

Categorical wall-crossing in Landau-Ginzburg models

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Dedicated to Professor Sergio Cecotti on the occasion of joining BIMSA as a faculty member

We describe how categorical BPS data including chain complexes of solitons, CPT pairings, and interior amplitudes jump across a wall of marginal stability in two-dimensional $\mathcal{N} = (2, 2)$ models. We show that our jump formulas hold if and only if the A_∞ -categories of $\frac{1}{2}$ -BPS branes constructed on either side of the wall are homotopy equivalent. These results can be viewed as categorical enhancements of the Cecotti-Vafa wall-crossing formula.

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1. Introduction and outline

BPS states have played an important role in many aspects of physical mathematics. As is very well-known, the spaces of BPS states can jump discontinuously as physical parameters are varied, a phenomenon known as wall-crossing. Investigations of BPS wall-crossing have led to a wide variety of very interesting developments. For some reviews of BPS wall-crossing see [Cec, KoSo2, KoSo3, M1, N, Pio].

BPS wall-crossing appears in two-dimensional quantum field theories with $\mathcal{N} = (2, 2)$ supersymmetry, where it was first discovered [CFIV, CV1] as well as in four-dimensional supergravity and field theory with $\mathcal{N} = 2$ supersymmetry [DM, Dor, GMN1, LY, SW]. It also appears in a more elaborate form in coupled 2d-4d systems [GMN3].

Indeed, there are quantitative formulae expressing how BPS indices change across walls of marginal stability. It is natural to ask if one can obtain more refined information about the spaces of BPS states. For example, if BPS states are identified with the cohomology of some chain complexes one would like to know how the chain complexes themselves jump across walls of marginal stability. One cannot expect an answer at the level of chain complexes *per se*, since homotopy equivalent chain complexes are also physically equivalent, but it is meaningful to ask how the equivalence class of the chain complexes (up to homotopy) changes.¹ In particular, relating the homotopy equivalence class of chain complexes across a marginal stability wall allows us, by taking cohomology, to answer *How do the BPS Hilbert spaces jump across a wall of marginal stability?* This is the question a categorified wall-crossing formula is meant to answer.

The present paper addresses the categorification of the renowned Cecotti-Vafa wall-crossing formula for BPS indices in two-dimensional $\mathcal{N} = (2, 2)$ quantum field theory. We have made use of a formalism developed in [GMW, GMWSh], specifically for the purpose of carrying out the program of categorification of wall-crossing formulae. Indeed, in [GMW, GMWSh] it was explained how to categorify the so-called “framed wall-crossing” or “S-wall-crossing” formulae in the two-dimensional models. The present paper adds to the story with an improved understanding of how to phrase the categorification of the Cecotti-Vafa wall-crossing formula.

¹Note that the homotopy class of a chain complex contains more information than the index. As a simple example, consider

$$(1) \quad C = (\mathbb{Z} \oplus \mathbb{Z}[1], d = 0)$$

and

$$(2) \quad C' = (\mathbb{Z} \oplus \mathbb{Z}[1], d')$$

where d' maps a generator of \mathbb{Z} to a generator of $\mathbb{Z}[1]$. Both have vanishing Euler characteristics

$$(3) \quad \chi(C) = \chi(C') = 0,$$

but their cohomology is different so they are not homotopy equivalent.

Much remains to be done in the program of the categorification of wall-crossing formulae. In particular, the categorification of the four-dimensional wall-crossing formula of Kontsevich-Soibelman is not known.² We believe an important step forward is to include twisted masses in two-dimensional Landau-Ginzburg models. This is work in progress and we hope to post a paper on the subject in the near future.

In the remainder of this introduction we outline in more detail the difficulties which must be overcome to categorify the Cecotti-Vafa wall-crossing formula, and how we will achieve this.

1.1. A failure of naive categorification

Supposing that i, j, k denote distinct massive vacua of a two-dimensional $\mathcal{N} = (2, 2)$ theory, recall that the Cecotti-Vafa wall-crossing formula states that across a wall of marginal stability of type ijk , the BPS indices μ and μ' on either side of the wall are related by

$$(4) \quad \mu'_{ij} = \mu_{ij},$$

$$(5) \quad \mu'_{jk} = \mu_{jk},$$

$$(6) \quad \mu'_{ik} = \mu_{ik} \pm \mu_{ij}\mu_{jk},$$

the sign accounting for which way the wall-crossing occurred. As a first step in categorification, it's indeed encouraging, as we recall in Section 3, that for Landau-Ginzburg models one can formulate finite-dimensional chain complexes (R_{ij}, d_{ij}) such that the BPS index μ_{ij} is given by a graded trace

$$(7) \quad \mu_{ij} = \text{Tr}_{R_{ij}}(-1)^F.$$

The BPS Hilbert space³ of type ij is isomorphic to the d_{ij} -cohomology,

$$(8) \quad \mathcal{H}_{ij}^{\text{BPS}} = H^\bullet(R_{ij}, d_{ij}).$$

²The change in the 4d BPS state spaces is nicely understood using the halo formalism of [ADJM, DM, GMN2]. In some sense, this answers the question of the categorification of wall-crossing formulae, but the categorification program is more ambitious, and seeks to describe the full set of BPS states on either side of the wall in homotopical algebra terms.

³Throughout this paper, we have factored out the (super)translational mode of the soliton. With it included the chain complex will be

$$(8) \quad \tilde{R}_{ij} = R_{ij} \otimes (\mathbb{Z}[-1] \oplus \mathbb{Z}),$$

and the BPS index would be the “new index” $\text{Tr}_{\tilde{R}_{ij}}(F(-1)^F)$ of [CFIV]. The spectrum of F on R_{ij} lies in a \mathbb{Z} -torsor, so after a suitable phase redefinition, the μ_{ij} will be integers.

A categorified wall-crossing formula should then relate the BPS chain complexes (R'_{ij}, d'_{ij}) upon crossing a wall of marginal stability to the original chain complexes (R_{ij}, d_{ij}) . The simplest guess consistent with (6) is to say that the underlying vector spaces of the chain complexes are related by

$$(10) \quad R'_{ij} = R_{ij},$$

$$(11) \quad R'_{jk} = R_{jk},$$

$$(12) \quad R'_{ik} = R_{ik} \oplus (R_{ij} \otimes R_{jk}),$$

accompanied possibly with a degree shift on the $(R_{ij} \otimes R_{jk})$ summand to account for which way the wall-crossing occurred. The simplest differentials that one can guess on the primed spaces are

$$(13) \quad d'_{ij} = d_{ij},$$

$$(14) \quad d'_{jk} = d_{jk},$$

$$(15) \quad d'_{ik} = d_{ik} \oplus (d_{ij} \otimes 1 + 1 \otimes d_{jk}).$$

Indeed, the Cecotti-Vafa statement (6) would follow as a corollary from this guess, simply by taking graded traces. Under this formula for the differentials, the primed BPS Hilbert spaces are simply

$$(16) \quad (\mathcal{H}_{ij}^{\text{BPS}})' \cong \mathcal{H}_{ij}^{\text{BPS}},$$

$$(17) \quad (\mathcal{H}_{jk}^{\text{BPS}})' \cong \mathcal{H}_{jk}^{\text{BPS}},$$

$$(18) \quad (\mathcal{H}_{ik}^{\text{BPS}})' \cong \mathcal{H}_{ik}^{\text{BPS}} \oplus (\mathcal{H}_{ij}^{\text{BPS}} \otimes \mathcal{H}_{jk}^{\text{BPS}}).$$

Things are not so simple: it is very easy to construct counter-examples to this naive prediction of how BPS Hilbert spaces jump across a wall of marginal stability. Here is a simple one.

Consider the quartic Landau-Ginzburg model, namely the theory of a chiral superfield Φ with superpotential

$$(19) \quad W = \frac{1}{4}\Phi^4 - \Phi.$$

Denote the three vacua $\Phi_1 = e^{-2\pi i/3}$, $\Phi_2 = 1$, $\Phi_3 = e^{2\pi i/3}$ with corresponding critical values W_1, W_2, W_3 . One can show that the absolute number of solitons is 1 between each pair of distinct vacua. By taking into account the fermion degree we have that

$$(20) \quad R_{12} = \mathbb{Z},$$

$$(21) \quad R_{23} = \mathbb{Z},$$

$$(22) \quad R_{13} = \mathbb{Z},$$

with all differentials identically zero. We can vary the lower order terms of the superpotential (for instance we can turn on a quadratic term) so that W_2 passes through the line connecting W_1 and W_3 . The naive guess implies that upon this wall-crossing the chain complex R'_{13} is

$$(23) \quad R'_{13} = R_{13} \oplus (R_{12} \otimes R_{23})[1],$$

$$(24) \quad = \mathbb{Z} \oplus \mathbb{Z}[1].$$

Because every differential in sight acts trivially, we conclude that $(\mathcal{H}_{13}^{\text{BPS}})'$ is two-dimensional. On the other hand, every Landau-Ginzburg model with target space \mathbb{C} and a polynomial superpotential has an absolute number of solitons between each pair of critical points given by either 0 or 1.⁴ Thus the cohomology in such a model is either trivial or one-dimensional and we have found a contradiction. Our naive attempt at categorification has failed.

1.2. Missing instantons

The reason for the failure of the differential d'_{ik} (15) is simple, but also interesting: We have missed instantons.

The spaces R_{ij} are made of perturbative BPS states $|\phi_{ij}\rangle$ coming from quantizing around a classical soliton ϕ_{ij} . The differentials d_{ij} on R_{ij} are meant to encode matrix elements

$$(25) \quad \langle \phi_{ij}^b | \mathcal{Q}_{ij} | \phi_{ij}^a \rangle,$$

where the superscripts a, b label different classical solitons of type ij . When these are non-zero there is a difference between the exact ground states and the perturbative ones. We know from the relation between Morse theory and supersymmetry [Wit2], that the former are computed by considering suitable instantons between these perturbative ground states. Now within a fixed sector, say the ij -sector, solutions of such an instanton on the plane look as in Figure 1: The soliton ϕ_{ij}^a is stationary, sitting at a fixed point x_0 , whereas at an instant τ_0 , we transition from the ϕ_{ij}^a to ϕ_{ij}^b . Such a process will contribute to the matrix element if the fermion numbers of ϕ_{ij}^a and ϕ_{ij}^b differ by 1.

⁴For a proof see Appendix C.

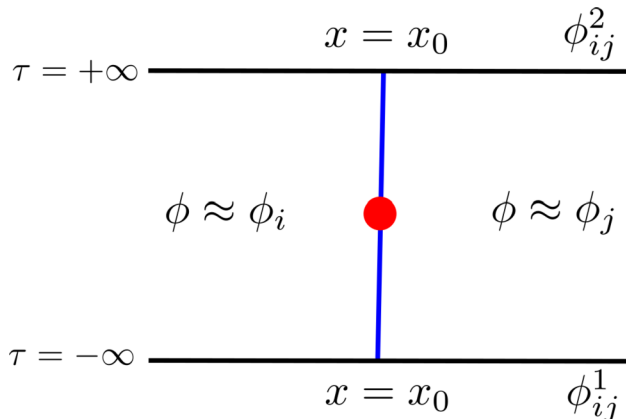


Figure 1: An instanton interpolating between two-different ij -solitons.

Close to a wall of marginal stability, it is reasonable to postulate that bound states of ij and jk -solitons give rise to an approximate ik -soliton, post wall-crossing, thus giving our guess (12). Instantons of the sort depicted in Figure 1, contribute to matrix elements of the type

$$(26) \quad \langle \phi_{ik}^b | \mathcal{Q}_{ik} | \phi_{ik}^a \rangle$$

and

$$(27) \quad \langle \phi_{ij}^a, \phi_{jk}^b | \mathcal{Q}_{ik} | \phi_{ij}^{a'}, \phi_{jk}^{b'} \rangle.$$

Such contributions are indeed reflected in our guess for the differential (15). Our formula for d'_{ik} has made an implicit assumption that the off-diagonal matrix element

$$(28) \quad \langle \phi_{ij}^a, \phi_{jk}^b | \mathcal{Q}_{ik} | \phi_{ik}^c \rangle$$

vanishes. However, it turns out, as we will explain in Section 3.2 that in addition to the familiar instanton of Figure 1, there can be a more interesting object, where a stationary ik -soliton can split into ij and jk solitons traveling at just the correct angles to preserve \mathcal{Q}_{ik} -supersymmetry. Such an instanton is depicted in Figure 2. Counting instantons of this type allows one to write down a corrected differential on R'_{ik} . This is the main new ingredient that enters the categorified wall-crossing formula.

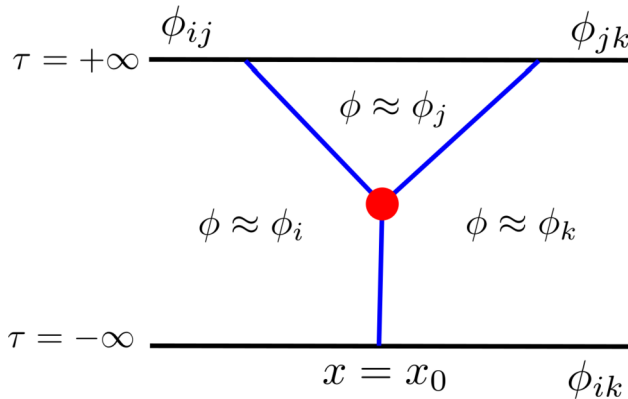


Figure 2: An instanton which contributes to an off-diagonal element of d'_{ik} .

1.3. Wall-crossing invariants

In order to derive wall-crossing formulas such as (6) it is extremely useful to introduce certain *wall-crossing invariants*. For Cecotti-Vafa wall-crossing an example of such a wall-crossing invariant is the spectrum generator⁵

$$(29) \quad S = \prod_{Z_{ij} \in \mathbb{H}}^{\curvearrowright} (\mathbf{1} + \mu_{ij} e_{ij}) \in SL(|\mathbb{V}|, \mathbb{Z})$$

which must be invariant under crossing marginal stability walls [KoSo1], so long as no BPS rays enter or exit the half-plane \mathbb{H} . The wall-crossing invariant S has a simple conceptual meaning. One can show that S_{ij} is the Witten index of the space of boundary local operators at a junction of thimbles of type i and j [GMW] (a related interpretation appeared in [HIV]), see Figure 3. Such a space is insensitive to marginal stability walls. Nonetheless the BPS indices S at a given point in parameter space allow the computation of the boundary Witten indices S . Comparing S on different sides of the wall of marginal stability leads to (6).

It is natural then to expect that a categorical wall-crossing invariant can also be constructed. The invariance of S is categorically enhanced as follows.

⁵Notation: \mathbb{V} is the vacuum set, assumed to be finite in this paper. \mathbb{H} is the upper-half plane, Z_{ij} are central charges and e_{ij} is the ij elementary matrix. \curvearrowright is meant to indicate a clockwise ordered product with respect to the central charges. Implicit in the notation is that an ordering on \mathbb{V} has been chosen.

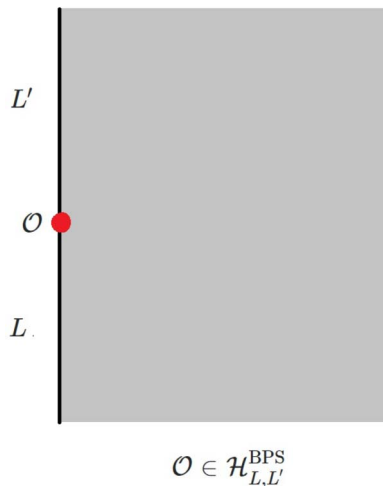


Figure 3: A boundary local operator \mathcal{O} between two branes L and L' .

The BPS chain complexes (R_{ij}, d_{ij}) , along with counts of ζ -instantons of the type depicted in Figure 2, allow for the construction of an A_∞ -category $\widehat{R}[X, W]$ whose objects can be thought of thimble branes⁶ and morphisms are vector spaces of boundary local operators at brane junctions [GMW].⁷ The categorical wall-crossing constraint is then formulated as follows.

The homotopy class of $\widehat{R}[X, W]$ is a wall-crossing invariant.

In the above statement *homotopy class* refers to the homotopy equivalence of A_∞ -categories which is defined in Appendix B. We show how our categorical wall-crossing formula can be derived from this wall-crossing constraint in Section 6.

Remark. Note that instead of $\widehat{R}[X, W]$, there are other wall-crossing invariants one could have used as a starting point. For instance instead of imposing

⁶Note that considering a category with only thimble objects is not restrictive. $\widehat{R}[X, W]$ can be enlarged to a triangulated A_∞ category for which the thimble objects provide a semi-orthogonal decomposition.

⁷The A_∞ category of [GMW] can be viewed as an infrared construction of the category of A-branes in a Landau-Ginzburg model, which to mathematicians is known as the Fukaya-Seidel category [Seid] of (X, W) , and is denoted by $FS[X, W]$. It is expected that $FS[X, W]$ and $\widehat{R}[X, W]$ are quasi-isomorphic as A_∞ -categories. An outline of a proof of this expectation was given in [GMW].

A_∞ -equivalence of the “open string algebra” $\widehat{R}[X, W]$ across a marginal stability wall like we do in this paper, one could have imposed L_∞ -equivalence of the closed string algebra R_c , defined in [GMW]. Another way of describing the categorical wall-crossing formula makes use of half-BPS interfaces. These can be used to construct a categorical notion of a flat parallel transport on a bundle of categories of boundary conditions over the space of Morse superpotentials [GMW]. The absence of monodromy around contractible cycles that intersect walls of marginal stability implies a categorified version of the invariance of S defined in equation (29). This categorical equation can in turn be reduced to categorified braid relations. For details see [GMW, M2]. These superficially distinct starting points are all expected to lead to the same eventual result.

1.4. Outline of the paper

The outline of this paper is as follows. In Section 2, we recall the standard discussion of wall-crossing at the level of BPS indices. This is followed in Section 3 by a discussion of how to formulate chain complexes that categorify the BPS indices. The crucial concept of a ζ -instanton with fan boundary conditions is discussed and we formulate the statement of categorical wall-crossing by using counts of certain trivalent instantons in Section 4. After reviewing the construction of the A_∞ category of half-BPS branes associated to a Landau-Ginzburg model in Section 5, we show the equivalence of the categorical wall-crossing formula to the homotopy equivalence of A_∞ categories constructed on either side of a marginal stability wall in Section 6. After a brief digression on fermion degrees of a ζ -instanton in Section 7, we turn our attention to some examples that illustrate our formulas in Section 8. We conclude with some speculations in Section 9 and review some aspects of A_∞ -theory and homological algebra in Appendices B and A.

2. Wall-crossing of BPS indices

While our formulas are expected to hold for arbitrary massive two-dimensional $\mathcal{N} = (2, 2)$ theories (with a non-anomalous $U(1)_R$ -symmetry), it is simplest to work in the setting of Landau-Ginzburg models. A Landau-Ginzburg model is a supersymmetric sigma model with a Kähler manifold target X and a potential of the form

$$(30) \quad V = |dW|^2,$$

where $W : X \rightarrow \mathbb{C}$ is a holomorphic function known as the superpotential. More precisely, working in two-dimensional $\mathcal{N} = 2$ -superspace, we can use the Kähler structure on X to write D-terms

$$(31) \quad L_D = \int d^4\theta K(\Phi, \overline{\Phi}),$$

and the holomorphicity of W to write F-terms

$$(32) \quad L_F = \int d^2\theta W(\Phi) + \int d^2\overline{\theta} \overline{W}(\overline{\Phi}),$$

to get a Lagrangian

$$(33) \quad L = L_D + L_F,$$

invariant under two-dimensional $\mathcal{N} = (2, 2)$ Poincaré supersymmetry. The reader is encouraged to consult [MS1], whose notation we adopt, for more details. Various non-renormalization theorems [Seib] of W tell us that one can get great mileage simply by studying the superpotential and its various properties. One use of the superpotential W is that it is sufficient to study many aspects of BPS states.

Supposing that W only has a finite number of isolated singularities, a familiar argument shows that the classical energy in such a theory obeys the BPS bound,

$$(34) \quad E \geq |Z_{ij}|$$

where

$$(35) \quad Z_{ij} = W_i - W_j$$

and W_i denotes the critical value $W(\phi_i)$ of the critical point ϕ_i . Denoting the bosonic fields of the LG model as ϕ , the standard Bogomolny trick leads to the BPS equation

$$(36) \quad \frac{d\phi}{dx} = \nabla \operatorname{Re}(\zeta^{-1}W),$$

known as the ζ -soliton equation, ζ being an arbitrary phase. Solutions on \mathbb{R} with prescribed vacua ϕ_i and ϕ_j at the ends of \mathbb{R} can only exist if

$$(37) \quad \zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}.$$

Using intersection theory of vanishing cycles, it is possible to get a well-defined *signed* count of the number of BPS solitons in the ij -sector. Let

$$(38) \quad L_i(\zeta) = \{p \in X \mid \lim_{x \rightarrow -\infty} f_x^\zeta(p) = \phi_i\},$$

$$(39) \quad R_i(\zeta) = \{p \in X \mid \lim_{x \rightarrow +\infty} f_x^\zeta(p) = \phi_i\}$$

be the ascending and descending manifolds respectively, emanating from the critical point ϕ_i of the Morse function $\text{Re}(\zeta^{-1}W)$. f_x^ζ denotes the one-parameter map $f_x^\zeta : X \rightarrow X$ defined by the gradient vector field of $\text{Re}(\zeta^{-1}W)$. We then set

$$(40) \quad \mu_{ij} = L_i^- \circ R_j^+,$$

where $L_i^- = L_i(\zeta_{ji}e^{-i\epsilon})$ and $R_j^+ = R_j(\zeta_{ji}e^{+i\epsilon})$ and ϵ is a small positive number. The infinitesimal rotation ensures that the intersection is transversal.

The significance of μ_{ij} from the perspective of the $\mathcal{N} = 2$ field theory defined by (X, W) is that one can show [CFIV, CV1] that

$$(41) \quad \mu_{ij} = \text{Tr}_{\mathcal{H}_{ij}^{\text{BPS}}}(-1)^F F,$$

where F is the fermion number and

$$(42) \quad \mathcal{H}_{ij}^{\text{BPS}} = \ker(\mathcal{Q}_{ij}) \cap \ker(\overline{\mathcal{Q}}_{ij}),$$

where

$$(43) \quad \mathcal{Q}_{ij} = Q_- - \zeta_{ij}^{-1} \overline{Q}_+.$$

μ_{ij} is thus a supersymmetry protected index that counts the degeneracy of BPS states of type ij . Some of its elementary properties are as follows.

Metric independence. While the BPS soliton equation does depend on the Kähler metric on X , the BPS index μ_{ij} is metric-independent.

CPT. Reversing $x \rightarrow -x$ takes $F \rightarrow -F$ so that $\mu_{ij} = -\mu_{ji}$.

It is familiar that supersymmetric indices such as the Witten index are quantities that are piecewise constant in parameter space. For instance, we can consider the one-dimensional system given by the real superpotential

$$(44) \quad h = x^4 + \alpha x^2 + \beta x.$$

While the conventional partition function $Z = \text{Tr}(e^{-\beta H})$ of the system will be a very non-trivial function of α and β , the Witten index $I = \text{Tr}(-1)^F e^{-\beta H}$ is simply equal to +1,

$$(45) \quad I = 1,$$

irrespective of α and β . In contrast the behavior of the BPS index is more subtle.

Historically⁸ wall-crossing was first noticed by considering points in the parameter space of the Landau-Ginzburg model with

$$(46) \quad W = X^4 + t_1 X^2 + t_2 X$$

with distinct symmetry groups. Supposing we start out at $(t_1, t_2) = (0, 1)$, where the model is \mathbb{Z}_3 -symmetric, the latter permuting the three vacua. We can show that there is indeed a single soliton between each pair of distinct critical points,

$$(47) \quad \mu_{12} = 1,$$

$$(48) \quad \mu_{23} = 1,$$

$$(49) \quad \mu_{13} = 1,$$

a spectrum consistent with the \mathbb{Z}_3 symmetry. If we move slightly away from this point, the collection of numbers doesn't change. On the other hand at $(t_1, t_2) = (1, 0)$, the superpotential has \mathbb{Z}_2 -symmetry. Requiring a \mathbb{Z}_2 -symmetric spectrum requires that one of the solitons disappears and the BPS indices are

$$(50) \quad \mu'_{12} = 1,$$

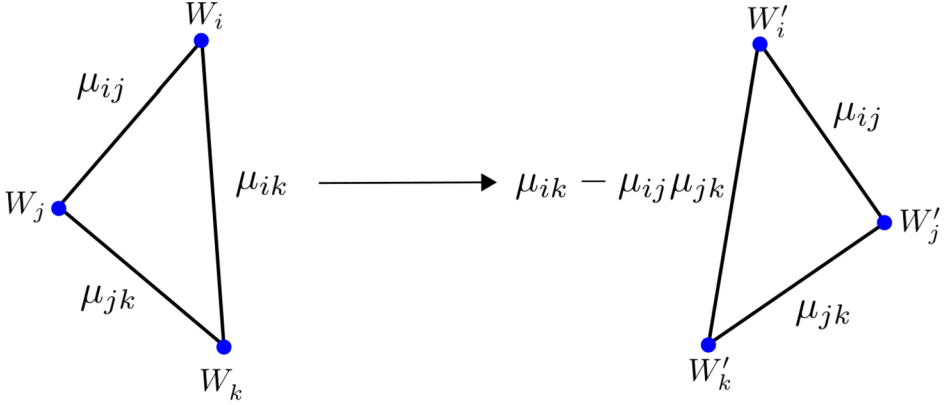
$$(51) \quad \mu'_{23} = 1,$$

$$(52) \quad \mu'_{13} = 0.$$

Thus BPS indices are examples of indices that are not constant but only piecewise constant.

The content of the Cecotti-Vafa formula is as follows. It first states that potential discontinuous jumps in the BPS spectrum can occur when three critical values W_i, W_j, W_k become co-linear as we vary parameters. This is the locus where $\text{Im}(Z_{ij}\overline{Z}_{jk}) = 0$. Next it gives an explicit formula for the

⁸We thank S. Cecotti for narrating this story.

Figure 4: Wall-crossing summarized in the W -plane.

quantitative nature of this jump: If μ and μ' denote BPS degeneracies on different sides of the wall of marginal stability, they must be related by

$$(53) \quad \mu'_{ij} = \mu_{ij},$$

$$(54) \quad \mu'_{jk} = \mu_{jk},$$

$$(55) \quad \mu'_{ik} = \mu_{ik} \pm \mu_{ij}\mu_{jk},$$

where the sign $-$ is picked in going from the negative side, where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to the positive side, where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ and the $+$ is picked in the reverse move. We summarize the formula from the perspective of the W -plane in Figure 4.

The trick in arguing for this is to consider not just BPS states, but rather to look at

$$(56) \quad \mathcal{Q}(\zeta) = Q_- - \zeta^{-1}\overline{Q}_+$$

preserving boundary conditions of our Landau-Ginzburg model when the latter is formulated on a half-space such as $(-\infty, 0] \times \mathbb{R}_t$. Such branes have been analyzed in great detail in references, [GMW, HIV]. One finds that the homology class of the support of these branes lives in the finite rank \mathbb{Z} -module

$$(57) \quad B(\zeta) := H_{\frac{1}{2}\dim(X)}^\perp(X, \text{Re}(\zeta^{-1}W) \rightarrow \infty; \mathbb{Z}).$$

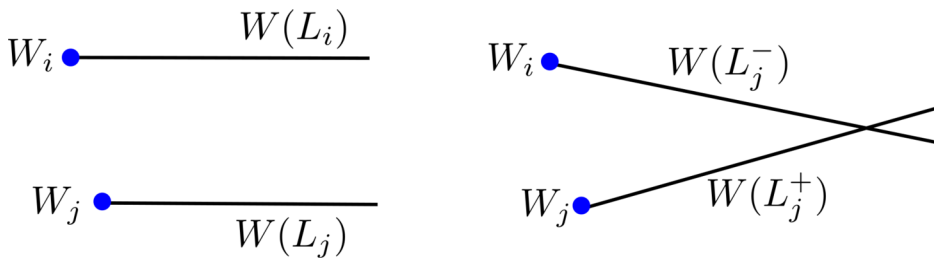


Figure 5: The topological intersection numbers $\hat{\mu}_{ij}$ obtained by looking at intersection numbers of slightly rotated thimbles.

We can equip $B(\zeta)$ with a natural bilinear form

$$(58) \quad \hat{\mu}^\zeta : B(\zeta) \times B(\zeta) \rightarrow \mathbb{Z},$$

defined as follows. When W is Morse, there is a natural \mathbb{Z} -module basis for $B(\zeta)$ given by the homology class of Lefschetz thimbles $\{[L_i(\zeta)]\}_{i \in \mathbb{V}}$. The thimble $L_i(\zeta)$ projects to half-infinite rays emanating from the critical value W_i in the ζ -direction. We then define

$$(59) \quad \hat{\mu}_{ij}^\zeta := \hat{\mu}(L_i, L_j) = L_i^- \circ L_j^+,$$

where L^\pm denote thimbles with phases slightly rotated by a small positive or negative angle respectively, as in Figure 5.

Some basic properties of $\hat{\mu}^\zeta$ are as follows. First: if i and j are distinct vacua, $\hat{\mu}_{ij}^\zeta$ and $\hat{\mu}_{ji}^\zeta$ cannot both be non-zero. In the case they are equal,

$$(60) \quad \hat{\mu}_{ii} = 1.$$

Finally, if the vacuum weights are ζ -generic,⁹ we can order the thimble basis in decreasing order of $\text{Im}(\zeta^{-1}W_i)$. Making this choice of ordering, we find that $\hat{\mu}^\zeta$ is an upper-triangular $|\mathbb{V}| \times |\mathbb{V}|$ matrix with +1 on the diagonal.

For definiteness and to avoid notational clutter we set $\zeta = 1$ and set $\hat{\mu} = \hat{\mu}^{\zeta=1}$. This is equivalent to choosing the half-plane in which we take phase ordered products to be the upper-half plane, as was done in (29).

The matrix representation $\hat{\mu}_{ij}$ for the bilinear form can be calculated from the BPS indices μ_{ij} by a nice rule expressed in terms of convex geometry.

⁹A set of critical values is called ζ -generic, following the terminology in [KKS], if none of the relative phases ζ_{ij} are equal to ζ .

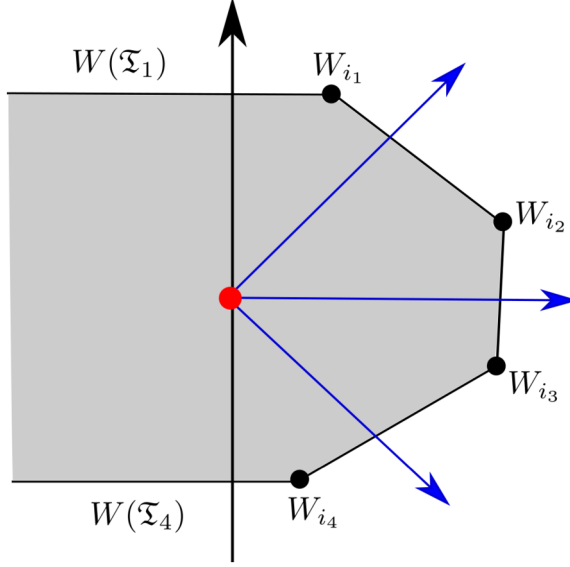


Figure 6: A half-plane fan $F_{i_1 i_4} = \{i_1, i_2, i_3, i_4\}$ for $\zeta = 1$ and the semi-infinite polygon it forms in the W -plane.

Definition. A half-plane fan F of phase ζ is a collection of vacua $F = \{i_1, \dots, i_n\}$ such that $W(F) = \{W_{i_1}, \dots, W_{i_n}\}$ are the clockwise-ordered vertices of a semi-infinite convex polygon going off to infinity in the $-\zeta$ -direction. See Figure 6 for an example with $n = 4$. The dual graph looks like a half-plane fan (and indeed has a space-time interpretation), hence the terminology.

To a given half-plane fan $F = \{i_1, i_2, i_3, \dots, i_n\}$ assign the number

$$(61) \quad \mu_F = \mu_{i_1 i_2} \mu_{i_2 i_3} \cdots \mu_{i_{n-1} i_n}.$$

We then make the

Claim.

$$(62) \quad \hat{\mu}_{ij} = \sum_{\substack{F_{ij} = \{i, i_1, \dots, i_k, j\} \\ F_{ij} \text{ half-plane fan}}} \mu_{i i_1} \cdots \mu_{i_k j}.$$

Proof. The proof is a straightforward inductive argument, where we induct on distance between i and j . To show the base case, for two neighboring

vacua $i < j$, one has $\widehat{\mu}_{ij} = \mu_{ij}$ due to (40).¹⁰ On the other hand there's only one polygon between two neighboring vacua, whose finite segment is given by the segment connecting them, to which we also assign μ_{ij} . For the inductive step, assume that the polygon rule (62) holds for vacua that are up to n units apart and consider a pair of vacua $\{i, j\}$ that are $n + 1$ units apart. We know that

$$(63) \quad L_i \circ \widetilde{L}_j = \mu_{ij}$$

where $\widetilde{L}_j := L_j(\zeta_{ij}e^{-i\epsilon})$. We thus want to compare \widetilde{L}_j with L_j^+ namely we must rotate this thimble in a clockwise direction by the phase of ζ_{ij} . In doing this rotation we pick up Picard-Lefschetz discontinuities: For each critical value W_k such that $\{i, k, j\}$ forms half-plane fan, we pick up a contribution of $L_k^+ \mu_{ki}$. Summing these up we get

$$(64) \quad L_j^+ = \widetilde{L}_j + \sum_{\substack{k \\ \{i, k, j\} \text{ is a fan}}} L_k^+ \mu_{ki}.$$

Thus we can compute that

$$(65) \quad \widehat{\mu}_{ij} = \mu_{ij} + \sum_{\substack{k \text{ s.t.} \\ \{i, k, j\} \text{ is a fan}}} \widehat{\mu}_{ik} \mu_{kj}.$$

The polygon rule applies to $\widehat{\mu}_{ik}$ so that

$$(66) \quad \widehat{\mu}_{ik} \mu_{kj} = \sum_{F_{ik}} \mu_{F_{ik}} \mu_{kj}.$$

On the other hand if $\{i, k, j\}$ is a fan, we can form an ij half-plane fan by taking the fan $F_{ik} = \{i, \dots, k\}$ and putting j at the end $\{i, \dots, k, j\}$. To this one precisely assigns $\mu_{F_{ik}} \mu_{kj}$. Conversely, every ij fan can be obtained in this way. \square

To see that this implies the wall-crossing formula, consider $\widehat{\mu}$ restricted to the three-dimensional $\{i, j, k\}$ space and note that if we are on the left side of Figure 4 then there is only one half-plane of type $ij \ ik \ jk$ respectively,

¹⁰Note that for ζ being the phase of an ij -soliton left-right intersection number of (40) agrees with the left-left intersection number of (59).

so that

$$(67) \quad \widehat{\mu} = \begin{pmatrix} 1 & \mu_{ij} & \mu_{ik} \\ 0 & 1 & \mu_{jk} \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other side of the wall we have two half-plane fans of type ik , depicted in Figure 10, leading us to write

$$(68) \quad \widehat{\mu} = \begin{pmatrix} 1 & \mu'_{ij} & \mu'_{ik} + \mu'_{ij}\mu'_{jk} \\ 0 & 1 & \mu'_{jk} \\ 0 & 0 & 1 \end{pmatrix}.$$

The two expressions for $\widehat{\mu}$ are equal if and only if the wall-crossing formula holds.

More generally suppose that $\{l, m\}$ is any pair of vacua such that there is a fan

$$(69) \quad F_{lm} = \{l, i_1, \dots, i, k, \dots, i_n, m\}$$

in which $\{i, k\}$ appears as a subset of consecutive vacua. Then on the other side of the wall, for every such fan, the set of lm -fans gains an additional fan obtained by taking F_{lm} and inserting j in between i and k . Moreover these are the only additional fans we gain, assuming we cross no other marginal stability walls in the move. Thus we compare

$$(70) \quad \mu_{li_1} \dots \mu_{ik} \dots \mu_{i_n m}$$

with

$$(71) \quad \mu_{li_1} \dots \mu'_{ik} \dots \mu_{i_n m} + \mu_{li_1} \dots \mu'_{ij} \mu'_{jk} \dots \mu_{i_n m}$$

and the two are equal if and only if the wall-crossing formula holds. Therefore we conclude that the wall-crossing formula is equivalent to the invariance of the bilinear form $\widehat{\mu}$ across a wall of marginal stability.

3. BPS chain complexes and ζ -instantons

3.1. BPS chain complexes

The chain complexes R_{ij} that categorify μ_{ij} can be formulated by using an infinite-dimensional version of Morse theory. Suppose that the symplectic

form ω on X is exact and choose a Liouville form λ so that $\omega = d\lambda$. We consider the (family of) “Morse” functions

$$(72) \quad h_\zeta[\phi] = \int_{\mathbb{R}} \phi^*(\lambda) + \operatorname{Im}(\zeta^{-1}W(\phi))dx$$

acting on the space

$$(73) \quad \mathcal{X}_{ij} = \{\phi : \mathbb{R} \rightarrow X \mid \lim_{x \rightarrow -\infty} \phi(x) = \phi_i \quad \lim_{x \rightarrow \infty} \phi(x) = \phi_j\}.$$

Generators. The critical points are the points where $\delta h_\zeta = 0$ which are solutions of the ζ -soliton equation

$$(74) \quad \frac{d\phi^I}{dx} = \frac{\zeta}{2} g^{I\bar{J}} \frac{\partial \overline{W}}{\partial \overline{\phi}^{\bar{J}}},$$

and so the critical point set is non-empty only for $\zeta = \zeta_{ji}$. The Morse function is actually not Morse because of the translational invariance of the soliton equation but we can mod out the solution space by this \mathbb{R} -action to obtain a (generically) finite set of critical points, in one-to-one correspondence with intersection points

$$(75) \quad L_i(\zeta_{ji}e^{-i\epsilon}) \cap R_j(\zeta_{ji}e^{i\epsilon}).$$

Thus we look to the pair

$$(76) \quad (\mathcal{X}_{ij}, h_{-\zeta_{ij}})$$

and assign a \mathbb{Z} -module R_{ij} with one generator for each solution of the ζ_{ji} -soliton equation

$$(77) \quad R_{ij} = \bigoplus_{p \in L_i^- \cap R_j^+} \mathbb{Z}\langle \phi_{ij}^p \rangle.$$

Gradations. Next we come to the subtle business of defining gradations on R_{ij} . The Fermion number, or homological degree of a generator in the Morse complex for a Morse function f as reviewed in [GMW, MS1] is given by

$$(78) \quad -\frac{1}{2} \sum_{\lambda \in \operatorname{Spec} \operatorname{Hess}(f(p))} \operatorname{sign}(\lambda),$$

where p is the critical point of f whose degree we're computing. To assign a degree to a ζ -soliton we must therefore compute the second derivative $\delta^2 h_\zeta$. Equivalently we may linearize the ζ -soliton equation (74) which leads to

$$(79) \quad D_x^{(1,0)} \delta \phi^I = \frac{\zeta}{2} g^{I\bar{J}} D_{\bar{J}} \partial_{\bar{K}} \bar{W} \delta \bar{\phi}^{\bar{K}}$$

where

$$(80) \quad D_x^{(1,0)} \delta \phi^I = \frac{\partial}{\partial x} \delta \phi^I + \Gamma_{JK}^I \frac{\partial \phi^J}{\partial x} \delta \phi^K$$

is the pullback connection on $\phi^*(T^{(1,0)}X)$. By considering also the complex conjugate of (79), we can write the linearized soliton equation as

$$(81) \quad D_\zeta \delta \phi = 0$$

where D_ζ is a Dirac type operator

$$(82) \quad D_\zeta : \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X)) \rightarrow \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X)).$$

Writing

$$(83) \quad \delta \phi \in \Gamma(\phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X))$$

as a column vector

$$(84) \quad \delta \phi = \begin{pmatrix} \delta \phi^I \\ \delta \bar{\phi}^{\bar{I}} \end{pmatrix}$$

the operator D_ζ reads¹¹

$$(85) \quad D_\zeta = \begin{pmatrix} \delta_J^I \partial_x + \Gamma_{JK}^I \partial_x \phi^K & 0 \\ 0 & \delta_{\bar{J}}^{\bar{I}} \partial_x + \Gamma_{\bar{J}\bar{K}}^{\bar{I}} \partial_x \bar{\phi}^{\bar{K}} \end{pmatrix} - \begin{pmatrix} 0 & \frac{\zeta}{2} g^{I\bar{K}} D_{\bar{K}} \partial_{\bar{J}} \bar{W} \\ \frac{\zeta^{-1}}{2} g^{\bar{I}K} D_K \partial_J W & 0 \end{pmatrix}.$$

The operator D_ζ is expressed a little more compactly by identifying

$$(86) \quad \phi^*(T^{(1,0)}X) \oplus \phi^*(T^{(0,1)}X) \cong \phi^*(TX),$$

¹¹Note that the operator (85) differs from that given in equation 12.6 of [GMW], v1. The authors of [GMW] forgot to include covariant derivatives.

where TX denotes the complexified tangent bundle. Choosing real coordinates indexed by $a = 1, \dots, \dim_{\mathbb{R}}(X)$, we can write

$$(87) \quad D_{\zeta} = \delta_b^a D_x - g^{ac} D_b \partial_c \operatorname{Re}(\zeta^{-1} W),$$

where

$$(88) \quad D_x \delta \phi^a = \partial_x \delta \phi^a + \Gamma_{bc}^a \partial_x \phi^b \delta \phi^c$$

is now the pullback connection on $\phi^*(TX)$. The Fermion number of an ij -soliton ϕ should thus be given by a regularized version of (78):

$$(89) \quad F(\phi) = - \lim_{\epsilon \rightarrow 0} \frac{1}{2} \sum_{\lambda \in \text{Eigenvalues}(D_{\zeta_{ji}}(\phi))} \operatorname{sign}(\lambda) e^{-\epsilon |\lambda|}$$

$$(90) \quad = -\frac{1}{2} \eta(D_{\zeta_{ij}}(\phi)).$$

One wants chain complexes $R_{ij}^{(1)}, R_{ij}^{(2)}$ constructed from two different choices of Kähler metrics $g^{(1)}, g^{(2)}$ (namely by a different choice of D-terms) to be homotopy equivalent

$$(91) \quad R_{ij}^{(1)} \simeq R_{ij}^{(2)}.$$

A necessary condition for this is that if we continuously interpolate between the metrics $g^{(1)}$ and $g^{(2)}$ and evolve the soliton $\phi^{(1)}$ solving the ζ -soliton equation for $g^{(1)}$ to $\phi^{(2)}$ a soliton for $g^{(2)}$ then their Fermion degrees must match. However the variational formula for the η -invariant says that

$$(92) \quad \frac{1}{2} \eta(D(\phi^{(1)}, g^{(1)})) - \frac{1}{2} \eta(D(\phi^{(2)}, g^{(2)})) = 2 \int_{\mathbb{R} \times [0,1]} \tilde{\phi}^* \left(\frac{1}{2\pi} \operatorname{Tr} \mathcal{R} \right),$$

where

$$(93) \quad \tilde{\phi} : \mathbb{R} \times [0, 1] \rightarrow X$$

is a path in \mathcal{X}_{ij} interpolating between $\phi^{(1)}$ and $\phi^{(2)}$, and

$$(94) \quad \frac{1}{2\pi} \operatorname{Tr} \mathcal{R}$$

is the Chern-Weil representative of $c_1(TX)$. This is nothing but a reminder that the LG model has an axial anomaly for arbitrary Kähler target. The

axial anomaly is traditionally expressed as the statement that the right hand side of (92) measures the net violation of Fermion number. The factor of two comes from taking into account the individual violations of both left and right moving fermions. Thus gradations are unchanged under metric variations only if X is Calabi-Yau. Otherwise to ensure this property we must grade R_{ij} by a cyclic group \mathbb{Z}_N such that the image of $2c_1(X)$ in $H^2(X, \mathbb{Z}_N)$ vanishes.

Differential. The differential d_{ij} is provided by counting (with signs) solutions of the ζ_{ji} -instanton equation

$$(95) \quad \partial_{\bar{s}} \phi^I = \frac{\zeta_{ji}}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

interpolating between solitons of fermion number differing by a unit. Here $s = x + i\tau$, where τ is the Euclidean time. Thus we get well-defined chain complexes (R_{ij}, d_{ij}) from which we can construct $\mathcal{H}_{ij}^{\text{BPS}}$ by taking cohomology

$$(96) \quad \mathcal{H}_{ij}^{\text{BPS}} \cong H^\bullet(R_{ij}, d_{ij}).$$

A ζ -instanton which contributes to the differential d_{ij} in spacetime looks like Figure 1. Physically we expect the following properties.

Metric dependence. BPS chain complexes constructed from two different choices of Kähler metrics should be homotopy equivalent.

CPT. Reversing the spatial coordinate, i.e the path $\phi_{ij}^p(-x)$ says that for every basis element ϕ_{ij}^p of R_{ij} we get an element ϕ_{ji}^p such that

$$(97) \quad \deg(\phi_{ji}^p) = 1 - \deg(\phi_{ij}^p).$$

The shift in degree by $+1$ is a technical consequence of factoring out the translational mode of the soliton. For more details on this point see the discussion in section 12.3 in [GMW]. In basis independent terms, CPT says that we have a degree -1 non-degenerate pairing

$$(98) \quad K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}.$$

3.2. ζ -instantons and interior amplitudes

As alluded to in the introduction, a categorified wall-crossing formula will involve certain “off-diagonal” maps

$$(99) \quad M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}$$

which allow construction of the correct differential. The construction of this map involves counting ζ -instantons with fan boundary conditions, which we now discuss.

We consider solutions of the ζ -instanton equation

$$(100) \quad \bar{\partial}_{\bar{s}} \phi^I = \frac{\zeta}{4} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

which look like a collection of “boosted solitons” at infinity. See [GMW] sections 14.1-14.2 and Appendix E for more details on such boundary conditions. Let

$$(101) \quad I = \{i_1, \dots, i_n\}$$

be a cyclic fan of vacua and

$$(102) \quad \phi = \{\phi_{i_1 i_2}, \dots, \phi_{i_n i_1}\}$$

be a fan of solitons. We want to consider ζ -instantons which support these particular solitons on the edges. I is a fan if and only if the critical values

$$(103) \quad W_I = \{W_{i_1}, \dots, W_{i_n}\}$$

are the clockwise ordered vertices of a convex polygon in the W -plane. Solutions of the ζ -instanton equation with fan boundary conditions are known as a domain-wall junctions and have been studied in [CHT, GT, INOS], and elsewhere. In particular, it was noted in [CHT], that just the way a ζ_{ij} -soliton maps to a line connecting W_i and W_j in the W -plane, a ζ -instanton maps to the interior of the convex polygon with W_I as vertices. See Figure 7 for an example with $n = 5$. This fact motivates the terminology BPS or gradient polygon for ϕ , as was introduced in [KKS].

Solutions of the ζ -instanton equation modulo translations with a fixed fan and fixed soliton collection ϕ supported on edges form a moduli space $\mathcal{M}_{\zeta}(\phi)$. Its dimension is given by forming the vector

$$(104) \quad e_{\phi} := \phi_{i_1 i_2} \otimes \dots \otimes \phi_{i_n i_1}$$

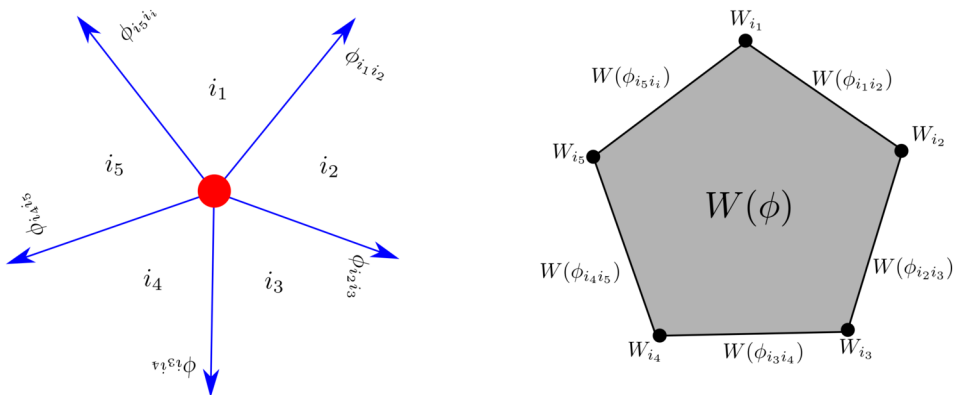


Figure 7: Boundary conditions for a ζ -instanton in the (x, τ) -plane on the left, and the image of such a ζ -instanton in the W -plane on the right.

in the cyclic tensor product

$$(105) \quad R_I = R_{i_1 i_2} \otimes \cdots \otimes R_{i_n i_1}$$

and considering its degree

$$(106) \quad F(\phi) := \deg(e_\phi).$$

The (virtual) dimension of these moduli spaces is [GMW]

$$(107) \quad \dim(\mathcal{M}_\zeta(\phi)) = F(\phi) - 2.$$

Moreover $\mathcal{M}_\zeta(\phi)$ can be oriented. In particular if $F(\phi) = 2$, we learn that the moduli space $\mathcal{M}_\zeta(\phi)$ is a collection of oriented points and thus we can get a well-defined signed count of ζ -instantons

$$(108) \quad N_\zeta(\phi) := \#\mathcal{M}_\zeta(\phi).$$

The integers $N(\phi)$ ¹² satisfy some miraculous identities. There is an identity corresponding to each cyclic fan.

¹²We can safely drop the ζ -subscript from the notation because the integers $N_\zeta(\phi)$ are ζ -independent

For a cyclic fan of length two, $\{i, j\}$ we have

$$(109) \quad \sum_{\substack{\chi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ij}, \chi_{ji})=2 \\ F(\chi_{ij}, \psi_{ji})=2}} N(\phi_{ij}, \chi_{ji}) N(\chi_{ij}, \psi_{ji}) = 0.$$

This is nothing but the identity that the differential d_{ij} counting ζ -instantons between ij -solitons is nilpotent, which is a familiar fact from Morse theory. It involves the fact that the moduli space $\mathcal{M}(\phi_{ij}, \psi_{ji})$ such that $d(\phi_{ij}, \psi_{ji}) = 3$ has ends corresponding to broken flow lines gluing intermediate instantons.

For $\{i, k, j\}$ a cyclic fan of vacua of length three, we have the identity

$$(110) \quad \begin{aligned} & \sum_{\substack{\xi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ik}, \psi_{kj}, \xi_{ji})=2 \\ F(\xi_{ij}, \chi_{ji})=2}} N(\phi_{ik}, \psi_{kj}, \xi_{ji}) N(\xi_{ij}, \chi_{ji}) \\ & + \sum_{\substack{\xi_{jk} \in L_j^- \cap R_k^+ \\ F(\chi_{ji}, \phi_{ik}, \xi_{kj})=2 \\ F(\xi_{jk}, \psi_{kj})=2}} N(\chi_{ji}, \phi_{ik}, \xi_{kj}) N(\xi_{jk}, \psi_{kj}) \\ & + \sum_{\substack{\xi_{ik} \in L_i^- \cap R_k^+ \\ F(\xi_{ik}, \psi_{kj}, \chi_{ji})=2 \\ F(\phi_{ik}, \xi_{ki})=2}} N(\phi_{ik}, \xi_{ki}) N(\xi_{ik}, \psi_{kj}, \chi_{ji}) = 0. \end{aligned}$$

The argument for this involves looking at the ends of the moduli space

$$(111) \quad \mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji})$$

of a fan of solitons such that $F(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 3$. There are three types of ends, where a rigid instanton of type $\{i, k\}$ is glued to a rigid instanton of type $\{i, k, j\}$, similarly for $\{i, j\}$ and $\{j, k\}$. See Figure 8. Such “broken flows” give

$$(112) \quad \begin{aligned} \partial \mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji}) = & \bigsqcup_{\substack{\xi_{ij} \in L_i^- \cap R_j^+ \\ F(\phi_{ik}, \psi_{kj}, \xi_{ji})=2 \\ F(\xi_{ij}, \chi_{ji})=2}} \mathcal{M}(\phi_{ik}, \psi_{kj}, \xi_{ji}) \times \mathcal{M}(\xi_{ij}, \chi_{ji}) \\ & \sqcup \bigsqcup_{\substack{\xi_{jk} \in L_j^- \cap R_k^+ \\ F(\chi_{ji}, \phi_{ik}, \xi_{kj})=2 \\ F(\xi_{jk}, \psi_{kj})=2}} \mathcal{M}(\chi_{ji}, \phi_{ik}, \xi_{kj}) \times \mathcal{M}(\xi_{jk}, \psi_{kj}) \end{aligned}$$

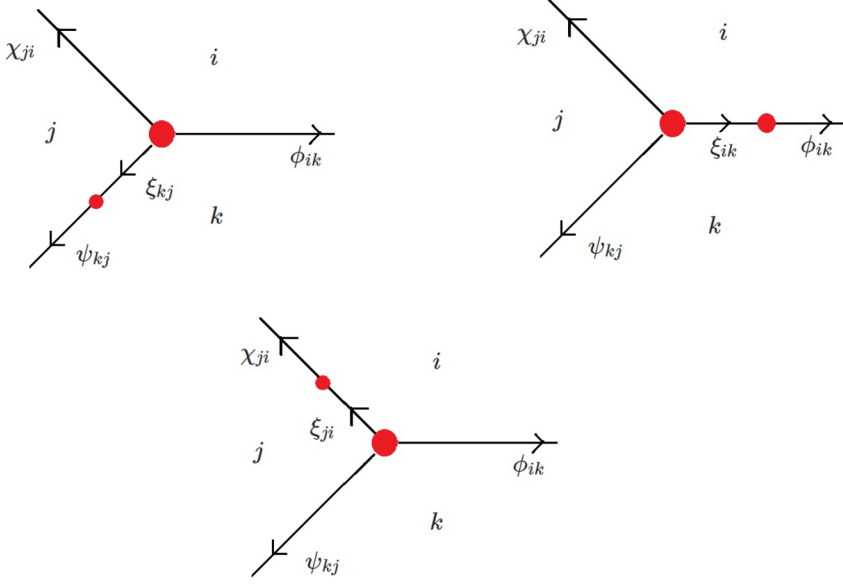


Figure 8: The various ends of $\mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji})$ where $F(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 3$.

$$\sqcup \bigsqcup_{\substack{\xi_{ik} \in L_i^- \cap R_k^+ \\ F(\xi_{ik}, \psi_{kj}, \chi_{ji})=2 \\ F(\phi_{ik}, \xi_{ki})=2}} \mathcal{M}(\phi_{ik}, \xi_{ki}) \times \mathcal{M}(\xi_{ik}, \psi_{kj}, \chi_{ji}).$$

(110) then follows from

$$(113) \quad \#\partial\mathcal{M}(\phi_{ik}, \psi_{kj}, \chi_{ji}) = 0.$$

More generally, one expects that the moduli spaces $\mathcal{M}_\zeta(\phi)$ can be compactified, such that the compactified moduli space $\overline{\mathcal{M}}_\zeta(\phi)$ has strata labeled by web diagrams of the type in Figure 8.

Although the identities (109) and (110) are all we need for categorical wall-crossing, we should mention for completeness that there are more complicated identities involving fans of longer length which can be deduced from the web combinatorics of [GMW]. The summary is that all identities follow from a single L_∞ -Maurer-Cartan equation. Form the vector space¹³

$$(114) \quad R_c = \oplus_I R_I$$

¹³ $R_i \cong \mathbb{Z}$.

$$(115) \quad = \oplus_{i \in \mathbb{V}} R_i \oplus_{i \neq j} (R_{ij} \otimes R_{ji}) \oplus \cdots$$

corresponding to taking all possible cyclic tensor products. R_c has the structure of an L_∞ -algebra. Namely there are maps

$$(116) \quad \rho(\mathfrak{t}) : S_+ R_c \rightarrow R_c,$$

where $S_+ R_c$ denotes (the positive part of) the symmetric algebra, satisfying L_∞ -axioms. $\rho(\mathfrak{t})$ is defined through **taut webs** as in [GMW]. Define

$$(117) \quad \beta_I := \sum_{\substack{\phi \text{ gradient polygons for } I \\ F(\phi)=2}} N(\phi) e_\phi,$$

and let

$$(118) \quad \beta := \sum_I \beta_I \in R_c.$$

One of the main results of [GMW] is that analysis of various moduli spaces leads one to conclude that β is a Maurer-Cartan element for the L_∞ -structure. Namely it satisfies the L_∞ Maurer-Cartan equation

$$(119) \quad \rho(\mathfrak{t})(e^\beta) = 0.$$

β was called the **interior amplitude** in [GMW]. The identities (109), (110) are some simple equations that come from unpacking the L_∞ Maurer-Cartan equation.

Remark. In general interior amplitudes will have components associated to arbitrary fans

$$(120) \quad \beta_{i_1 i_2 \dots i_n} \in R_{i_1 i_2} \otimes R_{i_2 i_3} \otimes \cdots \otimes R_{i_n i_1}.$$

However, only the trivalent components associated to the “wall-crossing triangle”; β_{ikj} on one side and β'_{ijk} on the other, enter the discussion in categorical wall-crossing.

3.3. Homotopy equivalence of BPS data

We have discussed the construction of the BPS chain complexes

$$(121) \quad \{(R_{ij}, d_{ij})\},$$

the contraction maps

$$(122) \quad \{K_{ij}\}$$

and the important vector encoding counts of rigid ζ -instantons

$$(123) \quad \beta \in R_c.$$

We have noted however that the BPS complexes by themselves are not physical observables, only their homotopy equivalence class is. It is natural to try to extend the notion of homotopy equivalence from the BPS complexes, to the full categorical BPS data, namely to introduce a natural notion of homotopy equivalence for the contraction pairings and interior amplitudes. We briefly formulate such a notion in this sub-section.

Suppose we are given another collection of BPS data $(\{S_{ij}\}, \{L_{ij}\}, \gamma)$ where S_{ij} denote complexes L_{ij} contraction maps, and γ is now a Maurer-Cartan element of the L_∞ -algebra S_c , constructed from S_{ij} and L_{ij} . We say that the BPS data

$$(124) \quad (\{R_{ij}\}, \{K_{ij}\}, \beta) \text{ and } (\{S_{ij}\}, \{L_{ij}\}, \gamma)$$

are **homotopy equivalent** if there are homotopy equivalences of chain complexes

$$(125) \quad f_{ij} : R_{ij} \rightarrow S_{ij}$$

that fit into a collection of maps

$$(126) \quad f_n : R_c^{\otimes n} \rightarrow S_c$$

with f_1 being induced canonically from the collection $\{f_{ij}\}$ that together define an L_∞ -equivalence from R_c to S_c . The maps $\{f_{ij}\}$ and the L_∞ -morphism $\{f_n\}$ must be such that the diagram

$$(127) \quad \begin{array}{ccc} R_{ij} \otimes R_{ji} & \xrightarrow{f_{ij} \otimes f_{ji}} & S_{ij} \otimes S_{ji} \\ & \searrow K_{ij} & \downarrow L_{ij} \\ & & \mathbb{Z} \end{array}$$

commutes up to homotopy, and the Maurer-Cartan element transports naturally:

$$(128) \quad f(e^\beta) \sim \gamma,$$

where \sim denotes gauge equivalence of Maurer-Cartan elements, defined in Appendix B.

The general philosophy of this paper is that we should only consider homotopy equivalence classes of the categorical BPS data. For example a D-term variation will only result in homotopy equivalent BPS data. The equivalence in this section can be viewed as a relaxation of the notion of strict isomorphism of categorical BPS data as defined in [GMW, section 4.1.1].

4. Statement of categorical wall-crossing

Notation. Given an element $r_{ik} \otimes r_{kj} \otimes r_{ji} \in R_{ik} \otimes R_{kj} \otimes R_{ji}$ we can define

$$(129) \quad M[r_{ik} \otimes r_{kj} \otimes r_{ji}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}$$

by using the contraction maps

$$(130) \quad M[r_{ik} \otimes r_{kj} \otimes r_{ji}](r'_{ij} \otimes r'_{jk}) = K_{ji}(r_{ji}, r'_{ij})K_{kj}(r_{kj}, r'_{jk})r_{ik}.$$

Similarly we define

$$(131) \quad M'[r_{ik} \otimes r_{kj} \otimes r_{ji}] : R_{ki} \rightarrow R_{kj} \otimes R_{ji}$$

by contracting the ik factor using K_{ik} , and using the Koszul sign rule. Finally the natural product rule differential on a tensor product chain complex of the form as $R_{ij} \otimes R_{jk}$ is denoted as d_{ijk} :

$$(132) \quad d_{ijk} = d_{ij} \otimes 1 \pm 1 \otimes d_{jk}.$$

When we write \pm it means we are not being precise about the exact sign.

Marginal stability wall. Recall an ijk wall of marginal stability is the locus where

$$(133) \quad \text{Im}(Z_{ij}\overline{Z}_{jk}) = 0.$$

See Figure 9.

Main statement. Let

$$(134) \quad (R_{ij}, R_{jk}, R_{ik}, \beta_{ikj})$$

be the chain complexes and interior amplitude component in a region where

$$(135) \quad \text{Im}(Z_{ij} \overline{Z}_{jk}) < 0,$$

and

$$(136) \quad (R'_{ij}, R'_{jk}, R'_{ik}, \beta'_{ijk})$$

be the chain complexes and interior amplitude component in a region where

$$(137) \quad \text{Im}(Z'_{ij} \overline{Z}'_{jk}) > 0.$$

Note that β_{ikj} defines a chain¹⁴ map

$$(138) \quad M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik},$$

and β'_{ijk} defines a chain map

$$(139) \quad M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}.$$

The categorical wall-crossing formula states that

$$(140) \quad R'_{ij} \simeq R_{ij},$$

$$(141) \quad R'_{jk} \simeq R_{jk},$$

$$(142) \quad R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}).$$

Furthermore, letting (P, Q) be the chain maps that implement the homotopy equivalence between the primed and unprimed sides, it states that the diagrams

$$(143) \quad \begin{array}{ccc} R'_{ik}[1] & \xrightarrow{M'[\beta'_{ijk}]} & R'_{ij} \otimes R'_{jk} \\ P \downarrow & & \downarrow P \otimes P \\ \text{Cone}(M[\beta_{ikj}])[1] & \xrightarrow{\pi} & R_{ij} \otimes R_{jk} \end{array}$$

¹⁴This follows from β being an interior amplitude, or equivalently, identity (110). The taut webs involved in this identity are the ones in Figure 8.

and

$$(144) \quad \begin{array}{ccc} R'_{ik}[1] & \xrightarrow{M'[\beta'_{ijk}]} & R'_{ij} \otimes R'_{jk} \\ Q \uparrow & & \uparrow Q \otimes Q \\ \text{Cone}(M[\beta_{ikj}])[1] & \xrightarrow{\pi} & R_{ij} \otimes R_{jk} \end{array}$$

commute up to homotopy.

Equivalently,

$$(145) \quad R_{ij} \simeq R'_{ij},$$

$$(146) \quad R_{jk} \simeq R'_{jk},$$

$$(147) \quad R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}),$$

and letting (S, T) be the chain maps implementing homotopy equivalence between the two sides, the diagrams

$$(148) \quad \begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik}, \\ T \otimes T \downarrow & & \downarrow T \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array}$$

and

$$(149) \quad \begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik}, \\ S \otimes S \uparrow & & \uparrow S \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array}$$

commute up to homotopy.

These formulas are also sufficient to relate the contraction maps. Given chain complexes

$$(150) \quad (R_{ij}, R_{jk}, R_{ik}, \beta_{ikj})$$

such that

$$(151) \quad \text{Im}(Z_{ij} \overline{Z}_{jk}) < 0,$$

the dual complexes (R_{kj}, R_{ji}, R_{ki}) will be a triple such that

$$(152) \quad \text{Im}(Z_{kj}\overline{Z}_{ji}) > 0.$$

Therefore the formulas for going from $\text{Im}(\cdots) > 0$ to $\text{Im}(\cdots) < 0$ imply that

$$(153) \quad R'_{kj} \simeq R_{kj},$$

$$(154) \quad R'_{ji} \simeq R_{ji},$$

$$(155) \quad R'_{ki} \simeq \text{Cone}(M'[\beta_{ikj}] : R_{ki}[1] \rightarrow R_{kj} \otimes R_{ji}).$$

Note that there is a canonical degree -1 map

$$(156) \quad L : \text{Cone}(M[\beta_{ikj}]) \otimes \text{Cone}(M'[\beta_{ijk}]) \rightarrow \mathbb{Z}$$

given by

$$(157) \quad L = \begin{pmatrix} 0 & K_{ik} \\ K_{ij} \otimes K_{jk} & 0 \end{pmatrix}.$$

Denote the chain maps implementing the homotopy equivalence as \tilde{P}, \tilde{Q} . With this, the final part of categorical wall-crossing also determines the homotopy class of the contraction maps, by stating that the diagrams

$$(158) \quad \begin{array}{ccc} R_{ij} \otimes R_{ji} & \xrightarrow{K_{ij}} & \mathbb{Z} \\ \uparrow P \otimes \tilde{P} & \nearrow K'_{ij} & \\ R'_{ij} \otimes R'_{ji} & & \end{array}$$

$$(159) \quad \begin{array}{ccc} R_{jk} \otimes R_{kj} & \xrightarrow{K_{jk}} & \mathbb{Z} \\ \uparrow P \otimes \tilde{P} & \nearrow K'_{jk} & \\ R'_{jk} \otimes R'_{kj} & & \end{array}$$

$$(160) \quad \begin{array}{ccc} C(M) \otimes C(M') & \xrightarrow{L} & \mathbb{Z} \\ \uparrow P \otimes \tilde{P} & \nearrow K'_{ik} & \\ R'_{ik} \otimes R'_{ki} & & \end{array}$$

commute up to homotopy. In the above we have abbreviated $\text{Cone}(M[\beta_{ikj}])$ and $\text{Cone}(M'[\beta'_{ijk}])$ as $C(M)$ and $C(M')$ respectively. There will be similar diagrams with (Q, \tilde{Q}) .

Canonical representatives. In practice given the chain complexes on one side, one wants to work with specific representatives within the homotopy equivalence class of chain complexes (and chain maps) for the other. There is a canonical choice for this. Suppose we treat the primed side as unknown. Then the canonical representatives for the primed complexes are

$$(161) \quad R'_{ij} = R_{ij},$$

$$(162) \quad R'_{jk} = R_{jk},$$

$$(163) \quad R'_{ik} = \text{Cone}(M[\beta_{ikj}]).$$

By letting P, Q to be identity maps, we can then make the diagrams (143), (144), strictly commute by letting

$$(164) \quad M'[\beta'_{ijk}] = \pi,$$

which is equivalent to saying that

$$(165) \quad \beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1}.$$

The canonical representatives for the dual complexes are

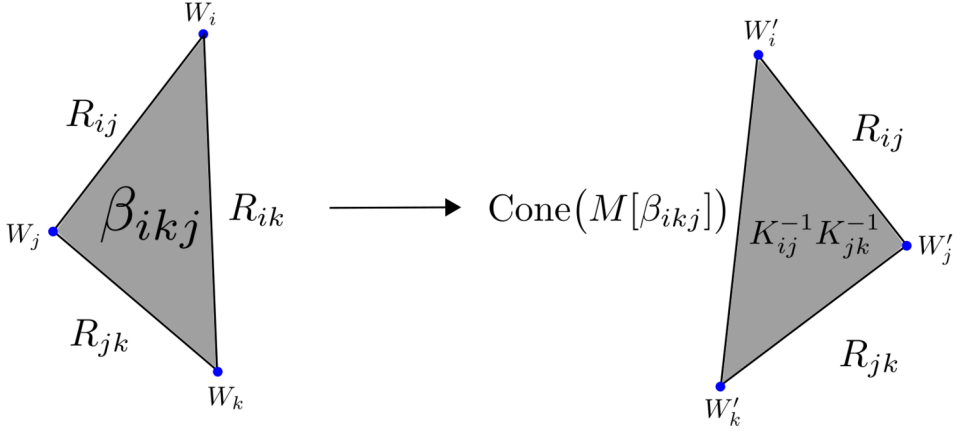
$$(166) \quad R'_{kj} = R_{kj},$$

$$(167) \quad R'_{ji} = R_{ji},$$

$$(168) \quad R'_{ki} = \text{Cone}(M'[\beta_{ikj}])$$

and one can then set the contraction maps to be

$$(169) \quad K'_{ij} = K_{ij},$$

Figure 9: Categorical wall-crossing summarized in the W -plane.

$$(170) \quad K'_{jk} = K_{jk},$$

$$(171) \quad K'_{ik} = \begin{pmatrix} 0 & K_{ik} \\ K_{ij} \otimes K_{jk} & 0 \end{pmatrix}.$$

Figure 9 summarizes the categorical wall-crossing formula for going from a point in parameter space with $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ from the perspective of the W -plane. The formulas and the figure summarizing the specific representatives in the inverse move would look similar. These straightforward details are left for the reader.

Remark: consistency check. A consistency check our formulas must pass is whether jumping from the negative side of the wall of marginal stability where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$ to the positive side where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ and then jumping back to the negative side is equivalent to doing nothing. We work with the canonical representatives. Starting from the complex R_{ik} the wall-crossing formula says that

$$(172) \quad R'_{ik} = \text{Cone}(M[\beta_{ijk}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}),$$

and

$$(173) \quad \beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1}.$$

Jumping back to the right side, gives us

$$(174) \quad R''_{ik} = \text{Cone}(M'[K_{ij}^{-1} K_{jk}^{-1}] : \text{Cone}(M[\beta_{ikj}])[1] \rightarrow R_{ij} \otimes R_{jk}).$$

But

$$(175) \quad M'[\beta'_{ijk}] = \pi$$

and therefore we have

$$(176) \quad R''_{ik} = \text{Cone}(\pi : \text{Cone}(M[\beta] : R_{ij} \otimes R_{jk} \rightarrow R_{ik})[1] \rightarrow R_{ij} \otimes R_{jk})$$

$$(177) \quad = \text{Cyl}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik})$$

$$(178) \quad \simeq R_{ik}.$$

The cylinder construction of homological algebra, used above is described in Appendix A. Therefore we end up with a complex canonically homotopy equivalent to the original complex. A similar check can be performed for β''_{ikj} . One shows that the diagram

$$(179) \quad \begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik} \\ & \searrow M[\beta''_{ikj}] & \downarrow i \\ & & \text{Cyl}(M[\beta_{ikj}]) \end{array}$$

commutes up to homotopy. This shows clearly the need to work at the level of homotopy equivalence.

In the next two sections we show how these conditions word-for-word are the homotopy equivalence of A_∞ categories constructed at a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) > 0$ compared to a point where $\text{Im}(Z_{ij}\overline{Z}_{jk}) < 0$.

5. ζ -instantons and brane categories

5.1. Bare thimble category

While the chain complex R_{ij} categorifies μ_{ij} , categorification of $\hat{\mu}_{ij}$ leads to more interesting structure. The correct viewpoint will be that B must be upgraded to a category, and $\hat{\mu}_{ij}$ will be categorified to vector spaces of morphisms.

The construction of the “bare” thimble category $\widehat{R}^{\text{bare}}$ proceeds as follows.

Objects. The objects are an ordered collection of thimbles

$$(180) \quad \mathfrak{T}_1, \dots, \mathfrak{T}_n,$$

one for each critical point $i \in \text{Crit}(W)$. They are ordered by $\text{Im}(-W)$ so that $i > j$ if $\text{Im}(W_i) < \text{Im}(W_j)$.

Morphisms. The morphisms are given as follows. In order to define¹⁵

$$(181) \quad \widehat{R}_{ij} := \text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j)$$

we look at all half-plane fans with “top” vacuum i and “bottom” vacuum j . To an edge separating i and j assign the vector space R_{ij} and take the (ordered) tensor product along each edge. Thus to each half-plane fan F_{ij} of this type we assign a vector space $R_{F_{ij}}$. The morphism space is then defined by taking direct sums over all F_{ij} half-plane fans

$$(182) \quad \widehat{R}_{ij} = \bigoplus_{F_{ij}} R_{F_{ij}}.$$

See Figure 10 for an example of a morphism space where two fans contribute. Note that

$$(183) \quad \widehat{R}_{ii} = \text{Hop}(\mathfrak{T}_i, \mathfrak{T}_i) = \mathbb{Z}.$$

If there are no half-plane fans then

$$(184) \quad \widehat{R}_{ij} = 0,$$

so that the objects $\{\mathfrak{T}_1, \dots, \mathfrak{T}_n\}$ are an exceptional collection; the matrix of morphism spaces \widehat{R}_{ij} is an upper-triangular matrix with \mathbb{Z} on the diagonal.

Compositions. An associative composition law

$$(185) \quad m_{ijk} : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}_{ik}$$

is given simply by looking at whether F_{jk} can be placed below F_{ij} to form a fan F_{ik} . If so, we take the tensor product of the vectors in $R_{F_{ij}}$ and $R_{F_{jk}}$ to get a vector in $R_{F_{ik}}$. If not, we set it equal to zero.

¹⁵ $\text{Hop}(A, B) := \text{Hom}^{\text{opp}}(A, B) = \text{Hom}(B, A).$

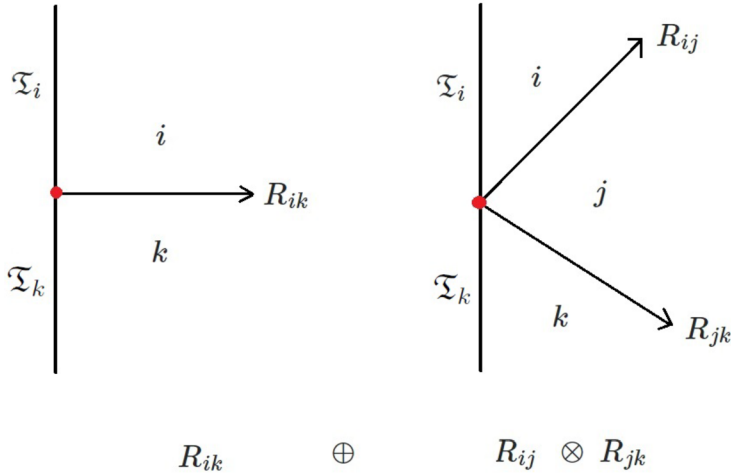


Figure 10: Contribution of some half-plane fans to $\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_k)$.

Differentials. Finally the differential \hat{d}_{ij} on

$$(186) \quad \hat{R}_{ij} = R_{ij} \oplus (R_{ik} \otimes R_{kj}) \oplus \cdots$$

will be inherited from the differentials on the complexes in the obvious way

$$(187) \quad \hat{d}_{ij} = d_{ij} \oplus (d_{ik} \otimes 1 + 1 \otimes d_{kj}) \oplus \cdots$$

Remark. The differential-graded algebra

$$(188) \quad \text{End}(\oplus_i \mathfrak{T}_i) = \bigoplus_{i,j} \hat{R}_{ij}$$

in which the algebra multiplication is specified by the morphisms as defined above, as explained in Appendix B, carries the same information as the category \hat{R} and so we often use the terms algebra and category interchangeably in what follows.

5.2. Interior amplitudes and deformations of \hat{R}

While \hat{R}^{bare} indeed gives $\hat{\mu}$ as its matrix of Euler characters, the cohomology space $H^\bullet(\hat{R}_{ij}, \hat{d}_{ij})$ is not very physically meaningful. In particular, it is not isomorphic to the space of boundary BPS local operators at a \mathfrak{T}_i - \mathfrak{T}_j brane junction, like we would want it to be. The reason for this is similar to the

failure of our naive categorification: we have not taken into account all ζ -instantons. In particular these ζ -instantons will correct the differential (187) and the composition law (185) described in the previous section.

The precise way to take ζ -instantons into account again uses the interior amplitude β . Similar to how one can use taut webs with n vertices to define L_∞ -maps,

$$(189) \quad \rho(\mathbf{t}^{(n)}) : S^n R_c \rightarrow R_c,$$

we can use taut half-plane webs with p boundary vertices and q bulk vertices to define maps

$$(190) \quad \rho(\mathbf{t}_{\mathcal{H}}^{(p,q)}) : (\widehat{R})^{\otimes p} \otimes (R_c)^{\otimes q} \rightarrow \widehat{R}$$

which satisfy the LA_∞ -axioms [GMW] (these are also known as the axioms of an open-closed homotopy algebra, see [KS]). We now make use of the

Theorem. Suppose (A, L) is an open-closed homotopy algebra with structure maps

$$(191) \quad m_{k,l} : A^{\otimes k} \otimes L^{\otimes l} \rightarrow A, \quad k \geq 1, l \geq 0$$

and suppose $\gamma \in L$ is a Maurer-Cartan element for the L_∞ algebra L . Then the collection of maps

$$(192) \quad m_k[\gamma] : A^{\otimes k} \rightarrow A,$$

defined by

$$(193) \quad m_k[\gamma](-, \dots, -) := \sum_{l \geq 0} \frac{1}{l!} m_{k,l}(-, \dots, -, \gamma^{\otimes l})$$

give a (new) A_∞ -structure on A .

Thus we use the ζ -instanton counting element β to deform the dg-category $\widehat{R}^{\text{bare}}$ to an A_∞ -category denoted by $\widehat{R}[X, W]$. The deformed category $\widehat{R}[X, W]$ is proposed as the physical brane category of the Landau-Ginzburg model associated to the pair (X, W) . In particular, we correct the differential \widehat{d}_{ij} to $\widehat{d}_{ij}[\beta]$ via (193) with $k = 1$, so that the cohomology

$$(194) \quad H^\bullet(\widehat{R}_{ij}, \widehat{d}_{ij}[\beta])$$

is isomorphic to the space of $\frac{1}{2}$ -boundary BPS local operators at a $(\mathfrak{T}_i, \mathfrak{T}_j)$ -brane junction. In addition $k = 2$ of (193) also modifies the bilinear composition (185). As a result of (193) higher operations

$$(195) \quad \{m_k[\beta]\}_{k \geq 2}$$

are also introduced. Together these operations turn $\widehat{R}[X, W]$ into a genuine A_∞ -category.

6. Homotopy equivalence of brane categories

The categorical wall-crossing constraint is formulated as follows.

Categorical wall-crossing constraint. Suppose W and W' are superpotentials on different sides of a wall of marginal stability. Then the β -deformed thimble categories on either side of the wall are homotopy equivalent

$$(196) \quad \widehat{R}[X, W] \simeq \widehat{R}'[X, W']$$

as A_∞ -categories.

We now relate our categorical wall-crossing formulas with the categorical wall-crossing constraint. First we construct the left and right $\{i, j, k\}$ -subcategories. As an instructive first check, we verify that the canonical representatives indeed give homotopy equivalent categories. Finally we unpack the axioms for A_∞ equivalence and show how the general statement follows.

6.1. Left configuration

Let us first construct the $\{i, j, k\}$ sub-algebra of \widehat{R} for the configuration on the left of Figure 4. The soliton complexes are

$$(197) \quad (R_{ij}, d_{ij}), (R_{ik}, d_{ik}), (R_{jk}, d_{jk}).$$

Because there are no half-plane fans with more than one edge emanating from the boundary, the morphism spaces are simply

$$(198) \quad \widehat{R}_{ij} = R_{ij},$$

$$(199) \quad \widehat{R}_{jk} = R_{jk},$$

$$(200) \quad \widehat{R}_{ik} = R_{ik}.$$

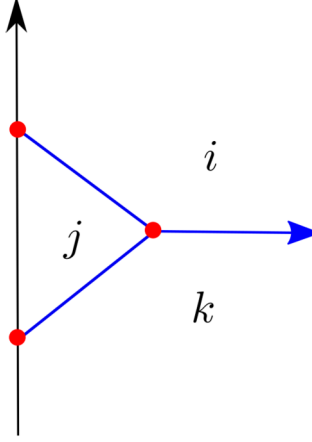


Figure 11: A taut half-plane web which contributes to a non-trivial bilinear product, by inserting the interior amplitude β in the bulk vertex.

In the undeformed algebra, there are no non-trivial multiplications.

Now consider the interior amplitude component

$$(201) \quad \beta_{ikj} \in R_{ik} \otimes R_{kj} \otimes R_{ji},$$

and consider the β -deformed algebra $\widehat{R}(X, W)$. In $\widehat{R}(X, W)$ we see that the taut half-plane web shown in Figure 11 now gives rise to a non-trivial morphism

$$(202) \quad M[\beta_{ikj}] : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}_{ik}$$

given precisely by (130) applied to β_{ikj} . The differential \widehat{d} remains uncorrected.

The only A_∞ axiom to check is that

$$(203) \quad d_{ik}(M[\beta_{ikj}](r_{ij}, r_{jk})) = M[\beta_{ikj}](dr_{ij}, r_{jk}) \pm M[\beta_{ikj}](r_{ij}, dr_{jk})$$

which follows from β_{ikj} being an interior amplitude component.

6.2. Right configuration

Suppose the BPS chain complexes on the right configuration are

$$(204) \quad (R'_{ij}, d'_{ij}), (R'_{jk}, d'_{jk}), (R'_{ik}, d'_{ik}).$$

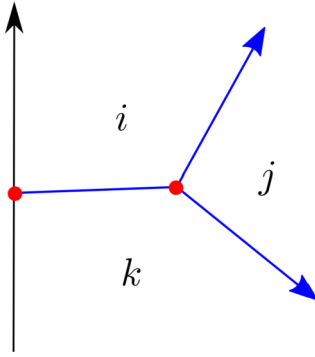


Figure 12: A taut half-plane web which contributes to an off-diagonal element in the differential by inserting the interior amplitude β' in the bulk vertex.

There are now two half-plane fans of type ik , shown in Figure 10 with one and two edges emanating from the boundary vertex respectively. This gives that the morphism spaces are

$$(205) \quad \widehat{R}'_{ij} = R'_{ij},$$

$$(206) \quad \widehat{R}'_{jk} = R'_{jk},$$

$$(207) \quad \widehat{R}'_{ik} = R'_{ik} \oplus (R'_{ij} \otimes R'_{jk}).$$

Denote the interior amplitude on the right configuration to be

$$(208) \quad \beta'_{ijk} \in R'_{ij} \otimes R'_{jk} \otimes R'_{ki}.$$

Writing an element of \widehat{R}'_{ik} as a column vector $\begin{pmatrix} r'_{ik} \\ r'_{ij} r'_{jk} \end{pmatrix}$ the differential on \widehat{R}'_{ik} is of the form

$$(209) \quad \widehat{d}'_{ik}[\beta'] = \begin{pmatrix} d'_{ik} & 0 \\ M'[\beta'_{ijk}] & d'_{ijk} \end{pmatrix}$$

where

$$(210) \quad M'[\beta'_{ijk}] : R'_{ik} \rightarrow R'_{ij} \otimes R'_{jk}$$

is a degree +1 map defined by Figure 12. Nilpotence of $\widehat{d}'_{ik}[\beta']$ holds if

$$(211) \quad d'_{ijk} M'[\beta'_{ijk}] + M'[\beta'_{ijk}] d'_{ik} = 0,$$

therefore we may equivalently view $M'[\beta'_{ijk}]$ as a chain map

$$(212) \quad M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}$$

and we can rewrite

$$(213) \quad \widehat{R}'_{ik} = \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}).$$

The only non-trivial multiplication map is

$$(214) \quad i : \widehat{R}'_{ij} \otimes \widehat{R}'_{jk} \rightarrow \widehat{R}'_{ik}$$

given by inclusion. The A_∞ axiom says that i is a chain map with respect to d'_{ijk} , the product rule differential on $R'_{ij} \otimes R'_{jk}$ and $\widehat{d}'_{ik}[\beta'] = d_{M'}$ the mapping cone differential on \widehat{R}'_{ik} .

6.3. Canonical representatives satisfy wall-crossing constraint

In this section we show that the canonical representatives (161), (162), (163), (165) satisfy the categorical wall-crossing constraint.

Claim. Suppose the primed complexes

$$(215) \quad (R'_{ij}, d'_{ij}), (R'_{jk}, d'_{jk}), (R'_{ik}, d'_{ik})$$

and interior amplitude

$$(216) \quad \beta'_{ijk} \in R'_{ij} \otimes R'_{jk} \otimes R'_{kj}$$

are given as in (161), (162), (163) and (165). Then there is a functor

$$(217) \quad T : \widehat{R} \rightarrow \widehat{R}'$$

which defines a quasi-isomorphism of A_∞ -categories. We call T the wall-crossing functor.

Proof. By virtue of the categorical wall-crossing statement, we have the primed morphism spaces

$$(218) \quad \widehat{R}'_{ij} = R_{ij},$$

$$(219) \quad \widehat{R}'_{jk} = R_{jk},$$

$$(220) \quad \widehat{R}'_{ik} = R_{ik} \oplus (R_{ij} \otimes R_{jk})[-1] \oplus (R_{ij} \otimes R_{jk}).$$

The differentials deformed by the interior amplitude component β'_{ijk} are of the form

$$(221) \quad \widehat{d}'_{ik}[\beta'] = \widehat{d}_{ik},$$

$$(222) \quad \widehat{d}'_{jk}[\beta'] = \widehat{d}_{jk},$$

$$(223) \quad \widehat{d}'_{ik}[\beta'] = \begin{pmatrix} d_{ik} & M[\beta_{ikj}] & 0 \\ 0 & d_{ijk}^{[-1]} & 0 \\ M'_1[\beta'_{ijk}] & M'_2[\beta'_{ijk}] & d_{ijk} \end{pmatrix},$$

where $M[\beta_{ikj}]$ was defined as before and

$$(224) \quad M'_1[\beta'_{ijk}] : R_{ik} \rightarrow R_{ij} \otimes R_{jk},$$

$$(225) \quad M'_2[\beta'_{ijk}] : (R_{ij} \otimes R_{jk})[-1] \rightarrow R_{ij} \otimes R_{jk}$$

are the different components of the maps defined by Figure 12 by inserting β' in the bulk vertex. The functor T can then be defined as follows. On objects we simply have the identity map. On morphism spaces we define

$$(226) \quad T_1 : \widehat{R}_{ij} \rightarrow \widehat{R}'_{ij},$$

$$(227) \quad T_1 : \widehat{R}_{kj} \rightarrow \widehat{R}'_{kj}$$

as identity maps, whereas

$$(228) \quad T_1 : \widehat{R}_{ik} \rightarrow \widehat{R}'_{ik}$$

is defined as inclusion,

$$(229) \quad T_1(r_{ik}) = \begin{pmatrix} r_{ik} \\ 0 \\ 0 \end{pmatrix}$$

Furthermore

$$(230) \quad T_2 : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}'_{ik}$$

is again defined to be inclusion, but into the summand with shifted degree,

$$(231) \quad T_2(r_{ij}r_{jk}) = \begin{pmatrix} 0 \\ (r_{ij}r_{jk})^{[-1]} \\ 0 \end{pmatrix}.$$

Indeed (T_1, T_2) have degrees $(0, -1)$ respectively. The higher maps T_n are set to be zero for $n \geq 3$.

First we have to show the axioms of an A_∞ -morphism are satisfied. Here there are just two axioms to check. At $n = 1$ we have to check if T_1 is a chain map. The only non-trivial check is on the ik -component of T_1 and it follows that we have a chain map from the form of the differential (223). At $n = 2$ we must check

$$(232) \quad \begin{aligned} T_1(m_2(r_{ij}, r_{jk})) - m'_2(T_1(r_{ij}), T_1(r_{jk})) \\ = T_2(r_{ij}, dr_{jk}) \pm T_2(dr_{ij}, r_{jk}) \pm d'(T_2(r_{ij}, r_{jk})). \end{aligned}$$

This follows from the following simplification for the expression of $\widehat{d}'_{ik}[\beta']$. The explicit form of β'

$$(233) \quad \beta'_{ijk} = K_{ij}^{-1} K_{jk}^{-1},$$

from (165), implies that the off-diagonal maps $M'_{1,2}[\beta'_{ijk}]$ are

$$(234) \quad M'_1[\beta'_{ijk}] = 0$$

$$(235) \quad M'_2[\beta'_{ijk}] = \text{id}.$$

Note that the identity map M'_2 has degree $+1$ due to the degree shift on the domain. Thus we can rewrite the differential as

$$(236) \quad \widehat{d}'_{ik}[\beta'] = \begin{pmatrix} d_{ik} & M[\beta_{ikj}] & 0 \\ 0 & d_{ijk}^{[-1]} & 0 \\ 0 & \text{id} & d_{ijk} \end{pmatrix}.$$

Using this expression for d' on the right hand side, the axiom easily follows. Thus T defines an A_∞ -functor.

Finally, we must show that the wall-crossing functor T is a quasi-isomorphism. Again this is non-trivial only on the ik -component. The simplification of $\widehat{d}'[\beta']$ in fact allows us to relate this to the mapping cylinder construction: similar to (163) one can recognize \widehat{R}'_{ik} as the mapping cone of the projection map

$$(237) \quad \pi : R'_{ik}[1] = \text{Cone}(M[\beta_{ikj}])[1] \rightarrow R_{ij} \otimes R_{jk}.$$

In other words we can rewrite

$$(238) \quad \widehat{R}'_{ik} = \text{Cyl}(M[\beta_{ikj}]).$$

Applying the **Proposition** about mapping cylinders from Appendix A to $f = M[\beta_{ikj}]$ yields that T is a quasi-isomorphism. \square

Remark. Two A_∞ -algebras are homotopy equivalent if and only if they are quasi-isomorphic (this is a theorem of Prouté, [Pro]). We can thus say

$$(239) \quad \widehat{R}[X, W] \simeq \widehat{R}'[X, W']$$

where \simeq is meant to be understood as homotopy equivalence.

6.4. Homotopy equivalence \implies categorical WCF

Finally we come to the main claim.

Claim. The categorical wall-crossing constraint, namely the homotopy equivalence of A_∞ -categories

$$(240) \quad \widehat{R}[X, W] \simeq \widehat{R}[X, W']$$

implies the categorical wall-crossing formula

$$(241) \quad R'_{ij} \simeq R_{ij},$$

$$(242) \quad R'_{jk} \simeq R_{jk},$$

$$(243) \quad R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}).$$

Consider first the A_∞ morphism

$$(244) \quad T : \widehat{R}[X, W] \rightarrow \widehat{R}'[X, W'].$$

This in particular means that there are chain maps

$$(245) \quad T_1 : \widehat{R}_{ij} \rightarrow \widehat{R}'_{ij},$$

$$(246) \quad T_1 : \widehat{R}_{jk} \rightarrow \widehat{R}'_{jk},$$

$$(247) \quad T_1 : \widehat{R}_{ik} \rightarrow \widehat{R}'_{ik}.$$

We showed in Sections 6.1, 6.2 that the hatted and un-hatted spaces coincide as chain complexes except for \widehat{R}'_{ik} which is of the form

$$(248) \quad \widehat{R}'_{ik} = \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R_{jk}).$$

Therefore we have chain maps

$$(249) \quad T_1 : R_{ij} \rightarrow R'_{ij},$$

$$(250) \quad T_1 : R_{jk} \rightarrow R'_{jk},$$

$$(251) \quad T_1 : R_{ik} \rightarrow \text{Cone}(M'[\beta'_{ikj}]).$$

In addition the A_∞ -morphism T provides a degree -1 map

$$(252) \quad T_2 : \widehat{R}_{ij} \otimes \widehat{R}_{jk} \rightarrow \widehat{R}_{ik} = \text{Cone}(M'[\beta'_{ikj}])$$

such that the second A_∞ -morphism axiom, (406), which in the present case reads

$$(253) \quad \begin{aligned} T_1(M[\beta_{ikj}](r_{ij}, r_{jk})) \pm M'_2(T_1(r_{ij}), T_1(r_{jk})) \\ = \widehat{d}'_{ik}[\beta']T_2(r_{ij}, r_{jk}) \pm T_2(dr_{ij}, r_{jk}) \pm T_2(r_{ij}, dr_{jk}), \end{aligned}$$

holds. We showed that M'_2 the bilinear multiplication

$$(254) \quad M'_2 : \widehat{R}'_{ij} \otimes \widehat{R}'_{jk} \rightarrow \text{Cone}(M'[\beta'_{ijk}])$$

is given simply by the inclusion map i in Section 6.2. We therefore see that the conceptual way to interpret this axiom is that it is saying that the square

$$(255) \quad \begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ikj}]} & R_{ik} \\ T_1 \otimes T_1 \downarrow & & \downarrow T_1 \\ R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \end{array}$$

commutes up to homotopy¹⁶

$$(257) \quad i(T_1 \otimes T_1) \simeq T_1(M[\beta_{ikj}])$$

with T_2 providing the chain homotopy. This condition is precisely (148).

¹⁶Note that the the compositions are chain maps

$$(256) \quad R_{ik} \otimes R_{jk} \xrightarrow[T_1 \circ M[\beta]]{i \circ T_1 \otimes T_1} \text{Cone}(M'[\beta'_{ijk}]),$$

Let the morphism in the other direction be

$$(258) \quad S : \widehat{R}'[X, W'] \rightarrow \widehat{R}[X, W]$$

which in particular says that we have chain maps

$$(259) \quad S_1 : R'_{ij} \rightarrow R_{ij},$$

$$(260) \quad S_1 : R'_{jk} \rightarrow R_{jk},$$

$$(261) \quad S_1 : \text{Cone}(M'[\beta'_{ijk}]) \rightarrow R_{ik}$$

that provide homotopy inverses to the T_1 's. S also provides us with a degree -1 map

$$(262) \quad S_2 : R'_{ij} \otimes R'_{ik} \rightarrow R_{ik}$$

that satisfies the second A_∞ axiom which in this case says that the square

$$(263) \quad \begin{array}{ccc} R'_{ij} \otimes R'_{jk} & \xrightarrow{i} & \text{Cone}(M'[\beta'_{ijk}]) \\ S_1 \otimes S_1 \downarrow & & \downarrow S_1 \\ R_{ij} \otimes R_{jk} & \xrightarrow{M[\beta_{ijk}]} & R_{ik} \end{array}$$

commutes up to homotopy, with S_2 providing the chain homotopy.

$$(264) \quad R'_{ij} \otimes R'_{jk} \xrightarrow[M[\beta] \circ S_1 \otimes S_1]{S_1 \circ i} R_{ik}.$$

In particular the existence of (S_1, T_1) implies that

$$(265) \quad R_{ik} \simeq R'_{ik},$$

$$(266) \quad R_{jk} \simeq R'_{jk},$$

$$(267) \quad R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk}),$$

which are precisely the homotopy equivalences (145), (146), (147) asserted in the categorical wall-crossing statement. The statement that these are homotopy equivalences follows from the definition of homotopy equivalence of A_∞ -algebras. Similarly the commutative square above is precisely (149).

Finally we use the Triangularity Lemma from Appendix A.

We found above that

$$(268) \quad R_{ik} \simeq \text{Cone}(M'[\beta'_{ijk}] : R'_{ik}[1] \rightarrow R'_{ij} \otimes R'_{jk})$$

so an application of the Triangularity Lemma implies that

$$(269) \quad R'_{ik} \simeq \text{Cone}(S_1 \circ i : R'_{ij} \otimes R'_{jk} \rightarrow R_{ik}).$$

Next we recall that the A_∞ -axiom for S_2 implies that

$$(270) \quad S_1 \circ i \simeq M[\beta_{ikj}] \circ (S_1 \otimes S_1)$$

and so their cones are homotopy equivalent. This gives

$$(271) \quad R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] \circ (S_1 \otimes S_1) : R'_{ik} \otimes R'_{jk} \rightarrow R_{ik}).$$

Finally since

$$(272) \quad S_1 : R'_{ij} \rightarrow R_{ij},$$

$$(273) \quad S_1 : R'_{jk} \rightarrow R_{jk}$$

are individually homotopy equivalences, so is

$$(274) \quad S_1 \otimes S_1 : R'_{ij} \otimes R'_{jk} \rightarrow R_{ij} \otimes R_{jk}.$$

Therefore the latter part has a trivial mapping cone and can be “factored out” to conclude that

$$(275) \quad R'_{ik} \simeq \text{Cone}(M[\beta_{ikj}] : R_{ij} \otimes R_{jk} \rightarrow R_{ik}),$$

the result to be shown.

7. The fermion degree of a ζ -instanton

Recall that a ζ -instanton with boundary conditions labeled by the triple of solitons

$$(276) \quad \phi = (\phi_{ik}, \phi_{kj}, \phi_{ji})$$

that occupy the edges of an ikj wall-crossing triangle contributes to the differential in a categorical wall-crossing process if and only if

$$(277) \quad F(\phi_{ik} \otimes \phi_{kj} \otimes \phi_{ji}) = 2.$$

Therefore it is quite important to determine the degree of a given gradient polygon.

By definition the Fermion number is the index of the Dirac operator

$$(278) \quad \mathcal{D}_\zeta : \Gamma(\phi^*(TX)) \rightarrow \Gamma(\phi^*(TX))$$

given by

$$(279) \quad \mathcal{D}_\zeta = \begin{pmatrix} \delta_J^I D_{\bar{s}}^{(1,0)} & 0 \\ 0 & \delta_{\bar{J}}^{\bar{I}} D_s^{(0,1)} \end{pmatrix} - \begin{pmatrix} 0 & \frac{\zeta}{2} g^{I\bar{K}} D_{\bar{K}} \partial_{\bar{J}} \overline{W} \\ \frac{\zeta^{-1}}{2} g^{\bar{I}K} D_K \partial_J W & 0 \end{pmatrix}.$$

in the background of a ζ -instanton ϕ with ϕ boundary conditions. Clearly such an index will be difficult to compute if we work directly with \mathcal{D} .¹⁷ However a Maslov index type construction, described in [KKS], gives a more geometric prescription to obtain a well-defined integer $d(\phi)$ which is expected to agree with the index of \mathcal{D} up to an overall shift. It would be interesting to prove the equality of $d(\phi)$ with the index of \mathcal{D} , but this would take us too far afield in the present paper. We proceed assuming the equality holds and use the geometric prescription in what follows. The Maslov index construction also assumes that X is equipped with a nowhere vanishing holomorphic volume form Ω .

Starting from a convex gradient polygon

$$(280) \quad \phi = (\phi_{i_0 i_1}, \dots, \phi_{i_n i_0})$$

the Maslov index prescription gives us $d(\phi) \in \mathbb{Z}$ as follows. The main step consists of assigning to the gradient polygon ϕ a (homotopy class of a) loop in the Lagrangian Grassmannian of X ,

$$(281) \quad \text{Lag}(TX) = \{(p, E) | p \in X, E \text{ Lagrangian subspace of } T_p X\},$$

constructed as follows.

First to each soliton ϕ_{ij} we associate an open path γ in $\text{Lag}(TX)$ simply by taking a point p along the soliton trajectory and assigning to it the Lagrangian subspace

$$(282) \quad T_p L_i(\zeta_{ij}) \subset T_p X$$

¹⁷Moreover the question of whether \mathcal{D} is even Fredholm is a very delicate one, [CGGLPFZ].

as the fiber. Let γ_k denote the open path assigned to $\phi_{i_{k-1}i_k}$ in this way. One notices that the endpoint of γ_k and the starting point of γ_{k+1} have the same base point, the k th critical point i_k , but the Lagrangians fibers differ. The endpoint of γ_k has fiber

$$(283) \quad \ell_k := T_{i_k} L_{i_{k-1}}(\zeta_{i_{k-1}i_k})$$

whereas the starting point of γ_{k+1} has the fiber

$$(284) \quad \ell_{k+1} := T_{i_k} L_{i_k}(\zeta_{i_k i_{k+1}}).$$

ℓ_k, ℓ_{k+1} are Lagrangians living in the same ambient space $T_{i_k}X$. Between any two Lagrangian subspaces L_1, L_2 in a symplectic vector space V , there is a canonical homotopy class of paths κ_{L_1, L_2} in $\text{Lag}(V)$ that connects these points, known as the **symplectic bridge**¹⁸ connecting L_1 and L_2 . For instance if $\dim(V) = 2$, the Lagrangians are specified by points θ_1, θ_2 in $\mathbb{RP}^1 \cong S^1$ and $\kappa_{\theta_1, \theta_2}$ is the circular arc going in the counter-clockwise direction between these two angles. Therefore there is a well-defined way to connect the open path γ_k to γ_{k+1} . Going around the gradient polygon by gluing adjacent open paths via symplectic bridges, one obtains a loop in $\text{Lag}(TX)$.

Next we need to define a winding number of the loop γ . Let $\bar{\gamma}$ be the loop in X obtained by projecting γ to X . Thus, if $\bar{\gamma}(t) = p \in X$ then $\gamma(t) \subset T_p X$ is a maximal Lagrangian subspace. Let $2n$ denote the rank of TX considered as a real vector bundle over X . Then $\gamma(t)$ is a real vector space of dimension n . The n^{th} exterior product of this space is a real line associated to the point p . Now, recall that TX can also be considered to be a complex vector bundle of rank n . Therefore, the n^{th} exterior power of TX as a complex vector bundle is a complex line associated to p . Indeed, this is the fiber of the canonical bundle at p , denoted \mathcal{K}_p . Note that $\Lambda^n \gamma(t) \subset \mathcal{K}_p$ is a real line inside a complex line. Finally we use Ω to trivialize the canonical bundle and therefore get a real line $\ell_p \subset \mathbb{C}$. That is, to the loop $\gamma : S^1 \rightarrow \text{Lag}(TX)$ we associate a loop in $\text{Lag}(\mathbb{C}) = \mathbb{RP}^1 \cong S^1$. All-in-all we get a map

$$(285) \quad \psi(\phi) : S^1 \rightarrow \text{Lag}(\mathbb{C}) \cong S^1.$$

The integer $d(\phi)$ is defined to be the winding number of $\psi(\phi)$. The fermion number is then

$$(286) \quad F(\phi) = d(\phi) + 1.$$

¹⁸This is also known as the **canonical short path**, see for instance [Aur].

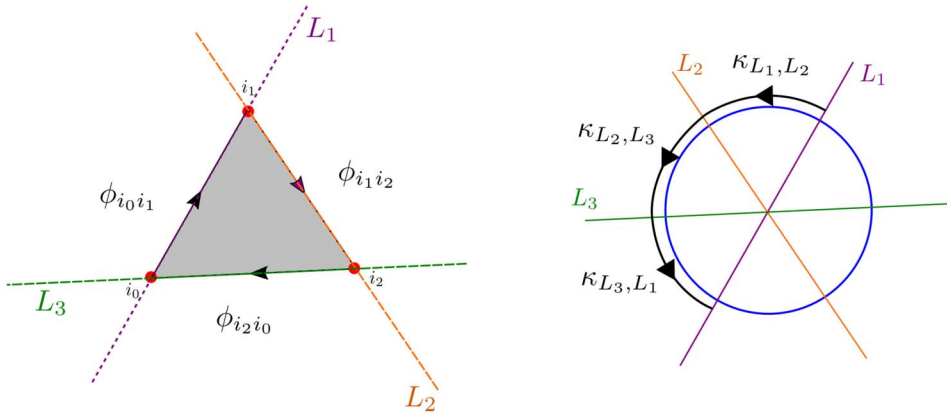


Figure 13: The left shows the gradient polygon $\phi = (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \phi_{i_2 i_0})$ assumed to trace out straight lines on the complex plane. The dashed lines depict the Lagrangians tangent to these solitons. On the right we show the symplectic bridges $\kappa_{L_i, L_{i+1}}$ connecting these Lagrangians. The winding number of the total path in $\text{Lag}(\mathbb{C}) = S^1/\mathbb{Z}_2$ is $+1$, therefore $d(\phi) = 1$.

We illustrate the computation of $d(\phi)$ in some examples.

7.1. Gradient polygons in \mathbb{C}

Suppose our target space is the complex plane, and say for simplicity that the solitons trace out straight lines so that the gradient polygon $\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_n i_0})$ traces out the boundary of an $(n+1)$ -gon. This boundary can be clockwise or counter-clockwise oriented and we analyze each case.

For the case of clockwise oriented boundaries, the tangent Lagrangian does not vary along the soliton. The symplectic bridge between $\phi_{i_{k-1} i_k}$ and $\phi_{i_k i_{k+1}}$ chooses to take the route that takes θ_k radians where θ_k is an internal angle of the polygon. Adding up these angles gives one a total winding number in S^1/\mathbb{Z}_2 of

$$(287) \quad d(\phi) = \frac{((n+1) - 2)\pi}{\pi}$$

$$(288) \quad = n - 1$$

where in the first equality we divide by π (not 2π) because of the \mathbb{Z}_2 quotient. See Figure 13 for the case of $n = 2$.

For counter-clockwise oriented (convex) polygons,¹⁹ the symplectic bridge chooses to connect adjacent Lagrangians via the route that takes $\pi - \theta_k$ radians. This gives one

$$(289) \quad d(\phi) = 2,$$

an index independent of n .

That clockwise versus counterclockwise give such different answers might be a bit puzzling first, but its origin is clarified if one thinks about the analogous situation in Morse theory. Suppose that $\mathcal{M}(x_a, x_b)$ denotes the reduced moduli space of solutions of the gradient flow equation

$$(290) \quad \frac{d\phi^I}{dx} = g^{IJ} \frac{\partial h}{\partial \phi^J},$$

between two critical points x_a, x_b of h with Morse indices μ_a, μ_b .²⁰ Then supposing $\mu_b > \mu_a$ we have

$$(291) \quad \dim \mathcal{M}(x_a, x_b) = \mu_b - \mu_a - 1.$$

On the other hand,

$$(292) \quad \dim \mathcal{M}(x_b, x_a) = 0.$$

$\mathcal{M}(x_b, x_a)$ is in fact empty, as a consequence of the ascending property of the gradient flow. Thus it should not be very surprising that the moduli space of ζ -instantons is not very well-behaved under orientation reversal of a cyclic fan.

7.2. Paths in \mathbb{C}^*

Let's now consider a gradient polygon of solitons in the punctured complex plane \mathbb{C}^* so that the total path winds around the origin. We choose the holomorphic volume form that trivializes $T\mathbb{C}^*$ to be

$$(293) \quad \Omega = \frac{dX}{X}.$$

¹⁹We don't know any examples of $W(\phi)$ where this happens, although we don't see a reason why it cannot happen in principle.

²⁰Not to be confused with the BPS index μ_{ij} .

One can show that a loop that winds around the origin, by virtue of this trivialization satisfies

$$(294) \quad d(\phi) = 0.$$

This will be useful for the trigonometric Landau-Ginzburg models.

7.3. Fermion degrees for \mathbb{Z}_N -symmetric models

We can use the observations above to determine (integral part of) the fermion degrees of solitons in at least two interesting \mathbb{Z}_N -symmetric family of models. These are

1. $W = \frac{1}{N+1}\phi^{N+1} - t\phi$, the deformed A_{N-1} model.
2. $W = \phi + \frac{1}{N-1}\phi^{-(N-1)}$, the \mathbb{Z}_N invariant “trigonometric” LG model.

Let’s analyze each one.

7.3.1. Deformed A_{N-1} -model. The model of a single chiral superfield ϕ with superpotential

$$(295) \quad W = \frac{1}{N+1}\phi^{N+1} - t\phi$$

is a well-studied one. The critical points are

$$(296) \quad \phi_k = t^{\frac{1}{N}} e^{\frac{2\pi i k}{N}}$$

for $k = 0, 1, \dots, N-1$, with critical values

$$(297) \quad W_k = -\frac{N}{N+1} t^{\frac{N+1}{N}} e^{\frac{2\pi i k}{N}}.$$

It is well-known that there is a unique soliton ϕ_{ij} interpolating between each pair (ϕ_i, ϕ_j) of distinct critical points. Therefore

$$(298) \quad R_{ij} = \mathbb{Z}\langle\phi_{ij}\rangle.$$

The degree F_{ij} of ϕ_{ij} is of the form

$$(299) \quad F_{ij} = n_{ij} + f_{ij}$$

where n_{ij} is the integral part and f_{ij} is the fractional part (for which we have a universal formula). It remains to determine n_{ij} .

For this we use the constraint coming from the Maslov index: Let $\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_k i_0})$ be a convex gradient polygon. Then

$$(300) \quad n_{i_0 i_1} + n_{i_1 i_2} + \dots + n_{i_k i_0} = d(\phi) + 1.$$

For the present model, we have that

$$(301) \quad (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_k i_0})$$

is a gradient polygon if and only if $i_0 > i_1 > i_2 \dots > i_n$ up to cyclic re-ordering. In the complex plane the gradient polygon traces out a clockwise oriented closed path with k -segments, and thus the computation in Section 7.1 implies

$$(302) \quad d(\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_k i_0}) = k - 2.$$

We thus get the constraint

$$(303) \quad n_{i_0 i_1} + n_{i_1 i_2} + \dots + n_{i_k i_0} = k - 1,$$

which is satisfied by a particularly simple solution:

$$(304) \quad n_{ij} = 1 \quad \text{for } i > j,$$

$$(305) \quad n_{ij} = 0 \quad \text{for } i < j.$$

By induction on k we see the solution is unique up to shifts

$$(306) \quad n_{ij} \rightarrow n_{ij} + n_i - n_j.$$

Therefore we conclude that

$$(307) \quad R_{ij} = \mathbb{Z}[1] \quad \text{for } i > j,$$

$$(308) \quad R_{ij} = \mathbb{Z} \quad \text{for } i < j.$$

7.3.2. Trigonometric models. We can do a similar analysis for the \mathbb{Z}_N -symmetric trigonometric Landau-Ginzburg models. These have target space \mathbb{C}^* and superpotential

$$(309) \quad W = \phi + \frac{1}{N-1} \phi^{-(N-1)}.$$

The critical points are again located at the roots of unity

$$(310) \quad \phi_k = e^{2\pi i k/N}$$

for $k = 0, 1, \dots, N-1$ and the critical values are

$$(311) \quad W_k = \frac{N}{N-1} e^{\frac{2\pi i k}{N}}.$$

The soliton spectrum of this model is also known (this model is example 3 in section 8.1 of [CV1]): There is a unique soliton between each nearest neighbor pair $(\phi_i, \phi_{i+1}), (\phi_i, \phi_{i-1})$ and none between the other pairs. Therefore the only gradient polygon ϕ with more than 2 solitons consists of the full N -gon

$$(312) \quad \phi = (\phi_{N-1, N-2}, \phi_{N-2, N-3}, \dots, \phi_{0, N-1}).$$

The paths these solitons trace out in \mathbb{C}^* consists of round arcs that together wind around the origin once in the clockwise direction. The computation of the Maslov index for paths in \mathbb{C}^* allows us to conclude that $d(\phi) = 0$ and therefore

$$(313) \quad n_{N-1, N-2} + n_{N-2, N-3} + \dots + n_{0, N-1} = 1.$$

We choose the solution

$$(314) \quad n_{i, i-1} = 0,$$

$$(315) \quad n_{i, i+1} = 1.$$

Thus the non-zero BPS chain complexes with this solution read

$$(316) \quad R_{i, i+1} = \mathbb{Z}\langle \phi_{i, i+1} \rangle \cong \mathbb{Z},$$

$$(317) \quad R_{i, i-1} = \mathbb{Z}\langle \phi_{i, i-1} \rangle \cong \mathbb{Z}[1].$$

8. Examples

Finally let's illustrate categorical wall-crossing in a few examples.

8.1. Quartic LG model

Let's return to the quartic Landau-Ginzburg model that was alluded to in the introduction. The target space is the complex plane \mathbb{C} and the superpotential is

$$(318) \quad W = \frac{1}{4}\phi^4 - \frac{t_1}{2}\phi^2 - t_2\phi.$$

Consider the point $(t_1, t_2) = (0, 1)$ where the critical points are

$$(319) \quad \phi_1 = e^{-\frac{2\pi i}{3}}, \quad \phi_2 = 1, \quad \phi_3 = e^{\frac{2\pi i}{3}}$$

with critical values

$$(320) \quad W_1 = -\frac{3}{4}e^{-\frac{2\pi i}{3}}, \quad W_2 = -\frac{3}{4}, \quad W_3 = -\frac{3}{4}e^{\frac{2\pi i}{3}}.$$

The BPS chain complexes consist of

$$(321) \quad R_{12} = \mathbb{Z}\langle\phi_{12}\rangle,$$

$$(322) \quad R_{13} = \mathbb{Z}\langle\phi_{13}\rangle,$$

$$(323) \quad R_{23} = \mathbb{Z}\langle\phi_{23}\rangle,$$

where ϕ_{ij} is the unique soliton interpolating between ϕ_i and ϕ_j . As discussed in Section 7.3.1, an assignment of degrees consistent with the Maslov index is that all three spaces are concentrated in degree zero.

Now we must count ζ -instantons. Consider the cyclic fan $\{1, 3, 2\}$ which has degree $+2$. It is argued in papers on domain wall junctions [GT] that there is indeed a solution with no reduced moduli with these trivalent fan boundary conditions. Therefore we have

$$(324) \quad N(\phi_{13}, \phi_{32}, \phi_{21}) = 1.$$

The image swept out by this instanton $\phi(\mathbb{C})$ is depicted in Figure 14.

Crossing the wall of marginal stability we consider $(t_1, t_2) = (1, \epsilon)$ where ϵ is some small number. Categorical wall-crossing says that the chain complex is

$$(325) \quad R'_{13} = \mathbb{Z}\langle(\phi_{12}\phi_{23})^{[-1]}\rangle \oplus \mathbb{Z}\langle\phi_{13}\rangle.$$

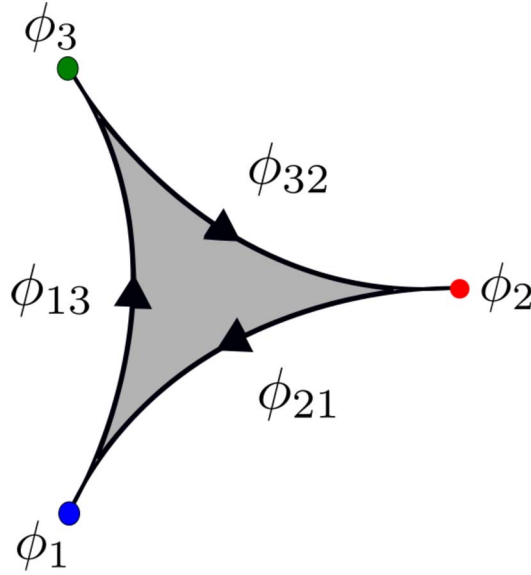


Figure 14: Image of the ζ -instanton with fan boundary conditions $\{1, 3, 2\}$ in the X -plane. It sweeps out a region bounded by the soliton paths.

The differential reads

$$(326) \quad d'_{13}((\phi_{12}\phi_{23})^{[-1]}) = \phi_{13},$$

$$(327) \quad d'_{13}(\phi_{13}) = 0,$$

by virtue of the ζ -instanton of Figure 14. Therefore the cohomology is trivial

$$(328) \quad H^\bullet(R'_{13}, d'_{13}) = 0.$$

Indeed this is the correct BPS Hilbert space on the other side of the wall.

8.2. Trigonometric LG model

Next we consider the model with target space the complex cylinder \mathbb{C}^* with coordinate ϕ . The family of superpotentials we consider is

$$(329) \quad W = \phi + \lambda\phi^{-1} + \frac{1}{2}\phi^{-2}.$$

The model at $\lambda = 0$ is known in [CV1] as the Bullough-Dodd model and that's where we begin our analysis. Here we have the critical points

$$(330) \quad \phi_1 = e^{\frac{2\pi i}{3}}, \quad \phi_2 = 1, \quad \phi_3 = e^{-\frac{2\pi i}{3}}$$

with critical values $W_i = \frac{3}{2}X_i$. As discussed in Section 7.3.2, there is a single soliton between each pair of vacua and so the BPS chain complexes read

$$(331) \quad R_{12} = \mathbb{Z}\langle\phi_{12}\rangle,$$

$$(332) \quad R_{23} = \mathbb{Z}\langle\phi_{23}\rangle,$$

$$(333) \quad R_{13} = \mathbb{Z}\langle\phi_{12}\rangle.$$

As discussed in Section 7.2, consistent with the Maslov index is to choose these spaces to be concentrated in degree zero (the vacua have been relabeled compared to that section). Note that there's a crucial difference with the quartic Landau-Ginzburg model. The vector space associated to the cyclic fan $\{1, 2, 3\}$ is one-dimensional but now concentrated in degree $+1$. The interior amplitude must therefore be trivial

$$(334) \quad \beta = 0.$$

Therefore, there are no ζ -instantons.

The absence of ζ -instantons with trivalent boundary conditions may also be geometrically argued as follows. The cyclic fan of solitons sweep out a path that winds around the origin. Were a ζ -instanton to exist, its image would be a region bounded by this path. However, the latter region contains the singular point $\phi = 0$, which means that the ζ -instanton blows up at finite (x, τ) .

We now vary λ by taking it to be purely imaginary and increasing the magnitude from the \mathbb{Z}_3 symmetric point $\lambda = 0$. The wall of marginal stability is crossed at $\lambda \sim 1.5i$. W_1 passes through the line between W_2 and W_3 . Therefore R_{23} jumps. We have

$$(335) \quad R'_{23} = (R_{21} \otimes R_{13})[-1] \otimes R_{23},$$

$$(336) \quad = \mathbb{Z}\langle(\phi_{21}\phi_{13})^{[-1]}\rangle \oplus \mathbb{Z}\langle\phi_{23}\rangle,$$

$$(337) \quad \cong \mathbb{Z}^2.$$

Trivial β implies that this is also the cohomology. We see that the 23 sector has gained a bound state of the 21 and 13 sectors.

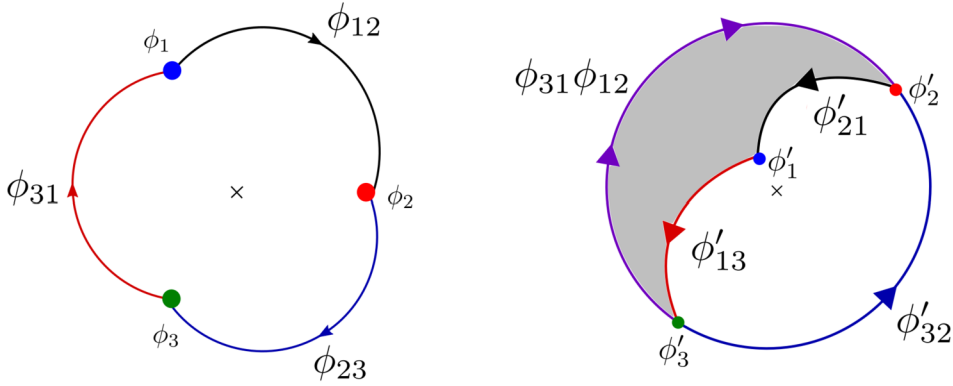


Figure 15: Left: the solitons in the $\lambda = 0$ model. There is one between each pair of vacua. On the right we cross the wall of marginal stability and go to $\lambda = 2i$. There are now two solitons in the 32 sector. We also gain a non-trivial ζ -instanton contributing to the interior amplitude.

These two states post wall-crossing have a simple interpretation. When λ is large the theory consists of the \mathbb{CP}^1 mirror along with a vacuum W_1 running away to infinity. The solitons between 2 and 3 are the solitons of this model.

Categorical wall-crossing also predicts the interior amplitude after wall-crossing. Formula (165) says that the interior amplitude should be

$$(338) \quad \beta'_{132} = (\phi_{31}\phi_{12}) \otimes \phi_{21} \otimes \phi_{13}.$$

Indeed the geometry of solitons allows the region between the new soliton that appears, $\phi_{31}\phi_{12}$, between 3 and 2 and the the old solitons ϕ_{21} and ϕ_{13} to be filled up by a ζ -instanton. See Figure 15.

8.3. Elliptic LG model

Let the target space be $T^2_\tau \setminus \{0\}$ and

$$(339) \quad W = \wp(\phi, \tau).$$

We study the wall-crossing properties as we vary τ , the complex structure parameter of the torus.²¹ The critical points are the familiar half-

²¹The moduli space of models is the stack $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ where \mathbb{H} is the upper-half plane. The moduli space of models with marked vacua is $\mathbb{H}/\Gamma(2)$ where $\Gamma(2)$ is the

periods

$$(340) \quad \left\{ \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right\} \bmod (\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with critical values being the elliptic constants

$$(341) \quad \{e_1(\tau), e_2(\tau), e_3(\tau)\}.$$

It is well-known [CV1, CV2] that this model has precisely two solitons between each pair of critical points, independent of the value of τ . On the other hand, there are still marginal stability walls. For example when τ is pure imaginary the $e_i(\tau)$ are all real and hence co-linear, so the imaginary axis and its $\mathrm{PSL}(2, \mathbb{Z})$ -images are marginal stability walls in the upper-half plane. The fact that there are two solitons in any chamber, is explained at the level of BPS indices by the equations

$$(342) \quad 2 = -2 + 2 \cdot 2, \text{ or,}$$

$$(343) \quad -2 = 2 - 2 \cdot 2.$$

We will now see what happens at the level of chain complexes.

First work at the \mathbb{Z}_3 symmetric point $\tau_0 = e^{\frac{2\pi i}{3}}$. We set

$$(344) \quad \phi_1 = \frac{\tau_0}{2}, \quad \phi_2 = \frac{1}{2}, \quad \phi_3 = \frac{1+\tau_0}{2}.$$

The homogeneity property of $\wp(\phi, \tau)$ at the \mathbb{Z}_3 symmetric point implies that the critical values are proportional to the cubic roots of unity

$$(345) \quad W_1 = W_0 e^{\frac{2\pi i}{3}}, \quad W_2 = W_0, \quad W_3 = W_0 e^{\frac{-2\pi i}{3}},$$

where the proportionality constant is, according to [DLMF]:

$$(346) \quad W_0 = \left(\frac{\Gamma^3(\frac{1}{3})}{2^{\frac{1}{3}} 2\pi} \right)^2.$$

The chain complexes are

$$(347) \quad R_{12} = \mathbb{Z}\langle\phi_{12}^1\rangle \oplus \mathbb{Z}\langle\phi_{12}^2\rangle \cong \mathbb{Z}^2[1],$$

level 2 principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. See [BC] for further examples of this type.

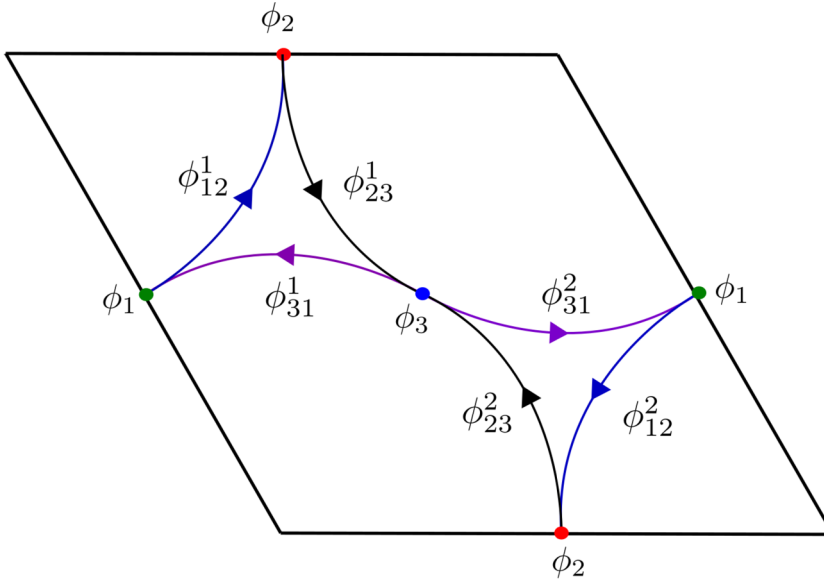


Figure 16: BPS solitons in the $W = \wp(\phi, \tau)$ model with $\tau = e^{\frac{2\pi i}{3}}$. There are two solitons between each pair of vacua, the paths they trace out are depicted.

$$(348) \quad R_{13} = \mathbb{Z}\langle\phi_{13}^1\rangle \oplus \mathbb{Z}\langle\phi_{13}^2\rangle \cong \mathbb{Z}^2[1],$$

$$(349) \quad R_{23} = \mathbb{Z}\langle\phi_{23}^1\rangle \oplus \mathbb{Z}\langle\phi_{23}^2\rangle \cong \mathbb{Z}^2[1].$$

A computation similar to the deformed A_{N-1} -minimal models can be performed to conclude that these chain complexes are all concentrated in degree +1 and so all the individual differentials d_{ij} vanish. The trajectories these solitons trace out on $T_{\tau_0}^2$ are depicted in Figure 16.

The anti-particles are associated to the BPS complexes

$$(350) \quad R_{21} = \mathbb{Z}\langle\phi_{21}^1\rangle \oplus \mathbb{Z}\langle\phi_{21}^2\rangle \cong \mathbb{Z}^2,$$

$$(351) \quad R_{31} = \mathbb{Z}\langle\phi_{31}^1\rangle \oplus \mathbb{Z}\langle\phi_{31}^2\rangle \cong \mathbb{Z}^2,$$

$$(352) \quad R_{32} = \mathbb{Z}\langle\phi_{32}^1\rangle \oplus \mathbb{Z}\langle\phi_{32}^2\rangle \cong \mathbb{Z}^2.$$

The pairings K_{12}, K_{13}, K_{23} are diagonal in this basis of solitons.

Let's now consider ζ -instantons. The vector space corresponding to the cyclic fan

$$(353) \quad \{1, 2, 3\},$$

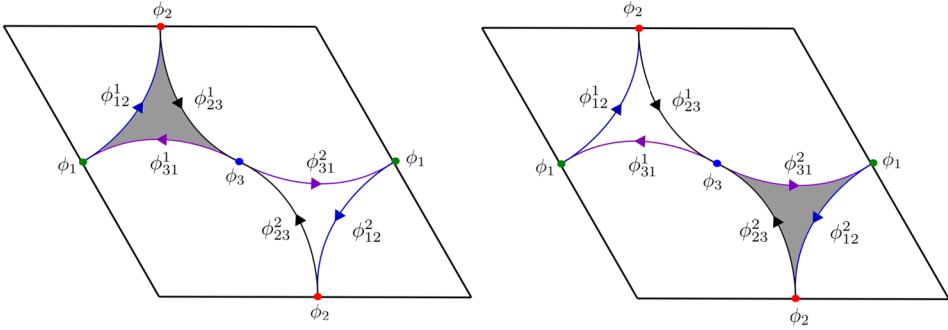


Figure 17: ζ -instantons in the $W = \wp(\phi, \tau)$ model with $\tau = e^{\frac{2\pi i}{3}}$.

$R_{12} \otimes R_{23} \otimes R_{31}$ is concentrated in degree +2 and so this model allows rigid instantons. There are eight possible gradient polygons $\phi^{a,b,c} = (\phi_{12}^a, \phi_{23}^b, \phi_{31}^c)$ for $a, b, c = 1, 2$ which could a-priori be occupied. However, the model has additional flavor symmetries whose charges are associated with the winding numbers around the torus.²² These symmetries reduce the number of possibilities as follows. Denoting q_1, q_2 the fugacities for the cycles that (half)-wind around the horizontal and τ -direction respectively, the solitons have the following (exponentiated) winding numbers: States in R_{12} have winding numbers $q_1 q_2$ and $(q_1 q_2)^{-1}$, in R_{23} they have q_2, q_2^{-1} , and in R_{13} they have q_1, q_1^{-1} . On the other hand β must have zero winding charge. This cuts down the allowed gradient polygons that can be occupied to

$$(354) \quad \phi^1 = (\phi_{12}^1, \phi_{23}^1, \phi_{31}^1),$$

$$(355) \quad \phi^2 = (\phi_{12}^2, \phi_{23}^2, \phi_{31}^2).$$

The simplest non-trivial guess is to posit that these polygons indeed support ζ -instantons with degeneracies

$$(356) \quad N(\phi^1) = 1,$$

$$(357) \quad N(\phi^2) = 1.$$

Thus we predict the interior amplitude for this model is

$$(358) \quad \beta = \phi_{12}^1 \otimes \phi_{23}^1 \otimes \phi_{31}^1 + \phi_{12}^2 \otimes \phi_{23}^2 \otimes \phi_{31}^2.$$

²²More precisely this symmetry doesn't come from translational invariance, since the pole in the superpotential distinguishes a point in the torus (there is a puncture at $X = 0$). Nevertheless we can form a conserved current for each harmonic one-form α given by $j = *\phi^*(\alpha)$.

Assuming this is indeed the case, we now evolve from $\tau_0 = e^{2\pi i/3}$ to, a point of the form $\tau_1 = ie^{-i\epsilon}$ with $\epsilon > 0$. In doing so we must cross the wall at $\text{Re}(\tau) = 0$. In such a move, one can check (numerically for instance) that the point W_3 passes through the line connecting W_1 and W_2 . Therefore the chain complexes R_{13}, R_{32} remain the same as before

$$(359) \quad R'_{13} = \mathbb{Z}^2[1],$$

$$(360) \quad R'_{32} = \mathbb{Z}^2,$$

but R_{12} can jump:

$$(361) \quad R'_{12} = (R_{13} \otimes R_{32})[-1] \oplus R_{12}$$

$$(362) \quad = (\mathbb{Z}\langle\phi_{13}^1, \phi_{13}^2\rangle \otimes \mathbb{Z}\langle\phi_{32}^1, \phi_{32}^2\rangle)[-1] \oplus \mathbb{Z}\langle\phi_{12}^1, \phi_{12}^2\rangle.$$

The first summand is concentrated in degree zero whereas the second factor is in degree one. The ζ -instanton count imply that the differentials act as follows.

$$(363) \quad d'_{12}((\phi_{13}^1\phi_{32}^1)^{[-1]}) = \phi_{12}^1,$$

$$(364) \quad d'_{12}((\phi_{13}^1\phi_{32}^2)^{[-1]}) = 0,$$

$$(365) \quad d'_{12}((\phi_{13}^2\phi_{32}^1)^{[-1]}) = 0,$$

$$(366) \quad d'_{12}((\phi_{13}^2\phi_{32}^2)^{[-1]}) = \phi_{12}^2.$$

Thus the cohomology is

$$(367) \quad H^\bullet(R'_{12}, d'_{12}) = \mathbb{Z}\langle(\phi_{13}^1\phi_{32}^2)^{[-1]}, (\phi_{13}^2\phi_{32}^1)^{[-1]}\rangle,$$

which is two-dimensional as expected. Categorical wall-crossing has allowed us to see that there has been a non-trivial reorganization of the BPS states in the 12-sector: in particular their winding numbers jump. This was not visible at the level of ordinary BPS indices.²³

9. Conclusions and future directions

There are various future directions that might be worth pursuing. While staying in the two-dimensional world, it is desirable to categorify more general wall-crossing statements. In particular the presence of twisted masses

²³Of course a *refined* index could have still detected this. In particular upgrading μ_{ij} to a character valued index $\mu_{ij}(q_1, q_2)$ and applying Cecotti-Vafa does the job in this example. In general such a refinement might not always be available.

leads to interesting new phenomena. These new phenomena and how they affect the discussion of categorical wall-crossing will be the subject of a separate paper. Similarly, another interesting direction would be to categorify the beautiful formula of Kontsevich and Soibelman, perhaps by constructing the category of infrared line defects in four-dimensional $\mathcal{N} = 2$ theories as a first step.

In a more speculative direction one might wonder about the following. We were studying two-dimensional theories, both in spacetime and from the perspective of the W -plane. Edges between vacua in the latter were initially supported by BPS indices, which are integers, and in particular we can use these edges to form a wall-crossing triangle. Categorifying upgraded these integers to chain complexes, but a lesson we learned is that information about these chain complexes by themselves is not sufficient to describe categorical wall-crossing: they must be accompanied by integers associated to the interior of the wall-crossing triangle. In a higher-dimensional generalization of the formalism, let's say three dimensions, we can imagine having a tetrahedron, whose edges carry categories, faces carry chain complexes and whose interior carries the data of integers. See Figure 18. Wall-crossing would occur when the vertices of the tetrahedron lie on a common plane followed by the apex switching sides as viewed from the base. It would be interesting to spell out the wall-crossing structure of this hierarchy of categories, vector spaces and integers that lie on the various faces of the tetrahedron. Even more compelling would be to find a quantum field theoretic realization of such a higher-dimensional “wall-crossing simplex.”

In the process of categorifying the simplest wall-crossing formula, we have been lead to an interesting blend of mathematics and physics. The physics of domain wall junctions and their moduli spaces allows one to construct canonical objects in homological algebra: the mapping cone and mapping cylinder. These mathematical objects allow us to compactly express the answer to the question we had initially asked. This is the very essence of physical mathematics.

Appendix A. Some basic homological algebra

The categorical wall-crossing formula is most cleanly stated using some standard homological algebra. We summarize the concepts we need below and refer the reader to [Weib] for further details.

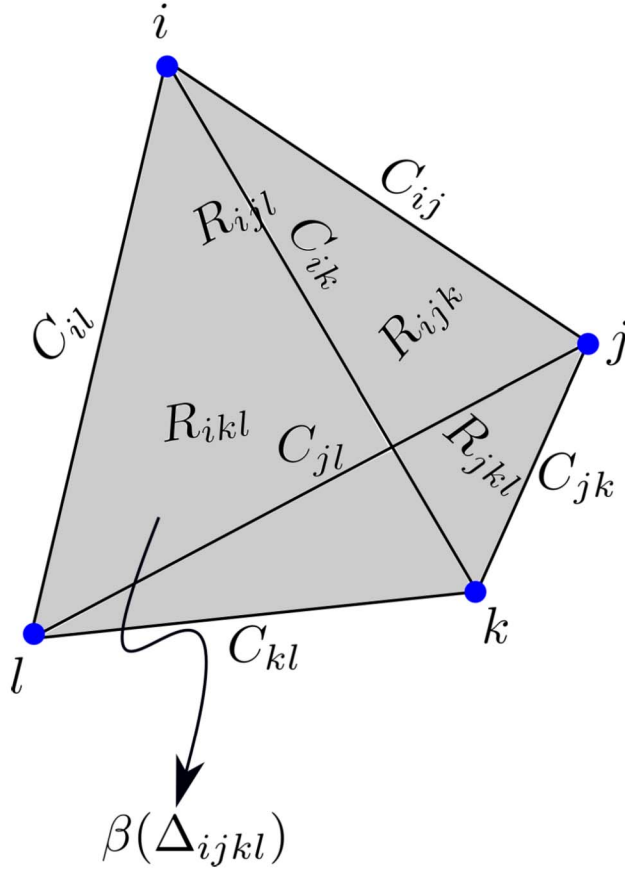


Figure 18: A speculative wall-crossing simplex. C_{ij} etc denote categories associated to edges, R_{ijk} etc denote chain complexes associated to faces and $\beta(\Delta_{ijkl})$ denotes a collection of numbers associated to the interior. Wall-crossing would occur when i passes through the base jkl triangle and moves over to the other side.

Homotopy equivalence of complexes. Two complexes (C, d) and (C', d') are said to be homotopy equivalent if there are chain maps $f : C \rightarrow C'$ and $g : C' \rightarrow C$ such that

$$(368) \quad gf = 1_C + \{d, s\},$$

$$(369) \quad fg = 1_{C'} + \{d', s'\},$$

for some degree -1 maps $s : C' \rightarrow C$ and $s' : C \rightarrow C'$. s and s' are known as chain homotopies.

Mapping cone recollection. Given two chain complexes (A^\bullet, d_A) and (B^\bullet, d_B) along with a chain map

$$(370) \quad f : A^\bullet \rightarrow B^\bullet,$$

there is a canonical chain complex $\text{Cone}(f)$ defined as follows. The underlying space consists of

$$(371) \quad \text{Cone}(f) = B \oplus A[-1].$$

Writing an element of $\text{Cone}(f)$ as a column vector

$$(372) \quad \begin{pmatrix} b \\ a^{[-1]} \end{pmatrix},$$

the differential on $\text{Cone}(f)$ is

$$(373) \quad d[f] = \begin{pmatrix} d_B & f \\ 0 & -d_A \end{pmatrix}.$$

$d[f]$ is nilpotent as a consequence of f being a chain map. The projection map

$$(374) \quad \pi : \text{Cone}(f) \rightarrow A[-1],$$

and the inclusion map

$$(375) \quad i : B \rightarrow \text{Cone}(f),$$

are chain maps that fit into the exact sequence

$$(376) \quad 0 \longrightarrow B \xrightarrow{i} \text{Cone}(f) \xrightarrow{\pi} A[-1] \longrightarrow 0.$$

Mapping cylinder recollection. Suppose we are in the setting of the mapping cone of a morphism $f : A \rightarrow B$, i.e consider $\text{Cone}(f)$. Note that the projection map

$$(377) \quad \pi : (\text{Cone}(f))[1] \rightarrow A$$

is a chain map. The **mapping cylinder** of f is then by definition

$$(378) \quad \text{Cyl}(f) := \text{Cone}(\pi).$$

More explicitly, we can write

$$(379) \quad \text{Cyl}(f) = B \oplus A[-1] \oplus A.$$

The differential on $\text{Cyl}(f)$ reads

$$(380) \quad d = \begin{pmatrix} d_B & f & 0 \\ 0 & -d_A & 0 \\ 0 & \text{id} & d_A \end{pmatrix}.$$

The following is standard in homological algebra and topology (for instance see Lemma 1.5.6 in Weibel [Weib]).

Proposition. Suppose $(A, d_A), (B, d_B)$ are chain complexes and $f : A \rightarrow B$ is a chain map. Then B and $\text{Cyl}(f)$ are canonically homotopy equivalent. The map $i : B \rightarrow \text{Cyl}(f)$ is given by inclusion and its homotopy inverse $j : \text{Cyl}(f) \rightarrow B$ is given by

$$(381) \quad j \begin{pmatrix} b \\ a^{[-1]} \\ a' \end{pmatrix} = b + f(a').$$

Remark. The mapping cone and mapping cylinder constructions have their origins in topology. If $f : (X, p_*) \rightarrow (Y, q_*)$ is a continuous map of topological spaces we can define topological spaces

$$(382) \quad \text{Cyl}(f) = (X \times I) \cup Y / (x, 1) \sim f(x),$$

$$(383) \quad \text{Cone}(f) = \text{Cyl}(f) / (x, 0) \sim p_*.$$

These spaces are related to the previous constructions as follows. If $C_*(X), C_*(Y)$ denote the singular chain complexes of X and Y , then

$$(384) \quad C_*(\text{Cyl}(f)) \cong \text{Cyl}(f_* : C_*(X) \rightarrow C_*(Y)),$$

$$(385) \quad C_*(\text{Cone}(f)) \cong \text{Cone}(f_* : C_*(X) \rightarrow C_*(Y)),$$

f_* being the induced map on complexes.

Triangularity lemma. Let A, B, C be chain complexes and $f : A \rightarrow B$ be a chain map. Suppose that

$$(386) \quad C \simeq \text{Cone}(f : A \rightarrow B).$$

Then we can construct chain maps

$$(387) \quad g : B \rightarrow C,$$

$$(388) \quad h : C[1] \rightarrow A$$

such that

$$(389) \quad A[-1] \simeq \text{Cone}(g : B \rightarrow C),$$

$$(390) \quad B \simeq \text{Cone}(h : C[1] \rightarrow A).$$

The maps g and h can be written down explicitly. We set

$$(391) \quad g = u \circ i$$

where $u : \text{Cone}(f) \rightarrow C$ is one of the maps provided by homotopy equivalence and $i : B \rightarrow \text{Cone}(f)$ is the inclusion map (also a chain map). Similarly

$$(392) \quad h = \pi \circ v$$

where $v : C \rightarrow \text{Cone}(f)$ is the homotopy inverse of u and $\pi : \text{Cone}(f) \rightarrow A[-1]$ is the projection map (also a chain map). These maps may be remembered from the commutative diagram

$$(393) \quad \begin{array}{ccccccc} & & & C & & & \\ & & & \downarrow v & & & \\ 0 & \longrightarrow & B & \xrightarrow{i} & \text{Cone}(f) & \xrightarrow{\pi} & A[-1] \longrightarrow 0. \\ & & & \uparrow u & & & \\ & & & C & & & \end{array}$$

Appendix B. A_∞ algebras and morphisms

This appendix serves as a reminder of some elementary formulas in A_∞ theory. We refer the reader to the (unpublished) book of Kontsevich-Soibelman [KoSo4], Keller's notes [Kel], and appendix A of [GMW] for more details.

A_∞ -algebra. Given a graded vector space A , denote by $T^\bullet(A)$ the tensor algebra of A , and $T_+^\bullet(A)$ the positive part of the tensor algebra:

$$(394) \quad T^\bullet(A) = \bigoplus_{n \geq 0} A^{\otimes n},$$

$$(395) \quad T_+^\bullet(A) = \oplus_{n \geq 1} A^{\otimes n}.$$

A is called an A_∞ -algebra if there is a square-zero, degree one derivation,²⁴

$$(396) \quad \delta : T_+^\bullet(A^*[1]) \rightarrow T_+^\bullet(A^*[1]).$$

Extracting Taylor coefficients amounts to a collection maps

$$(397) \quad m_n : A^{\otimes n} \rightarrow A$$

of degree $2 - n$ satisfying the A_∞ -associativity axioms: for each $d \geq 1$ we have

$$(398) \quad \sum_{\substack{k+l=d+1 \\ 1 \leq i \leq k}} (-1)^{d_1 + \dots + d_{i-1} - i + 1} m_k(a_1, \dots, a_{i-1}, m_l(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_d) = 0.$$

a_i is a homogeneous element and $d_i = \deg(a_i)$.

A_∞ -morphism. Given two A_∞ -algebras

$$(399) \quad (A_1, \{m_n\}) \text{ and } (A_2, \{\mu_n\})$$

an A_∞ -morphism

$$(400) \quad f : A_1 \rightarrow A_2$$

is an algebra homomorphism (respects tensor algebra structure)

$$(401) \quad f : T_+^\bullet(A_2^*[1]) \rightarrow T_+^\bullet(A_1^*[1])$$

that is also a chain map: namely f is degree 0 map satisfying

$$(402) \quad f\delta_2 = \delta_1 f.$$

Again expanding out Taylor coefficients we get a collection of maps

$$(403) \quad f_n : (A_1)^{\otimes n} \rightarrow A_2$$

²⁴Meaning δ is both a derivation of the tensor algebra $\delta(X^a X^b) = \delta X^a X^b \pm X^a \delta X^b$, and a differential, a degree one map such that $\delta^2 = 0$.

of degree $1 - n$ satisfying the A_∞ -morphism axioms

$$\begin{aligned}
 (404) \quad & \sum_{\substack{k+l=d+1 \\ 1 \leq i \leq k}} (-1)^{d_1 + \dots + d_{i-1} - i + 1} f_k(a_1, \dots, a_{i-1}, m_l(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_d) \\
 &= \sum_{\substack{n_1 + \dots + n_k = d \\ k \geq 1}} \mu_k(f_{n_1}(a_1, \dots, a_{n_1}), \dots, f_{n_k}(a_{d-n_k+1}, \dots, a_d)).
 \end{aligned}$$

The $d = 1$ relation is

$$(405) \quad \mu_1(f_1(a_1)) = f_1(m_1(a_1))$$

which simply says that f_1 is a chain map.

The $d = 2$ relation is

$$\begin{aligned}
 (406) \quad & f_1(m_2(a_1, a_2)) \pm \mu_2(f_1(a_1), f_1(a_2)) \\
 &= f_2(m_1(a_1), a_2) \pm f_2(a_1, m_1(a_2)) \pm \mu_1(f_2(a_1, a_2))
 \end{aligned}$$

where the precise signs can be restored via (404). This says that the diagram

$$(407) \quad \begin{array}{ccc} A_1^{\otimes 2} & \xrightarrow{f_1 \otimes f_1} & A_2^{\otimes 2} \\ m_2 \downarrow & & \downarrow \mu_2 \\ A_1 & \xrightarrow{f_1} & A_2, \end{array}$$

commutes up to homotopy, with f_2 providing the chain homotopy.

Quasi-isomorphism of A_∞ -algebras. An A_∞ -morphism $\{f_n\}_{n \geq 1}$ is said to be a quasi-isomorphism if $f_1 : (A_1, m_1) \rightarrow (A_2, \mu_1)$ is a quasi-isomorphism of chain complexes.

Homotopy equivalence of A_∞ -algebras. Two A_∞ -morphisms $f, g : A_1 \rightarrow A_2$, between A_∞ -algebras are said to be homotopic $f \simeq g$, if there is a degree -1 map

$$(408) \quad S : T_+^\bullet(A_2^*[1]) \rightarrow T_+^\bullet(A_1^*[1])$$

such that

$$(409) \quad f - g = S\delta_2 + \delta_1 S.$$

That is S provides a homotopy between the parent maps f, g of the tensor algebra. A_1 and A_2 are said to be homotopy equivalent A_∞ algebras if there are A_∞ -morphisms $f : A_1 \rightarrow A_2$ and $g : A_2 \rightarrow A_1$ such that the compositions in either direction are homotopic to the identities on the tensor algebras:

$$(410) \quad g \circ f \simeq 1_{T_+ A_2^*},$$

$$(411) \quad f \circ g \simeq 1_{T_+ A_1^*}.$$

In particular, (A_1, m_1) and (A_2, μ_1) are homotopy equivalent chain complexes.

L_∞ -algebra. A graded vector space L is called an L_∞ -algebra if there is a derivation differential

$$\delta : S_+^\bullet(L^*[2]) \rightarrow S_+^\bullet(L^*[2]).$$

Extracting coefficients gives us that we have a collection of maps

$$(412) \quad \lambda_n : L^{\otimes n} \rightarrow L$$

of degree $3-2n$ which are graded symmetric, and satisfy the L_∞ -associativity axioms: for each $d \geq 1$ we have

$$(413) \quad \sum_{\substack{k+l=d+1 \\ \sigma \in \text{Sh}_2(k-1, l)}} \epsilon(\sigma, \vec{\ell}) \lambda_k(\lambda_l(\ell_{\sigma(1)}, \dots, \ell_{\sigma(l)}), \ell_{\sigma(l+1)}, \dots, \ell_{\sigma(d)}) = 0.$$

In the above $\sigma \in \text{Sh}_2(k, l)$ denotes a permutation $\sigma \in S_{k+l}$ such that

$$(414) \quad \sigma(1) < \dots < \sigma(k), \quad \sigma(k+1) < \dots < \sigma(k+l).$$

L_∞ -morphism. Given

$$(415) \quad (L_1, \{\lambda_n\}), \quad (L_2, \{\kappa_n\})$$

two L_∞ -algebras an L_∞ -morphism $f : L_1 \rightarrow L_2$ is an algebra homomorphism

$$(416) \quad f : S_+^\bullet(L_1^*[2]) \rightarrow S_+^\bullet(L_2^*[2])$$

that is also a chain map with respect to the L_∞ -structures. Extracting coefficients we get a collection of maps

$$(417) \quad f_n : (L_1)^{\otimes n} \rightarrow L_2$$

of degree $1 - n$ satisfying axioms for an L_∞ -morphism: for each $d \geq 1$

(418)

$$\begin{aligned} & \sum_{\substack{k+l=d+1 \\ \sigma \in \text{Sh}_2(k-1, l)}} \epsilon(\sigma, \vec{\ell}) f_k(\lambda_l(\ell_{\sigma(1)}, \dots, \ell_{\sigma(l)}), \ell_{\sigma(l+1)}, \dots, \ell_{\sigma(d)}) \\ &= \sum_{\substack{n_1 + \dots + n_k \\ \sigma \in \text{Sh}_k(n_1, \dots, n_k) \\ k \geq 1}} \frac{1}{k!} \epsilon'(\sigma) \kappa_n(f_{n_1}(\ell_{\sigma(1)}, \dots, \ell_{\sigma(n_1)}), \dots, f_{n_k}(\ell_{\sigma(d-n_k+1)}, \dots, \ell_{\sigma(d)})), \end{aligned}$$

and $\epsilon(\sigma, \vec{\ell})$ and $\epsilon'(\sigma)$ are suitable signs.

Quasi-isomorphism of L_∞ -algebras. An L_∞ -morphism $\{f_n\}_{n \geq 1}$ from $(L_1, \{\lambda_n\})$ and $(L_2, \{\kappa_n\})$ is said to be a quasi-isomorphism if

$$(419) \quad f_1 : (L_1, \lambda_1) \rightarrow (L_2, \kappa_2)$$

is a quasi-isomorphism of chain complexes.

Maurer-Cartan elements of L_∞ -algebras. A Maurer-Cartan element γ of an L_∞ algebra $(L, \{\lambda_n\})$ is a degree two element that solves the L_∞ Maurer-Cartan equation

$$(420) \quad \sum_{n \geq 1} \frac{1}{n!} \lambda_n(\gamma, \dots, \gamma) = 0.$$

An infinitesimal gauge transformation of a Maurer-Cartan element γ is written as

$$(421) \quad \delta_\epsilon \gamma = \sum_{n \geq 1} \frac{1}{n!} \lambda_n(\gamma^{\otimes(n-1)}, \epsilon)$$

where ϵ is any degree one element of L . Indeed one checks that $\gamma + \delta_\epsilon \gamma$ solves the Maurer-Cartan equation to first order in ϵ .

Terminology: algebras vs categories. In the bulk text of this paper we have often used the terms “algebra” and “category” interchangeably. This is justified because we can go between the two in a precise manner. Following the discussion in chapter 6 of [KoSo4], given a linear category with a finite object set S , we can define a unital algebra to be

$$(422) \quad A = \oplus_{r, s \in S} \text{Hom}(r, s),$$

with the unit being the direct sum of identity compositions and multiplications given by compositions of morphisms. Conversely, if a unital algebra A is equipped with commuting idempotents $\{\Pi_i\}_{i \in I}$ such that $1_A = \oplus_i \Pi_i$, then we can construct a category \mathcal{C} by setting the object set to be I and letting

$$(423) \quad \text{Hom}(i, j) = \Pi_i A \Pi_j.$$

Appendix C. $N_{ij} \in \{0, 1\}$ for $W \in \mathbb{C}[X]$

We give a proof of the assertion that a Landau-Ginzburg model with target \mathbb{C} and $W(X)$ a Morse polynomial has at most a single soliton between any pair of critical points. For this we consider the relative homology group

$$(424) \quad V = H_1(\mathbb{C}, \text{Re}(\zeta^{-1}W) \rightarrow \infty; \mathbb{Z})$$

where ζ is a phase not equal to any of the critical phases. V is easily constructed. Supposing that the degree of W is n , we divide the complex plane \mathbb{C} into $2n$ wedges of equal angle $\frac{2\pi}{n}$ and shade alternating regions R_1, \dots, R_n . A basis for V is provided by cycles $\gamma_{a,a+1}$ that connect R_a and R_{a+1} for $a = 1, \dots, n-1$. On the other hand, Picard-Lefschetz theory says that the homology class of the Lefschetz thimbles $L_i(\zeta)$ for $i = 1, \dots, n-1$ critical points of W must also form a \mathbb{Z} -module basis for V . In particular this implies that if $L_i(\zeta)$ connects R_a, R_b and $L_j(\zeta)$ connects R_c, R_d then $\{a, b\} \neq \{c, d\}$ since otherwise they will be multiples of each other by ± 1 in homology, and thus linearly dependent elements of V . Considering a point p on the ζ -ray emanating from W_i far out enough, $W^{-1}(p) \cap L_i(\zeta)$ is a pair of points lying in distinct regions R_a, R_b which are connected by L_i . Therefore $|L_i(\zeta_{ji}e^{-i\epsilon}) \cap L_j(\zeta_{ji}e^{i\epsilon})|$ is at most one, concluding the proof.

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