in \mathcal{C}^\infty(P, G)^G$ which satisfies:

$$(36) \quad u(p) = pf(p) \forall p \in P \forall g \in G.$$

Since the action of G on P is free and we have $\Gamma \equiv \text{Aut}(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_t) \simeq \text{Aut}(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$ for all $m \in M$ while the reduced adjoint representation $\bar{\rho}$ is faithful, every automorphism $\varphi \in \text{Aut}(\Delta)$ determines a map $\bar{\varphi} : P \rightarrow \Gamma$ which satisfies:

$$(37) \quad \varphi([p, v]) = [p, \bar{\rho}(\bar{\varphi}(p))(v)] \forall p \in P \forall v \in \mathfrak{g}$$

as well as:

$$\bar{\varphi}(pg) = q(g)^{-1} \bar{\varphi}(p) q(g) \forall p \in P \forall g \in G.$$

The last relation follows from (37) and from the condition $\varphi([pg, v]) = \varphi([p, \bar{\rho}(q(g))(v)])$ of invariance under change of representative of the equivalence class, where we used (30). Let $u \in \text{Aut}_b(P)$ be the based automorphism of P which corresponds to the G -equivariant map $f : P \rightarrow G$ defined through:

$$f(p) \stackrel{\text{def.}}{=} (0_A, \bar{\varphi}(p)) \in G = A \rtimes \Gamma \forall p \in P.$$

For any $p \in P$ and $v \in \mathfrak{g}$, we have:

$$\begin{aligned} \text{ad}_P(u)([p, v]) &= [u(p), v] = [pf(p), v] \\ &= [p, \text{Ad}(f(p))(v)] = [p, \bar{\rho}(\bar{\varphi}(p))(v)] = \varphi([p, v]), \end{aligned}$$

where we used (36) and (37). This shows that $\text{ad}_P(u) = \varphi$. Since $\varphi \in \text{Aut}_b(\Delta)$ is arbitrary, we conclude that $\text{ad}_P(\text{Aut}_b(P)) = \text{Aut}_b(\Delta)$. \square

The previous proposition clarifies the geometric origin of electromagnetic duality as a ‘discrete remnant’ of gauge symmetry, a notion which is discussed in more detail in [35]. In particular, $\text{Aut}_b(P)$ is an extension of $\text{Aut}_b(\Delta)$ by the continuous group $C(P)$. Intuitively, elements of the latter correspond to the gauge transformations of a principal torus bundle.

4.2. Duality groups for Siegel bundles with trivial monodromy

Let M be a connected and oriented four-manifold.

Lemma 4.12. *Let P be a trivial principal G -bundle over M . Then any trivialization of P induces an isomorphism of groups:*

$$\text{Aut}(P) \simeq \mathcal{C}^\infty(M, G) \rtimes_\alpha \text{Diff}(M),$$

where $\alpha : \text{Diff}(M) \rightarrow \text{Aut}(\mathcal{C}^\infty(M, G))$ is the morphisms of groups defined through:

$$\alpha(\varphi)(f) \stackrel{\text{def.}}{=} f \circ \varphi^{-1}, \quad \forall \varphi \in \text{Diff}(M), \quad \forall f \in \mathcal{C}^\infty(M, G).$$

In particular, we have a short exact sequence of groups:

$$1 \rightarrow \mathcal{C}^\infty(M, G) \rightarrow \text{Aut}(P) \rightarrow \text{Diff}(M) \rightarrow 1$$

which is split from the right.

Proof. Let $\tau : P \xrightarrow{\sim} M \times G$ be a trivialization of P . Then the map $\text{Ad}(\tau) : \text{Aut}(P) \rightarrow \text{Aut}(M \times G)$ defined through:

$$\text{Ad}(\tau)(f) \stackrel{\text{def.}}{=} \tau \circ f \circ \tau^{-1}, \quad \forall f \in \text{Aut}(P),$$

is an isomorphism of groups. Let $f \in \text{Aut}(P)$ be an unbased automorphism of P which covers the diffeomorphism $\varphi \in \text{Diff}(M)$. Then $\text{Ad}(\tau)(f)$ is an unbased automorphism of $M \times G$ which covers φ and hence we have:

$$\text{Ad}(\tau)(f)(m, g) = (\varphi(m), \hat{f}(m, g)), \quad \forall (m, g) \in M \times G,$$

where $\hat{f} : M \times G \rightarrow G$ is a smooth map which satisfies:

$$\hat{f}(m, g_1 g_2) = \hat{f}(m, g_1) g_2, \quad \forall m \in M, \quad \forall g_1, g_2 \in G.$$

The last relation is equivalent with the condition that \hat{f} has the form:

$$\hat{f}(m, g) = \tilde{f}(m)g, \quad \forall (m, g) \in M \times G,$$

where $\tilde{f} : M \rightarrow G$ is a smooth function which can be recovered from \hat{f} through the relation:

$$\tilde{f}(m) = \hat{f}(m, 1), \quad \forall m \in M.$$

Setting $h \stackrel{\text{def.}}{=} \tilde{f} \circ \varphi^{-1} \in \mathcal{C}^\infty(M, G)$, we have:

$$(38) \quad \text{Ad}(\tau)(f)(m, g) = (\varphi(m), h(\varphi(m))g), \quad \forall (m, g) \in M \times G,$$

and the correspondence $f \rightarrow (h, \varphi)$ gives a bijection between $\text{Aut}(P) \simeq \text{Aut}(M \times G)$ and the set $\mathcal{C}^\infty(M, G) \times \text{Diff}(M)$. If $f_1, f_2 \in \text{Aut}(P)$ correspond through this map to the pairs $(h_1, \varphi_1), (h_2, \varphi_2) \in \mathcal{C}^\infty(M, G) \times \text{Diff}(M)$, then direct computation using (38) gives:

$$\text{Ad}(\tau)(f_1 \circ f_2)(m, g) = ((\varphi_1 \circ \varphi_2)(m), h_1(m)(h_2 \circ \varphi_1^{-1})(m)g),$$

showing that $f_1 \circ f_2$ corresponds to the pair $(h_1 \cdot \alpha(\varphi_1)(h_2), \varphi_1 \circ \varphi_2)$. □

Let P be a Siegel bundle or rank n and type $\mathfrak{t} \in \text{Div}^n$ on (M, g) and let $\Delta = (\Delta, Z)$ be the integral duality structure defined by P . Suppose that P is topologically trivial and that $Z = Z(P)$ has trivial monodromy, so that Δ is holonomy trivial. Choosing a trivialization of P gives:

$$P \equiv M \times \text{Aff}_{\mathfrak{t}}, \quad \Delta \equiv (M \times \mathbb{R}^{2n}, \omega_{2n}, d, M \times \Lambda_{\mathfrak{t}})$$

and:

$$\text{Aut}_b(P) \equiv \mathcal{C}^\infty(M, \text{Aff}_{\mathfrak{t}}), \quad \text{Aut}(P) \equiv \mathcal{C}^\infty(M, \text{Aff}_{\mathfrak{t}}) \rtimes_{\alpha} \text{Diff}(M),$$

where the last identification follows from Lemma 4.12. Since $\text{Aff}_{\mathfrak{t}} = \text{U}(1)^{2n} \rtimes \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ and $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ is discrete, we have:

$$\mathcal{C}^\infty(M, \text{Aff}_{\mathfrak{t}}) = \mathcal{C}^\infty(M, \text{U}(1)^{2n}) \rtimes \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}).$$

In particular, maps $h \in \mathcal{C}^\infty(M, \text{Aff}_{\mathfrak{t}})$ can be identified with pairs (f, γ) , where $f \in \mathcal{C}^\infty(M, \text{U}(1)^{2n})$ and $\gamma \in \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. The unbased gauge duality group is given by:

$$\text{Aut}(g, P) \equiv \mathcal{C}^\infty(M, \text{Aff}_{\mathfrak{t}}) \rtimes_{\alpha} \text{Iso}(M, g).$$

The integral pseudo-duality, relative duality and duality groups of Δ are in turn given by:

$$\begin{aligned} \text{Aut}(\Delta) &\equiv \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \times \text{Diff}(M), \\ \text{Aut}(g, \Delta) &\equiv \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \times \text{Iso}(M, g), \\ \text{Aut}_b(\Delta) &\equiv \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}), \end{aligned}$$

and we have short exact sequences:

$$\begin{aligned} 1 &\rightarrow \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \rightarrow \text{Aut}(P) \rightarrow \text{Aut}(\mathbf{\Delta}) \rightarrow 1, \\ 1 &\rightarrow \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \rightarrow \text{Aut}(g, P) \rightarrow \text{Aut}(g, \mathbf{\Delta}) \rightarrow 1, \\ 1 &\rightarrow \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \rightarrow \text{Aut}_b(P) \rightarrow \text{Aut}_b(\mathbf{\Delta}) \rightarrow 1. \end{aligned}$$

Let us fix a taming of $\mathbf{\Delta}$, which we view as a map $\mathcal{J} \in \mathcal{C}^\infty(M, \text{Sp}(2n, \mathbb{R}))$. Then the unbased unitary gauge group of the tamed Siegel bundle $\mathbf{P} = (P, \mathcal{J})$ is:

$$\text{Aut}(g, \mathbf{P}) \equiv \left\{ (f, \gamma, \varphi) \in [\mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \times \text{Sp}_t(2n, \mathbb{Z})] \rtimes_\alpha \text{Iso}(M, g) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \circ \varphi \right\},$$

while its unitary gauge group is:

$$\text{Aut}_b(\mathbf{P}) \equiv \left\{ (f, \gamma) \in \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \times \text{Sp}_t(2n, \mathbb{Z}) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \right\}.$$

The integral unbased unitary duality group of the integral electromagnetic structure $\mathbf{\Xi} = (\mathbf{\Delta}, \mathcal{J})$ is:

$$\text{Aut}(g, \mathbf{\Xi}) \equiv \left\{ (\gamma, \varphi) \in \text{Sp}_t(2n, \mathbb{Z}) \times \text{Iso}(M, g) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \circ \varphi \right\},$$

while the integral unitary duality group of $\mathbf{\Xi}$ is:

$$\text{Aut}_b(\mathbf{\Xi}) \equiv \left\{ \gamma \in \text{Sp}_t(2n, \mathbb{Z}) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J} \right\} \subset \text{Sp}_t(2n, \mathbb{Z}).$$

We have short exact sequences:

$$\begin{aligned} 1 &\rightarrow \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \rightarrow \text{Aut}(g, \mathbf{P}) \rightarrow \text{Aut}(g, \mathbf{\Xi}) \rightarrow 1 \\ 1 &\rightarrow \mathcal{C}^\infty(M, \mathbb{U}(1)^{2n}) \rightarrow \text{Aut}_b(\mathbf{P}) \rightarrow \text{Aut}_b(\mathbf{\Xi}) \rightarrow 1. \end{aligned}$$

5. Time-like dimensional reduction and polarized Bogomolny equations

This section investigates the time-like dimensional reduction of the equations of motion of abelian gauge theory on an oriented static space-time (M, g) of the form:

$$(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus h),$$

where t is the global coordinate on \mathbb{R} and (Σ, h) is an oriented Riemannian three-manifold. We show that the reduction produces an equation of Bogomolny type, similar to the dimensional reduction of the ordinary self-duality

equation on a Riemannian four-manifold. Unlike that well-known case, here we reduce the *polarized* self-duality condition on a *Lorentzian* four-manifold.

5.1. Preparations

Consider the time-like exact one-form $\theta = dt \in \Omega^1(M)$. Let ν_h be the volume form of (Σ, h) and orient (M, g) such that its volume form is given by:

$$\nu_g = \theta \wedge \nu_h .$$

Let $*_g$ and $*_h$ be the Hodge operators of (M, g) and (Σ, h) and $\langle \cdot, \cdot \rangle_g, \langle \cdot, \cdot \rangle_h$ be the non-degenerate bilinear pairings induced by g and h on $\Omega^*(M)$ and $\Omega^*(\Sigma)$. Let $p : M \rightarrow \Sigma$ be the projection of $M = \mathbb{R} \times \Sigma$ on the second factor and consider the distribution $\mathcal{D} = p^*(T\Sigma) \subset TM$, endowed with the fiberwise Euclidean pairing given by the bundle pullback h^p of h . This distribution is integrable with leaves gives by the spacelike hypersurfaces:

$$M_t \stackrel{\text{def.}}{=} \{t\} \times \Sigma, \quad \forall t \in \mathbb{R},$$

on which h^p restricts to the metric induced by g . Using h^p , we extend $\langle \cdot, \cdot \rangle_h$ and $*_h$ in the obvious manner to the space $\mathcal{C}^\infty(M, \wedge^* \mathcal{D}^*) \subset \Omega^*(M)$. We have:

$$\mathcal{C}^\infty(M, \wedge^* \mathcal{D}^*) = \{\omega \in \Omega^*(M) \mid \iota_{\partial_t} \omega = 0\} .$$

Since g has signature $(3, 1)$ while h has signature $(3, 0)$, we have:

$$*_g \circ *_g = -\pi, \quad *_h \circ *_h = \text{id}_{\mathcal{C}^\infty(M, \wedge^* \mathcal{D}^*)},$$

where $\pi \stackrel{\text{def.}}{=} \bigoplus_{k=0}^4 (-1)^k \text{id}_{\Omega^k(M)}$ is the signature automorphism of the exterior algebra $(\Omega^*(M), \wedge)$ (see [36]). Moreover, we have $\langle \theta, \theta \rangle_g = -1$ and:

$$\langle \nu_g, \nu_g \rangle_g = -1, \quad \langle \nu_h, \nu_h \rangle_h = +1 .$$

Any polyform $\omega \in \Omega^*(M, g)$ has a unique decomposition:

$$(39) \quad \omega = \omega_{\parallel} + \omega_{\perp} ,$$

such that $\omega_{\parallel}, \omega_{\perp} \in \Omega^*(M)$ satisfy [36]:

$$(40) \quad \theta \wedge \omega_{\parallel} = 0, \quad \iota_{\partial_t} \omega_{\perp} = 0 .$$

The second of these conditions amounts to the requirement that $\omega_{\perp} \in \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^*)$, while the first is solved by:

$$\omega_{\parallel} = \theta \wedge \omega_{\top} \text{ where } \omega_{\top} \stackrel{\text{def.}}{=} -\iota_{\partial_t} \omega \in \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^*).$$

As shown in loc. cit., the map $\omega \rightarrow (\omega_{\top}, \omega_{\perp})$ gives a linear isomorphism between $\Omega^*(M)$ and $\mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^*)^{\oplus 2}$. For any $k = 0, \dots, 4$ and any $\omega, \eta \in \Omega^k(M)$, we have:

$$\langle \omega, \eta \rangle_g = -\langle \omega_{\top}, \eta_{\top} \rangle_h + \langle \omega_{\perp}, \eta_{\perp} \rangle_h, \quad \langle \omega_{\parallel}, \eta_{\parallel} \rangle_g = -\langle \omega_{\top}, \eta_{\top} \rangle_h.$$

Hence the isomorphism above identifies the quadratic space $(\Omega^*(M), \langle \cdot, \cdot \rangle_h)$ with the direct sum of quadratic spaces $(\mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^*), -\langle \cdot, \cdot \rangle_h) \oplus (\mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^*), \langle \cdot, \cdot \rangle_h)$. Notice that $\nu_h = \iota_{\partial_t} \nu_g = -(\nu_g)_{\top}$. An easy computation gives:

Lemma 5.1. *For any polyform $\omega \in \Omega^*(M)$, we have:*

$$(41) \quad (*_g \omega)_{\top} = *_h \pi(\omega_{\perp}), \quad (*_g \omega)_{\perp} = - *_h \omega_{\top}$$

and hence:

$$(42) \quad *_g \omega = - *_h \omega_{\top} + \theta \wedge *_h \pi(\omega_{\perp}).$$

Given any vector bundle V defined on M , the Hodge operators of g and h extend trivially to operators $*_g : \Omega^*(M, V) \rightarrow \Omega^*(M, V)$ and $*_h : \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^* \otimes V) \rightarrow \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^* \otimes V)$. The decomposition (39) holds for any $\omega \in \Omega^*(M, V)$, with components $\omega_{\parallel}, \omega_{\perp} \in \Omega^*(M, V)$ satisfying (40). We have $\omega_{\perp} \in \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^* \otimes V)$ and $\omega_{\parallel} = \theta \wedge \omega_{\top}$ with $\omega_{\top} \stackrel{\text{def.}}{=} -\iota_{\partial_t} \omega \in \mathcal{C}^{\infty}(M, \wedge^* \mathcal{D}^* \otimes V)$. Finally, Lemma 5.1 holds for any $\omega \in \Omega^*(M, V)$.

5.2. Timelike dimensional reduction of abelian gauge theory

Let P be a Siegel bundle of type $\mathfrak{t} \in \text{Div}^n$ defined on Σ , whose projection we denote by $\pi : P \rightarrow \Sigma$. Let $\hat{P} \stackrel{\text{def.}}{=} p^*(P)$ be the p -pullback of P to M , whose projection we denote by $\hat{\pi}$. The map $\varphi : \hat{P} \rightarrow \mathbb{R} \times P$ defined through:

$$\varphi(t, \sigma, y) \stackrel{\text{def.}}{=} (t, y), \quad \forall (t, \sigma) \in \mathbb{R} \times \Sigma, \quad \forall y \in P_{\sigma} = \pi^{-1}(\sigma)$$

allows us to identify \hat{P} with the principal $\text{Aff}_{\mathfrak{t}}$ -bundle with total space given by $\mathbb{R} \times P$, base $M = \mathbb{R} \times \Sigma$ and projection given by $\mathbb{R} \times P \ni (t, y) \rightarrow$

$(t, \pi(y)) \in M$. We make this identification in what follows. Accordingly, we have:

$$\hat{\pi}(t, y) = (t, \pi(y)), \quad \forall (t, y) \in \hat{P} \equiv \mathbb{R} \times P.$$

Let $\tau : \hat{P} \rightarrow P$ be the unbased morphism of principal Aff_t -bundles given by projection on the second factor:

$$\tau(t, y) \stackrel{\text{def.}}{=} y, \quad \forall (t, y) \in \hat{P},$$

which covers the map $p : M \rightarrow \Sigma$:

$$\pi \circ \tau = p \circ \hat{\pi}.$$

Consider the action $\rho : \mathbb{R} \rightarrow \text{Aut}(\hat{P})$ of $(\mathbb{R}, +)$ through unbased automorphisms of \hat{P} given by:

$$(t, y) \rightarrow \rho(a)(t, y) \stackrel{\text{def.}}{=} (t + a, y), \quad \forall (t, y) \in \hat{P} \equiv \mathbb{R} \times P.$$

This covers the action of \mathbb{R} on $M = \mathbb{R} \times \Sigma$ given by time translations:

$$\hat{\pi}(\rho(a)(t, y)) = (t + a, \pi(y)), \quad \forall (t, y) \in \hat{P} \equiv \mathbb{R} \times P.$$

Definition 5.2. A principal connection on $\hat{\mathcal{A}} \in \text{Conn}(\hat{P})$ is called *time-invariant* if it satisfies:

$$\rho(a)^*(\hat{\mathcal{A}}) = \hat{\mathcal{A}}, \quad \forall a \in \mathbb{R}.$$

Notice that time-invariant principal connections defined on \hat{P} form an affine subspace $\text{Conn}^s(\hat{P})$ of $\text{Conn}(\hat{P})$. The timelike one-form $\theta \in \Omega^1(M)$ pulls back through $\hat{\pi}$ to an exact one-form defined on \hat{P} which we denote by $\hat{\theta} \stackrel{\text{def.}}{=} p^*(\theta) \in \Omega^1(\hat{P})$.

Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be the duality structure defined by P on Σ . Then it is easy to see that the duality structure $\hat{\Delta} = (\hat{\mathcal{S}}, \hat{\omega}, \hat{\mathcal{D}})$ defined by \hat{P} on M is given by:

$$\hat{\mathcal{S}} = p^*(\mathcal{S}), \quad \hat{\omega} = p^*(\omega), \quad \hat{\mathcal{D}} = p^*(\mathcal{D}).$$

Lemma 5.3. *A connection $\hat{\mathcal{A}} \in \text{Conn}(\hat{P})$ is time-invariant if and only if it can be written as:*

$$(43) \quad \hat{\mathcal{A}} = -(\Psi^\pi \circ \tau)\hat{\theta} + \tau^*(\mathcal{A})$$

for some $\Psi \in \mathcal{C}^\infty(\Sigma, \mathcal{S})$ and some $\mathcal{A} \in \text{Conn}(P)$. In this case, Ψ and \mathcal{A} are determined uniquely by $\hat{\mathcal{A}}$ and any pair $(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P)$ determines a time-invariant connection on \hat{P} through this relation. Moreover, the curvature of $\hat{\mathcal{A}}$ is given by:

$$(44) \quad \mathcal{V}_{\hat{\mathcal{A}}} = \theta \wedge p^*(d_{\mathcal{D}}\Psi) + p^*(\mathcal{V}_{\mathcal{A}}) \in \Omega^2(\Sigma, \hat{\mathcal{S}})$$

and we have:

$$(45) \quad d_{\mathcal{D}}\mathcal{V}_{\mathcal{A}} = 0.$$

Remark 5.4. Relation (44) gives the decomposition (39) of $\mathcal{V}_{\hat{\mathcal{A}}}$ since $\iota_{\partial_t} p^*(\mathcal{V}_{\mathcal{A}}) = \iota_{\partial_t} p^*(d_{\mathcal{D}}\Psi) = 0$. Thus:

$$(46) \quad (\mathcal{V}_{\hat{\mathcal{A}}})_{\top} = p^*(d_{\mathcal{D}}\Psi), \quad (\mathcal{V}_{\hat{\mathcal{A}}})_{\perp} = p^*(\mathcal{V}_{\mathcal{A}}).$$

Proof. Any principal connection $\mathcal{A} \in \text{Conn}(\hat{P}) \subset \Omega^1(\hat{P}, \mathfrak{a})$ decomposes uniquely as:

$$\hat{\mathcal{A}} = -\Phi\hat{\theta} + \mathcal{A}_{\perp},$$

where $\Phi \in \Omega^0_{\text{Ad}}(\hat{P}, \mathfrak{a})$ and $\mathcal{A}_{\perp} \in \text{Conn}(\hat{P}, \mathfrak{a})$ satisfies $\mathcal{A}_{\perp}(\partial_t) = 0$. It is clear that $\hat{\mathcal{A}}$ is time-invariant if and only if $\Phi = \Psi' \circ \tau$ for some $\Psi' \in \Omega^0_{\text{Ad}}(P, \mathfrak{a})$ and $\hat{\mathcal{A}}_{\perp} = \tau^*(\mathcal{A})$ for some $\mathcal{A} \in \text{Conn}(P)$. Since $\Omega^0_{\text{Ad}}(P, \mathfrak{a}) \simeq \mathcal{C}^\infty(\Sigma, \mathcal{S})$, we have $\Psi' = \Psi^\pi$ for some $\Psi \in \mathcal{C}^\infty(\Sigma, \mathcal{S})$. Since $d\hat{\theta} = 0$, the principal curvature of $\hat{\mathcal{A}}$ reads:

$$\Omega_{\hat{\mathcal{A}}} = d\hat{\mathcal{A}} = d\Phi \wedge \hat{\theta} + \tau^*(\Omega_{\mathcal{A}}),$$

which is equivalent with (44). Relation (45) follows from the results of [35]. The remaining statements are immediate. □

Definition 5.5. The *Bogomolny pair* of a time-invariant connection $\hat{\mathcal{A}} \in \text{Conn}^s(\hat{P})$ is the pair $(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P)$ defined in Lemma 5.3. The section $\Psi \in \mathcal{C}^\infty(\Sigma, \mathcal{S})$ is called the *Higgs field* of the pair.

Let \mathcal{J} be a taming of Δ . Then the p -pullback $\hat{\mathcal{J}}$ of \mathcal{J} defines a time-invariant taming of $\hat{\mathcal{S}}$, thus $(\hat{P}, \hat{\mathcal{J}})$ is a polarized Siegel bundle. Let $\star_{h, \mathcal{J}} = \star_h \otimes \mathcal{J}$ be

the polarized Hodge operator defined by h and \mathcal{J} . Since $\mathcal{J}^2 = -\text{id}_{\mathcal{S}}$ while $*_h$ squares to the identity on $\Omega^*(\Sigma)$, we have:

$$*_h, \mathcal{J} \circ *_h, \mathcal{J} = -\text{id}_{\Omega^*(\Sigma, \mathcal{S})}.$$

Proposition 5.6. *A time-invariant connection $\hat{\mathcal{A}} \in \text{Conn}^s(\hat{P})$ is polarized self-dual with respect to $\hat{\mathcal{J}}$ if and only if its Bogomolny pair (Ψ, \mathcal{A}) satisfies the polarized Bogomolny equation with respect to \mathcal{J} :*

$$(47) \quad *_h, \mathcal{J} \mathcal{V}_{\mathcal{A}} = d_{\mathcal{D}} \Psi \iff *_h, \mathcal{J} d_{\mathcal{D}} \Psi = -\mathcal{V}_{\mathcal{A}}.$$

A polarized Bogomolny pair (Ψ, \mathcal{A}) which satisfies this equation is called a polarized abelian dyon relative to \mathcal{J} .

Proof. Relations (46) and (42) give:

$$*_g \mathcal{V}_{\hat{\mathcal{A}}} = -p^*(*_h d_{\mathcal{D}} \Psi) + \theta \wedge p^*(*_h \mathcal{V}_{\mathcal{A}}),$$

which implies:

$$*_g, \hat{\mathcal{J}} \mathcal{V}_{\hat{\mathcal{A}}} = -p^*(*_h, \mathcal{J} d_{\mathcal{D}} \Psi) + \theta \wedge p^*(*_h, \mathcal{J} \mathcal{V}_{\mathcal{A}}).$$

Comparing this with (44) shows that the polarized self-duality condition for $\mathcal{V}_{\hat{\mathcal{A}}}$ amounts to (47), where we used uniqueness of the decomposition (39). □

Let:

$$\text{Dyons}(\Sigma, h, \mathbf{P}) \stackrel{\text{def.}}{=} \{(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P) \mid *_h, \mathcal{J} \mathcal{V}_{\mathcal{A}} = d_{\mathcal{D}} \Psi\}$$

be the set of all polarized abelian dyons relative to \mathcal{J} .

Remark 5.7. Equation (47) is reminiscent of the usual Bogomolny equations obtained by dimensional reduction of the self-duality equations for a connection on a principal bundle over a four-dimensional Riemannian manifold. However, it differs from the latter in two crucial respects:

- The usual Bogomolny equations arise by dimensional reduction of the self-duality equations (which are first order equations for a connection) in four *Euclidean* dimensions. By contrast, equation (47) is the reduction along a timelike direction of the complete second order equations of motion defining abelian gauge theory in four *Lorentzian* dimensions, once these

equations have been re-written as first-order equations by doubling the number of variables through the inclusion of both electromagnetic and magnetoelectric gauge potentials. In particular, our reduction yields a system of first-order differential equations, despite originating in a theory that was initially defined by local second-order PDEs (see Appendix A).

- Equation (47) is modified by the action of the taming \mathcal{J} , which is absent in the usual Bogomolny equations.

5.3. Gauge transformations of polarized abelian dyons

As explained in Subsection 4.1 (see [35] for more detail), the gauge group $\text{Aut}_b(P)$ of the Siegel bundle P over Σ has an action:

$$\text{ad}_P : \text{Aut}_b(P) \rightarrow \text{Aut}_b(\mathcal{S})$$

through based automorphisms of the vector bundle $\mathcal{S} = \text{ad}(P)$. This action agrees through the adjoint curvature map with the pushforward action:

$$\mathbb{A} : \text{Aut}_b(P) \rightarrow \text{Aff}(\text{Conn}(P))$$

of the gauge group on the space of principal connections defined on P . For any principal connection $\mathcal{A} \in \text{Conn}(P)$ and any $u \in \text{Aut}_b(P)$, we have:

$$\mathcal{V}_{\mathbb{A}_u(\mathcal{A})} = \text{ad}_P(u)(\mathcal{V}_{\mathcal{A}}).$$

Similar statements hold for the gauge group of the Siegel bundle \hat{P} defined on $M = \mathbb{R} \times \Sigma$.

Definition 5.8. A gauge transformation $\hat{u} \in \text{Aut}_b(\hat{P})$ of \hat{P} is called *time-invariant* if:

$$u \circ \rho(a) = \rho(a) \circ u, \quad \forall a \in \mathbb{R}.$$

Notice that time-invariant gauge transformations of \hat{P} form a subgroup of $\text{Aut}_b(\hat{P})$, which we denote by $\text{Aut}_b^s(\hat{P})$. Such transformations stabilize the affine subspace $\text{Conn}^s(P)$ of time-invariant principal connections defined on \hat{P} . Since $\hat{P} = p^*(P)$, we have $\text{Ad}_G(\hat{P}) = p^*(\text{Ad}_P)$. It is easy to see that $\hat{u} \in \text{Aut}_b(\hat{P})$ is time-invariant if and only if the corresponding section $\sigma_{\hat{u}} \in \mathcal{C}^\infty(M, \text{Ad}_G(\hat{P}))$ is the bundle pull-back by p of a section $\sigma \in \mathcal{C}^\infty(\Sigma, \text{Ad}_G(P))$. The latter corresponds to a gauge transformation of

P which we denote by $u \in \text{Aut}_b(P)$. We have:

$$(48) \quad \sigma_{\hat{u}} = (\sigma_u)^p,$$

(where the subscript p denotes bundle pullback by p) as well as:

$$(49) \quad \tau \circ \hat{u} = u \circ \tau.$$

Conversely, any gauge transformation u of P determines a gauge transformation \hat{u} of \hat{P} by relation (48) and \hat{u} satisfies (49). The correspondence $u \rightarrow \hat{u}$ gives an isomorphism of groups between $\text{Aut}_b(P)$ and $\text{Aut}_b^s(\hat{P})$.

The following proposition shows that the map which takes a time-invariant principal connection defined on \hat{P} to its Bogomolny pair intertwines the action of $\text{Aut}_b^s(\hat{P}) \simeq \text{Aut}(P)$ on $\text{Conn}^s(P)$ with the action of $\text{Aut}_b(P)$ on the set $\mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P)$ given by:

$$\mu \stackrel{\text{def.}}{=} \text{ad}_P \times \mathbb{A} : \text{Aut}_b(P) \rightarrow \text{Aut}_{\mathbb{R}}(\mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Aff}(\text{Conn}(P))).$$

Proposition 5.9. *Let $\hat{\mathcal{A}}$ be a time-invariant principal connection on \hat{P} and $(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P)$ be the Bogomolny pair defined by $\hat{\mathcal{A}}$. For any $u \in \text{Aut}_b(P)$, the Bogomolny pair (Ψ_u, \mathcal{A}_u) of the time-invariant connection $\mathbb{A}_{\hat{u}}(\hat{\mathcal{A}})$ obtained from $\hat{\mathcal{A}}$ by applying the time-invariant gauge transformation \hat{u} is given by:*

$$\Psi_u = \text{ad}_P(u)(\Psi), \quad \mathcal{A}_u = \mathbb{A}_u(\mathcal{A}).$$

In particular, we have:

$$\mathcal{V}_{\mathcal{A}_u} = \text{ad}_P(u)(\mathcal{V}_{\mathcal{A}}).$$

Proof. We have $\mathbb{A}_{\hat{u}}(\hat{\mathcal{A}}) = (\hat{u}^{-1})^*(\hat{\mathcal{A}})$. Using Lemma 5.3, this gives:

$$\mathbb{A}_{\hat{u}}(\hat{\mathcal{A}}) = -(\Psi^\pi \circ u^{-1} \circ \tau)\hat{\theta} + \tau^*((u^{-1})^*(\mathcal{A})) = -(\Psi_u^\pi \circ \tau)\hat{\theta} + \tau^*(\mathbb{A}_u(\mathcal{A})),$$

where we used relation (49) and noticed that $(\hat{u}^{-1})^*(\hat{\theta}) = \theta$ since $\hat{\theta} = \hat{\pi}^*(\theta)$ and $\hat{\pi} \circ \hat{u}^{-1} = \hat{u}$ because \hat{u}^{-1} is a based automorphism of \hat{P} . \square

Notice that the discrete remnant (see [35]) of any gauge transformation of \hat{P} is time-invariant since Σ (and hence M) is connected and therefore any discrete gauge transformation of \hat{P} is constant on M . In particular, the groups of discrete gauge transformations of \hat{P} and P can be identified. As

explained in [35], we have:

$$\text{ad}_P(u) = \text{ad}_{\Gamma(P)}(\bar{u}), \quad \forall u \in \text{Aut}_b(P),$$

where \bar{u} is the discrete remnant of u . This implies that $\text{Aut}_b(P)$ acts on \mathcal{S} through based automorphism of the integral duality structure Δ determined by P :

$$\text{ad}_P(u) \in \text{Aut}_b(\Delta), \quad \forall u \in \text{Aut}_b(P)$$

and hence induces an integral duality transformation of Ψ .

Proposition 5.10. *Let \mathcal{J} be a taming of \mathcal{S} . Then $(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathcal{S}) \times \text{Conn}(P)$ is a polarized abelian dyon with respect to \mathcal{J} if and only if (Ψ_u, \mathcal{A}_u) is a polarized abelian dyon with respect to the taming $\mathcal{J}_u \stackrel{\text{def.}}{=} \text{ad}_P(u) \circ \mathcal{J} \circ \text{ad}_P(u)^{-1}$. In particular, the action μ of $\text{Aut}_b(P)$ restricts to bijections:*

$$\mu(u) : \text{Dyons}(M, g, P, \mathcal{J}) \xrightarrow{\sim} \text{Dyons}(\Sigma, g, P, \mathcal{J}_u), \quad \forall u \in \text{Aut}_b(P).$$

Proof. Applying $\text{ad}_P(u)$ shows that the polarized Bogomolny equation (47) relative to \mathcal{J} is equivalent with the polarized Bogomolny equation relative to \mathcal{J}_u :

$$\star_{h, \mathcal{J}_u} \mathcal{V}_{\mathcal{A}_u} = \text{d}_{\mathcal{D}} \Psi_u,$$

where we used Proposition 5.9 and the fact that $\text{ad}_P(u)$ commutes with \mathcal{D} , which is proved in [35]. □

5.4. The case when $Z(P)$ has trivial monodromy on Σ

Suppose that $Z(P)$ has trivial monodromy on Σ , so the duality structure $\Delta := \Delta(P)$ is holonomy-trivial and its Siegel system is trivial. Then there exists a flat trivialization of Δ which identifies the integral duality structure $\Delta = (\Delta, \mathcal{L})$ of P with $(\Sigma \times \mathbb{R}^{2n}, \omega_{2n}, \text{d}, \Sigma \times \Lambda_t)$. Notice that a Higgs field identifies with a smooth map from Σ to \mathbb{R}^{2n} , which we decompose into maps $\Phi, \Upsilon : \Sigma \rightarrow \mathbb{R}^n$ according to:

$$(50) \quad \Psi = \begin{pmatrix} -\Phi \\ \Upsilon \end{pmatrix}$$

Proposition 5.11. *A pair $(\Psi, \mathcal{A}) \in \mathcal{C}^\infty(\Sigma, \mathbb{R}^{2n}) \times \text{Conn}(P)$ satisfies the polarized Bogomolny equations iff:*

$$(51) \quad d\Psi = \begin{pmatrix} E \\ -\mathcal{R}E - \mathcal{I} *_h B \end{pmatrix},$$

where $E \in \Omega^1(\Sigma, \mathbb{R}^n)$ and $B \in \Omega^2(\Sigma, \mathbb{R}^n)$ are determined by $\mathcal{V}_{\mathcal{A}}$ through the relation:

$$(52) \quad \mathcal{V}_{\mathcal{A}} = \begin{pmatrix} B \\ -\mathcal{R}B + \mathcal{I} *_h E \end{pmatrix}.$$

Proof. In the chosen trivialization of Δ , tamings identify with taming maps $\mathcal{J} : \Sigma \rightarrow \text{GL}(2n, \mathbb{R})$. By Proposition A.7, the latter have the form:

$$\mathcal{J} = \begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix},$$

where $\mathcal{N} = \mathcal{R} + i\mathcal{I} : \Sigma \rightarrow \mathbb{S}\mathbb{H}^n$ is the corresponding period matrix map. A similar equation relates the taming $\hat{\mathcal{J}} = \mathcal{J} \circ p$ of $\hat{\Delta}$ to the period matrix map $\hat{\mathcal{N}} \stackrel{\text{def.}}{=} \mathcal{N} \circ p = \hat{\mathcal{R}} + i\hat{\mathcal{I}} : M \rightarrow \text{GL}(2n, \mathbb{R})$ (where $\hat{\mathcal{R}} \stackrel{\text{def.}}{=} \mathcal{R} \circ p$ and $\hat{\mathcal{I}} \stackrel{\text{def.}}{=} \mathcal{I} \circ p$) defined on M . By Lemma A.4, the adjoint curvature of \mathcal{A} is twisted self-dual with respect to $\hat{\mathcal{J}}$ if and only if it has the form:

$$(53) \quad \mathcal{V}_{\hat{\mathcal{A}}} = \begin{pmatrix} \hat{F} \\ G_g(\hat{\mathcal{N}}, \hat{F}) \end{pmatrix}$$

for some $\hat{F} \in \Omega^2(M, \mathbb{R}^n)$, where:

$$G_g(\hat{\mathcal{N}}, \hat{F}) \stackrel{\text{def.}}{=} -\hat{\mathcal{R}}\hat{F} - \hat{\mathcal{I}} *_g \hat{F}.$$

Lemma 5.1 gives:

$$G_g(\hat{\mathcal{N}}, \hat{F})_{\top} = -\hat{\mathcal{R}}\hat{F}_{\top} - \hat{\mathcal{I}} *_h \hat{F}_{\perp}, \quad G_g(\hat{\mathcal{N}}, \hat{F})_{\perp} = -\hat{\mathcal{R}}\hat{F}_{\perp} + \hat{\mathcal{I}} *_h \hat{F}_{\top}.$$

Thus (53) amounts to:

$$(54) \quad (\mathcal{V}_{\hat{\mathcal{A}}})_{\top} = \begin{pmatrix} \hat{F}_{\top} \\ -\hat{\mathcal{R}}\hat{F}_{\top} - \hat{\mathcal{I}} *_h \hat{F}_{\perp} \end{pmatrix}, \quad (\mathcal{V}_{\hat{\mathcal{A}}})_{\perp} = \begin{pmatrix} \hat{F}_{\perp} \\ -\hat{\mathcal{R}}\hat{F}_{\perp} + \hat{\mathcal{I}} *_h \hat{F}_{\top} \end{pmatrix}.$$

These relations imply:

$$\hat{F}_{\top} = p^*(E), \quad \hat{F}_{\perp} = p^*(B)$$

for some $E \in \Omega^1(\Sigma, \mathbb{R}^n)$ and $B \in \Omega^2(\Sigma, \mathbb{R}^n)$. Using (54), we conclude that (46) is equivalent with:

$$d\Psi = \begin{pmatrix} E \\ -\mathcal{R}E - \mathcal{I} *_h B \end{pmatrix}, \quad \mathcal{V}_A = \begin{pmatrix} B \\ -\mathcal{R}B + \mathcal{I} *_h E \end{pmatrix}.$$

Notice that the pair (E, B) is uniquely determined by \mathcal{V}_A through the second of these relations. □

We will refer to the forms $E \in \Omega^1(\Sigma, \mathbb{R}^n)$ and $B \in \Omega^2(\Sigma, \mathbb{R}^n)$ as the *electrostatic* and *magnetostatic* field strengths of \mathcal{A} . Equation (51) is equivalent to the system:

$$d\Phi = E, \quad d\Upsilon = -\mathcal{R}E - \mathcal{I} *_h B,$$

which in turn amounts to:

$$(55) \quad \vec{E} = -\text{grad}_h \Phi, \quad \mathcal{I} \vec{B} + \mathcal{R} \vec{E} = -\text{grad}_h \Upsilon,$$

where we defined $\vec{E}, \vec{B} \in \mathcal{C}^\infty(\Sigma, T\Sigma) \otimes \mathbb{R}^n$ through:

$$\vec{E} = E^\sharp, \quad \vec{B} = (*_h B)^\sharp.$$

Here \sharp denotes the musical isomorphism given by raising of indices with the metric h . Equation $d\mathcal{V}_A = 0$ amounts to the system:

$$(56) \quad dB = 0, \quad d(-\mathcal{R}B + \mathcal{I} *_h E) = 0 \iff \text{div}_h \vec{B} = 0, \quad \text{div}_h (\mathcal{R} \vec{B} + \mathcal{I} \vec{E}) = 0.$$

5.5. Polarized dyons in prequantum electrodynamics

Prequantum electrodynamics defined on M corresponds to setting $n = 1$ and $\mathcal{R} = \frac{\theta}{2\pi}, \mathcal{I} = \frac{4\pi}{g^2}$ with constant $\theta \in \mathbb{R}$ in the previous subsection (see Appendix A). Then:

$$\mathcal{J} = \mathcal{J}_\theta \stackrel{\text{def.}}{=} \begin{pmatrix} \frac{g^2 \theta}{8\pi^2} & \frac{g^2}{4\pi} \\ -\frac{4\pi}{g^2} & -\frac{g^2 \theta}{16\pi^3} - \frac{g^2 \theta}{8\pi^2} \end{pmatrix},$$

and relation (52) becomes:

$$(57) \quad \mathcal{V}_A = \begin{pmatrix} B \\ -\frac{\theta}{2\pi} B + \frac{4\pi}{g^2} *_h E \end{pmatrix}.$$

In this case, relations (55) reduce to:

$$(58) \quad \vec{E} = -\text{grad}_h \Phi, \quad \frac{4\pi}{g^2} \vec{B} + \frac{\theta}{2\pi} \vec{E} = -\text{grad}_h \Upsilon,$$

which imply $\vec{B} = -\frac{g^2}{4\pi} \text{grad}_h(\Upsilon - \frac{\theta}{2\pi} \Phi)$ and:

$$(59) \quad \text{curl}_h \vec{E} = \text{curl}_h \vec{B} = 0.$$

On the other hand, relations (56) become:

$$(60) \quad dB = d(*_h E) = 0 \iff \text{div}_h \vec{B} = \text{div}_h \vec{E} = 0.$$

Equations (59) and (60) describe source-free Maxwell electromagnetostatics on Σ , where the vector fields $\vec{E}, \vec{B} \in \mathcal{A}(\Sigma)$ are the classical static electric and magnetic fields. Relation (58) shows that Φ and $\frac{g^2}{4\pi} \Upsilon - \frac{g^2 \theta}{8\pi^2} \Phi$ are globally-defined classical electrostatic and magnetostatic scalar potentials. Since $H^1(\Sigma, \mathbb{R})$ need not vanish, relation (59) need not imply (58). Hence polarized dyons describe special potential electromagnetostatic configurations, i.e. solutions of the static Maxwell equations on Σ which admit both a globally-defined electric scalar potential and a globally-defined magnetic scalar potential. Even though \vec{E} and \vec{B} satisfy (60), such solutions need *not* admit a globally-defined vector electric or vector magnetic potential, since $H^2(\Sigma, \mathbb{R})$ may be non-zero. The condition that the configuration (\vec{E}, \vec{B}) admits globally-defined electric and magnetic scalar potentials is a consequence of the fact that a polarized dyon originates from a principal connection defined on \hat{P} , which itself is a consequence of the DSZ integrality condition. We formalize this as follows.

Definition 5.12. An *electromagnetostatic configuration* defined on (Σ, h) is a pair of vector fields $(\vec{E}, \vec{B}) \in \mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma)$ which satisfies the static source-free Maxwell equations (59) and (60). Such a configuration is said to be *potential* if there exist real-valued smooth functions Φ, Υ defined on Σ such that relations (58) hold.

Let $\text{EMC}(\Sigma, h)$ denote the vector space of all electromagnetostatic configurations defined on (Σ, h) and $\text{EMC}_{\text{pot}}(\Sigma, h)$ denote the subspace of those configurations which are potential. Consider the map:

$$H_\theta : \text{Dyons}(\Sigma, h, P, \mathcal{J}_\theta) \rightarrow \text{EMC}_{\text{pot}}(\Sigma, h)$$

which associates the electromagnetostatic configuration (\vec{E}, \vec{B}) defined through relation (57) to the polarized dyon (Ψ, \mathcal{A}) . As in Subsection 2.10, the chosen trivialization of Δ induces isomorphisms:

$$H^2_{\mathcal{D}}(\Sigma, \mathcal{S}) \simeq H^2(M, \mathbb{R}^{2n}), \quad H^2(\Sigma, \mathcal{L}) \simeq H^2(\Sigma, \Lambda_t),$$

which allow us to identify the morphism $j : H^2(\Sigma, \mathcal{L}) \rightarrow H^2_{\mathcal{D}}(\Sigma, \mathcal{S})$ with the morphism ι appearing in the long exact sequence:

$$\dots \rightarrow H^1(\Sigma, A) \rightarrow H^2(\Sigma, \Lambda_t) \xrightarrow{i} H^2(\Sigma, \mathbb{R}^{2n}) \rightarrow H^2(\Sigma, A) \rightarrow \dots$$

induced by the exponential sequence:

$$0 \rightarrow \Lambda_t \rightarrow \mathbb{R}^{2n} \rightarrow A \rightarrow 0.$$

The relation $[\mathcal{V}_{\mathcal{A}}]_{\mathcal{D}} = 2\pi j_*(c(P))$ becomes:

$$[\mathcal{V}_{\mathcal{A}}] = 2\pi i_*(c(P)),$$

where $[\omega]$ denotes the de Rham cohomology class of a form $\omega \in \Omega^k(M, S)$. Conversely, any closed two-form $\mathcal{V} \in \Omega^2_{\text{cl}}(\Sigma, \mathbb{R}^{2n})$ which satisfies $[\mathcal{V}] = 2\pi i_*(c(P))$ identifies with the curvature of some principal connection \mathcal{A} on P . For any $(\vec{E}, \vec{B}) \in \text{EMC}(\Sigma, h)$, let:

$$\mathcal{V}^{\theta}_{\vec{E}, \vec{B}} \stackrel{\text{def.}}{=} \begin{pmatrix} B \\ -\frac{\theta}{2\pi} B + \frac{4\pi}{g^2} *_h E \end{pmatrix},$$

where $E \stackrel{\text{def.}}{=} \vec{E}^{\flat}$ and $B \stackrel{\text{def.}}{=} - *_h(\vec{B}^{\flat})$.

Theorem 5.13. *The image of the map H_{θ} coincides with the set:*

$$\text{EMC}_{\text{pot}}(\Sigma, h; \theta, c(P)) \stackrel{\text{def.}}{=} \{(\vec{E}, \vec{B}) \in \text{EMC}_{\text{pot}}(\Sigma, h) \mid [\mathcal{V}^{\theta}_{\vec{E}, \vec{B}}] = 2\pi \iota_*(c(P))\}.$$

Moreover, the H_{θ} -preimage of a configuration

$$(\vec{E}, \vec{B}) \in \text{EMC}_{\text{pot}}(\Sigma, h; \theta, c(P))$$

is given by:

$$H_{\theta}^{-1}(\{(\vec{E}, \vec{B})\}) = \left\{ (\Psi, A) \in \text{Dyons}(\Sigma, h, \mathcal{J}_{\theta}) \mid \mathcal{V}_{\mathcal{A}} = \mathcal{V}^{\theta}_{\vec{E}, \vec{B}} \ \& \ \text{grad}_h \Psi = \left(-\frac{\theta}{2\pi} \vec{E} - \frac{4\pi}{g^2} \vec{B} \right) \right\}.$$

Proof. Let (Ψ, \mathcal{A}) be a polarized abelian dyon. Then $H_\theta(\Psi, \mathcal{A}) \in \text{EMC}_{\text{pot}}(\Sigma, h)$ and by equation (57), there exists an electromagnetostatic configuration (\vec{E}, \vec{B}) such that:

$$\mathcal{V}_\mathcal{A}^\theta = \left(\begin{array}{c} B \\ -\frac{\theta}{2\pi}B + \frac{4\pi}{g^2} *_h E \end{array} \right).$$

Since $\mathcal{V}_\mathcal{A}^\theta$ is the curvature of a connection \mathcal{A} on P , it satisfies $[\mathcal{V}_\mathcal{A}^\theta]_{\mathcal{D}} = 2\pi j_*(c(P))$ and hence $(\vec{E}, \vec{B}) \in \text{EMC}_{\text{pot}}(\Sigma, h; \theta, c(P))$. On the other hand, if $(\vec{E}, \vec{B}) \in \text{EMC}_{\text{pot}}(\Sigma, h; \theta, c(P))$, equation $[\mathcal{V}_{\vec{E}, \vec{B}}^\theta] = \iota_*(c(P))$ implies that there exists a connection \mathcal{A} on P whose curvature is $\mathcal{V}_{\vec{E}, \vec{B}}^\theta$ and since (\vec{E}, \vec{B}) is a potential electromagnetostatic solution, choosing potentials Φ and Υ and defining:

$$(61) \quad \Psi = \left(\begin{array}{c} -\Phi \\ \Upsilon \end{array} \right)$$

we conclude that (Ψ, \mathcal{A}) is a polarized abelian dyon whence $(\vec{E}, \vec{B}) \in \text{Im}(H_\theta)$. It is now clear that the only freedom in choosing a preimage of (\vec{E}, \vec{B}) by H_θ lies only in choosing the potentials Φ and Υ as prescribed in the statement of the theorem and hence we conclude. \square

5.6. Polarized Bogomolny equations on the punctured Euclidean 3-space

In this subsection, we construct families of solutions (which generalize the dyons of ordinary electromagnetism) on the punctured Euclidean space $\Sigma = \mathbb{R}_0^3 \stackrel{\text{def.}}{=} \mathbb{R}^3 \setminus \{0\}$. Spherical coordinates give a diffeomorphism:

$$\mathbb{R}_0^3 \simeq \mathbb{R}_{>0} \times \mathbb{S}^2,$$

and we denote by $r \in \mathbb{R}_{>0}$ the radial coordinate. The metric h reads:

$$h = dr^2 + r^2 h_{\mathbb{S}^2},$$

where $h_{\mathbb{S}^2}$ is the unit round metric of \mathbb{S}^2 . We have $\nu_h = r^2 dr \wedge \nu_{\mathbb{S}^2}$, where $\nu_{\mathbb{S}^2}$ is the volume form of \mathbb{S}^2 . Let $q : \mathbb{R}_0^3 \rightarrow \mathbb{S}^2$ be the map defined though:

$$q(x) \stackrel{\text{def.}}{=} \frac{x}{\|x\|},$$

where $\|x\|$ is the Euclidean norm of x . Up to isomorphism, any Siegel bundle of rank n and type $\mathfrak{t} \in \text{Div}^n$ defined on P on \mathbb{R}_0^3 is the pull-back through q of a Siegel bundle of the same rank and type defined on S^2 , which reduces to a principal torus bundle since S^2 is simply connected. Accordingly, the duality structure Δ of P is holonomy trivial and we can fix a global flat trivialization in which Δ identifies with:

$$(62) \quad \Delta \equiv (\Sigma \times \mathbb{R}^{2n}, \omega_{2n}, d),$$

and the Dirac system \mathcal{L} determined by P identifies with $\Sigma \times \Lambda_{\mathfrak{t}}$. Hence a Bogomolny pair consists of a map $\Psi \in C^\infty(\Sigma, \mathbb{R}^{2n})$ and a connection $\mathcal{A} \in \text{Conn}(P)$ whose curvature can be viewed as a vector-valued two-form $\mathcal{V}_{\mathcal{A}} \in \Omega^2(\Sigma, \mathbb{R}^{2n})$. Moreover, a taming of Δ can be viewed as a map $\mathcal{J} \in C^\infty(\Sigma, \text{GL}(2n, \mathbb{R}))$ which squares to minus the identity everywhere and satisfies $\mathcal{J}^t(\sigma)\hat{\omega}_{2n}\mathcal{J}(\sigma) = \hat{\omega}_{2n}$ for all $\sigma \in \Sigma$.

Let us assume that \mathcal{J} and Ψ depend only on r . Then $d\Psi = \partial_r\Psi dr$ and $*_h(dr) = \iota_{\partial_r}\nu_h = r^2 q^*(\nu_{S^2})$, hence the polarized Bogomolny equation reads:

$$(63) \quad \mathcal{V}_{\mathcal{A}} = -r^2 \mathcal{J}(r) \frac{d\Psi}{dr} q^*(\nu_{S^2}),$$

Since $\mathcal{V}_{\mathcal{A}}$ and ν_{S^2} are closed, a necessary condition for this equation to admit solutions $\mathcal{A} \in \text{Conn}(P)$ is:

$$\frac{d}{dr}(r^2 \mathcal{J} \frac{d}{dr} \Psi) = 0.$$

The general solution of this integrability condition is:

$$\Psi = - \int \frac{\mathcal{J}v}{2r^2} dr + v',$$

for constant vectors $v, v' \in \mathbb{R}^{2n}$. Using this in (63) gives:

$$\mathcal{V}_{\mathcal{A}} = -\frac{1}{2} v q^*(\nu_{S^2}).$$

The last relation determines \mathcal{A} up to a transformation of the form:

$$\mathcal{A} \rightarrow \mathcal{A} + \omega$$

where $\omega \in \Omega^1_{\text{Ad}}(P, \mathbb{R}^{2n})$ is closed and hence corresponds to a closed 1-form $\omega' \in \Omega^1(\Sigma, \mathbb{R}^{2n})$ (recall that $d_{\mathcal{D}} = d$ in our trivialization of \mathcal{S}). Notice that ω' need not be exact since $\Sigma = \mathbb{R}_0^3$ is not contractible.

5.6.1. The integrality condition for v . Since S^2 is a deformation retract of Σ , the map $q^* : H^*(S^2, \mathbb{R}^{2n}/\Lambda_t) \rightarrow H^*(\Sigma, \mathbb{R}^{2n}/\Lambda_t)$ induced by q on cohomology is an isomorphism of groups for any abelian group of coefficients A . Since $H_1(S^2, \mathbb{Z}) = 0$, the universal coefficient theorem for cohomology gives:

$$(64) \quad H^2(S^2, \Lambda_t) \simeq_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(H_2(S^2, \mathbb{Z}), \Lambda_t) \simeq_{\mathbb{Z}} \Lambda_t,$$

where the last isomorphism is given by evaluation on the fundamental class $[S^2] \in H_2(S^2, \mathbb{Z})$. This allows us to view the characteristic class $c(P) = q^*(c(P_0))$ as an element of Λ_t . Moreover, we have isomorphisms of vector spaces:

$$(65) \quad H^2(S^2, \mathbb{R}^{2n}) \simeq_{\mathbb{R}} H^2(S^2, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}^{2n} \simeq_{\mathbb{R}} \mathbb{R}^{2n},$$

the last of which takes $u \otimes_{\mathbb{Z}} x$ to x for all $x \in \mathbb{R}^{2n}$. Through the isomorphisms (64) and (65), the map $i_* : H^2(S^2, \Lambda_t) \rightarrow H^2(S^2, \mathbb{R}^{2n})$ corresponds to the inclusion $\Lambda_t \subset \mathbb{R}^{2n}$ and $2\pi c(P)$ identifies with the de Rham cohomology class of $\mathcal{V}_{\mathcal{A}}$. The free abelian group $H^2(S^2, \mathbb{Z})$ is generated by half of the Euler class $e(S^2)$ of S^2 , which satisfies:

$$\int_{S^2} e(S^2) = \chi(S^2) = 1 + (-1)^2 = 2.$$

On the other hand, the de Rham cohomology class $u \stackrel{\text{def.}}{=} \left[\frac{\nu_{S^2}}{4\pi} \right] \in H^2(S^2, \mathbb{R}^{2n})$ satisfies:

$$u = i_* \left(\frac{1}{2} e(S^2) \right).$$

We have:

$$[\mathcal{V}_{\mathcal{A}}] = 2\pi i_*(c(P)),$$

which implies $v \in \Lambda_t$ as well as:

$$c(P_0) = \frac{1}{2} v e(S^2) \text{ and } i_*(c(P_0)) = vu.$$

Hence we obtain an integrality condition for the integration constant v , which guarantees the existence of a solution and determines through the previous equation the topological type of the bundle P carrying the solution.

Appendix A. Local abelian gauge theory

This appendix discusses the duality-covariant formulation and global symmetries of source-free $U(1)^n$ abelian gauge theory on a *contractible* oriented four-manifold M endowed with an arbitrary metric of signature $(3, 1)$, with the goal of motivating the geometric model introduced in Section 1 and of making contact with the physics literature. The local study of electromagnetic duality for Maxwell electrodynamics in four Lorentzian dimensions has a long history, see [21, 22] as well as [41] and its references and citations. Let $*_g$ be the Hodge operator of (M, g) .

The prototypical example of local abelian gauge theory is given by classical electrodynamics, which is defined by the Lagrangian density functional:

$$\begin{aligned} \mathfrak{L}[A] &= -\frac{4\pi}{g^2}(F_A)_{ab}(F_A)^{ab} + \frac{\theta}{2\pi}(F_A)_{ab}(*_g F_A)^{ab}, \\ F &= dA \text{ with } A \in \Omega^1(M), \end{aligned}$$

where $\theta \in \mathbb{R}$ is the theta angle. Classical Maxwell theory is obtained for $\theta = 0$. Below, we discuss the more general construction of local abelian gauge theories, which is motivated by the local structure of four-dimensional supergravity and supersymmetric field theory and which involves a finite number of abelian gauge fields.

A.1. Lagrangian formulation

Given $n \in \mathbb{Z}_{>0}$ and a Lorentzian metric g on M , consider the following generalization of the previous Lagrangian density, which allows all couplings to be smooth real-valued functions:

$$\mathfrak{L}[A^1, \dots, A^n] = -\mathcal{I}_{\Lambda\Sigma} \langle F_A^\Lambda, F_A^\Sigma \rangle_g + \mathcal{R}_{\Lambda\Sigma} \langle F_A^\Lambda, *_g F_A^\Sigma \rangle_g.$$

Here $\Lambda, \Sigma = 1, \dots, n$ and :

$$F_A^\Lambda = dA^\Lambda, \quad \Lambda = 1, \dots, n,$$

where $A^\Lambda \in \Omega^1(M)$ and $\mathcal{R}_{\Lambda\Sigma}, \mathcal{I}_{\Lambda\Sigma} \in \mathcal{C}^\infty(M)$ are the components of an \mathbb{R}^n -valued one-form $A \in \Omega^1(M, \mathbb{R}^n)$ and of functions $\mathcal{R}, \mathcal{I} \in \mathcal{C}^\infty(M, \text{Mat}_s(n, \mathbb{R}))$ valued in the space $\text{Mat}_s(n, \mathbb{R})$ of symmetric square matrices of size n with real entries. To guarantee a positive-definite kinetic term we must assume that \mathcal{I} is positive-definite everywhere. Notice that classical electrodynamics corresponds to $n = 1$ with $\mathcal{R} = \frac{\theta}{2\pi}$ and $\mathcal{I} = \frac{4\pi}{g^2}$. Since $*_g^2 F^\Lambda = -F^\Lambda$, the

action is equivalent to:

$$\begin{aligned} \mathcal{S}[A^1, \dots, A^n] &= \int \mathfrak{L}[A^1, \dots, A^n] \text{vol}_g \\ &= - \int (\mathcal{I}_{\Lambda\Sigma} F_A^\Lambda \wedge *_g F_A^\Sigma + \mathcal{R}_{\Lambda\Sigma} F_A^\Lambda \wedge F_A^\Sigma) . \end{aligned}$$

The partial differential equations obtained as critical points of the variational problem with respect to compactly supported variations are:

$$d(\mathcal{R}_{\Lambda\Sigma} F_A^\Sigma + \mathcal{I}_{\Lambda\Sigma} *_g F_A^\Sigma) = 0 ,$$

or, in matrix notation:

$$d(\mathcal{R}F_A + \mathcal{I} *_g F_A) = 0 ,$$

where $F_A = dA \in \Omega_{\text{cl}}^2(M, \mathbb{R}^n) = \Omega_{\text{ex}}^2(M, \mathbb{R}^n)$ is an \mathbb{R}^n -valued closed (and hence exact) two-form. These equations define *classical local abelian gauge theory*. Since both \mathcal{R} and \mathcal{I} are symmetric and \mathcal{I} is positive definite, the pair $(\mathcal{R}, \mathcal{I})$ is equivalent to a map:

$$\mathcal{N} \stackrel{\text{def.}}{=} \mathcal{R} + \mathbf{i}\mathcal{I} : M \rightarrow \mathbb{S}\mathbb{H}_n ,$$

where $\mathbb{S}\mathbb{H}_n$ denotes the Siegel upper half space of symmetric $n \times n$ complex matrices with positive definite imaginary part.

Definition A.1. A *period matrix map* of size n on M is a smooth function $\mathcal{N} \in \mathcal{C}^\infty(M, \mathbb{S}\mathbb{H}_n)$ valued in $\mathbb{S}\mathbb{H}_n$. We denote the set of such maps by $\text{Per}_n(M)$.

When the metric g is fixed, classical local abelian gauge theory is uniquely determined by a choice of period matrix map.

Definition A.2. Let $\mathcal{N} = \mathcal{R} + \mathbf{i}\mathcal{I}$ be a period matrix map of size n . The *local abelian gauge theory* associated to \mathcal{N} is defined through the following system of equations:

$$(A.1) \quad d(\mathcal{R} F_A + \mathcal{I} *_g F_A) = 0 ,$$

with unknowns given by the vector valued one-form $A \in \Omega^1(M, \mathbb{R}^n)$, where $F_A \stackrel{\text{def.}}{=} dA \in \Omega_{\text{cl}}^2(M, \mathbb{R}^n)$.

Remark A.3. The equations of motion of local abelian gauge theory have a *gauge symmetry* consisting of transformations of the type:

$$A \mapsto A + d\alpha, \quad \alpha \in \mathcal{C}^\infty(M, \mathbb{R}^n).$$

Variables $A \in \Omega^1(M, \mathbb{R}^n)$ related by such gauge transformations should be viewed as physically equivalent.

Let us write the equations of motion (A.1) in a form amenable to geometric interpretation. Given a period matrix map $\mathcal{N} = \mathcal{R} + \mathbf{i}\mathcal{I}$ and a vector of two-forms $F \in \Omega^2(M, \mathbb{R}^n)$, define:

$$G_g(\mathcal{N}, F) \stackrel{\text{def.}}{=} -\mathcal{R}F - \mathcal{I} *_g F.$$

Then the condition $dF = 0$ together with the equations of motion (A.1) are equivalent with the single equation:

$$(A.2) \quad d\mathcal{V} = 0,$$

where the \mathbb{R}^{2n} -valued two-form \mathcal{V} is related to F by:

$$(A.3) \quad \mathcal{V} = \begin{pmatrix} F \\ G_g(\mathcal{N}, F) \end{pmatrix} \in \Omega^2(M, \mathbb{R}^{2n}) = \Omega^2(M, \mathbb{R}^n \oplus \mathbb{R}^n).$$

As the following lemma shows, not every vector-valued two-form $\mathcal{V} \in \Omega^2(M, \mathbb{R}^{2n})$ can be written as prescribed by equation (A.3).

Lemma A.4. *Let $\mathcal{N} = \mathcal{R} + \mathbf{i}\mathcal{I} \in \text{Per}_n(M)$ be a period matrix map of size n on M . A vector valued two-form $\mathcal{V} \in \Omega^2(M, \mathbb{R}^{2n})$ can be written as:*

$$\mathcal{V} = \begin{pmatrix} F \\ G_g(\mathcal{N}, F) \end{pmatrix}$$

for some $F \in \Omega^2(M, \mathbb{R}^n)$ if and only if:

$$(A.4) \quad *_g \mathcal{V} = -\mathcal{J}\mathcal{V},$$

where $\mathcal{J} \in \mathcal{C}^\infty(M, \text{GL}(2n, \mathbb{R}))$ is the matrix-valued map defined through:

$$\mathcal{J} = \begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix}.$$

We have $\mathcal{J}^2 = -1$. Moreover, F with the property above is uniquely determined by \mathcal{V} .

Remark A.5. Notice that \mathcal{J} can be viewed as a complex structure defined on the trivial real vector bundle of rank $2n$ over M . For classical electrodynamics, the taming map is constant and given by:

$$\mathcal{J} = \begin{pmatrix} \frac{g^2\theta}{8\pi^2} & \frac{g^2}{4\pi} \\ -\frac{4\pi}{g^2} - \frac{g^2\theta^2}{16\pi^3} & -\frac{g^2\theta}{8\pi^2} \end{pmatrix}.$$

In this case, the period matrix map is constant and given by:

$$\mathcal{N} = \frac{4\pi}{g^2} + \mathbf{i}\frac{\theta}{2\pi},$$

being traditionally denoted by τ .

Proof. If:

$$\mathcal{V} = \begin{pmatrix} F \\ G_g(\mathcal{N}, F) \end{pmatrix},$$

then direct computation using the fact that $*_g^2 = -1$ on two-forms shows that \mathcal{V} satisfies $*_g\mathcal{V} = -\mathcal{J}\mathcal{V}$. On the other hand, writing $\mathcal{V} = \begin{pmatrix} F \\ G \end{pmatrix}$ with $F, G \in \Omega^2(M, \mathbb{R}^n)$ shows that the equation $*_g\mathcal{V} = -\mathcal{J}\mathcal{V}$ is equivalent to:

$$\begin{pmatrix} *_gF \\ *_gG \end{pmatrix} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R}F - \mathcal{I}^{-1}G \\ -(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})F - \mathcal{R}\mathcal{I}^{-1}G \end{pmatrix},$$

which in turn amounts to $G = G_g(\mathcal{N}, F)$. □

Let ω_{2n} be the standard symplectic form on \mathbb{R}^{2n} , which in our conventions has the following matrix in the canonical basis $\mathcal{E} = (e_1, \dots, e_n, f_1, \dots, f_n)$ of the latter:

$$(A.5) \quad \hat{\omega}_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n is the identity matrix of size n . We have:

$$(A.6) \quad \omega_{2n} \stackrel{\text{def.}}{=} \sum_{a=1}^n e_a^* \wedge f_a^*,$$

where $\mathcal{E}^* = (e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*)$ is the basis dual to

$$\mathcal{E} = (e_1, \dots, e_n, f_1, \dots, f_n).$$

The following result gives a geometric interpretation of the equations of motion (A.4). Recall that an almost complex structure J on \mathbb{R}^{2n} is called a *taming* of the standard symplectic form ω_{2n} if:

$$\omega_{2n}(J\xi_1, J\xi_2) = \omega_{2n}(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in \mathbb{R}^{2n},$$

and:

$$\omega_{2n}(J\xi, \xi) > 0, \quad \forall \xi \in \mathbb{R}^{2n} \setminus \{0\}.$$

Definition A.6. A *taming map* of size $2n$ defined on M is a smooth map $\mathcal{J} \in C^\infty(M, \text{GL}(2n, \mathbb{R}))$ such that $\mathcal{J}(m)$ is a taming of ω_{2n} for every $m \in M$. We denote the set of all such maps by $\mathfrak{J}_n(M)$.

Proposition A.7. A *matrix-valued map* $\mathcal{J} \in C^\infty(M, \text{GL}(2n, \mathbb{R}))$ can be written as:

$$(A.7) \quad \mathcal{J} = \begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix}$$

in terms of a *period matrix map* $\mathcal{N} = \mathcal{R} + i\mathcal{I} \in \text{Per}_n(M)$ iff $\mathcal{J} \in \mathfrak{J}_n(M)$.

Proof. If \mathcal{J} is taken as in equation (A.7) for a period matrix map \mathcal{N} then direct computation shows that $\mathcal{J}(m)$ is a taming of ω_{2n} for all $m \in M$. For the converse, assume that $\mathcal{J}(m) \in \text{GL}(2n, \mathbb{R})$ is a taming of ω_{2n} for all $m \in M$ (we omit to indicate the evaluation at m for ease of notation). Let $\mathcal{E} = (e_1, \dots, e_n, f_1, \dots, f_n)$ the canonical basis of \mathbb{R}^{2n} . The vectors $\mathcal{E}_f = (f_1, \dots, f_n)$ form a basis over \mathbb{C} of the complex vector space $(\mathbb{R}^{2n}, \mathcal{J}(m)) \simeq \mathbb{C}^n$, hence there exists a unique map $\tau \in C^\infty(M, \text{Mat}(n, \mathbb{C}))$ which satisfies:

$$(A.8) \quad e_a = \tau(m)_{ab} f_b, \quad \forall a = 1, \dots, n,$$

where we use Einstein summation over repeated indices. Thus:

$$\delta_{ab} = \omega_{2n}(e_a, f_b) = \omega_{2n}(\tau(m)_{ac} f_c, f_b) = \text{Im}(\tau(m))_{ac} \omega_{2n}(\mathcal{J}(m) f_c, f_b),$$

which implies that $\text{Im}(\tau(m))$ is symmetric and positive-definite. Using the previous equation and compatibility of $\mathcal{J}(m)$ with ω_{2n} , we compute:

$$\begin{aligned} 0 &= \omega_{2n}(e_a, e_b) = \text{Re}(\tau(m))_{ba} - \text{Im}(\tau(m))_{bc} \omega_{2n}(\mathcal{J}(m)(e_a), f_c) \\ &= \text{Re}(\tau(m))_{ba} - \text{Re}(\tau(m))_{ab}, \end{aligned}$$

which shows that $\text{Re}(\tau(m))$ is symmetric. Hence the smooth map $\mathcal{N} \in \mathcal{C}^\infty(M, \mathbb{S}\mathbb{H}^n)$ defined through $\mathcal{N} = \mathcal{R} + \mathbf{i}\mathcal{I}$, where:

$$\mathcal{R} \stackrel{\text{def.}}{=} \text{Re}(\tau), \quad \mathcal{I} \stackrel{\text{def.}}{=} \text{Im}(\tau),$$

is a period matrix map. Equation (A.8) gives:

$$\begin{aligned} \mathcal{J}(m)(e_a) &= \mathcal{R}(m)_{ab}\mathcal{I}^{-1}(m)_{bc} e_c - \mathcal{R}(m)_{ab}\mathcal{I}(m)_{bc}^{-1}\mathcal{R}(m)_{cd} f_d \\ &\quad - \mathcal{I}(m)_{ad} f_d \\ \mathcal{J}(m)(f_a) &= \mathcal{I}(m)_{ab}^{-1} e_b - \mathcal{I}(m)_{ab}^{-1}\mathcal{R}(m)_{bc} f_c, \end{aligned}$$

which is equivalent to (A.7). □

Proposition A.8. *The map $\Theta: \text{Per}_n(M) \rightarrow \mathfrak{J}_n(M)$ defined through:*

$$\text{Per}_n(M) \ni \mathcal{N} = \mathcal{R} + \mathbf{i}\mathcal{I} \mapsto \Theta(\mathcal{N}) \stackrel{\text{def.}}{=} \begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \in \mathfrak{J}_n(M)$$

is a bijection between $\text{Per}_n(M)$ and $\mathfrak{J}_n(M)$.

Proof. Follows directly from the proof of Proposition A.7. The inverse of Θ takes a taming map $\mathcal{J} \in \mathfrak{J}_n(M)$ to the period matrix $\Theta^{-1}(\mathcal{J}) = \text{Re}(\tau) + \mathbf{i}\text{Im}(\tau)$, where, for all $m \in M$, $\tau(m)$ is the complex symmetric matrix of size n uniquely determined by the expansion $e_a = \tau_{ab}f_b$ of e_a over \mathbb{C} when \mathbb{R}^{2n} is endowed with the complex structure $\mathcal{J}(m)$. □

Since M is contractible, we have $\Omega_{\text{cl}}^2(M, \mathbb{R}^{2n}) = \Omega_{\text{ex}}^2(M, \mathbb{R}^{2n})$. By the discussion above, this implies that local abelian gauge theory can be formulated equivalently as a theory of closed \mathbb{R}^{2n} -valued two-forms $\mathcal{V} \in \Omega_{\text{cl}}^2(M, \mathbb{R}^{2n})$ satisfying the condition:

$$*_g\mathcal{V} = -\mathcal{J}\mathcal{V}$$

with respect to a fixed taming map $\mathcal{J} \in \mathfrak{J}_n(M)$. Consequently, the theory is uniquely determined by the choice of taming map. The condition $d\mathcal{V} = 0$ is equivalent with $\mathcal{V} = d\mathcal{A}$, where $\mathcal{A} \in \Omega^1(M, \mathbb{R}^{2n})$ is considered modulo gauge transformations $\mathcal{A} \mapsto \mathcal{A} + d\alpha$ with $\alpha \in \mathcal{C}^\infty(M, \mathbb{R}^{2n})$. The map $[\mathcal{A}] \mapsto d\mathcal{A}$ gives a well-defined bijection $\Omega^1(M, \mathbb{R}^{2n})/\Omega_{\text{cl}}^1(M, \mathbb{R}^{2n}) \xrightarrow{\sim} \Omega_{\text{cl}}^2(M, \mathbb{R}^{2n})$. Thus we can formulate classical local abelian gauge theory either in terms of *electromagnetic gauge potentials* $\mathcal{A} \in \Omega^1(M, \mathbb{R}^{2n})$ taken modulo gauge-equivalence or in terms of *electromagnetic field strengths* $\mathcal{V} \in \Omega_{\text{cl}}^2(M, \mathbb{R}^{2n})$.

Definition A.9. Let $\mathcal{J} \in \mathfrak{J}_n(M)$ be a taming map. The space of *electromagnetic gauge configurations* of the $U(1)^n$ local abelian gauge is $\Omega^1(M, \mathbb{R}^{2n})$.

Two gauge configurations are called *gauge equivalent* if they differ by an exact one-form. The theory is defined by the *polarized self-duality condition* for $\mathcal{A} \in \Omega^1(M, \mathbb{R}^{2n})$:

$$(A.9) \quad *_g \mathcal{V}_{\mathcal{A}} = -\mathcal{J}\mathcal{V}_{\mathcal{A}}, \text{ where } \mathcal{V}_{\mathcal{A}} \stackrel{\text{def.}}{=} d\mathcal{A}.$$

The *space of electromagnetic gauge fields* (or *electromagnetic gauge potentials*) of the theory is the linear subspace of $\Omega^1(M, \mathbb{R}^{2n})$ consisting of those elements which satisfy (A.9):

$$\mathfrak{Sol}_n(M, g, \mathcal{J}) \stackrel{\text{def.}}{=} \{ \mathcal{A} \in \Omega^1(M, \mathbb{R}^{2n}) \mid *_g \mathcal{V}_{\mathcal{A}} = -\mathcal{J}\mathcal{V}_{\mathcal{A}} \}.$$

Elements $\mathcal{A} \in \Omega^1(M, \mathbb{R}^{2n})$ are called (electromagnetic) *gauge potentials* or *gauge fields*. The space of *field strength configurations* is the vector space:

$$\text{Conf}_n(M) \stackrel{\text{def.}}{=} \Omega_{\text{cl}}^2(M, \mathbb{R}^{2n}),$$

while the space of *field strengths* is defined through:

$$\text{Sol}_n(M, g, \mathcal{J}) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \text{Conf}_n(M) \mid *_g \mathcal{V} = \mathcal{J}\mathcal{V} \}.$$

The map $[\mathcal{A}] \mapsto d\mathcal{A}$ gives a bijection $\Omega^1(M, \mathbb{R}^{2n})/\Omega_{\text{cl}}^1(M, \mathbb{R}^{2n}) \xrightarrow{\sim} \text{Conf}_n(M)$, which restricts to a bijection $\mathfrak{Sol}_n(M, g, \mathcal{J})/\Omega_{\text{cl}}^1(M, \mathbb{R}^{2n}) \xrightarrow{\sim} \text{Sol}_n(M, g, \mathcal{J})$.

A.2. Duality groups

Let $\text{Diff}(M)$ be the group of orientation-preserving diffeomorphisms of M and $\mathcal{J} \in \mathfrak{J}_n(M)$ be a taming map of rank $2n$ defined on M . For $(\gamma, f) \in \text{GL}(2n, \mathbb{R}) \times \text{Diff}(M)$, consider the linear isomorphism:

$$(A.10) \quad \mathbb{A}_{\gamma, f}: \Omega^k(M, \mathbb{R}^{2n}) \xrightarrow{\sim} \Omega^k(M, \mathbb{R}^{2n}), \quad \omega \mapsto \gamma(f_*\omega),$$

where $f_*: \Omega^k(M, \mathbb{R}^{2n}) \rightarrow \Omega^k(M, \mathbb{R}^{2n})$ is the push-forward through the diffeomorphism f . This gives a linear action of $\text{GL}(2n, \mathbb{R}) \times \text{Diff}(M)$ on $\Omega^k(M, \mathbb{R}^{2n})$. Since this action commutes with the exterior derivative, it preserves the space $\text{Conf}_n(M)$ of field strength configurations.

For any $\gamma \in \text{Sp}(2n, \mathbb{R})$, the map:

$$\mathcal{J}_{\gamma, f} \stackrel{\text{def.}}{=} \gamma(\mathcal{J} \circ f^{-1})\gamma^{-1},$$

is a taming map. This gives an action μ of $\text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)$ on $\mathfrak{J}_n(M)$ defined through:

$$\mu(\gamma, f)(\mathcal{J}) \stackrel{\text{def.}}{=} \mathcal{J}_{\gamma, f}, \quad \forall (\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M).$$

Proposition A.10. *For every $(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)$, the map $\mathbb{A}_{\gamma, f}$ induces by restriction a linear isomorphism:*

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\simeq} \text{Sol}_n(M, f_*(g), \mathcal{J}_{\gamma, f}),$$

where $f_*(g)$ denotes the push-forward of g by $f \in \text{Diff}(M)$.

Remark A.11. If we consider a pair $(\gamma, f) \in \text{GL}(2n, \mathbb{R}) \times \text{Diff}(M)$ with $\gamma \notin \text{Sp}(2n, \mathbb{R})$, then $\mathcal{J}_{\gamma, f}$ is not a taming map, so it does not define a local abelian gauge theory. From a different point of view, such a transformation would not preserve the energy momentum tensor of the theory and its Lagrangian formulation. See [41] and references therein for more details about this point.

Proof. For any $\mathcal{V} \in \Omega_{\text{cl}}^2(M, \mathbb{R}^{2n})$, we have:

$$\begin{aligned} *_g \mathcal{V} = -\mathcal{J} \mathcal{V} &\iff \mathbb{A}_{\gamma, f}(*_g \mathcal{V}) = -\mathbb{A}_{\gamma, f}(\mathcal{J} \mathcal{V}) \\ &\iff *_f(g)(\mathbb{A}_{\gamma, f} \mathcal{V}) = -\mathcal{J}_{\gamma, f} \mathbb{A}_{\gamma, f}(\mathcal{V}). \end{aligned}$$

□

Consider the infinite rank vector bundle with total space:

$$\text{Sol}_n(M) \stackrel{\text{def.}}{=} \coprod_{(g, \mathcal{J}) \in \text{Met}_{3,1}(M) \times \mathfrak{J}_n(M)} \text{Sol}_n(M, g, \mathcal{J}),$$

and infinite-dimensional base $B_n(M) \stackrel{\text{def.}}{=} \text{Met}_{3,1}(M) \times \mathfrak{J}_n(M)$, with the natural projection. Let σ be the action of $\text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)$ on $B_n(M)$ defined though $\sigma = f_* \times \mu$, i.e.:

$$\sigma(\gamma, f)(g, \mathcal{J}) = (f_*(g), \mathcal{J}_{\gamma, f}).$$

Then Proposition A.10 shows that the restriction of \mathbb{A} gives a linearization of σ on the vector bundle $\text{Sol}_n(M)$. In particular, each fiber of $\text{Sol}_n(M)$ carries

a linear representation of the isotropy group of the corresponding point in the base. Let $\text{Iso}(M, g)$ be the group of orientation-preserving isometries of (M, g) . Then:

$$\text{Stab}_{\text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)}(g, \mathcal{J}) = \{(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M, g) \mid \mathcal{J}_{\gamma, f} = \mathcal{J}\}$$

and we have:

Corollary A.12. *Let $(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M, g)$ such that $\mathcal{J}_{\gamma, f} = \mathcal{J}$, i.e. $\mathcal{J} \circ f = \gamma \mathcal{J} \gamma^{-1}$. Then $\mathbb{A}_{\gamma, f}$ is a linear automorphism of $\text{Sol}_n(M, g, \mathcal{J})$.*

Definition A.13. Let $\mathcal{J} \in \mathfrak{J}_n(M)$ be a taming map and g be a Lorentzian metric on M .

- The group $\text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)$ is called the *unbased pseudo-duality group*. The linear isomorphism:

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, f_*(g), \mathcal{J}_{\gamma, f}),$$

induced by an element of this group is called a *unbased pseudo-duality transformation*.

- The group $\text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M, g)$ is called the *unbased duality group*. The linear isomorphism:

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, g, \mathcal{J}_{\gamma, f}),$$

induced by an element (γ, f) of this group is called a *unbased duality transformation*.

- The group $\text{Sp}(2n, \mathbb{R})$ is called the *duality group*. The linear isomorphism:

$$\mathbb{A}_{\gamma, \text{id}_M}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, g, \mathcal{J}_\gamma),$$

where $\mathcal{J}_\gamma \stackrel{\text{def.}}{=} \mathcal{J}_{\gamma, \text{id}_M} = \gamma \mathcal{J} \gamma^{-1}$, is called a *classical duality transformation*.

Definition A.14. Let $\mathcal{J} \in \mathfrak{J}_n(M)$ be a taming map and g be a Lorentzian metric on M .

- The stabilizer:

$$(A.11) \quad \mathcal{U}(M, \mathcal{J}) \stackrel{\text{def.}}{=} \{(\gamma, f) \in \text{Diff}(M, g) \times \text{Sp}(2n, \mathbb{R}) \mid \mathcal{J} \circ f = \gamma \mathcal{J} \gamma^{-1}\},$$

of \mathcal{J} in $\text{Sp}(2n, \mathbb{R}) \times \text{Diff}(M)$ with respect to the representation μ is called the *unbased unitary pseudo-duality group*. The linear isomorphism:

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, f_*(g), \mathcal{J}),$$

induced by an element of this group is called an *unbased unitary pseudo-duality transformation*.

- The stabilizer:

$$(A.12) \quad \mathcal{U}(M, g, \mathcal{J}) \stackrel{\text{def.}}{=} \{(\gamma, f) \in \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M, g) \mid \mathcal{J} \circ f = \gamma \mathcal{J} \gamma^{-1}\},$$

of \mathcal{J} in $\text{Sp}(2n, \mathbb{R}) \times \text{Iso}(M)$ with respect to the representation μ is called the *unbased unitary duality group*. The linear isomorphism:

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, g, \mathcal{J}),$$

induced by an element of this group is called an *unbased unitary duality transformation*.

- The stabilizer:

$$\mathcal{U}_{\mathcal{J}}(n) \stackrel{\text{def.}}{=} \{\gamma \in \text{Sp}(2n, \mathbb{R}) \mid \gamma \mathcal{J} \gamma^{-1} = \mathcal{J}\}.$$

of \mathcal{J} in $\text{Sp}(2n, \mathbb{R})$ with respect to the action $\mathcal{J} \rightarrow \gamma \mathcal{J} \gamma^{-1}$ is called the *unitary duality group*. The linear automorphism:

$$\mathbb{A}_{\gamma, f}: \text{Sol}_n(M, g, \mathcal{J}) \xrightarrow{\sim} \text{Sol}_n(M, g, \mathcal{J})$$

of $\text{Sol}_n(M, g, \mathcal{J})$ induced by an element of this group is called a *unitary duality transformation*.

We have inclusions:

$$U_{\mathcal{J}}(n) \subset \mathcal{U}(M, g, \mathcal{J}) \subset \mathcal{U}(M, \mathcal{J})$$

and short exact sequences:

$$(A.13) \quad 1 \rightarrow U_{\mathcal{J}}(n) \rightarrow \mathcal{U}(M, g, \mathcal{J}) \rightarrow \text{Iso}_{\mathcal{J}}(M, g) \rightarrow 1,$$

$$(A.14) \quad 1 \rightarrow \text{Iso}(M, g)_{\mathcal{J}} \rightarrow \mathcal{U}(M, \mathcal{J}) \rightarrow \text{Sp}_{\mathcal{J}}(2n, \mathbb{R}) \rightarrow 1,$$

where $\text{Iso}_{\mathcal{J}}(M, g)$ is the subgroup of those $f \in \text{Iso}(M, g)$ for which there exists $\gamma \in \text{Sp}(n, \mathbb{R})$ such that $\mathcal{J} \circ f = \gamma \mathcal{J} \gamma^{-1}$, while $\text{Sp}_{\mathcal{J}}(2n, \mathbb{R})$ is the subgroup of those $\gamma \in \text{Sp}(2n, \mathbb{R})$ for which there exists $f \in \text{Iso}(M, g)$ such that $\mathcal{J} \circ f = \gamma \mathcal{J} \gamma^{-1}$. Finally, the group:

$$\text{Iso}(M, g)_{\mathcal{J}} \stackrel{\text{def.}}{=} \{f \in \text{Iso}(M, g) \mid \mathcal{J} \circ f = \mathcal{J}\}$$

is the stabilizer of \mathcal{J} in $\text{Iso}(M, g)$. In particular, we have:

Corollary A.15. *If $U_{\mathcal{J}}(n) = 1$ then $\mathcal{U}(M, g, \mathcal{J}) = \text{Iso}_{\mathcal{J}}(M, g)$. If $\text{Iso}(M, g)_{\mathcal{J}} = 1$ then $\mathcal{U}(M, g, \mathcal{J}) = \text{Sp}_{\mathcal{J}}(2n, \mathbb{R})$.*

A.3. Gluing local abelian gauge theories

Let now M be an arbitrary oriented manifold admitting Lorentzian metrics. Let $\mathcal{U} = (U_{\alpha})_{\alpha \in I}$ be a good open cover of M , where I is an index set. Denote by g_{α} the restriction of g to U_{α} . Roughly speaking, the definition of abelian gauge theory on (M, g) given in Section 1 is the result of gluing the local $U(1)^n$ abelian gauge theories defined on the contractible Lorentzian four-manifolds (U_{α}, g_{α}) using electromagnetic dualities. In order to implement this idea, we choose a locally constant $\text{Sp}(2n, \mathbb{R})$ -valued Čech cocycle for \mathcal{U} :

$$u_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \rightarrow \text{Sp}(2n, \mathbb{R})$$

and a family of taming maps $\mathcal{J}_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{2n}$ for ω_{2n} such that:

$$\mathcal{J}_{\beta} = u_{\alpha\beta} \mathcal{J}_{\alpha} u_{\alpha\beta}^{-1}$$

on double overlaps. The collection:

$$\{(U_{\alpha})_{\alpha \in I}, (g_{\alpha})_{\alpha \in I}, (\mathcal{J}_{\alpha})_{\alpha \in I}, (u_{\alpha\beta})_{\alpha, \beta \in I}\},$$

is equivalent to a flat symplectic vector bundle $(\mathcal{S}, \omega, \mathcal{D})$ with symplectic form ω and symplectic flat connection \mathcal{D} , equipped with an almost complex

structure \mathcal{J} which is a taming of ω . A family $(\mathcal{V}_\alpha)_{\alpha \in I}$ of solutions of the local abelian gauge theories defined by $(\mathcal{J}_\alpha)_{\alpha \in I}$ on (U_α, g_α) which satisfies:

$$\mathcal{V}_\beta = u_{\alpha\beta} \mathcal{V}_\alpha$$

corresponds to an \mathcal{S} -valued two-form $\mathcal{V} \in \Omega^2(M, \mathcal{S})$ which obeys:

$$d_{\mathcal{D}} \mathcal{V} = 0,$$

where $d_{\mathcal{D}}: \Omega^*(M, \mathcal{S}) \rightarrow \Omega^*(M, \mathcal{S})$ is the exterior differential twisted by \mathcal{D} . This construction motivates the global geometric model introduced in [32] and further elaborated in Section 1.

Appendix B. Integral symplectic spaces and integral symplectic tori

This appendix recalls some notions from the theory of symplectic lattices and symplectic tori which are used throughout the paper. We also introduce the notion of integral symplectic torus.

Definition B.1. An integral symplectic space is a triple (V, ω, Λ) such that:

- (V, ω) is a finite-dimensional symplectic vector space over \mathbb{R} .
- $\Lambda \subset V$ is full lattice in V , i.e. a lattice in V such that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.
- ω is integral with respect to Λ , i.e. we have $\omega(\Lambda, \Lambda) \subset \mathbb{Z}$.

An isomorphism of integral symplectic spaces $f: (V_1, \omega_1, \Lambda_1) \rightarrow (V_2, \omega_2, \Lambda_2)$ is a bijective symplectomorphism from (V_1, ω_1) to (V_2, ω_2) which satisfies:

$$f(\Lambda_1) = \Lambda_2.$$

Denote by $\text{Symp}_{\mathbb{Z}}$ the groupoid of integral symplectic spaces and isomorphisms of such. Let $\text{Aut}(V)$ be the group of linear automorphisms of the vector space V and $\text{Sp}(V, \omega) \subset \text{Aut}(V)$ be the subgroup of symplectic transformations. Then the automorphism group of the integral symplectic space (V, ω, Λ) is denoted by:

$$\text{Sp}(V, \omega, \Lambda) = \{T \in \text{Sp}(V, \omega) \mid T(\Lambda) = \Lambda\}.$$

Definition B.2. An *integral symplectic basis* of a $2n$ -dimensional integral symplectic space (V, ω, Λ) is a basis $\mathcal{E} = (\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n)$ of Λ (as a free

\mathbb{Z} -module) such that:

$$\begin{aligned} \omega(\xi_i, \xi_j) &= \omega(\zeta_i, \zeta_j) = 0, \\ \omega(\xi_i, \zeta_j) &= t_i \delta_{ij}, \quad \omega(\zeta_i, \xi_j) = -t_i \delta_{ij}, \quad \forall i, j = 1, \dots, n, \end{aligned}$$

where $t_1, \dots, t_n \in \mathbb{Z}$ are strictly positive integers satisfying the divisibility conditions:

$$t_1 | t_2 | \dots | t_n.$$

By the *elementary divisor theorem*, see [19, Chapter VI], every integral symplectic space admits an integral symplectic basis and the positive integers t_1, \dots, t_n (which are called the *elementary divisors* of (V, ω, Λ)) do not depend on the choice of such a basis. Define:

$$\text{Div}^n \stackrel{\text{def.}}{=} \{ (t_1, \dots, t_n) \in \mathbb{Z}_{>0}^n \mid t_1 | t_2 | \dots | t_n \},$$

and:

$$\delta(n) \stackrel{\text{def.}}{=} (1, \dots, 1) \in \text{Div}^n.$$

Let \leq be the partial order relation on Div^n defined through:

$$(t_1, \dots, t_n) \leq (t'_1, \dots, t'_n) \text{ iff } t_i | t'_i \forall i = 1, \dots, n.$$

Then $\delta(n)$ is the least element of the ordered set (Div^n, \leq) . Notice that this ordered set is directed, since any two elements $t, t' \in \text{Div}^n$ have an upper bound given by $(t_1 t'_1, \dots, t_n t'_n)$. In fact, (Div^n, \leq) is a lattice with join and meet given by:

$$t \vee t' = (\text{lcm}(t_1, t'_1), \dots, \text{lcm}(t_n, t'_n)), \quad t \wedge t' = (\text{gcd}(t_1, t'_1), \dots, \text{gcd}(t_n, t'_n)).$$

This lattice is semi-bounded from below with bottom element given by $\delta(n)$ and it is complete for meets (i.e., it is a complete meet semi-lattice).

Definition B.3. The *type* of an integral symplectic space (V, ω, Λ) is the ordered system of elementary divisors of (V, ω, Λ) , which we denote by:

$$\mathfrak{t}(V, \omega, \Lambda) = (t_1, \dots, t_n) \in \text{Div}^n.$$

The integral symplectic space (V, ω, Λ) is called *principal* if:

$$\mathfrak{t}(V, \omega, \Lambda) = \delta(n) \in \text{Div}^n.$$

Let ω_{2n} denotes the standard symplectic pairing on \mathbb{R}^{2n} .

Proposition B.4. *Two integral symplectic spaces have the same type if and only if they are isomorphic. Moreover, every element of Div^n is the type of an integral symplectic space. Hence the type induces a bijection between the set of isomorphism classes of integral symplectic spaces and the set Div^n .*

Proof. The first statement is obvious. For the second statement, fix $\mathfrak{t} \stackrel{\text{def.}}{=} (t_1, \dots, t_n) \in \text{Div}^n$. Consider the full lattice $\Lambda_{\mathfrak{t}} \subseteq \mathbb{R}^{2n}$ defined as follows:

$$(B.15) \quad \Lambda_{\mathfrak{t}} \stackrel{\text{def.}}{=} \{(l_1, \dots, l_n, t_1 l_{n+1}, \dots, t_n l_{2n}) \mid l_1, \dots, l_{2n} \in \mathbb{Z}\} .$$

Then $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ is an integral symplectic space of type \mathfrak{t} . □

Definition B.5. The lattice $\Lambda_{\mathfrak{t}}$ defined in (B.15) is called the *standard symplectic lattice* of type \mathfrak{t} and $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ is called the *standard integral symplectic space* of type \mathfrak{t} .

We have $\Lambda_{\delta(n)} = \mathbb{Z}^{2n}$. Moreover, $\Lambda_{\mathfrak{t}}$ is a sub-lattice of \mathbb{Z}^{2n} and we have $\mathbb{Z}^{2n}/\Lambda_{\mathfrak{t}} \simeq \mathbb{Z}_{t_1} \times \dots \times \mathbb{Z}_{t_n}$ for all $\mathfrak{t} \in \text{Div}(n)$. For $\mathfrak{t}, \mathfrak{t}' \in \text{Div}^n$, we have $\Lambda_{\mathfrak{t}'} \subset \Lambda_{\mathfrak{t}}$ if and only if $\mathfrak{t} \leq \mathfrak{t}'$. The lattice $\Lambda_{\mathfrak{t}}$ admits the basis:

$$\begin{aligned} \xi_1 = e_1 = (1, 0, \dots, 0), \dots, \xi_n = e_n = (0, \dots, 0, 1, 0, \dots, 0) \\ \zeta_1 = t_1 f_1 = (0, \dots, 0, t_1, 0, \dots, 0), \dots, \zeta_n = t_n f_n = (0, \dots, 0, t_n), \end{aligned}$$

in which the standard symplectic form of \mathbb{R}^{2n} has coefficients:

$$\begin{aligned} \omega_{2n}(\xi_i, \xi_j) = \omega_{2n}(\zeta_i, \zeta_j) = 0 \\ \omega_{2n}(\xi_i, \zeta_j) = t_i \delta_{ij}, \quad \omega_{2n}(\zeta_i, \xi_j) = -t_i \delta_{ij} . \end{aligned}$$

The isomorphism which takes ξ_i to e_i and ζ_j to f_j identifies $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ with the integral symplectic space $(\mathbb{R}^{2n}, \omega_{\mathfrak{t}}, \mathbb{Z}^{2n})$, where $\omega_{\mathfrak{t}}$ is the symplectic pairing defined on \mathbb{R}^{2n} by:

$$(B.16) \quad \begin{aligned} \omega_{\mathfrak{t}}(e_i, e_j) = \omega_{\mathfrak{t}}(f_i, f_j) = 0, \\ \omega_{\mathfrak{t}}(e_i, f_j) = \delta_{ij}, \quad \omega_{\mathfrak{t}}(f_i, e_j) = -\delta_{ij}, \quad \forall i = 1, \dots, n. \end{aligned}$$

Given $\mathfrak{t} = (t_1, \dots, t_n) \in \text{Div}^n$, consider the diagonal $n \times n$ matrix:

$$D_{\mathfrak{t}} \stackrel{\text{def.}}{=} \text{diag}(t_1, \dots, t_n) \in \text{Mat}(n, \mathbb{Z}),$$

as well as:

$$\Gamma_{\mathfrak{t}} \stackrel{\text{def.}}{=} \begin{pmatrix} \mathbf{I}_n & 0 \\ 0 & \mathbf{D}_{\mathfrak{t}} \end{pmatrix} \in \text{Mat}(2n, \mathbb{Z}).$$

Definition B.6. The *modified Siegel modular group* of type $\mathfrak{t} \in \text{Div}^n$ is the subgroup of $\text{Aut}(\mathbb{R}^{2n}, \omega_{2n}) \simeq \text{Sp}(2n, \mathbb{R})$ defined through:

$$\begin{aligned} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) &\stackrel{\text{def.}}{=} \{T \in \text{Aut}(\mathbb{R}^{2n}, \omega_{2n}) \mid T(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{t}}\} \\ &\simeq \{T \in \text{Sp}(2n, \mathbb{R}) \mid \Gamma_{\mathfrak{t}} T \Gamma_{\mathfrak{t}}^{-1} = T\}. \end{aligned}$$

Since $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}}) \simeq (\mathbb{R}^{2n}, \omega_{\mathfrak{t}}, \mathbb{Z}^{2n})$, we have

$$\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \simeq \text{Aut}(\mathbb{R}^{2n}, \omega_{\mathfrak{t}}, \mathbb{Z}^{2n}).$$

Hence $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ is a subgroup of $\text{GL}(2n, \mathbb{Z})$. The remarks above give:

Proposition B.7. [32, Proposition F.12] *Let (V, ω, Λ) be an integral symplectic space of dimension $2n$. Any integral symplectic basis of this space induces an isomorphism of integral symplectic spaces between (V, ω, Λ) and $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_{\mathfrak{t}})$ as well as an isomorphism of groups between $\text{Sp}(V, \omega, \Lambda)$ and $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$.*

We have $\text{Sp}_{\delta(n)}(2n, \mathbb{Z}) = \text{Sp}(2n, \mathbb{Z})$ and $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subseteq \text{Sp}_{\mathfrak{t}'}(2n, \mathbb{Z})$ when $\mathfrak{t} \leq \mathfrak{t}'$. Hence $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ forms a direct system of groups and we have $\text{Sp}(2n, \mathbb{Z}) \subseteq \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ for all $\mathfrak{t} \in \text{Div}^n$. The direct limit $\varinjlim_{\mathfrak{t} \in \text{Div}^n} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ identifies with the following subgroup of $\text{Sp}(2n, \mathbb{R})$:

$$\text{Sp}_{\infty}(2n, \mathbb{Z}) \stackrel{\text{def.}}{=} \{T \in \text{GL}(2n, \mathbb{R}) \mid \exists \mathfrak{t}_T \in \text{Div}^n : T \in \text{Sp}_{\mathfrak{t}_T}(2n, \mathbb{Z})\},$$

through the isomorphism of groups $\varphi : \text{Sp}_{\infty}(2n, \mathbb{Z}) \rightarrow \varinjlim_{\mathfrak{t} \in \text{Div}^n} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ which sends $T \in \text{Sp}_{\infty}(2n, \mathbb{Z})$ to the equivalence class $[\alpha(T)] \in \varinjlim_{\mathfrak{t} \in \text{Div}^n} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ of the family $\alpha(T) \in \sqcup_{\mathfrak{t} \in \text{Div}^n} \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ defined through:

$$\alpha(T)_{\mathfrak{t}} \stackrel{\text{def.}}{=} \begin{cases} T & \text{if } \mathfrak{t}_T \leq \mathfrak{t} \\ 1 & \text{if } \mathfrak{t}_T \not\leq \mathfrak{t} \end{cases}.$$

Notice that $\text{Sp}(2n, \mathbb{Z}) = \text{Sp}_{\delta(n)}(2n, \mathbb{Z})$ is a subgroup of $\text{Sp}_{\infty}(2n, \mathbb{Z})$.

Definition B.8. The *type* of an element $T \in \text{Sp}_{\infty}(2n, \mathbb{Z})$ is defined as the greatest lower bound $\mathfrak{t}(T) \in \text{Div}^n$ of the finite set $\{\mathfrak{t} \in \text{Div}^n \mid \mathfrak{t} \geq \mathfrak{t}_T\}$, where \mathfrak{t}_T is any element of Div^n such that $T \in \text{Sp}_{\mathfrak{t}_T}(2n, \mathbb{Z})$.

Notice that the type of T is well-defined and that we have $T \in \text{Sp}_{\mathfrak{t}(T)}(2n, \mathbb{Z})$.

B.1. Integral symplectic tori

The following definition distinguishes between a few closely related notions.

Definition B.9. A d -dimensional *torus* is a smooth manifold T diffeomorphic with the *standard d -torus* $T^d \stackrel{\text{def.}}{=} (S^1)^d$. A d -dimensional *torus group* is a compact abelian Lie group A which is isomorphic with the *standard d -dimensional torus group* $U(1)^d$ as a Lie group. A d -dimensional *affine torus* is a principal homogeneous space \mathbb{A} for a d -dimensional torus group. The *standard affine d -torus* is the d -dimensional affine torus \mathbb{A}_d defined by the right action of $U(1)^d$ on itself.

The underlying manifold of the standard d -dimensional torus group is the standard d -torus while the underlying manifold of a d -dimensional torus group is a d -torus. Moreover, any d -dimensional affine torus group is isomorphic with a standard affine d -torus. The transformations of an affine torus given the right action of its underlying group will be called *translations*. Choosing a distinguished point in any affine torus makes it into a torus group having that point as zero element. Conversely, any torus group defines an affine torus obtained by ‘forgetting’ its zero element. The singular homology and cohomology groups of a d -torus T are the free abelian groups given by:

$$\begin{aligned} H_k(T, \mathbb{Z}) &\simeq \wedge^k H_1(T, \mathbb{Z}) \\ H^k(T, \mathbb{Z}) &\simeq \wedge^k H^1(T, \mathbb{Z}) \simeq \wedge^k H_1(T, \mathbb{Z})^\vee \end{aligned}$$

for all $k = 0, \dots, d$, where $H_1(T, \mathbb{Z}) \simeq H^1(T, \mathbb{Z}) \simeq \mathbb{Z}^d$. The underlying torus group of any affine torus \mathbb{A} is isomorphic with $A \stackrel{\text{def.}}{=} H_1(\mathbb{A}, \mathbb{R})/H_1(\mathbb{A}, \mathbb{Z})$. The group of automorphisms of a d -dimensional torus group A is given by:

$$\text{Aut}(A) = \text{Aut}(H_1(A, \mathbb{R}), H_1(A, \mathbb{Z})) \simeq \text{GL}(d, \mathbb{Z}).$$

Note that the group of automorphisms of the d -dimensional affine torus is isomorphic to $U(1)^d \rtimes \text{Aut}(U(1)^d) \simeq U(1)^d \rtimes \text{GL}(2n, \mathbb{Z})$, where $\text{GL}(2n, \mathbb{Z})$ acts on $U(1)^d$ through the morphism of groups $\rho: \text{GL}(2n, \mathbb{Z}) \rightarrow \text{Aut}(A)$ given by:

$$(B.17) \quad \rho(T)(\exp(2\pi i x)) = \exp(2\pi i T(x)), \quad \forall T \in \text{GL}(d, \mathbb{Z}), \quad \forall x \in \mathbb{R}^d.$$

Here $\exp: \mathbb{R}^d \rightarrow U(1)^d$ is the exponential map of $U(1)^d$, which is given by:

$$\exp(v) = (e^{v_1}, \dots, e^{v_d}), \quad \forall v = (v_1, \dots, v_d) \in \mathbb{R}^d.$$

Definition B.10. Let T be a torus of dimension at least two. A subtorus $T' \subset T$ is called *primitive* if $H_1(T', \mathbb{Z})$ is a primitive sub-lattice of $H_1(T, \mathbb{Z})$, i.e. if the abelian group $H_1(T, \mathbb{Z})/H_1(T', \mathbb{Z})$ is torsion-free.

Definition B.11. A *symplectic torus* is a pair $\mathbf{T} = (T, \Omega)$, where T is an even-dimensional torus and Ω is a symplectic form defined on T . A *symplectic torus group* is a pair $\mathbf{A} = (A, \Omega)$, where A is an even-dimensional torus group and Ω is a symplectic form defined on the underlying torus which is invariant under translations by all elements of A . An *affine symplectic torus* is a pair $\mathbb{A} = (\mathbb{A}, \Omega)$, where \mathbb{A} is an even-dimensional affine torus and Ω is a symplectic form on \mathbb{A} which is invariant under translations.

Definition B.12. A symplectic torus $\mathbf{T} = (T, \Omega)$ is called *integral* if the symplectic area $\int_T \Omega$ of any of its primitive two-dimensional subtori T' is an integer.

Let (T, Ω) be a symplectic torus. The cohomology class of Ω is a non-degenerate element $\omega \in H^2(T, \mathbb{R}) \simeq \wedge^2 H_1(T, \mathbb{R})^\vee$, i.e. a symplectic pairing on the vector space $H_1(T, \mathbb{R})$. The symplectic torus (T, Ω) is integral if and only if the triplet $(H_1(T, \mathbb{R}), H_1(T, \mathbb{Z}), \omega)$ is an integral symplectic space. In this case, ω descends to a symplectic form $\hat{\Omega}$ which makes $H_1(T, \mathbb{R})/H_1(T, \mathbb{Z})$ into an integral symplectic torus group. If $\mathbb{A} = (\mathbb{A}, \Omega)$ is an integral affine symplectic torus, then Ω is determined by its cohomology class ω , hence \mathbb{A} can also be viewed as a pair (\mathbb{A}, ω) where \mathbb{A} is an affine torus and ω is a symplectic form on $H_1(\mathbb{A}, \mathbb{R})$ which is integral for the lattice $H_1(\mathbb{A}, \mathbb{Z})$. In this case, any choice of a point in \mathbb{T} allows us to identify \mathbb{A} with the integral symplectic torus group $(H_1(\mathbb{A}, \mathbb{R})/H_1(\mathbb{A}, \mathbb{Z}), \hat{\Omega})$.

Let $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_t)$ be the standard integral symplectic space of type $t \in \text{Div}^n$ and Ω_t be the translationally invariant symplectic form induced by ω_{2n} on the torus group $\mathbb{R}^{2n}/\Lambda_t$. Then the symplectic torus group $(\mathbb{R}^{2n}/\Lambda_t, \Omega_t)$ is integral.

Definition B.13. The $2n$ -dimensional *standard integral symplectic torus group* of type $t \in \text{Div}^n$ is:

$$\mathbf{A}_t \stackrel{\text{def.}}{=} (\mathbb{R}^{2n}/\Lambda_t, \Omega_t) .$$

The integral affine symplectic torus \mathbb{A}_t obtained from \mathbf{A}_t by forgetting the origin is the *standard integral affine symplectic torus* of type t .

Note that every integral affine symplectic torus \mathbb{A} is affinely symplectomorphic to a standard affine symplectic torus $\mathbb{A}_{\mathfrak{t}}$, whose type \mathfrak{t} is uniquely-determined and called the *type* of \mathbb{A} . Similarly, every integral symplectic torus group \mathbf{A} is isomorphic with a standard integral symplectic torus group $\mathbf{A}_{\mathfrak{t}}$ whose type \mathfrak{t} is uniquely determined by \mathbf{A} and called the *type* of \mathbf{A} . The group of automorphisms of $\mathbf{A}_{\mathfrak{t}}$ for $\mathfrak{t} \in \text{Div}^n$ is given by:

$$\text{Aut}(\mathbf{A}_{\mathfrak{t}}) = \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}),$$

while the group of automorphisms of an integral symplectic affine torus $\mathbb{A}_{\mathfrak{t}} = (\mathbb{A}_{\mathfrak{t}}, \Omega)$ of type $\mathfrak{t} \in \text{Div}^n$ is:

$$\text{Aut}(\mathbb{A}_{\mathfrak{t}}) = A \rtimes \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}),$$

where $A = H_1(\mathbb{A}, \mathbb{R})/H_1(\mathbb{A}, \mathbb{Z})$ is the underlying torus group of \mathbb{A} and the action of $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subset \text{GL}(2n, \mathbb{Z}) \simeq \text{Aut}(A)$ on A coincides with that induced from the action on $H_1(\mathbb{A}, \mathbb{R}) \simeq \mathbb{R}^{2n}$. We denote by $\text{Aff}_{\mathfrak{t}} \stackrel{\text{def.}}{=} \text{Aut}(\mathbb{A}_{\mathfrak{t}})$ the automorphism group of the integral affine symplectic torus of type \mathfrak{t} . We have:

$$\text{Aff}_{\mathfrak{t}} = \text{U}(1)^{2n} \rtimes \text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}),$$

where $\text{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \subset \text{GL}(2n, \mathbb{Z})$ acts on $\text{U}(1)^{2n}$ through the restriction of (B.17).

Definition B.14. Let $\mathbf{A} = (A, \Omega)$ and $\mathbf{A}' = (A', \Omega')$ be two integral symplectic torus groups. A *symplectic isogeny* from \mathbf{A} to \mathbf{A}' is a surjective morphism of groups $f : \mathbf{A} \rightarrow \mathbf{A}'$ with finite kernel such that $f^*(\Omega') = \Omega$.

The following statement is immediate.

Proposition B.15. Let $\mathfrak{t}, \mathfrak{t}' \in \text{Div}^n$ be such that $\mathfrak{t} \leq \mathfrak{t}'$, namely $t'_i = q_i t_i$ (where $q_i \in \mathbb{Z}_{>0}$) for all $i = 1, \dots, n$. Then the map $f : \mathbf{A}_{\mathfrak{t}'} \rightarrow \mathbf{A}_{\mathfrak{t}}$ defined through:

$$f(x + \Lambda_{\mathfrak{t}'}) = x + \Lambda_{\mathfrak{t}} \forall x \in \mathbb{R}^{2n}$$

is a symplectic isogeny whose kernel is given by:

$$\ker(f) \simeq \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_n}.$$

In particular, $\mathbf{A}_{\mathfrak{t}}$ is isogenous with $\mathbf{A}_{\delta(n)}$ for all $\mathfrak{t} \in \text{Div}^n$.

B.2. Tamings

Definition B.16. A *tamed integral symplectic space* is a quadruple (V, ω, Λ, J) , where (V, ω, Λ) is an integral symplectic space and J is a taming of the symplectic space (V, ω) . The *type* of a tamed integral symplectic space is the type of its underlying integral symplectic space.

Given a tamed integral symplectic space (V, ω, Λ, J) of type $\mathfrak{t} \in \text{Div}^n$, the taming J makes V into a n -dimensional complex vector space, which we denote by V_J . The symplectic pairing ω induces a Kähler form Ω which makes the complex torus V_J/Λ into a (generally non-principal) polarized abelian variety whose underlying symplectic torus coincides with $\mathbf{A}_{\mathfrak{t}}$. We refer the reader to [32, Appendix F] for details on the relation between tamed integral symplectic spaces and (generally non-principal) abelian varieties.

References

- [1] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Phys. Rev. **115** (1959), 485–491.
- [2] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre and T. Magri, *$N=2$ supergravity and $N=2$ superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23** (1997), 111–189.
- [3] L. Andrianopoli, R. D’Auria and S. Ferrara, *U duality and central charges in various dimensions revisited*, Int. J. Mod. Phys. A **13** (1998), 431–490.
- [4] P. C. Argyres and M. Martone, *$4d \mathcal{N} = 2$ theories with disconnected gauge groups*, JHEP **03** (2017), 145.
- [5] P. C. Argyres, A. Bourget and M. Martone, *On the moduli spaces of $4d \mathcal{N} = 3$ SCFTs I: triple special Kähler structure*, preprint, arXiv:1912.04926.
- [6] P. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part I: physical constraints on relevant deformations*, JHEP **02** (2018), 001.
- [7] P. C. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part II: construction of special Kähler geometries and RG flows*, JHEP **02** (2018), 002.

- [8] P. C. Argyres and M. Martone, *Towards a classification of rank r $\mathcal{N} = 2$ SCFTs Part II: special Kahler stratification of the Coulomb branch*, preprint, [arXiv:2007.00012](https://arxiv.org/abs/2007.00012).
- [9] P. Aschieri, S. Ferrara and B. Zumino, *Duality Rotations in Nonlinear Electrodynamics and in Extended Supergravity*, Riv. Nuovo Cim. **31** (2008), 625–708.
- [10] M. F. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton Legacy Library, Porter Lectures, 1988.
- [11] D. Baraglia, *Topological T-duality for general circle bundles*, Pure Appl. Math. Q. **10** (2014), 3, 367–438.
- [12] D. Baraglia, *Topological T-duality for torus bundles with monodromy*, Rev. Math. Phys. **27** (2015), 3, 1550008.
- [13] C. Becker, M. Benini, A. Schenkel and R. J. Szabo, *Abelian duality on globally hyperbolic spacetimes*, Commun. Math. Phys. **349** (2017), no. 1, 361–392.
- [14] M. Caorsi and S. Cecotti, *Geometric classification of $4d$ $\mathcal{N} = 2$ SCFTs*, JHEP **07** (2018), 138.
- [15] M. Caorsi and S. Cecotti, *Homological classification of $4d$ $\mathcal{N} = 2$ QFT. Rank-1 revisited*, JHEP **10** (2019), 013.
- [16] S. Cecotti, *Supersymmetric Field Theories: Geometric Structures and Dualities*, Cambridge, 2015.
- [17] I. Chatterji, J. Mislin, C. Pittet, *Flat bundles with complex analytic holonomy*, Quart. J. Math. **00** (2016), 1–13.
- [18] V. Cortés, C. I. Lazaroiu and C. S. Shahbazi, *$\mathcal{N} = 1$ Geometric Supergravity and chiral triples on Riemann surfaces*, Commun. Math. Phys. **375** (2020), 429–478.
- [19] O. Debarre, *Tores et variétés abéliennes complexes*, EDP Sciences, 2000.
- [20] P. Deligne, *Extensions centrales non résiduellement finies de groupes arithmétiques*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), A203–A208.
- [21] S. Deser and C. Teitelboim, *Duality Transformations of abelian and Nonabelian Gauge Fields*, Phys. Rev. D **13** (1976), 1592–1597.
- [22] S. Deser, *Off-Shell Electromagnetic Duality Invariance*, J. Phys. A **15** (1982), 1053–1054.

- [23] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc. Lond. A **133** (1931), 821, 60–72.
- [24] S. K. Donaldson, P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs, 1997.
- [25] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, Nucl. Phys. B **193** (1981), 221–244.
- [26] S. Eilenberg, *Homology of Spaces with Operators I*, Trans. AMS **61** (1947), 378–417.
- [27] S. Gitler, *Cohomology operations with local coefficients*, Amer. J. Math, **85** (1963), 2, 156–188.
- [28] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.
- [29] A. Hatcher, *Homology with local coefficients and characteristic classes*, Homology, Homotopy and Applications, **8**(2) (2006), 91–103.
- [30] D. Husemöller, *Fiber bundles*, Graduate Texts in Mathematics, 3rd ed., Springer, 1994.
- [31] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, 1995.
- [32] C. I. Lazaroiu and C. S. Shahbazi, *Generalized Einstein-Scalar-Maxwell theories and locally geometric U-folds*, Rev. Math. Phys. Vol. 30, No. 05, 1850012 (2018).
- [33] C. I. Lazaroiu and C. S. Shahbazi, *Section sigma models coupled to symplectic duality bundles on Lorentzian four-manifolds*, J. Geom. Phys. **128** (2018), 58–86.
- [34] C. I. Lazaroiu and C. S. Shahbazi, *The global formulation of generalized Einstein-Scalar-Maxwell theories*, Springer Proceedings in Mathematics & Statistics, Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics, vol. 2 (2018), 217–231
- [35] C. I. Lazaroiu, C. S. Shahbazi, *The classification of weakly abelian principal bundles*, in preparation.
- [36] C. I. Lazaroiu, E. M. Babalic, I.-A. Coman, *Geometric algebra techniques in flux compactifications*, Adv. High Energy Phys. 2016, 7292534.
- [37] W. S. Massey, *Obstructions to the existence of almost complex structures*, Bull. Amer. Math. Soc. vol. 67 **6** (1961), 559–564.

- [38] P. W. Michor, *Topics in differential geometry*, A.M.S., 2008.
- [39] J. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223.
- [40] J. Millson, *Real vector bundles with discrete structure group*, Elsevier, NY, Topology **18** (1979), 83–89.
- [41] D. I. Olive, *Exact electromagnetic duality*, Nucl. Phys. Proc. Suppl. **45A** (1996), 88–102.
- [42] J. S. Schwinger, *Magnetic charge and quantum field theory*, Phys. Rev. **144** (1966), 1087–1093.
- [43] E. H. Spanier, *Algebraic Topology*, Springer, 1966.
- [44] N. E. Steenrod, *Homology with Local Coefficients*, Annals of Math. **44** (1943), 610–627.
- [45] R. J. Szabo, *Quantization of Higher abelian Gauge Theory in Generalized Differential Cohomology*, PoS ICMP2012 (2012), 9–67.
- [46] T. Szamuely, *Galois groups and fundamental groups*, Cambridge U.P., 2009.
- [47] D. Zwanziger, *Quantum field theory of particles with both electric and magnetic charges*, Phys. Rev. **176** (1968), 1489–1495.
- [48] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics **61**, Springer, 1978.

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