

Homological mirror symmetry of \mathbb{F}_1 via Morse homotopy

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This is a sequel to our paper [11], where we proposed a definition of the Morse homotopy of the moment polytope of toric manifolds. Using this as the substitute of the Fukaya category, we proved a version of homological mirror symmetry for the projective spaces and their products via Strominger-Yau-Zaslow construction of the mirror dual Landau-Ginzburg model.

In this paper we go this way further and extend our previous result to the case of the Hirzebruch surface \mathbb{F}_1 .

1. Introduction

In [18], Strominger-Yau-Zaslow proposed a construction of mirror dual Calabi-Yau manifolds via dual torus fibrations on a closed manifold. Kontsevich-Soibelman [14] proposed a framework to systematically prove homological mirror symmetry by interpolating a variant of the Morse homotopy. Morse homotopy was first introduced for closed manifolds as a Morse theoretic (dimensionally reduced) model of the Fukaya category by Fukaya [7]. He and Oh showed that Morse homotopy fully faithfully embeds into the Fukaya category of the cotangent bundles [9]. It is hard to extend Kontsevich-Soibelman's program to general Calabi-Yau's because of the existence of singular fibers of the SYZ fibration. See for example Fukaya [8] for an outline of the whole program on this line.

SYZ picture is also applicable to the case of toric Fano's and their Landau-Ginzburg mirrors, in which the ends of the total space of the

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Landau-Ginzburg model corresponds to the toric divisors. It was first discussed in mathematical-symplectic geometric context by Auroux [2]. Abouzaid [1] formulated and proved a version of the homological mirror symmetry for toric Fano's by employing the tropical geometric setting on the Fukaya-Morse side, and proved an A_∞ embedding by using an abstract framework of Čech category on the complex side.

Based on the differential geometric formulation by Leung-Yau-Zaslow [17] and Leung [16], Fang [6] studied homological mirror symmetry for $\mathbb{C}P^n$ using the mirror transform associated with the SYZ fibration. Chan [4] studied the case of more general projective torics to determine which Lagrangians in the Landau-Ginzburg mirror correspond to holomorphic line bundles.

We further investigate this kind of formulation to more directly realize the SYZ picture in the toric Fano cases in our previous paper [11]. We proposed a definition of the Morse homotopy $Mo(P)$ for the moment polytope P and proved a version of the homological mirror symmetry for projective spaces and their products. This enables us to give a concrete description of the functorial mirror transform.

We go this way further in this paper and compute the case of the Hirzebruch surface \mathbb{F}_1 . Namely, we prove a version of the homological mirror symmetry:

Theorem 1.1 (Corollary 3.3). *We have an equivalence of triangulated categories*

$$Tr(Mo_{\mathcal{E}}(P)) \simeq D^b(\text{coh}(\mathbb{F}_1)),$$

where P is the moment polytope of \mathbb{F}_1 and \mathcal{E} is the collection of Lagrangian sections mirror to the chosen full strongly exceptional collection of holomorphic line bundles on \mathbb{F}_1 .

This paper is organized as follows. In Section 2, we recall some basic settings and definitions from our previous paper [11] without going into details. In Section 3.1, we recall Hirzebruch surfaces in homogeneous coordinates. In Section 3.2, we realize holomorphic line bundles on \mathbb{F}_1 in a geometric way and construct the DG category $DG(\mathbb{F}_1)$. The corresponding Lagrangian sections are obtained explicitly in subsection 3.4. In subsection 3.6, we compute the Morse homotopy and prove the main theorem.

While we do some calculations for \mathbb{F}_k with general $k \geq 1$, we restrict ourselves in this paper to the case $k = 1$ since \mathbb{F}_k is Fano if and only if $k = 0$ or 1 , where $\mathbb{F}_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ is already discussed in [11]. In fact, in our setting, we mainly keep in mind that the toric manifold is Fano, where any line bundle is guaranteed to be an exceptional object by the Kodaira vanishing theorem. However, we should note that a similar equivalence may exist for \mathbb{F}_k with $k \geq 2$ since it is known that $D^b(\text{coh}(\mathbb{F}_k))$ has full strongly exceptional collections of line bundles [12]. Another reason for us to restrict the case $k = 1$ in this paper is that we can construct the category $Mo(P)$ and enjoy the equivalence above explicitly in this case. See subsection 3.4 for more details.

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2. Preliminaries on the SYZ fibrations and homological mirror symmetry

In this section, we review some notions and settings from our previous paper [11].

2.1. Hessian manifold and dual torus fibrations

Let B be a tropical affine manifold: it is equipped with an affine open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ whose transition functions are affine with integral linear part. We assume for simplicity that all nonempty intersections of U_λ 's are contractible. Namely, the coordinate transformation is of the form

$$x_{(\mu)} = \varphi_{\lambda\mu}x_{(\lambda)} + \psi_{\lambda\mu}$$

with $\varphi_{\lambda\mu} \in GL(n; \mathbb{Z})$ and $\psi_{\lambda\mu} \in \mathbb{R}^n$, where $x_{(\lambda)} = (x_{(\lambda)}^1, \dots, x_{(\lambda)}^n)^t$ and $x_{(\mu)} = (x_{(\mu)}^1, \dots, x_{(\mu)}^n)^t$ denote the local coordinates on U_λ and U_μ respectively. We omit the suffix (λ) when no confusion may occur.

We call B Hessian when it is equipped with a metric g locally expressed as

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$$

for some smooth local function ϕ . Hereafter we assume that B is Hessian.

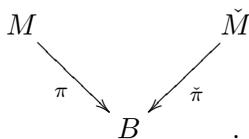
Using the metric g we first define the dual affine coordinates on the base space as follows: since $\sum_{j=1}^n g_{ij} dx^j$ is closed if (B, g) is Hessian, there exists a function $x_i := \phi_i$ of x for each i such that

$$(1) \quad dx_i = \sum_{j=1}^n g_{ij} dx^j.$$

We thus obtain the dual coordinates $\check{x}^{(\lambda)} := (x_1^{(\lambda)}, \dots, x_n^{(\lambda)})^t$.

We then denote the fiber coordinates on $T^*U_\lambda = T^*B|_{U_\lambda}$ dual to $x^{(\lambda)}$ by $(y_1^{(\lambda)}, \dots, y_n^{(\lambda)})$. Denote by $(y_1^{(\lambda)}, \dots, y_n^{(\lambda)})$ the fiber coordinates on $TB|_{U_\lambda}$ which corresponds to $(y_1^{(\lambda)}, \dots, y_n^{(\lambda)})$ via the isomorphism $TB \xrightarrow{\cong} T^*B$ induced by g . The cotangent bundle T^*B is equipped with the standard symplectic form $\omega_{T^*B} := \sum_{i=1}^n dx^i \wedge dy_i$. The tangent bundle TB is a complex manifold, where $z^i = x^i + \mathbf{i}y^i$'s form the complex coordinates. We can further equipped TB with the symplectic form $\omega_{TB} := \sum_{i,j=1}^n g_{ij} dx^i \wedge dy^j$ and T^*B with the complex structure given by the complex coordinates $z_i = x_i + \mathbf{i}y_i$'s. These structures in turn give the Kähler structures on both TB and T^*B .

Next we consider \mathbb{Z}^n -actions on TB and T^*B . The action of $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$ is defined by $y^i \mapsto y^i + 2\pi$ and $y_i \mapsto y_i + 2\pi$ respectively. This is well-defined because B is affine and the linear part $\varphi_{\lambda\mu}$ of the transition functions are integral. Therefore we can divide TB and T^*B by this action of \mathbb{Z}^n to get a pair of Kähler manifolds $M = TB/\mathbb{Z}^n$ and $\check{M} = T^*B/\mathbb{Z}^n$, and dual torus fibrations:



2.2. The categories DG and \mathcal{V}

Now let X be a smooth compact toric manifold. Consider the complement of toric divisors $\check{M} := X \setminus \mu^{-1}(\partial P)$ for the moment map $\mu: X \rightarrow P \subset \mathbb{R}^n$. By fixing an appropriate structure of the Hessian manifold on $B := \text{Int}P$, we get an affine torus fibration on $\check{M} \rightarrow B$ whose Kähler structure coincides with that coming from the given one on X . Applying the construction in the previous subsection, we get the dual torus fibration $M \rightarrow B$ with a

Kähler structure on the total space. We thus get the structure of affine torus fibration $\check{M} \rightarrow B$ and its dual torus fibration $M \rightarrow B$ with Kähler structures. We describe such affine structure of \mathbb{F}_k concretely in Section 3.1.

We consider the correspondence between Lagrangian submanifolds in M and holomorphic vector bundles with $U(1)$ -connection on \check{M} . The following calculation is based on a version of the Fourier-Mukai transform and came from Leung-Yau-Zaslow [17] and Leung [16] (but with different conventions). Let L be a Lagrangian section $\underline{y}: B \rightarrow M$. Then \underline{y} can locally expressed as df for some smooth function f . We associate to it a line bundle V on \check{M} with the $U(1)$ -connection

$$(2) \quad D := d - \frac{\mathbf{i}}{2\pi} \sum_{i=1}^n y^i(x) dy_i,$$

where $y = (y^1, \dots, y^n)$ is a lift to the covering space TB of M . This is holomorphic since \underline{y} is a Lagrangian section.

We first define the category \mathcal{V} associated with \check{M} . It is the DG category consisting of pairs of holomorphic line bundles and $U(1)$ -connections of the form (2). More precisely, for objects $y_a = (V_a, D_a)$ and $y_b = (V_b, D_b)$ it has the hom space

$$\mathcal{V}(y_a, y_b) := \Gamma(V_a, V_b) \otimes_{C^\infty(\check{M})} \Omega^{0,*}(\check{M})$$

where $\Gamma(V_a, V_b)$ denotes the space of homomorphisms from V_a to V_b . This space is \mathbb{Z} -graded with the degree of the anti holomorphic differential forms and we denote the degree r part by $\mathcal{V}^r(y_a, y_b)$. Decompose $D_a = D_a^{(1,0)} + D_a^{(0,1)}$ and set $d_a := 2D_a^{(0,1)}$. The differential d on $\mathcal{V}(y_a, y_b)$ is then defined as

$$d_{ab}(\psi) := d_b\psi - (-1)^r \psi d_a$$

for $\psi \in \mathcal{V}^r(y_a, y_b)$. The product structure is given by combining the composition of bundle homomorphisms and the wedge product:

$$m(\psi_{ab}, \psi_{bc}) := (-1)^{r_{ab}, r_{bc}} \psi_{bc} \wedge \psi_{ab}.$$

We then define the DG category $DG(X)$ of holomorphic line bundles on the toric manifold X . For a line bundle V on X , we take a holomorphic connection D whose restriction to \check{M} is isomorphic to a line bundle on \check{M}

with a connection of the form

$$d - \frac{\mathbf{i}}{2\pi} \sum_{i=1}^n y^i(x) dy_i.$$

We set the objects of $DG(X)$ as such pairs (V, D) . The space $DG(X)(y_a, y_b)$ of morphisms is defined as the graded vector space each graded piece of which is given by

$$DG^r(X)(y_a, y_b) := \Gamma(V_a, V_b) \otimes \Omega^{0,r}(X)$$

with $\Gamma(V_a, V_b)$ being the space of smooth bundle morphism from V_a to V_b . The composition of morphisms is defined in a similar way as that in $\mathcal{V}(\tilde{M})$ above. The differential

$$d_{ab} : DG^r(X)(y_a, y_b) \rightarrow DG^{r+1}(X)(y_a, y_b)$$

is defined by

$$d_{ab}(\tilde{\psi}) := 2 \left(D_b^{0,1} \tilde{\psi} - (-1)^r \tilde{\psi} D_a^{0,1} \right).$$

We then have a faithful embedding $\mathcal{I} : DG(X) \rightarrow \mathcal{V}$ by restricting line bundles on X to \tilde{M} . We define \mathcal{V}' to be the image $\mathcal{I}(DG(X))$ of $DG(X)$ under \mathcal{I} .

For a full exceptional collection \mathcal{E} of $DG(X)$, we denote the corresponding full subcategories consisting of \mathcal{E} by $DG_{\mathcal{E}}(X) \subset DG(X)$ and $\mathcal{V}'_{\mathcal{E}} \subset \mathcal{V}'$ respectively.

2.3. The Morse homotopy $Mo(P)$

In the case of the moment polytopes, the objects of $Mo(P)$ are Lagrangian sections $\underline{y} : B \rightarrow M$ which corresponds to objects of $DG(X)$ described in the previous subsection. We shall see explicitly later in Section 3.4 in the case of \mathbb{F}_1 . Note that (i) they intersect cleanly, i.e. there exists an open set \tilde{B} such that $\bar{B} \subset \tilde{B}$ and L, L' over B can be extended to graphs of smooth sections over \tilde{B} so that they intersect cleanly, and (ii) for each L , we can locally take a Morse function f_L on \tilde{B} so that L is the graph of df_L .

For a given pair (L, L') , we assign a grading $|V|$ for each connected component V of the intersection $\pi(L \cap L')$ in $P = \bar{B}$ as the dimension of the

stable manifold $S_v \subset \tilde{B}$ of the gradient vector field $-\text{grad}(f_L - f_{L'})$ with a point $v \in V$. This does not depend on the choice of the point $v \in V$. The space $Mo(P)(L, L')$ of morphisms is then set to be the \mathbb{Z} -graded vector space spanned by the connected components V of $\pi(L \cap L') \in P$ such that there exists a point $v \in V$ which is an interior point of $S_v \cap P$.¹ Note that, by this definition, the space $Mo(P)(L, L)$ is generated by P , which is of degree zero and forms the identity morphism for any object L .

Rather than going into full details, we only explain \mathfrak{m}_2 because of the following reasons: firstly, the Morse homotopy for \mathbb{F}_1 is minimal, i.e. with zero differential. Secondly, the set of objects \mathcal{E} we compute later forms strongly exceptional collection in $Tr(Mo_{\mathcal{E}}(P))$ and therefore we do not need to compute \mathfrak{m}_k with $k \geq 3$ to compute $Tr(Mo_{\mathcal{E}}(P))$. (For more comments, see [11], Section 4.5.)

Take a triple (L_1, L_2, L_3) , connected components of the intersections $V_{12} \subseteq L_1 \cap L_2$, $V_{23} \subseteq L_2 \cap L_3$, $V_{13} \subseteq L_1 \cap L_3$ and define $\mathcal{GT}(v_{12}, v_{23}; v_{13})$ to be the set of the trivalent gradient trees starting at $v_{12} \in V_{12}$, $v_{23} \in V_{23}$ and ending at $v_{13} \in V_{13}$. Define

$$\mathcal{GT}(V_{12}, V_{23}; V_{13}) := \cup_{v_{12} \in V_{12}, v_{23} \in V_{23}, v_{13} \in V_{13}} \mathcal{GT}(v_{12}, v_{23}; v_{13})$$

and

$$\mathcal{HGT}(V_{12}, V_{23}; V_{13}) := \mathcal{GT}(V_{12}, V_{23}; V_{13})/\text{smooth homotopy}.$$

This set becomes a finite set when $|V_{13}| = |V_{12}| + |V_{23}|$ and therefore we define $\mathfrak{m}_2 =$ composition of morphisms by

$$\begin{aligned} \mathfrak{m}_2: Mo(P)(L_1, L_2) \otimes Mo(P)(L_2, L_3) &\rightarrow Mo(P)(L_1, L_3) \\ (V_{12}, V_{13}) &\mapsto \sum_{|V_{13}|=|V_{12}|+|V_{23}|} \sum_{[\gamma] \in \mathcal{HGT}(V_{12}, V_{23}; V_{13})} e^{-A(\gamma)} V_{13} \end{aligned}$$

where $A(\gamma)$ is the symplectic area of disk obtained by lifting the gradient tree γ to M .

¹We consider the Morse cohomology degree instead of the Morse homology degree.

3. Homological mirror symmetry of \mathbb{F}_1

Following Hille-Perling[12], Elagin-Lunts[5], Kuznesov[15], etc, the triangulated category $D^b(\text{coh}(\mathbb{F}_1)) \simeq \text{Tr}(DG(\mathbb{F}_1))$ has a series of full strongly exceptional collections

$$\mathcal{E} := (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(c, 1), \mathcal{O}(1 + c, 1)),$$

where $\mathcal{O}(a, b)$ is a line bundle on \mathbb{F}_1 we shall define later in subsection 3.2.

We denote the corresponding full subcategories by $DG_{\mathcal{E}}(\mathbb{F}_1) \subset DG(\mathbb{F}_1)$, $\mathcal{V}'_{\mathcal{E}} \subset \mathcal{V}' = \mathcal{V}'(\mathbb{F}_1)$ and $Mo_{\mathcal{E}}(P) \subset Mo(P)$, where P is the moment polytope of \mathbb{F}_1 .

Then our main theorem is stated as follows.

Theorem 3.1. *For any fixed $c = 0, 1, \dots$, there exists a linear A_{∞} -equivalence*

$$\iota : Mo_{\mathcal{E}}(P) \rightarrow \mathcal{V}'_{\mathcal{E}}$$

such that for any generator $V \in Mo_{\mathcal{E}}(P)(L, L')$ with any $L, L' \in Mo_{\mathcal{E}}(P)$

- $\iota(V) \in (\mathcal{V}')^0(\iota(L), \iota(L')) \subset C^{\infty}(B)$ extends to a continuous function on $P = \bar{B}$ and
- we have

$$\max_{x \in P} |\iota(V)(x)| = 1, \quad \{x \in P \mid |\iota(V)(x)| = 1\} = V.$$

As in the cases for $X = \mathbb{C}P^n$ and $X = \mathbb{C}P^m \times \mathbb{C}P^n$, this theorem implies a version of homological mirror symmetry of \mathbb{F}_1 .

Corollary 3.2. *We have a linear A_{∞} -equivalence*

$$Mo_{\mathcal{E}}(P) \simeq DG_{\mathcal{E}}(\mathbb{F}_1).$$

Corollary 3.3. *We have an equivalence of triangulated categories*

$$\text{Tr}(Mo_{\mathcal{E}}(P)) \simeq D^b(\text{coh}(\mathbb{F}_1))$$

where Tr denotes the twisted complexes construction by Bondal-Kapranov [3] and Kontsevich [13].

As a biproduct of the proof of Theorem 3.1, we also show the following.

Proposition 3.4. *If $L \neq L'$, any generator $V \in Mo_{\mathcal{E}}(P)(L, L')$ belongs to the boundary $\partial(P)$.*

For given bases $V \in Mo_{\mathcal{E}}(L, L')$ and $V' \in Mo_{\mathcal{E}}(L', L'')$, the image $\gamma(T)$ by any gradient tree $\gamma \in \mathcal{GT}(V, V'; V'')$ with $V'' \in Mo_{\mathcal{E}}(P)(L, L'')$ belongs to the boundary $\partial(P)$ unless $L = L' = L''$.

In subsection 3.1, we explain how to treat Hirzebruch surfaces in our set-up. In subsection 3.2, we discuss line bundles on \mathbb{F}_k constructed from the toric divisors and construct the DG category $DG(\mathbb{F}_k)$ consisting of these line bundles. In subsection 3.3, we calculate the cohomology of the DG-category $DG(\mathbb{F}_1)$ of line bundles and in particular full subcategories $DG_{\mathcal{E}}(\mathbb{F}_1)$ consisting of full strongly exceptional collections \mathcal{E} of the triangulated category $Tr(DG(\mathbb{F}_1)) \simeq D^b(\text{coh}(\mathbb{F}_1))$ following the known technique in toric geometry. In subsection 3.4, we construct the Lagrangian sections which are SYZ mirror dual to the line bundles in $DG(\mathbb{F}_1)$ based on a geometric realization of the line bundles in subsection 3.2. The obtained Lagrangian sections will be the objects of $Mo(P)$. In subsection 3.5, we translate the cohomology $H(DG_{\mathcal{E}}(\mathbb{F}_1))$ to the cohomology $H(\mathcal{V}'_{\mathcal{E}})$. In subsection 3.6, we construct $Mo_{\mathcal{E}}(P)$ and show our main theorem by comparing the result with $H(\mathcal{V}'_{\mathcal{E}})$. At the end, we discuss a morphism of degree one in \mathcal{V}' and see that we have the corresponding morphism in $Mo(P)$ in subsection 3.7.

3.1. Hirzebruch surfaces

Though we need \mathbb{F}_1 only, in this subsection we treat \mathbb{F}_k with general $k \geq 1$ since the SYZ mirror of \mathbb{F}_k is obtained for any k in a similar way.

The Hirzebruch surface \mathbb{F}_k is defined by

$$\mathbb{F}_k := \{[s_0 : s_1], [t_0 : t_1 : t_2] \mid (s_0)^k t_0 = (s_1)^k t_1\} \subset \mathbb{C}P^1 \times \mathbb{C}P^2.$$

If $a \neq 0$, then $t_0 = (s_1/s_0)^k t_1$. Thus, for $a \neq 0$ and $x \neq 0$, we have a chart

$$U_1 := \{[1 : u], [u^k : 1 : v]\}$$

where $u = s_1/s_0$ and $v = t_2/t_1$. Similarly, for $s_0 \neq 0$ and $t_2 \neq 0$, we have a chart

$$U_3 := \{[1 : u], [u^k v : v : 1]\}$$

where $u = s_1/s_0$ and $v = t_1/t_2$. Similarly, we have a chart

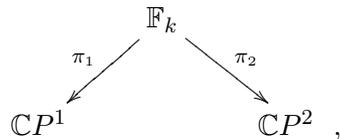
$$U_2 := \{[u : 1], [1 : u^k : v]\}, \quad u = s_0/s_1, \quad v = t_2/t_0$$

for $s_1 \neq 0$ and $t_0 \neq 0$, and a chart

$$U_4 := \{[u : 1], [v : u^k v : 1]\}, \quad u = s_0/s_1, \quad v = t_0/t_2$$

for $s_1 \neq 0$ and $t_2 \neq 0$. Thus, one sees that $\mathbb{F}_k \simeq P(\mathcal{O}(-k) \oplus \mathcal{O})$, where $\mathcal{O}(-k)$ and \mathcal{O} are the line bundles over $\mathbb{C}P^1 = \{[s_0 : s_1]\}$.

We have the natural projections



where π_1 gives the fibration structure of $P(\mathcal{O}(-k) \oplus \mathcal{O})$ over $\mathbb{C}P^1$.

A Kähler form ω is then obtained by

$$\omega = C_1 \pi_1^*(\omega_{\mathbb{C}P^1}) + C_2 \pi_2^*(\omega_{\mathbb{C}P^2}),$$

where $C_1 > 0$ and $C_2 > 0$ are real constants and $\omega_{\mathbb{C}P^n}$ is the Fubini-Study form (which we treat explicitly in [11, section 2.2]). Correspondingly, we have the moment map $\mu : \mathbb{F}_k \rightarrow \mathbb{R}^2$ defined by

$$\begin{aligned}
 & \mu([s_0 : s_1], [t_0 : t_1 : t_2]) \\
 & := 2 \left(C_1 \frac{|s_0|^2}{|s_0|^2 + |s_1|^2} + C_2 k \frac{|t_1|^2}{|t_0|^2 + |t_1|^2 + |t_2|^2}, C_2 \frac{|t_2|^2}{|t_0|^2 + |t_1|^2 + |t_2|^2} \right).
 \end{aligned}$$

The image $\mu(\mathbb{F}_k)$ is the trapezoid surrounded by the x^1 -axis, x^2 -axis, $x^2 = 2C_2$ and $x^1 + kx^2 = 2(C_2k + C_1)$. Namely, the moment polytope is

$$P := \{(x^1, x^2) \in \mathbb{R}^2 \mid 0 \leq x^1 \leq 2(C_2k + C_1) - kx^2, 0 \leq x^2 \leq 2C_2\}.$$

Now, we treat

$$\check{M} := U_1 \cap U_2 \cap U_3 \cap U_4 \rightarrow \text{Int}P =: B$$

as a torus fibration. We express this with the coordinates of $U_2 =: U$, where $\mu((u = 0, v = 0)) = (0, 0)$. When we further denote $u = e^{x_1 + iy_1}$ and $v = e^{x_2 + iy_2}$, then (y_1, y_2) is the coordinates of a fiber of \check{M} . We see that the restriction of μ to $\check{M} \subset \mathbb{F}_k$ gives the fibration structure $\tilde{\pi} := \mu|_{\check{M}}: \check{M} \rightarrow B = \text{Int}P$.

We further see that the dual coordinates of (x_1, x_2) is (x^1, x^2) above. Actually, the Kähler form ω is expressed as

$$\omega = -2id \left(C_1 \frac{\bar{u}du}{1 + u\bar{u}} + C_2 \frac{\bar{u}^k du^k + \bar{v}dv}{1 + (u\bar{u})^k + v\bar{v}} \right),$$

so one has

$$g^{-1} = 4 \begin{pmatrix} C_1 \frac{s}{(1+s)^2} + C_2 \frac{k^2 \cdot s^k (1+t)}{(1+s^k+t)^2} & C_2 \frac{-k \cdot s^k t}{(1+s^k+t)^2} \\ C_2 \frac{-k \cdot s^k t}{(1+s^k+t)^2} & C_2 \frac{t(1+s^k)}{(1+s^k+t)^2} \end{pmatrix},$$

where $s := u\bar{u}$ ($= e^{2x_1}$), $t := v\bar{v}$ ($= e^{2x_2}$). By (1), the dual coordinates are defined by

$$\begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix} = g^{-1} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = 4 \begin{pmatrix} C_1 \frac{s}{(1+s)^2} + C_2 \frac{k^2 \cdot s^k (1+t)}{(1+s^k+t)^2} & C_2 \frac{-k \cdot s^k t}{(1+s^k+t)^2} \\ C_2 \frac{-k \cdot s^k t}{(1+s^k+t)^2} & C_2 \frac{t(1+s^k)}{(1+s^k+t)^2} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix},$$

which is satisfied by

$$(3) \quad (x^1, x^2) = \left(C_1 \frac{2e^{2x_1}}{1 + e^{2x_1}} + C_2 k \frac{2e^{2kx_1}}{1 + e^{2kx_1} + e^{2x_2}}, C_2 \frac{2e^{2x_2}}{1 + e^{2kx_1} + e^{2x_2}} \right) \\ = \mu([e^{x_1 + iy_1} : 1], [1 : e^{k(x_1 + iy_1)} : e^{x_2 + iy_2}]).$$

Hereafter we fix $C_1 = C_2 = 1$ since the structure of the category $Mo(P)$ we shall construct is independent of these constants.

3.2. Line bundles on \mathbb{F}_k

Any line bundle over \mathbb{F}_k is constructed from a toric divisor, which is a linear combination of the following four divisors

$$\begin{aligned} D_{12} &= (t_2 = 0) \\ D_{24} &= (s_0 = t_1 = 0) \\ D_{13} &= (s_1 = t_0 = 0) \\ D_{34} &= (t_0 = t_1 = 0). \end{aligned}$$

Note that D_{24} is the fiber of π_1 at $[s_0 : s_1] = [0 : 1]$, D_{13} is the fiber of π_1 at $[1 : 0]$, and the remaining two are sections of π_1 . The corresponding Cartier divisors are as follows.

$$\begin{aligned} D_{12} &: \{(U_1, v_1), (U_2, v_2), (U_3, 1), (U_4, 1)\} \\ D_{24} &: \{(U_1, 1), (U_2, u_2), (U_3, 1), (U_4, u_4)\} \\ D_{13} &: \{(U_1, u_1), (U_2, 1), (U_3, u_3), (U_4, 1)\} \\ D_{34} &: \{(U_1, 1), (U_2, 1), (U_3, v_3), (U_4, v_4)\}. \end{aligned}$$

Now, the coordinate transformations are

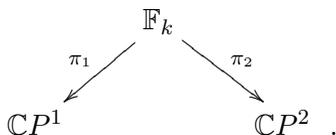
$$\begin{aligned} (u_1, v_1) &= (1/u_2, v_2/u_2^k), \\ (u_3, v_3) &= (1/u_2, u_2^k/v_2), \\ (u_4, v_4) &= (u_2, 1/v_2). \end{aligned}$$

The transition functions are then

$$\begin{aligned} D_{12} : \quad & \phi_{21} = v_1/v_2 = u_2^{-k}, & \phi_{23} = 1/v_2, & \phi_{24} = 1/v_2, \\ D_{24} : \quad & \phi_{21} = 1/u_2, & \phi_{23} = 1/u_2, & \phi_{24} = u_4/u_2 = 1, \\ D_{13} : \quad & \phi_{21} = u_1 = 1/u_2, & \phi_{23} = u_3 = 1/u_2, & \phi_{24} = 1, \\ D_{34} : \quad & \phi_{21} = 1, & \phi_{23} = v_3 = u_2^k/v_2, & \phi_{24} = v_4 = 1/v_2. \end{aligned}$$

Thus, we see that D_{13} and D_{24} define the same line bundle; $\mathcal{O}(D_{13}) = \mathcal{O}(D_{24})$, and $\mathcal{O}(D_{12}) = \mathcal{O}(D_{34} + kD_{24})$. In this sense, any line bundle over \mathbb{F}_k is generated either by (D_{24}, D_{34}) or by (D_{24}, D_{12}) .

On the other hand, we have line bundles $\pi_1^*(\mathcal{O}_{\mathbb{C}P^1}(1))$ and $\pi_2^*(\mathcal{O}_{\mathbb{C}P^2}(1))$ over \mathbb{F}_k via the pair of projections



The transition functions for them are obtained by the pullbacks of $\mathcal{O}_{\mathbb{C}P^1}(1) \rightarrow \mathbb{C}P^1$ and $\mathcal{O}_{\mathbb{C}P^2}(1) \rightarrow \mathbb{C}P^2$ by π_1^* and π_2^* , respectively. Then we can identify $\pi_1^*(\mathcal{O}_{\mathbb{C}P^1}(1)) = \mathcal{O}(D_{24})$ and $\pi_2^*(\mathcal{O}_{\mathbb{C}P^2}(1)) = \mathcal{O}(D_{12})$.

The connection one-forms for $\pi_1^*(\mathcal{O}_{\mathbb{C}P^1}(1))$ and $\pi_2^*(\mathcal{O}_{\mathbb{C}P^2}(1))$ are also obtained by the pullbacks. On U_2 , they are expressed as

$$\begin{aligned}
 \pi_1^*(A_{\mathbb{C}P^1}) &= -\frac{\bar{u}du}{1+u\bar{u}}, \\
 \pi_2^*(A_{\mathbb{C}P^2}) &= -\frac{\bar{u}^k d(u^k) + \bar{v}dv}{1+(u\bar{u})^k + v\bar{v}},
 \end{aligned}$$

where $A_{\mathbb{C}P^n}$ is the connection one-form for the line bundle $\mathcal{O}_{\mathbb{C}P^n}(1)$ on $\mathbb{C}P^n$ as is presented explicitly for instance in [11, section 3.3]. We again denote $s := u\bar{u}$, $t := v\bar{v}$, and then

$$\begin{aligned}
 \pi_1^*(A_{\mathbb{C}P^1}) &= -\frac{s(dx_1 + \mathbf{i}dy_1)}{1+s}, \\
 \pi_2^*(A_{\mathbb{C}P^2}) &= -\frac{ks^k(dx_1 + \mathbf{i}dy_1) + t(dx_2 + \mathbf{i}dy_2)}{1+s^k+t}.
 \end{aligned}$$

Similarly, the connection one-forms for $\mathcal{O}(a, b) := \mathcal{O}(aD_{24} + bD_{12})$ is given by

$$A_{(a,b)} := -a\frac{s(dx_1 + \mathbf{i}dy_1)}{1+s} - b\frac{ks^k(dx_1 + \mathbf{i}dy_1) + t(dx_2 + \mathbf{i}dy_2)}{1+s^k+t}.$$

In a similar way as for $\mathbb{C}P^n$ in [11, section 3.3], the dx_1 term and the dx_2 term are removed by the isomorphisms as follows.

$$\begin{aligned}
 (4) \quad \Psi_{(a,b)}^{-1}(d + A_{(a,b)})\Psi_{(a,b)} &= d - a\mathbf{i}\frac{sdy_1}{1+s} - b\mathbf{i}\frac{ks^kdy_1 + tdy_2}{1+s^k+t}, \\
 \Psi_{(a,b)} &:= (1+s)^{\frac{a}{2}} \left(1+s^k+t\right)^{\frac{b}{2}}.
 \end{aligned}$$

Note that the connection in the right hand side above is of the form (2). Thus, we set $DG(\mathbb{F}_k)$ as the DG-category consisting of the line bundles $\mathcal{O}(a, b)$ with any (a, b) .

3.3. Cohomologies of the DG category $DG(\mathbb{F}_1)$

We first discuss the global sections of line bundles $\mathcal{O}(a, b)$ over \mathbb{F}_k . The structure of the global sections is obtained by following [10, p.66]. In particular, we see that

$$\begin{aligned} d_{(a,b)} &:= \dim(\Gamma(\mathbb{F}_k, \mathcal{O}(a, b))) = (a + 1) + (a + 1 + k) + \cdots + (a + 1 + kb) \\ &= \frac{(b + 1)(2a + 2 + kb)}{2}. \end{aligned}$$

For instance, $d_{(1,0)} = 2$, $d_{(0,1)} = 2 + k$, $d_{(a,0)} = a + 1$. Using the coordinates (u, v) for U_2 , the generators of $\Gamma(\mathcal{O}(a, b))$ are expressed as

$$(5) \quad \begin{array}{ccccccc} u^0, & u^1, & \dots, & \dots, & u^{a+kb}, \\ u^0v^1, & u^1v^1, & \dots, & u^{a+k(b-1)}v^1, & & & \\ \vdots, & \vdots & & & & & \\ u^0v^b, & \dots & u^av^b & & & & \end{array} .$$

Namely,

$$\psi_{(i_1, i_2)} := u^{i_1}v^{i_2}$$

are the generators, where $0 \leq i_2 \leq b$ and $0 \leq i_1 \leq a + k(b - i_2)$.

Since each $\mathcal{O}(a, b)$ is a line bundle, we have

$$DG^0(\mathbb{F}_k)(\mathcal{O}(a_1, b_1), \mathcal{O}(a_2, b_2)) \simeq DG^0(\mathbb{F}_k)(\mathcal{O}, \mathcal{O}(a_2 - a_1, b_2 - b_1)).$$

Thus, we obtain any zero-th cohomology of the space of morphisms in $DG(\mathbb{F}_k)$ from $\Gamma(\mathcal{O}(a, b))$, $a, b \in \mathbb{Z}$, i.e.,

$$\begin{aligned} H^0(DG(\mathbb{F}_k)(\mathcal{O}(a_1, b_1), \mathcal{O}(a_2, b_2))) &\simeq H^0(DG(\mathbb{F}_k)(\mathcal{O}, \mathcal{O}(a_2 - a_1, b_2 - b_1))) \\ &\simeq \Gamma(\mathbb{F}_k, \mathcal{O}(a_2 - a_1, b_2 - a_1)). \end{aligned}$$

We can also calculate the cohomologies

$$H^r(DG(\mathbb{F}_k)(\mathcal{O}(a_1, b_1), \mathcal{O}(a_2, b_2))) \simeq H^r(DG(\mathbb{F}_k)(\mathcal{O}, \mathcal{O}(a_2 - a_1, b_2 - b_1)))$$

with $r > 0$ in the way written in [10, p.74]. Now we consider \mathbb{F}_1 . It is known (Hille-Perling [12]) that

$$\mathcal{E} := (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(c, 1), \mathcal{O}(1 + c, 1))$$

with a fixed $c = 0, 1, \dots$ forms a full strongly exceptional collection of $Tr(DG(\mathbb{F}_1)) \simeq D^b(\text{coh}(\mathbb{F}_1))$. Let us consider the full subcategory $DG_{\mathcal{E}}(\mathbb{F}_1) \subset DG(\mathbb{F}_1)$ consisting of \mathcal{E} . We already calculated the zero-th cohomologies of the space of morphisms in $DG_{\mathcal{E}}(\mathbb{F}_1)$. We can also check that we have no nontrivial cohomologies of the space of morphisms of degree $r > 0$ in $DG_{\mathcal{E}}(\mathbb{F}_1)$. These calculations give a direct confirmation of the fact that \mathcal{E} is actually a strongly exceptional collection in $Tr(DG_{\mathcal{E}}(\mathbb{F}_1))$, and agree, for instance, with the Euler bilinear form on $K_0(D^b(\text{coh}(\mathbb{F}_1)))$ presented in Kuznesov [15, Example 3.7].

3.4. Lagrangian sections $L(a, b)$

We continue to concentrate on the case \mathbb{F}_1 and let us discuss the Lagrangian section $L(a, b)$ in the fiber $M \rightarrow B$ corresponding to the line bundle $\mathcal{O}(a, b)$. Comparing the connection one-form in (4) with (2), we see that $L(a, b)$ is expressed as the graph of

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = 2\pi \begin{pmatrix} a \frac{s}{1+s} + b \frac{s}{1+s+t} \\ b \frac{t}{1+s+t} \end{pmatrix}$$

where $s = e^{2x_1}$ and $t = e^{2x_2}$. Now, let us rewrite s, t by x^1, x^2 . Recall that the dual coordinates are given in (3):

$$(6) \quad x^1 = \frac{2s}{1+s} + \frac{2s}{1+s+t},$$

$$(7) \quad x^2 = \frac{2t}{1+s+t}.$$

By (7), t is expressed as

$$t = \frac{x^2(1+s)}{2-x^2}.$$

Substituting this to (6) yields

$$x^1 = \frac{2s}{1+s} + \frac{2s}{(1+s) + \frac{x^2}{2-x^2}(1+s)} = \frac{(4-x^2)s}{1+s}$$

and hence we obtain

$$\frac{s}{1+s} = \frac{x^1}{4-x^2}.$$

So, $s = x^1/(4 - x^1 - x^2)$, $1 + s = (4 - x^2)/(4 - x^1 - x^2)$, and we get $t = (1 + s)x^2/(2 - x^2) = (4 - x^2)x^2/(2 - x^2)(4 - x^1 - x^2)$. To summarize, we obtain

$$\begin{aligned} \frac{s}{1+s} &= \frac{x^1}{4-x^2}, \\ \frac{s}{1+s+t} &= \frac{x^1(2-x^2)}{2(4-x^2)}, \\ \frac{t}{1+s+t} &= \frac{x^2}{2}. \end{aligned}$$

In particular, the Lagrangian $L(a, b)$ corresponding to $\mathcal{O}(a, b) := \mathcal{O}(aD_{24} + bD_{12})$ is given by

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} a \frac{x^1}{4-x^2} + b \frac{x^1(2-x^2)}{2(4-x^2)} \\ b \frac{x^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{(2a+(2-x^2)b)x^1}{2(4-x^2)} \\ \frac{bx^2}{2} \end{pmatrix}.$$

The corresponding Morse function f is given by

$$\begin{aligned} (8) \quad f &= \frac{a}{2} \log(1+s) + \frac{b}{2} \log(1+s^k+t) \\ &= +\frac{1}{2} \log \left(\frac{4-x^2}{4-x^1-x^2} \right)^a \left(\frac{2(4-x^2)}{(2-x^2)(4-x^1-x^2)} \right)^b. \end{aligned}$$

For \mathbb{F}_k with general $k > 1$, the Lagrangian section $L(a, b)$ should still be obtained in a similar way. However, we do not seem to obtain a closed formula for (s, t) in terms of (x^1, x^2) .

3.5. Cohomologies $H(\mathcal{V}'_{\mathcal{E}})$

We consider $DG(\mathbb{F}_1)$ and the faithful embedding $\mathcal{I} : DG(\mathbb{F}_1) \rightarrow \mathcal{V}$ where $\mathcal{V} = \mathcal{V}(\check{M})$ is the DG category of line bundles on \check{M} . The image is denoted by $\mathcal{V}' := \mathcal{I}(DG(\mathbb{F}_1))$. Then, each generator $\psi_{(i_1, i_2)}$ in (5) is sent to be

$$\begin{aligned} (9) \quad \Psi_{(a,b)}^{-1} \psi_{(i_1, i_2)} &= (4-x^1-x^2)^{\frac{a+b-i_1-i_2}{2}} (2-x^2)^{\frac{b-i_2}{2}} \\ &\quad \times (4-x^2)^{-\frac{a+b-i_2}{2}} (x^1)^{\frac{i_1}{2}} (x^2)^{\frac{i_2}{2}} e^{\mathbf{i}(i_1 y_1 + i_2 y_2)} \end{aligned}$$

in \mathcal{V}' . Namely, these form a basis of $H^0(\mathcal{V}'(\mathcal{O}, \mathcal{O}(a, b)))$. A basis of $H^0(\mathcal{V}'(\mathcal{O}(a_1, b_1), \mathcal{O}(a_1 + a), \mathcal{O}(b_1 + b)))$ is of the same form. We rescale each basis $\Psi_{(a,b)}^{-1} \psi_{(i_1, i_2)}$ by multiplying a positive number and denote it by $\mathbf{e}_{(a,b);(i_1, i_2)}$ so that

$$\max_{x \in P} |\mathbf{e}_{(a,b);(i_1, i_2)}(x)| = 1.$$

Note that $\mathbf{e}_{(a,b);(i_1, i_2)}$ is a function on B , but can be extended continuously to that on $P = \bar{B}$ since the exponents

$$\frac{a + b - i_1 - i_2}{2}, \quad \frac{b - i_2}{2}, \quad \frac{i_1}{2}, \quad \frac{i_2}{2}$$

in (9) are non-negative. This shows the former statement about the properties of $\iota(V)$ in Theorem 3.1.

3.6. $Mo_{\mathcal{E}}(P)$

The objects of $Mo(P)$ are the Lagrangian sections $L(a, b)$ obtained in subsection 3.4. Since we have

$$Mo(P)(L(a_1, b_1), L(a_2, b_2)) \simeq Mo(P)(L(0, 0), L(a_2 - a_1, b_2 - b_1)),$$

we concentrate on calculating the space $Mo(P)(L(0, 0), L(a, b))$. We discuss that when there exists a nonempty intersection of

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} y_{(a,b)}^1(x) \\ y_{(a,b)}^2(x) \end{pmatrix} = \begin{pmatrix} \frac{(2a+(2-x^2)b)x^1}{2(4-x^2)} \\ \frac{bx^2}{2} \end{pmatrix}$$

with

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

in the covering space of $\bar{M} \rightarrow P$.

We first consider $L(a, b)$ with $b \geq 0$. Since $0 \leq x^2 \leq 2$, we have

$$0 \leq \frac{bx^2}{2} = i_2 \leq b.$$

If we further assume $a + b - i_2 \geq 0$, then we also have

$$(10) \quad 0 \leq i_1 = \frac{(2a + (2 - x^2)b)x^1}{2(4 - x^2)} \leq \frac{(2a + (2 - x^2)b)}{2} = a + b - i_2,$$

where we used $4 - x^2 \geq x^1$ in the inequality. We will discuss the case $a + b - i_2 < 0$ later. By solving $y_{(a,b)}^j(x) = i_j$, $j = 1, 2$, we obtain the following.

Lemma 3.5. *We assume $b \geq 0$ and $(a, b) \neq (0, 0)$. For any (i_1, i_2) satisfying*

$$0 \leq i_2 \leq b, \quad 0 \leq i_1 \leq a + b - i_2,$$

the intersection V_I is nonempty and connected.

- *If $b \neq 0$ and $a + b - i_2 \neq 0$, then V_I consists of the point v_I such that*

$$x^1(v_I) = \frac{4 - 2i_2/b}{a + b - i_2} i_1, \quad x^2(v_I) = \frac{2i_2}{b}.$$

- *If $b = 0$, then $i_2 = 0$ and then the intersection is*

$$V_{(i_1, 0)} := \{(x^1, x^2) \in P \mid x^1 = \frac{4 - x^2}{a} i_1\}.$$

- *If $a + b - i_2 = 0$, then $i_1 = 0$ and the intersection is*

$$V_{(0, a+b)} := \{(x^1, 2 + \frac{2a}{b}) \in P\}.$$

Note that the condition $a + b - i_2 = 0$ is satisfied only when $a \leq 0$ and $b > 0$.

Lemma 3.6. *We assume $b \geq 0$ and $(a, b) \neq (0, 0)$. For any $I = (i_1, i_2)$ satisfying*

$$0 \leq i_2 \leq b, \quad 0 \leq i_1 \leq a + b - i_2,$$

the intersection V_I forms a generator of $Mo(P)(L(a_1, b_1), L(a_1 + a, b_1 + b))$ of degree zero.

Proof. The gradient vector field associated to V_I is of the form

$$(11) \quad \left(\frac{(a + b - bx^2/2)x^1}{4 - x^2} - i_1 \right) \frac{\partial}{\partial x^1} + \left(\frac{bx^2}{2} - i_2 \right) \frac{\partial}{\partial x^2}.$$

If $b \neq 0$ and $a + b - i_2 \neq 0$, then V_I consists of the point v_I , and the stable manifold S_{v_I} of the gradient vector field is $\{v_I\}$ itself, so V_I is a generator of

degree zero. If $b = 0$, the gradient vector field associated to $V_{(i_1,0)}$ is of the form

$$(12) \quad \left(\frac{ax^1}{4-x^2} - i_1 \right) \frac{\partial}{\partial x^1}.$$

For each $v \in V_{(i_1,0)}$, the stable manifold S_v of the gradient vector field is again $\{v\}$ itself, so $V_{(i_1,0)}$ gives a generator of degree zero. If $a + b - i_2 = 0$, then the gradient vector field associated to $V_{(0,a+b)}$ is of the form

$$(a + b - bx^2/2) \left(\frac{x^1}{4-x^2} \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \right).$$

Because of the term $-(a + b - bx^2/2)\partial/\partial x^2$, for each $v \in V_{(0,a+b)}$, the stable manifold S_v of the gradient vector field is again $\{v\}$ itself, so $V_{(0,a+b)}$ gives a generator of degree zero. \square

Now, we discuss the case $a + b - i_2 < 0$. This case occurs only if a is negative since $a < -(b - i_2) \leq 0$. We have

$$a + b - i_2 \leq i_1 \leq 0$$

instead of (10). The intersection V_I again consists of a point v_I given by

$$x^2 = \frac{2i_2}{b}, \quad x^1 = \frac{4 - 2i_2/b}{a + b - i_2} i_1$$

if $b \neq 0$, and

$$V_{(i_1,0)} := \{(x^1, x^2) \in P \mid x^1 = \frac{4 - x^2}{a} i_1\}$$

if $b = 0$.

Lemma 3.7. *We assume $b \geq 0$ and $(a, b) \neq (0, 0)$. For any $I = (i_1, i_2)$ satisfying*

$$0 \leq i_2 \leq b, \quad a + b - i_2 \leq i_1 \leq 0,$$

we consider the intersection V_I which is nonempty and connected.

- (i) *The intersection does not form a generator of $Mo(P)(L(a_1, b_1), L(a_1 + a, b_1 + b))$ of degree zero.*

(ii) *The intersection does not form a generator of $Mo(P)(L(a_1, b_1), L(a_1 + a, b_1 + b))$ (o any degree) if $a = -1$.*

Proof. The gradient vector field associated to V_I is also of the same form as in the proof of Lemma 3.6. If $b > 0$, then the gradient vector field is (11) but now the sign of the coefficient for x^1 is reversed compared to the case $a + b - i_2 > 0$. Thus, we have $|V_I| = 1$. Similarly, if $b = 0$, then the gradient vector field is (12), but again the sign of the coefficient for x^1 is reversed compared to the case $a + b - i_2 > 0$ and we have $|V_I| = 1$. Thus, the statement (i) is proved.

We consider the case $a = -1$, where $b = i_2$ holds since $a + b - i_2 < 0$ and $i_1 = 0, -1$. If $b > 0$, then V_I consists of the point $v_I = (-2i_1, 2)$. We see that the stable manifold is

$$S_{v_I} = \{x^2 = 2\}.$$

in both cases $I = (0, b)$ and $I = (-1, b)$. Thus, $v_{(0,b)} = (0, 2)$ (resp. $v_{(-1,b)} = (2, 2)$) is not an interior point of $S_{v_{(0,1)}} \cap P \subset S_{v_{(0,1)}}$ (resp. $S_{v_{(-1,b)}} \cap P \subset S_{v_{(-1,b)}}$). If $b = 0$, then we have $V_{(0,0)} = D_{24}$ and $V_{(-1,0)} = D_{13}$. For $v \in D_{24}$, the stable manifold is

$$S_v = \{x^2 = x^2(v)\},$$

so v is not an interior point of $S_v \cap P \subset S_v$. Similarly, any $v \in D_{13}$ is not an interior point of $S_v \cap P \subset S_v$. Thus, any V_I does not form a generator of $Mo(P)(L(a_1, b_1), L(a_1 + a, b_1 + b))$ if $a = -1$. □

By Lemma 3.6 and Lemma 3.7 (i), we see that each generator V_I of degree zero is in one-to-one correspondence with the generator $\mathbf{e}_{(a,b);I}$ in subsection 3.5.

Lemma 3.8. *We assume $b \geq 0$ and $(a, b) \neq (0, 0)$. Each generator $\mathbf{e}_{(a,b);I} \in H^0(\mathcal{V}'(\mathcal{O}(a_1, b_1), \mathcal{O}(a_1 + a, b_1 + b)))$ is expressed as the form*

$$\mathbf{e}_{(a,b);I}(x) = e^{-f_I} e^{iIy},$$

where f_I is the C^∞ function on P satisfying

$$df_I = \sum_{j=1}^2 \frac{\partial f_I}{\partial x_j} dx_j, \quad \frac{\partial f_I}{\partial x_j} = y_{(a,b)}^j - i_j$$

in B and $\min_{x \in P} f_I(x) = 0$. In particular, we have

$$\{x \in P \mid f_I(x) = 0\} = V_I.$$

Proof. The statement of the first half is guaranteed by our construction $DG(\mathbb{F}_1) \simeq \mathcal{V}' \subset \mathcal{V}$. Of course, we can also check it directly. By (8), the function f_I satisfying $\frac{\partial f_I}{\partial x_j} = y_{(a,b)}^j - i_j$, $j = 1, 2$, is given by

$$\frac{1}{2} \log \left(\frac{4 - x^2}{4 - x^1 - x^2} \right)^a \left(\frac{2(4 - x^2)}{(2 - x^2)(4 - x^1 - x^2)} \right)^b - i_1 x_1 - i_2 x_2,$$

where $i_1, i_2 \in \mathbb{Z}$. Since we have

$$\begin{aligned} x_1 &= \frac{1}{2} \log s = \frac{1}{2} \log \left(\frac{x^1}{4 - x^1 - x^2} \right), \\ x_2 &= \frac{1}{2} \log t = \frac{1}{2} \log \left(\frac{x^2(4 - x^2)}{(2 - x^2)(4 - x^1 - x^2)} \right), \end{aligned}$$

we obtain

$$\begin{aligned} f_I &= - \left(\log \left((4 - x^1 - x^2)^{\frac{a+b-i_1-i_2}{2}} (2 - x^2)^{\frac{b-i_2}{2}} (4 - x^2)^{-\frac{a+b-i_2}{2}} (x^1)^{\frac{i_1}{2}} (x^2)^{\frac{i_2}{2}} \right) \right) \\ &\quad + \text{const..} \end{aligned}$$

The latter half can be shown directly by rewriting

$$\begin{aligned} &(4 - x^1 - x^2)^{\frac{a+b-i_1-i_2}{2}} (2 - x^2)^{\frac{b-i_2}{2}} (4 - x^2)^{-\frac{a+b-i_2}{2}} (x^1)^{\frac{i_1}{2}} (x^2)^{\frac{i_2}{2}} \\ &= \left(1 - \frac{x^1}{4 - x^2} \right)^{\frac{a+b-i_1-i_2}{2}} \left(\frac{x^1}{4 - x^2} \right)^{\frac{i_1}{2}} (2 - x^2)^{\frac{b-i_2}{2}} (x^2)^{\frac{i_2}{2}} \end{aligned}$$

and regarding the result as a function in variables $x^1/(4 - x^2)$ and x^2 . \square

Now, we fix $c \geq 0$ and consider the full subcategory $Mo_{\mathcal{E}}(P) \subset Mo(P)$ consisting of $\mathcal{E} = (L(0, 0), L(1, 0), L(c, 1), L(1 + c, 1))$. We call an element of $Mo(P)(L, L')$ an **ordered** (resp. **non-ordered**) morphism if $\mathcal{E} = (\dots, L, \dots, L', \dots)$ (resp. $\mathcal{E} = (\dots, L', \dots, L, \dots)$). We see that the space

$$Mo_{\mathcal{E}}(P)(L(a_1, b_1), L(a_1 + a, b_1 + b)) \simeq Mo(P)(L(0, 0), L(a, b))$$

of ordered morphisms satisfies $0 \leq b \leq 1$. We call an ordered morphism with $b = 0$ (resp. $b = 1$) a morphism **of type** $b = 0$ (resp. $b = 1$).

Lemma 3.9. *In $Mo_{\mathcal{E}}(P)$, we have only morphisms of degree zero, where each generator $V_I \neq P$ belongs to $\partial(P)$. Then, the correspondence $V_I \mapsto \iota(V_I) = \mathbf{e}_{ab;I}$ gives a quasi-isomorphism*

$$\iota : Mo_{\mathcal{E}}(P)(L(a_1, b_1), L(a_2, b_2)) \rightarrow \mathcal{V}'_{\mathcal{E}}(\mathcal{O}(a_1, b_1), \mathcal{O}(a_2, b_2))$$

of complexes.

Proof. By Lemma 3.7 (ii), we see that restricting the correspondence in Lemma 3.8 to $Mo_{\mathcal{E}}(P)$ yields the quasi-isomorphism of this Lemma for each space of ordered morphisms.

In particular, an ordered morphism of type $b = 0$ has $a = 1$, and an ordered morphism of type $b = 1$ satisfies $a \geq -1$. In both cases, we can check directly that the generators given in Lemma 3.5 belongs to $\partial(P)$.

It remains to calculate non-ordered morphisms. Then, we may consider the opposite case to Lemma 3.5 in the sense that we consider the case $b < 0$ and

$$b \leq i_2 \leq 0, \quad a + b - i_2 \leq i_1 \leq 0.$$

We see that the corresponding intersection is the same as the one in Lemma 3.5. Namely, we have

$$V_{(-a,-b);-I} = V_{(a,b);I} (=: V_I)$$

for $b \geq 0$. The sign of the corresponding gradient vector field is reversed compared to that in the proof in 3.6. Thus, we have

- $|V_{(-a,-b);-I}| = 2$ if $b > 0$ and $a + b - i_2 > 0$,
- $|V_{(-a,-b);-I}| = 1$ if $b = 0$,
- $|V_{(-a,-b);-I}| = 1$ if $a + b - i_2 = 0$.

In particular, for $b = 0$ or $a + b - i_2 = 0$, the stable manifold S_v of a point $v \in V_{(-a,-b);-I}$ intersect transversally with $V_{(-a,-b);-I}$ at v . Since $V_{(-a,-b);-I} \in \partial(P)$ in all these three cases, we can conclude that $V_{(-a,-b);-I}$ cannot be a generator of $Mo_{\mathcal{E}}(L(a_1 + a, b_1 + b), L(a_1, b_1))$. □

Now we discuss the composition structure in $Mo_{\mathcal{E}}(P)$. Let us examine the gradient flows starting from a point in a generator $V_I \in$

$Mo(P)(L(0, 0), L(a, b))$ with $b = 0$ and $b = 1$. If $b = 0$, then the gradient vector field is of the form

$$\text{grad}f = \left(\frac{ax^1}{4 - x^2} - i_1 \right) \frac{\partial}{\partial x^1}$$

as we already saw, where we have $i_2 = 0$. Thus, the gradient trajectories starting from $v = (x^1(v), x^2(v)) \in V_I$ are always in the line $x^2 = x^2(v)$. If $b = 1$, then the gradient vector field is of the form

$$\text{grad}f = \left(\frac{(a + 1 - x^2/2)x^1}{4 - x^2} - i_1 \right) \frac{\partial}{\partial x^1} + \left(\frac{x^2}{2} - i_2 \right) \frac{\partial}{\partial x^2}.$$

In this case, we see that $V_{(i_1, i_2=0)}$ belongs to $D_{12} \subset \partial(P)$, (where $V_{(0,0)} = D_{12}$ if $a = -1$ and otherwise $V_{(i_1,0)}$ consists of the point $v_{(i_1,0)} := (4/(a + 1), 0)$), and $V_{(i_1, i_2=1)}$ belongs to $D_{34} \subset \partial(P)$. (Note that $V_{(0,1)} = D_{34}$ if $a = 0$, and otherwise $V_{(i_1,1)}$ consists of a point.) This implies that the gradient trajectories starting from $v_{(i_1,0)}$ are always in the line $x^2 = 0$, and the gradient trajectories starting from a point in $V_{(i_1,1)}$ are always in the line $x^2 = 2$.

Suppose that we have a composition in $Mo_{\mathcal{E}}(P)$

$$V_I \cdot W_J$$

of two generators such that $V_I \neq P$ and $W_J \neq P$. We have such a composition only when

- V_I is of type $b = 0$ and W_J is of type $b = 1$ or
- V_I is of type $b = 1$ and W_J is of type $b = 0$.

In both cases, the result $V_I \cdot W_J$ is generated by a generator Z_{I+J} of type $b = 1$ with index $I + J$ since indices are preserved by the composition. This means that Z_{I+J} belongs to D_{12} (resp. D_{34}) if the generator, V_I or W_J , of type $b = 1$ belongs to D_{12} (resp. D_{34}). Since the gradient trajectories starting from a point in the generator, V_I or W_J , of type $b = 0$ run horizontally, we see that the image $\gamma(T)$ of the gradient tree γ defining the product $V_I \cdot W_J$ belongs to D_{12} or D_{34} . This gives the proof of Proposition 3.4.

Let $v := V_I \cap \gamma(T)$, $w := W_J \cap \gamma(T)$, and $z := Z_{I+J} \cap \gamma(T)$. Note that v and w are the images of the two external vertices of the trivalent tree T by γ , whereas z is the image of the root vertex of T by γ . Then, z sits

on the interval vw and the image of the root edge of T by γ is $\{z\}$. If we express $\iota(V_I) = e^{-f_v} \cdot e^{iJy}$, then $f_v(v) = 0$ and $f_v(z)$ is the symplectic area of the triangle disk whose edges are the interval vz and the corresponding two Lagrangian sections on vz . Similarly, for $\iota(W_J) = e^{-f_w} \cdot e^{iJ\tilde{y}}$, the value $f_w(z)$ is the symplectic area of the corresponding triangle disk. This shows the compatibility

$$\iota(V_I \cdot W_J) = \iota_I(V_I) \cdot \iota(W_J).$$

This completes the proof of Theorem 3.1.

3.7. Example of morphisms of degree one: $Mo(L(0, 0), L(2, -2))$

So far, we do not see any morphism of higher degree. In this subsection, we calculate the space $Mo(L(0, 0), L(2, -2))$ and show that it includes a generator of degree one. The intersections of $\pi(L(\mathcal{O}))$ with $\pi(L(\mathcal{O}(2, -2)))$ are obtained by solving

$$y^1 = -2 \frac{x^1(2 - x^2)}{2(4 - x^2)} + 2 \frac{x^1}{4 - x^2} = \frac{x^1 x^2}{4 - x^2} = i_1$$

$$y^2 = -2 \frac{x^2}{2} = -x^2 = i_2.$$

We have the following intersections in $\pi^{-1}(P)$ as follows.

- If $y^2 = i_2 = 0$, then $x^2 = 0$, where $y^1 = i_1 = 0$ for any x^1 .
- If $y^2 = i_2 = -1$, then $x^2 = 1$, where $y^1 = x^1/3 = i_1$. In this case, we have two choices $(i_1 = 0, x^1 = 0)$ and $(i_1 = 1, x^1 = 3)$.
- If $y^2 = i_2 = -2$, then $x^2 = 2$, where $y^1 = x^1 = i_1$. In this case, we have three choices $i_1 = x^1 = 0, 1, 2$.

To summarize, we have

$$V_{(0,0)} = \{(x^1, 0) \in P\}$$

$$V_{(0,-1)} = \{(0, 1) \in P\}, \quad V_{(1,-1)} = \{(3, 1) \in P\}$$

$$V_{(0,-2)} = \{(0, 2) \in P\}, \quad V_{(1,-2)} = \{(1, 2) \in P\}, \quad V_{(2,-2)} = \{(2, 2) \in P\}.$$

We concentrate on the examples $V_{(0,-1)}, V_{(1,-1)}$, which turn out to be generators of morphisms of degree one. For $V_{(0,-1)}$, the associated gradient vector

field is

$$\frac{x^1x^2}{4-x^2} \frac{\partial}{\partial x^1} - (x^2-1) \frac{\partial}{\partial x^2},$$

so the stable manifold turns out to be

$$S_{(0,1)} = \{(0, x^2)\}.$$

The point $(0, 1)$ is an interior point of $S_{(0,1)} \cap P \subset S_{(0,1)}$, so $V_{(0,-1)}$ turns out to be a generator of degree one. Similarly, the gradient vector field associated to $V_{(1,-1)}$ is

$$\begin{aligned} &\left(\frac{x^1x^2}{4-x^2} - 1\right) \frac{\partial}{\partial x^1} - (x^2-1) \frac{\partial}{\partial x^2} \\ &= \left(\frac{(x^1-3)(x^2-1) + (x^1-3) + 4(x^2-1)}{3-(x^2-1)}\right) \frac{\partial}{\partial x^1} - (x^2-1) \frac{\partial}{\partial x^2}, \end{aligned}$$

so we see that $S_{(3,1)}$ is of one-dimension around $(3, 1)$. In particular, we can check that

$$S_{(3,1)} = \{(x^1, x^2) \mid 4 - x^1 - x^2 = 0\}$$

since

$$\left(\frac{x^1x^2}{4-x^2} - 1\right) \frac{\partial}{\partial x^1} - (x^2-1) \frac{\partial}{\partial x^2} = (x^2-1) \left(\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\right)$$

on $4 - x^1 - x^2 = 0$. Thus, $V_{(1,-1)}$ also forms a generator of degree one.

These results of course agree with the structure of $DG(\mathbb{F}_1)$ where $H^1(DG(\mathbb{F}_1)(\mathcal{O}(0,0), \mathcal{O}(2,-2)))$ is of two dimension.

References

- [1] M. Abouzaid. Morse homology, tropical geometry, and homological mirror symmetry for toric varieties. *Selecta Mathematica*, 15(2):189–270, 2009.
- [2] D. Auroux. Mirror symmetry and T -duality in the complement of an anticanonical divisor. *J. Gökova Geom. Topol. GGT 1* (2007), 51–91.
- [3] A. I. Bondal and M. M. Kapranov. Enhanced triangulated categories. *Math. USSR-Sb.*, 70:93–107, 1991.

- [4] K. Chan. Holomorphic line bundles on projective toric manifolds from Lagrangian sections of their mirrors by SYZ transformations. *International Mathematics Research Notices*, 2009(24):4686–4708, 2009.
- [5] A. Elagin and V. Lunts. On full exceptional collections of line bundles on del Pezzo surfaces. *Moscow Mathematical Journal* **16** (2016), no. 4, 691–709.
- [6] B. Fang. Homological mirror symmetry is T-duality for \mathbb{P}^n . *Commun. Number Theory Phys.* **2** (2008), no. 4, 719–742.
- [7] K. Fukaya. Morse homotopy, A^∞ -category, and Floer homologies. In *Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993)*, volume 18 of *Lecture Notes Ser.*, pages 1–102. Seoul Nat. Univ., Seoul, 1993.
- [8] K. Fukaya. Asymptotic analysis, multivalued morse theory, and mirror symmetry. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 205–278, Providence, RI, 2005. Amer. Math. Soc.
- [9] K. Fukaya and Y.-G. Oh. Zero-loop open strings in the cotangent bundle and morse homotopy. *Asian J. Math.*, 1:96–180, 1997.
- [10] W. Fulton. *Introduction to toric varieties*. Number 131. Princeton University Press, 1993.
- [11] M. Futaki and H. Kajiura. Homological mirror symmetry of $\mathbb{C}P^n$ and their products via Morse homotopy. *J. Math. Phys.*, 62(3):032307, 21, 2021.
- [12] L. Hille and M. Perling. Exceptional sequences of invertible sheaves on rational surfaces. *Compositio Mathematica*, 147(4):1230–1280, 2011.
- [13] M. Kontsevich. Homological algebra of mirror symmetry. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.
- [14] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publishing, River Edge, NJ, 2001.

- [15] A. G. Kuznetsov. Exceptional collections in surface-like categories. *Sbornik: Mathematics*, 208(9):1368, 2017.
- [16] N. C. Leung. Mirror symmetry without corrections. *Communications in Analysis and Geometry*, 13(2):287–331, 2005.
- [17] N. C. Leung, S.-T. Yau, and E. Zaslow. From special Lagrangian to hermitian-Yang-Mills via Fourier-Mukai transform. *Adv. Theor. Math. Phys.* 4 (2000), no. 6, 1319–1341.
- [18] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nucl. Phys. B*, 479:243–259, 1996.

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