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THE RICCI FLOW ON COMPLETE $\mathbb{R}^2$

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ABSTRACT. Given $\mathbb{R}^2$, with a "good" complete metric, we show that the unique solution of the Ricci flow approaches a soliton at time infinity. Solitons are solutions of the Ricci flow which move only by diffeomorphism.

INTRODUCTION

In this paper we will study the Ricci flow on $\mathbb{R}^2$ with a complete metric. It is interesting to note that the Ricci flow on $\mathbb{R}^2$ is the limiting case of the porous medium equation as $m \to 0$, which will be further discussed in the appendix by Sigurd Angenent and a short announcement [W-2].

The Ricci flow on a surface is to evolve the metric under

$$\frac{\partial}{\partial t} ds^2 = -R ds^2,$$

where R is the scalar curvature. For more detail see [H-1].

We say that $ds^2(t)$ is a *Ricci gradient soliton solution* if there exists a function f such that

$$\frac{\partial}{\partial t} ds^2(t) = -R ds^2(t) = L_{\nabla f} ds^2(t).$$

There are two types of *gradient solitons* on $\mathbb{R}^2$. Namely, the flat soliton and the cigar soliton. The flat soliton is the standard flat metric on $\mathbb{R}^2$. The cigar soliton is a metric which can be expressed as $ds^2 = \dfrac{dx^2 + dy^2}{1 + x^2 + y^2}$, where $\{x, y\}$ are rectangular coordinates on $\mathbb{R}^2$.

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On a complete $(\mathbb{R}^2, ds^2)$, *the circumference at infinity* is defined as

(0.1)
$$C_\infty(ds^2) = \sup_K \inf_{D^2}\{L(\partial D^2)|\forall \text{ compact set } K \subset \mathbb{R}^2, \forall \text{ open set } D^2 \supset K\};$$

and *the aperture* is defined as

$$A(ds^2) = \frac{1}{2\pi} \lim_{r\to\infty} \frac{L(\partial B_r)}{r}, \tag{0.2}$$

where B_r is a geodesic ball at any given point on $\mathbb{R}^2$ with radius r, and $L[\partial D^2]$ is the length of ∂D^2 with respect to ds^2. On a flat soliton, we have $C_\infty = \infty$ and $A(ds^2) = 1$; while on a cigar soliton C_∞ is finite and $A(ds^2) = 0$.

We say that the Ricci flow on M has modified subsequence convergence at time infinity if there exists a 1-parameter family of diffeomorphisms $\{\phi_t\}_{t\in[0,\infty)}$ on M such that for any sequence of times going to infinity there is a subsequence of times $\{t_j\}_{j=0}^{\infty}$ and the modified metrics $ds^2(\phi_{t_j}(\cdot), t_j)$ converges uniformly on every compact set as $j \to \infty$.

On $\mathbb{R}^2$, let $ds^2 = e^{u(x,y)}(dx^2 + dy^2)$ be a complete metric, where $\{x, y\}$ are rectangular coordinates. The main result in this paper is:

Main Theorem. *Given a complete* $(\mathbb{R}^2, ds^2(0))$ *with* $|R| \leq C$ *and* $|Du| \leq C$ *at* $t = 0$, *then the Ricci flow has modified subsequence convergence at time infinity to a limiting metric. Furthermore, in the case when the curvature is positive at time zero, the limiting metric is a cigar soliton if* $C_\infty(ds^2(0)) < \infty$, *or a flat metric if* $A(ds^2(0)) > 0$.

Note that: There is still a big class of Riemannian structures with $C_\infty = \infty$ and $A = 0$ which our method fails to classify the limit.

Sketch of the Proof. The evolution equation of $h = R + |Du|^2$ provides the infinite time existence and uniform bounds for $|Du|$, $|D^k u|$, R and $|D^k R|$ for all $k \geq 1$ after a short time. Finite total curvature and $C_\infty > 0$ imply that the curvature decays to zero at distance infinity. This yields that C_∞, $A(ds^2)$, and $\int R d\mu$ are preserved under the flow. Furthermore, the solution of the flow is unique and the positivity of the curvature is preserved.

The positivity of the curvature of an initial metric provides pointwise convergence of the function e^u at time infinity. All the uniform bounds of $|D^m u|$

yield that $\lim_{t\to\infty} e^u$ is a smooth function and is either identically zero or positive everywhere. In the case, when $\lim_{t\to\infty} e^u > 0$, the limiting solution is a flat metric.

Nevertheless, there is a 1-parameter family of diffeomorphisms $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that for any sequence of times going to infinity, there is a subsequence $\{t_j\}_{j=0}^{\infty}$ and $\lim_{j\to\infty} ds^2(\phi_{t_j}(\cdot), t_j)$ converges to a metric uniformly on every compact set. If $R > 0$ and $C_\infty < \infty$ at $t = 0$, then some integral bounds classify $\lim_{j\to\infty} ds^2(\phi_{t_j}(\cdot), t_j)$ as a cigar soliton with circumference no bigger than $C_\infty(0) < \infty$. If $R > 0$ and $A(ds^2) > 0$ at $t = 0$, then the Harnack's inequality classifies $\lim_{j\to\infty} ds^2(\phi_{t_j}(\cdot), t_j)$ as a flat metric.

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1. The Maximum Principle

In this section we will give an elementary proof of the maximum principle for a class of parabolic equations on $\mathbb{R}^2$ with changing metric $ds^2(t) = e^{u(x,y,t)}(dx^2 + dy^2)$. For more details, we refer the reader to [Ar] and [P-We].

Theorem 1.1 (Maximum Principle). *Given $(\mathbb{R}^2, ds^2(t))$ for all $t \in [0,T]$, and a function $f : \mathbb{R}^2 \times [0,T] \to R^1$ satisfying $\frac{\partial f}{\partial t} \le \Delta f$. Then if $f \le C$ at $t = 0$ and f is bounded $\forall t \in [0,T]$, we have $f \le C$, $\forall t \in [0,T]$.*

The idea of proving the maximum principle on $\mathbb{R}^2$ with an evolving metric is similar to the proof of the case when the metric is fixed. For clarity, we will first prove the maximum principle with a fixed metric.

Lemma 1.2. *Given $(\mathbb{R}^2, \overline{ds^2})$, let $f : \mathbb{R}^2 \times [0,T] \to R^1$ be a function which satisfies $\frac{\partial f}{\partial t} \le \overline{\Delta} f$. If $f \le C$ at $t = 0$ and f is bounded $\forall t \in [0,T]$, then $f \le C$, $\forall t \in [0,T]$.*

Proof. Let $\tau \in [0,T]$ be the largest time such that $f \leq 2C$ on $0 \leq t \leq \tau$. Define

$$f_\delta = f - \delta(x^2+y^2) - 4\delta t, \forall \delta > 0,$$

where x, y are the standard rectangular coordinates. Then we have

$$\frac{\partial}{\partial t} f_\delta \leq \overline{\Delta} f_\delta, \forall t \in [0,T],$$

and $f_\delta \leq C$ at $t = 0$. Since $f \leq 2C, \forall t \in [0,\tau]$ and $\sup_{x\in\mathbb{R}^2} f_\delta$ can only occur in the interior, we have $\frac{\partial}{\partial t} f_\delta|_{\max} \leq 0$. So $f_\delta \leq C$, $\forall 0 \leq t \leq \tau$, $\forall \delta$. That is,

$$f - \delta(x^2+y^2) - 4\delta t \leq C, \forall \delta.$$

As $\delta \to 0$, we get $f \leq C$ on $0 \leq t \leq \tau$. If $\tau < T$, there exists an $\epsilon > 0$ such that $f \leq 2C, \forall t \in [0, \tau+\epsilon] \subset [0,T]$. This contradicts the choice of τ, so $\tau = T$. □

Lemma 1.3. *On* $(\mathbb{R}^2, ds^2(t))$ *with* $t \in [0,T]$, *there exists a time-independent function* $h(r) = h(r,t)$ *on* $\mathbb{R}^2 \times [0,T]$, *such that* $\Delta h \leq 1$ *on* $\mathbb{R}^2 \times [0,T]$, *and* $h(r) = h(r,t) \to +\infty$, *as* $r \to \infty, \forall t \in [0,T]$.

Proof. Define a time-independent function $h(r) = h(r,t)$ on $\mathrm{M} \times [0,T]$:

$$h(r) = \int_{b=0}^{b=r} \frac{\int_{a=0}^{a=b} a e^{v(a)} da}{b} db,$$

where $v(r) = \min_{\substack{0\leq t\leq T \\ 0\leq\theta\leq 2\pi}} u(r,\theta,t)$. Obviously we have $h(0) = 0$, and

$$h(r) \leq \int_{b=0}^{b=r} \frac{\int_{a=0}^{a=b} a e^{u(a,\theta,t)} da}{b} db, \qquad \forall \theta, t.$$

Note that typically we expect u to approach infinity at distance infinity. Since $re^{u(r,\theta,t)} > 0$, there exists an $\epsilon_1 > 0$, such that $\int_{a=0}^{a=\epsilon_1} a e^{u(a,\theta,t)} da = C > 0$. This yields

$$\begin{aligned} h(r) &> \int_{b=\epsilon_1}^{b=r} \frac{\int_{a=0}^{a=r} a e^{v(a)} da}{b} db, \\ &> \int_{b=\epsilon_1}^{b=r} \frac{\int_{a=0}^{a=\epsilon_1} a e^{v(a)} da}{b} db \\ &> C \int_{b=\epsilon_1}^{b=r} \frac{db}{b} = C \ln b\Big|_{b=\epsilon_1}^{b=r} \\ &> C(\ln r - \ln \epsilon_1), \qquad \forall r > 0. \end{aligned}$$

In particular, we have $h = h(r) \to +\infty$, as $r \to \infty$. Since $h_r = \frac{\int_{a=0}^{a=r} ae^{v(a)}da}{r}$, we have

$$\Delta h = e^{-u}(\frac{\partial_r(rh_r)}{r}) = e^{-u}(\frac{re^{v(r)}}{r}) \leq 1. \quad \square$$

We will apply the same argument in Lemma 1.2 to prove Theorem 1.1.

Proof of Theorem 1.1. Take the function h obtained in Lemma 1.3, and let $f_\delta = f - \delta h - \delta t$. Then $\frac{\partial}{\partial t} f_\delta \leq \Delta f_\delta + \delta(\Delta h - 1) \leq \Delta f_\delta$. From the same argument in Lemma 1.2, we have $f \leq C$, $\forall t \in [0, T]$. $\square$

2. Long time existence

In this paper $(\mathbb{R}^2, ds^2)$ will be a complete, noncompact conformally flat surface. The evolving metric $ds^2(t) = e^{u(x,y,t)}(dx^2+dy^2) = e^{u(r,\theta,t)}(dr^2+r^2d\theta^2)$ is the solution of the Ricci flow, where $\{x, y\}$ and $\{r, \theta\}$ are rectangular and polar coordinates.

Proposition 2.1. *On* $(\mathbb{R}^2, ds^2)$, *the evolution equation of the Ricci flow*

$$\frac{\partial}{\partial t} ds^2 = -Rds^2. \tag{2.1}$$

is equivalent to:

$$\frac{\partial}{\partial t} u = \Delta u = -R. \tag{2.2}$$

From straightforward computations, one may obtain the following related evolution equations:

$$\frac{\partial}{\partial t} R = \Delta R + R^2 \tag{2.3}$$

$$\frac{\partial}{\partial t} |Du|^2 = \Delta |Du|^2 - 2|D_iD_ju|^2. \tag{2.4}$$

2.1. Short Time Existence. The short time existence for the Ricci flow can be obtained by bounds on the curvature at $t = 0$. ([Shi-2]) The bounds for $|D^kR|$ over a small time interval may be obtained by looking at some dilationally invariant terms.

Theorem 2.2 ([Shi-2, Theorem 1.1]). *On a complete, noncompact 2-manifold, if $|R| \le k_0$ at $t = 0$, then the Ricci Flow has short time existence. We can bound the derivatives of the curvature by $|D^m R| \le \frac{K_m(k_0)}{t^{m/2}}$, $\forall m \ge 0$ on a short time interval $[0, T_{k_0}]$, where T_{k_0} and $K_m(k_0)$ only depend on the bound k_0 for the curvature.*

Now we will use the same method as in [Shi-2] to prove the dilationally invariant bounds for higher derivatives of u.

Theorem 2.3. *On $(\mathbb{R}^2, ds^2)$ with $|R| < k_0$ and $|Du|^2 < D_0$ at $t = 0$, there exist positive constants T_{k_0} and $C_m(k_0, D_0)$ depending only on k_0 and D_0, such that under the Ricci Flow we have*

$$|D^m u|^2 \le \frac{C_m(k_0, D_0)}{t^{m-1}}, \qquad \forall t \in [0, T_{k_0}], \qquad \forall m \ge 1.$$

Proof. From Theorem 2.5 in [Shi-2], there exist positive constants $K_m(k_0)$ and T_{k_0} such that

$$|D^m R| \le \frac{K_m(k_0)}{\sqrt{t}^m}, \qquad \forall t \in [0, T_{k_0}], \forall j \ge 0.$$

In particular, we have $|R| \le K_0(k_0)$ $\forall t \in [0, T_{k_0}]$.

The evolution equation of the conformal factor under the Ricci flow (2.2) implies

$$\frac{\partial}{\partial t}|Du|^2 = -2D_i u D_i R + R|Du|^2,$$

then

$$\frac{\partial}{\partial t}|Du|^2 \le 2|Du||DR| + R|Du|^2 \le \frac{2K_1(k_0)}{\sqrt{t}}|Du| + K_0(k_0)|Du|^2.$$

This implies

$$|Du|^2 \le C(k_0, D_0)e^{C(k_0,D_0)t} \le C_1(k_0, D_0), \qquad \forall t \in [0, T_{k_0}],$$

where $C(k_0, D_0)$ and $C_1(k_0, D_0)$ are constants depending only on k_0 and D_0. Furthermore, we have

$$\frac{\partial}{\partial t}|Du|^2 = \Delta|Du|^2 - 2|D_i D_j u|^2,$$

$$\begin{aligned}\frac{\partial}{\partial t}|D_i D_j u|^2 &= \Delta|D_i D_j u|^2 - 2|D_i D_j D_k u|^2 - R(|D_i D_j u|^2 - (\Delta u)^2) \\ &\le \Delta|D_i D_j u|^2 - 2|D_i D_j D_k u|^2 - C^0_{k_0}|D^2 u|^2.\end{aligned}$$

That is

$$\frac{\partial}{\partial t}|D^2u|^2 \le \Delta|D^2u|^2 - 2|D^3u|^2 - C^0_{k_0}|D^2u|^2.$$

Then the same methods as in Lemma 7.1 in [Shi-2] can be applied directly to get the bounds for all the higher order derivatives. □

2.2. Long time existence. We will use the same method as in [H-1] to show the infinite time existence for solutions of the Ricci flow. We define a function $h = -\Delta u + |Du|^2 = R + |Du|^2$ then the evolution equation of h under the Ricci flow is

$$\frac{\partial h}{\partial t} = \Delta h - 2|M_{ij}|^2 \le \Delta h. \tag{2.5}$$

Theorem 2.4. *On $(\mathbb{R}^2, ds^2)$ with $|R| < k_0$ and $|Du|^2 < D_0$ at $t = 0$, under the Ricci flow we have*

(1) *The solution of the Ricci flow exists $\forall t \in [0, \infty)$.*
(2) *$|R| \le k_0 + D_0$ and $|Du|^2 \le D_0$ for all time.*

Proof. From (2.3) and the maximum principle, we know that the curvature is bounded below by $-k_0$ for all time. From Theorems 1.1 and 2.4, if $|R| \le k_0$ and $|Du| \le D_0$ at $t = 0$, there exists a constant T_{k_0} such that R and $|Du|$ are bounded on the time interval $[0, T_{k_0}]$. Then the maximum principle implies $h(x,t) \le \sup_{x\in\mathbb{R}^2} h(x,0), \forall t \in [0, T_{k_0}]$. That is,

$$h(x,t) \le \sup_{x\in\mathbb{R}^2} h(x,0) \le k_0 + D_0, \forall t \in [0, T_{k_0}].$$

Thus $R \le k_0 + D_0$ and $|Du|^2 \le 2k_0 + D_0$, $\forall t \in [0, T_{k_0}]$. In particular, combining with (2.3), (2.4) and the maximum principle we have $|R| \le k_0 + D_0$ and $|Du|^2 \le D_0, \forall t \in [0, T_{k_0}]$. Implement the same argument at time $t = T_{k_0}$, we have $|R| \le k_0 + D_0$ and $|Du|^2 \le D_0$ on $\mathbb{R}^2 \times [T_{k_0}, T_{k_0+D_0}]$. Hence, by repeating the above process n times, we have

$$|R| \le k_0 + D_0, \qquad \text{and} \qquad |Du|^2 \le D_0 \qquad \text{on } \mathbb{R}^2 \times [0, T_{k_0} + nT_{k_0+D_0}],$$

where $T_{k_0+D_0} > 0$ is a constant depending only on k_0 and D_0.

Let n approach infinity, then we have $|R| \le k_0 + D_0$, and $|Du|^2 \le D_0$ for all $t \in [0, \infty)$, and the long time existence follows. □

Corollary 2.5. *Given a complete* $(\mathbb{R}^2, ds^2)$ *with* $|R| < k_0$ *and* $|Du|^2 < D_0$ *at* $t = 0$*, under the Ricci flow,* $\forall \tau > 0$ *we have*

$$|D^k u|^2 \leq C_\tau^k, \qquad \forall t > \tau > 0, \qquad \forall k \geq 2. \tag{2.6}$$

In particular, we have uniform bounds for all the higher derivatives of the curvature after a short time.

3. Modified Subsequence Convergence

The following theorem tells us that even if the limit of the solution u exists it might not yield a metric at time infinity. Let $|\cdot|$ (resp. $\overline{|\cdot|}$) denote the norm with respect to $ds^2(t)$ (resp. $\overline{ds^2}$).

Theorem 3.1. *On a complete* $(\mathbb{R}^2, ds^2)$ *with* $0 < R < k_0$ *and* $|Du|^2 < D_0$ *at* $t = 0$*, then* $\lim_{t\to\infty} e^{u(x,y,t)}$ *converges uniformly on every compact set and* $\lim_{t\to\infty} e^{u(x,y,t)}$ *is a nonnegative constant. If* $\lim_{t\to\infty} e^{u(x,y,t)} > 0$*, then* $\lim_{t\to\infty} e^{u(x,y,t)}(dx^2 + dy^2)$ *induces a metric on* $\mathbb{R}^2$ *with curvature identically zero.*

Proof. Positive curvature implies that $\frac{\partial}{\partial t}u = -R < 0$ and $\frac{\partial}{\partial t}e^u = -Re^u < 0$. For each $\xi \in \mathbb{R}^2$, $e^{u(\xi,t)}$ is a decreasing function in t and $e^{u(\xi,t)} \geq 0$, thus $\lim_{t\to\infty} e^{u(\xi,t)}$ converges pointwisely.

Since $\overline{|De^u(\cdot,t)|^2} = e^{3u}|Du(\cdot,t)|^2$ is bounded, combining with the pointwise convergence, $\lim_{t\to\infty} e^{u(\cdot,t)}$ converges uniformly on every compact set. All higher derivative bounds of e^u imply that $\lim_{t\to\infty} e^{u(\cdot,t)}$ is a smooth function.

If there exists a point $p \in \mathbb{R}^2$, such that $\lim_{t\to\infty} e^{u(p,t)} = e^a > 0$, then $\lim_{t\to\infty} u(p,t) = a > -\infty$. On any compact set near p there is a bound for $\lim_{t\to\infty} \overline{|Du(\cdot,t)|}$ thus on every compact set, $\lim_{t\to\infty} u(\cdot,t)$ is away from negative infinity, i.e.

$$\lim_{t\to\infty} e^u(\cdot,t) > 0.$$

Furthermore, $|\int_{t=0}^{\infty} -R(\xi,t)dt| = |u(\xi,\infty) - u(\xi,0)| < \infty$. From the positivity of R, and the uniform bounds of R and $|DR|$, we have $\lim_{t\to\infty} R(\cdot,t) = 0$, in particular, $\lim_{t\to\infty} e^{u(\cdot,t)}$ is a positive constant. □

To avoid the problem of the limiting solution failing to yield a metric we will modify the solutions by a family of diffeomorphisms. To see why modifying the solution by diffeomorphism is needed to prove convergence for the Ricci flow, we will illustrate the following example.

EXAMPLE 3.1. Given the soliton metric $ds^2(0) = \dfrac{dx^2+dy^2}{1+x^2+y^2}$ on $\mathbb{R}^2$, then it is easy to compute that the solution of the Ricci flow with initial data $ds^2(0)$ is $ds^2(t) = \dfrac{dx^2+dy^2}{e^{4t}+x^2+y^2}$. Then $e^{u(x,y,t)} = \dfrac{1}{e^{4t}+x^2+y^2}$ goes to zero as time approaches infinity. Therefore, we can not claim that $\lim\limits_{t\to\infty} e^{u(x,y,t)}$ yields a metric on $\mathbb{R}^2$.

Nevertheless, if we let diffeomorphism $\phi_t(A,B) = (e^{2t}A, e^{2t}B) = (x,y)$, then

$$ds^2(x,y,t) = ds^2(\phi_t(A,B),t) = \frac{dA^2+dB^2}{1+A^2+B^2}.$$

Denote $\widehat{ds^2}(A,B,t) = ds^2(\phi_t(A,B),t)$ and $e^{\widehat{u}} = \dfrac{1}{1+A^2+B^2}$. Then $e^{\widehat{u}}$ is stationary in time.

As in Example 3.1, we will modify the solution of the Ricci flow by a 1-parameter family of diffeomorphisms. Choose a diffeomorphism $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\phi_t(A,B) = (e^{-\frac{u(0,0,t)}{2}}A, e^{-\frac{u(0,0,t)}{2}}B) = (x,y).$$

Then define the metric $\widehat{ds^2}$ as

$$\widehat{ds^2}(A,B,t) = ds^2(x,y,t) = ds^2(\phi_t(A,B),t) = e^{u(\phi_t(A,B),t)-u(0,0,t)}(dA^2+dB^2)$$

function $\widehat{u}(A,B,t) = u(\phi_t(A,B),t) - u(0,0,t)$, and hence $e^{\widehat{u}(0,0,t)} = 1$.

Let $|\cdot|_{\hat{t}}$ and $\widehat{d\mu}$ be the norm and the area form with respect to $\widehat{ds^2}(t)$, then

$$|D\widehat{u}|^2_{\hat{t}}(A,B,t) = |Du|^2_{\hat{t}}(A,B,t) = |Du|^2(x,y,t) \leq D_0$$

and $\widehat{u}(0,0,t) = 0$. By Azela-Ascoli theorem, for any sequence of times going to infinity there exists a subsequence $\{t_l\}$ such that $\lim\limits_{l\to\infty} \widehat{ds^2}(t_l)$ converges uniformly to a metric $\widehat{ds^2}(\infty)$ on every compact set. Furthermore, if $R > 0$, then any open set D we have $L_{\hat{t}}[\partial\phi_t^{-1}(D)] = L_t[\partial(D)]$ is a decreasing function in time. And $C_\infty(\widehat{ds^2}(\infty)) \leq C_\infty(0)$.

Theorem 3.2. *Given a complete $(\mathbb{R}^2, ds^2)$ with $|R| \leq k_0$ and $|Du|^2 \leq D_0$ at $t = 0$, then the Ricci flow has the modified subsequence convergence at time infinity to a limiting metric. If $R > 0$ at $t = 0$ then $C_\infty(\widehat{ds^2}(\infty)) \leq C_\infty(0)$.*

We will devote the rest of the paper to classifyingthe limiting metric.

4. Geometric Preliminaries

In order the classify the limit metric at time infinity, we need to investigate how some geometric quantities evolve under the Ricci flow. In this section, we will briefly review some of the geometric properties of complete, noncompact surfaces, which are mainly quoted from [L-T]. As described by K. Shiohama, the total curvature of a complete, non-compact surface (i.e. $\int Rd\mu$) is not a topological invariant but it depends upon the choice of Riemannian structures. A well studied class of Riemannian structures is the one with finite total absolute curvature. We say that a metric has finite total absolute curvature if $\int |R|d\mu$ is bounded. In fact, finite total absolute curvature on complete, noncompact surface is equivalent to finite total negative curvature (i.e. $\int R_- d\mu < \infty$, where $R_- = \max\{-R, 0\}$.)

Theorem 4.1 ([Hu]). *If M is a complete, noncompact surface with finite total negative curvature, then M is conformally equivalent to a compact Riemann surfaces with finitely many points deleted. Moreover,*

$$\int_M Rd\mu \leq 4\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. In particular, $\int_M |R|d\mu < \infty$.

The following proposition reveals the relation between the total curvature and the aperture. Let $p \in M$ be a fixed point. Let us denote the geodesic ball of radius r at center p by $B_r(p)$, and its boundary by $\partial B_r(p)$.

Proposition 4.2 ([Sh]). *Let (M, ds^2) be a complete surface with finite total curvature. If R is the scalar curvature of M, then $\forall p \in M$ we have*

$$4\pi\chi(M) - \int_M Rd\mu = 2 \lim_{r\to\infty} \frac{L(\partial B_r(p))}{r} = 4 \lim_{r\to\infty} \frac{Area(B_r(p))}{r^2}.$$

In particular,

$$4\pi\chi(M) - \int_M Rd\mu = 4\pi A(ds^2).$$

Note 4.1. The above proposition also implies that the aperture $A(ds^2)$ is independent of the choice of the base point p on a complete surface with finite total curvature.

Next we will show that a complete $(\mathbb{R}^2, ds^2)$ with bounded positive curvature has nonzero circumference at infinity.

Lemma 4.3. *Any complete $(\mathbb{R}^2, ds^2)$ with bounded positive curvature has nonzero circumference at infinity.*

Proof. Express the metric as $ds^2 = e^u(dr^2 + r^2 d\theta^2)$, where $\{r, \theta\}$ are polar coordinates. Choose open sets O_1 and O_2 such that the origin $(0,0) \in O_1 \subset O_2$ and $\text{dist}(\partial O_1, \partial O_2) \geq 1$. Let K_0 be a compact set such that $K_0 \supset O_2$.

If $C_\infty = 0$, we have $\inf\limits_{D \supset K_0} L[\partial D] = 0$ for all open sets D. Hence for all $n > 0$ there exists an open set $D(n) \supset K_0$ such that $L[\partial D(n)] \leq \frac{1}{n}$. Let $r_n = \max\limits_{x \in \partial D(n)} \text{dist}(0, 0, x)$.

For all $n \geq \dfrac{\sqrt{R_{\max}}}{\pi}$, $\partial D(n)$ can not be a closed geodesic loop, since any geodesic loop has length greater than $\frac{2\pi}{\sqrt{R_{\max}}}$. Let $\{p, q\}$ be two points on $\partial D(n)$ which divide $\partial D(n)$ into two equal length segments γ_1, γ_2. Perturb γ_1 and γ_2 with fixed end points $\{p, q\}$ to minimize the length. Then for each i, γ_i either reaches a minimizing geodesic segment with fixed end points in $D_{r_n+1}(0,0) - O_1$ or intersects $\partial O_1 \cup \partial D_{r_n+1}(0,0)$ and $L[\gamma_i] \geq \min\{2, \frac{\pi}{\sqrt{R_{\max}}}\} > 0$.

Hence, $C_\infty \geq 2\min\{2, \frac{\pi}{\sqrt{R_{\max}}}\} - \dfrac{1}{n} > 0$. □

5. The Geometric Properties under the Ricci Flow

For the rest of the paper, we will use the following abbreviation for the initial hypotheses.

(*1) $$0 < R < k_0 \quad \text{and} \quad |Du|^2 < D_0, \quad \text{at } t = 0;$$

(*2) $$|R| < k_0 \quad \text{and} \quad |Du|^2 < D_0, \quad \text{at } t = 0;$$

(*3) $$|R| < k_0, |Du|^2 < D_0, \int R_- d\mu < +\infty, \text{ and } C_\infty > 0 \quad \text{at } t = 0;$$

By lemma 4.3 and proposition 4.2, (*1) implies (*3).

In this section we will show that under the Ricci flow, the curvature decays to zero at distance infinity after a short time. As a consequence, C_∞, $A(ds^2)$, and $\int_{\mathbb{R}^2} R d\mu$ are preserved, and the uniqueness of the solution is also provided. Here we will first show that the completeness of the metrics with bounded curvature is preserved under the Ricci flow. Denote $C_\infty(t) = C_\infty\Big(ds^2(t)\Big)$ and $A(t) = A(ds^2(t))$. Let $L_t[\Gamma]$ be the length of curve Γ with respect to the metric $ds^2(t)$.

Lemma 5.1. *On $M_t = (\mathbb{R}^2, ds^2(t))$, with $|R| \le C$ for all $t \in [0,T]$, if M_0 is complete then M_t remains so for all $t \in [0,T]$.*

Proof. Let Γ be any curve on $\mathbb{R}^2$ with a time-independent parameter u. Then

$$\text{(5.1)}\quad \frac{d}{dt}L_t[\Gamma] = \frac{d}{dt}\int_\Gamma \sqrt{ds^2(t)(\frac{\partial}{\partial u},\frac{\partial}{\partial u})}du = \int_\Gamma -\frac{R}{2}\sqrt{ds^2(t)(\frac{\partial}{\partial u},\frac{\partial}{\partial u})}du$$

The bound for curvature gives us $|\frac{d}{dt}L_t[\Gamma]| \le \frac{C}{2}L_t[\Gamma]$, and

$$\text{(5.2)}\qquad e^{-\frac{C}{2}t}L_0[\Gamma] \le L_t[\Gamma] \le e^{\frac{C}{2}t}L_0[\Gamma].$$

The left inequality $e^{-\frac{C}{2}t}L_0[\Gamma] \le L_t[\Gamma]$ implies that any curve with infinite length with respect to $ds^2(0)$ has infinite length with respect to $ds^2(t)$ for $t > 0$. This implies that completeness is preserved. □

5.1. Injectivity Radius.

Lemma 5.2. *For any fixed time $t > 0$, on $M_t = \Big(\mathbb{R}^2, ds^2(t)\Big)$ with $|R(\cdot,t)| \le C$ and $C_\infty(t) > 0$, the injectivity radius $i(M_t) \ge \min\frac{1}{2}\{C_\infty(t), L[\Gamma_t], \frac{\pi}{\sqrt{R_{\max}(t)}}\} > 0$ where Γ_t is (one of) the shortest closed geodesic loop(s) in M_t with respect to $ds^2(t)$.*

Proof. From the same arguments as in [C-E], if

$$i(\mathrm{M}_t) < \min\frac{1}{2}\{C_\infty(t), \frac{\pi}{\sqrt{R_{\max}(t)}}\},$$

then $i(\mathrm{M}_t)$ can be realized by (one of) the shortest geodesic loop(s) Γ_t away from infinity. That is, Γ_t is a closed geodesic loop contained in a compact subset of M. In particular, this implies $L[\Gamma_t] > 0$. □

The same arguments as in [C] and [C-W] can be used to derive a positive lower bound for the injectivity radius under the flow.

Lemma 5.3. *On* $M_t = \left(\mathbb{R}^2, ds^2(t)\right)$ *with* $C_\infty(0) > 0$ *and* $|R| \le C$ $\forall t \in [0,T]$, *we have*

$$e^{Ct/2}C_\infty(0) \ge C_\infty(t) \ge e^{-Ct/2}C_\infty(0). \tag{5.3}$$

Hence

$$i(M_t) \ge \frac{1}{2}\min_{\tau\in[0,T]}\{C_\infty(\tau), e^{-\frac{C\tau}{2}}L_0[\Gamma_0], \frac{\pi}{\sqrt{C}}\} > 0, \tag{5.4}$$

where Γ_0 *is (one of) the shortest geodesic loop(s) on* M_0.

Proof. (5.3) follows from (5.2). For any curve Γ we have $L_t(\Gamma) \ge e^{-\frac{Ct}{2}}L_0(\Gamma)$, thus

$$\begin{aligned} i(M_t) &= \frac{1}{2}\min\{C_\infty(t), L_t(\Gamma_t), \frac{\pi}{\sqrt{C}}\} \\ &\ge \frac{1}{2}\min_{\tau\in[0,t]}\{e^{-\frac{C\tau}{2}}C_\infty(0), e^{-\frac{C\tau}{2}}L_0[\Gamma_\tau], \frac{\pi}{\sqrt{C}}\} \\ &\ge \frac{1}{2}\min_{\tau\in[0,t]}\{e^{-\frac{C\tau}{2}}C_\infty(0), e^{-\frac{C\tau}{2}}L_0[\Gamma_0], \frac{\pi}{\sqrt{C}}\} > 0. \quad \square \end{aligned} \tag{5.5}$$

5.2. Total Absolute Curvature is Nonincreasing.

Proposition 5.4. *On a complete* $(\mathbb{R}^2, ds^2)$ *with* (*3), *under the Ricci flow as long as the solution exists, we have*

$$\frac{d}{dt}\int_{\mathbb{R}^2}|R|d\mu \le 0.$$

This implies that $\int_{\mathbb{R}^2}|R|d\mu$ *is nonincreasing in time.*

Proof. From theorem 4.1, finite total negative curvature at $t = 0$ implies $\int_{\mathbb{R}^2}|R|d\mu < \infty$ at $t = 0$. Let $\mathrm{B}_+(t)$ be the region in $\mathbb{R}^2$ where $R(x,t) \ge 0$, $\mathrm{B}_-(t)$ be the region in $\mathbb{R}^2$ where $R(x,t) \le 0$, and the exterior unit normal vector ν_+ (resp. ν_-) of $\partial\mathrm{B}_+(t)$ (resp. $\partial\mathrm{B}_-(t)$) has $D_{\nu_+}R \le 0$ (resp. $D_{\nu_-}R \ge 0$.) Thus we have

$$\frac{d}{dt}\int_{\mathrm{B}_+(t)}Rd\mu = \int_{\mathrm{B}_+(t)}\Delta Rd\mu = \int_{\partial\mathrm{B}_+(t)}D_{\nu_+}Rds \le 0, \tag{5.6}$$

and

$$\frac{d}{dt}\int_{\mathrm{B}_-(t)}Rd\mu = \int_{\partial\mathrm{B}_-(t)}D_{\nu_-}Rds \ge 0. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} R_+ d\mu = \int_{\mathrm{B}_+(t)} R d\mu - \int_{\mathrm{B}_-(t)} R d\mu$$
$$= \int_{\partial\mathrm{B}_+(t)} D_\nu R ds - \int_{\partial\mathrm{B}_-(t)} D_\nu R ds \leq 0. \quad \square$$

5.3. Curvature at Distance Infinity.

Proposition 5.5. *On $(\mathbb{R}^2, ds^2(t))$ with (*3), under the Ricci flow, the curvature R decays to zero at distance infinity after a short time.*

Proof. From corollary 2.5, for all $\tau > 0$, there is a constant $C(\tau)$ such that $|DR| \leq C(\tau)$, for all $t > \tau$. Using the uniform bounds on R, $\int_{\mathbb{R}^2} |R| d\mu$, and $i(M_t)$ after any short time τ, we claim that the curvature R falls off to zero at distance infinity after the short time τ. Otherwise, there exists a time $t_\tau > \tau$, and a sequence of points $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} |x_n| = \infty$ such that

$$\lim_{n\to\infty} |R(x_n, t_\tau)| = a \neq 0. \tag{5.8}$$

We may extract a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that

$$|x_{n_{j+1}} - x_{n_j}| > \frac{a}{4C(\tau)}, \qquad \text{and} \qquad |R(x_{n_{j+1}}, t_\tau)| > a/2.$$

Let $r = \min\{\frac{a}{4C(\tau)}, \min_{t\in[0,t_0]} i(\mathrm{M}_t)\}$; from Lemma 4.3, we have $r > 0$. Let $B_r^{t_\tau}(x_{n_j})$ denote the geodesic ball centered at x_{n_j} with radius r with respect to the metric at time τ. Hence $\forall y \in B_r^{t_\tau}(x_{n_j})$, we have $|R(y, t_\tau) - R(x_{n_j}, t_\tau)| \leq a/4$ and $|R(y, t_\tau)| > a/4$.

From the choice of r, the exponential map at x_{n_j} in $B_r^{t_\tau}(0)$ is injective, which implies $\int_{B_r^{t_\tau}(x_{n_j})} d\mu \geq \overline{C}$, where $\overline{C}$ is a positive constant depending only on the bounds for the curvature and r. Furthermore, if $a \neq 0$, we have

$$C > \int_{\mathbb{R}^2} |R(x, t_\tau)| d\mu > \sum_{j=1}^{\infty} \int_{B_r^{t_\tau}(x_{n_j})} |R(\cdot, t_\tau)| d\mu > \sum_{j=1}^{\infty} \overline{C} a = +\infty.$$

This is a contradiction, so $a = 0$. $\square$

5.4. The Circumference at Infinity is Preserved. The idea of the following proof was suggested by Richard Hamilton.

Theorem 5.6. *On* $(\mathbb{R}^2, g_{ij}(t))$ *with* (*3), *then under the Ricci flow,* $C_\infty(t) = C_\infty(0)$ *for all* t.

Proof. For any fixed time T, for any given $\tau \in (0,T]$, since the curvature decays to zero at distance infinity, there exists a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$, such that

$$(5.9) \quad (1) \qquad K_n \subset K_{n+1}, \qquad \forall n \geq 1,$$

$$(5.10) \quad (2) \qquad \cup_{n=1}^{\infty} K_n = R^2,$$

$$(5.11) \quad (3) \qquad \forall x \in \mathbb{R}^2 - K_n, \qquad |R(x,t)| < \frac{1}{n}, \qquad \forall t \in [\tau, T].$$

It is easy to see that $C_\infty(t) \geq \sup_n \inf_{D \supset K_n} L_t[\partial D]$.

For any compact sets $C_1 \subset C_2$, $\inf_{D \in C_1} L[\partial D] \leq \inf_{D \in C_2} L[\partial D]$. Also for any compact set K there exists a K_n such that $K_n \supset K$, thus $C_\infty(t) \leq \sup_n \inf_{D \supset K_n} L_t[\partial D]$, hence

(5.12)
$$C_\infty(t) = \sup_{K_n} \inf_{D \supset K_n} L_t[\partial D] = \sup_n \inf_{D \supset K_n} L_t[\partial D] = \lim_{n \to \infty} \inf_{D \supset K_n} L_t[\partial D].$$

From (5.1) and (5.11), $\forall D \supset K_n$, we have $\left| \frac{d}{dt} L_t[\partial D] \right| \leq \frac{1}{2n} L_t[\partial D]$, $\forall t \in [\tau, T]$. This yields

$$e^{-\frac{1}{2n}(t-\tau)} \inf_{D \in X_n} L_\tau[\partial D] \leq \inf_{D \in X_n} L_t[\partial D] \leq e^{\frac{1}{2n}(t-\tau)} \inf_{D \in X_n} L_\tau[\partial D].$$

Therefore as $n \to \infty$, $\tau \to 0$ and $T \to \infty$, $C_\infty(t) = C_\infty(0)$ for all $t \geq 0$.

This illustrates that any finite circumference at infinity is preserved under the Ricci flow for any finite time interval. Hence any infinite circumference is also preserved, otherwise there is an obvious contradiction. □

Corollary 5.7. *On* $(\mathbb{R}^2, ds^2(t))$ *with* (*3), *under the Ricci flow, the injectivity radius has a positive lower bound.*

5.5. The Aperture and the Total Curvature are Preserved.

Theorem 5.8. *On $(\mathbb{R}^2, ds^2)$ with (*3), under the Ricci flow $\int_M Rd\mu$ and $A(ds^2)$ are constants in time.*

Proof. From Proposition 4.2, we only need to show that the aperture is preserved. If $C_\infty < \infty$, there exists a point p_0 such that $\lim_{r\to\infty} L(B_r(p_0)) = C_\infty$. Hence

$$A(ds^2) = \lim_{r\to\infty} \frac{L(r)}{r} = 0 \text{ and } \int_M Rd\mu = 4\pi\chi(M).$$

Since C_∞ is preserved, from Theorem 5.6, we have $A(g_{ij}) = 0$ is preserved.

If $C_\infty = \infty$, let T be any fixed positive time and let Exp_p^t be the exponential map at a fixed point p with respect to the metric $ds^2(t)$ and $B_r^t(p) = Exp_{*p}^t(B_r(0))$. Then $\forall \epsilon > 0$ there is a sequence $\{r_n\}_{n=1}^{\infty}$ with $\lim_{r_n\to\infty} r_n = \infty$ such that

$$|R(x,t)| < \frac{1}{n}, \qquad \forall x \in M - B_{r_n}^t(p), \forall t \in [\epsilon, T].$$

Hence $\left|\frac{d}{dt}(\frac{L_t[\partial B_{r_n}^t]}{r_n})\right| \le \frac{1}{n}(\frac{L_t[\partial B_{r_n}^t]}{r_n})$, and

$$e^{-\frac{1}{n}(t-\epsilon)} L_\epsilon[\partial B_{r_n}^\epsilon] \le \frac{L_t[\partial B_{r_n}^t]}{r_n} \le e^{\frac{1}{n}(t-\epsilon)} L_\epsilon[\partial B_{r_n}^\epsilon].$$

Thus

$$\lim_{n\to\infty} \frac{L_t[\partial B_{r_n}]}{r_n} = \lim_{n\to\infty} \frac{L_\epsilon[\partial B_{r_n}^\epsilon]}{r_n}, \qquad \forall t \in [\epsilon, T].$$

Let $\epsilon \to 0$ and $T \to \infty$, we have $\lim_{r\to\infty} \frac{L_t(r)}{r} = \lim_{r\to\infty} \frac{L_0(r)}{r}$, for all $t \ge 0$. □

5.6. Uniqueness of the Solution. Now we will use Proposition 5.5 to show the uniqueness of the solution.

Corollary 5.9. *On (R^2, ds^2) with (*3), the solution to the Ricci flow is unique.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two different solutions of the Ricci flow with the same initial data $u_1(x,0) = u_2(x,0)$. Let R_i denote the curvature for the metric generated by u_i, and $|R_i| \le k_0, \forall i = 1, 2, \forall t \in [0, 1]$. Then $\frac{\partial}{\partial t} u_1 = -R_1$, $\frac{\partial}{\partial t} u_2 = -R_2$, and $\frac{\partial}{\partial t}(u_1 - u_2) = -R_1 + R_2$. Furthermore, we have

$$\begin{aligned}\frac{\partial}{\partial t}(u_1 - u_2) &= (e^{-u_1}\overline{\Delta} u_1 - e^{-u_2}\overline{\Delta} u_2) \\ &= e^{-u_1}\overline{\Delta}(u_1 - u_2) + (e^{-u_1} - e^{-u_2})\overline{\Delta} u_2 \\ &= e^{-u_1}\overline{\Delta}(u_1 - u_2) + (e^{u_2 - u_1} - 1) R_2\end{aligned}$$

hence $\frac{\partial}{\partial t}(u_1 - u_2)(x,t) = e^{-u_1}\overline{\Delta}(u_1 - u_2)(x,t) + [e^{\tilde{u}}(u_2 - u_1)R_2](x,t)$ where $\tilde{u}(x,t)$ is some value between 0 and $(u_2 - u_1)(x,t)$ obtained from the mean value theorem.

The bounds on the curvatures yield $\frac{\partial}{\partial t}(u_1 - u_2) \le 2k_0$. For small $\epsilon > 0$, there exists a time $\tau(\epsilon) = \frac{\epsilon}{4k_0} \in (0, 1]$, such that $u_1 - u_2 \le \frac{\epsilon}{2}$, $\forall t \in [0, \tau(\epsilon)]$. From Theorem ??, there exists a compact set $K_\epsilon \subset \mathbb{R}^2$ such that

$$|R_i(x,t)| \le \frac{\epsilon}{10}, \qquad \forall x \in \mathbb{R}^2 - K_\epsilon, \qquad \forall t \in [\tau(\epsilon), 1]$$

for $i = 1, 2$. This implies that $\forall x \in \mathbb{R}^2 - K_\epsilon$, and $\forall t \in [\tau(\epsilon), 1]$,

$$u_1(x,t) - u_2(x,t) \le \frac{\epsilon}{2} + 2\frac{\epsilon}{10} \times (1 - \tau(\epsilon)) \le \frac{3\epsilon}{4}.$$

For all $t \in [\tau(\epsilon), 1]$, if $\max_{M_t}(u_1 - u_2) = (u_1 - u_2)(\overline{x}, t) > 0$ and $\overline{x} \in K_\epsilon$, then $\overline{\Delta}(u_1 - u_2)|_{\max} \le 0$, and

$$\frac{\partial}{\partial t}(u_1 - u_2)|_{\max} \le |[e^{\hat{u}}(u_2 - u_1)R_2](\overline{x}, t)|,$$

where $u_2(\overline{x}, t) - u_1(\overline{x}, t) \le \hat{u}(\overline{x}, t) \le 0$. Thus

$$\frac{\partial}{\partial t}(u_1 - u_2)|_{\max} \le C(u_1 - u_2)|_{\max}.$$

Combining all the above arguments, we have

$$\max_{x \in M}(u_1 - u_2)(x,t) \le \max\{\frac{\epsilon}{2}, \frac{3\epsilon}{4}\} \cdot \max\{1, e^{k_0(t - \tau(\epsilon))}\} \le \frac{3\epsilon}{4} e^{k_0}, \quad \forall t \in [\tau(\epsilon), 1].$$

That is,

$$u_1(x,t) - u_2(x,t) \le \epsilon e^{k_0}, \qquad \forall t \in [0, 1].$$

Let $\epsilon \to 0$, we have $(u_1 - u_2)(x,t) \le 0$, $\forall t \in [0, 1]$.

Repeating the same process for $u_2 - u_1$ and on any finite time interval, we have $u_1 = u_2$ on $\mathbb{R}^2 \times [0, \infty)$. □

5.7. The Positivity of the Curvature is Preserved.

Lemma 5.10. *On a complete $(\mathbb{R}^2, ds^2(t))$ with (*1), the positivity of the curvature is preserved. In particular, the metric can not become flat at any finite time.*

Proof. By the maximum principle and Proposition 5.5, if $R > 0$ at time zero, at any finite time either $R > 0$ or R is identically zero. On the other hand, theorem 5.8 implies that $\int R d\mu$ is preserved, hence the positivity of the curvature is preserved. □

6. Gradient Solitons and Expanding Gradient Solitons

A soliton is a solution of the Ricci flow which moves only by diffeomorphism, i.e. there exists a vector field V such that $\frac{\partial}{\partial t} g_{ij} = L_V g_{ij}$. Any compact 2-soliton is a gradient soliton; that is, the vector field V must be the gradient of some function [H-1].

If we let $ds^2 = g_{ij} dx^i dx^j$ and $M_{ij} = D_i D_j u + \frac{1}{2} R g_{ij}$, then

$$\frac{\partial}{\partial t} |M_{ij}|^2 = \Delta |M_{ij}|^2 - 2|D_k M_{ij}|^2 - 2R|M_{ij}|^2. \tag{6.1}$$

It is easy to see that ds^2 is a gradient soliton if $M_{ij} = 0$.

In this section we will show that there are only two types of gradient solitons on R^2 with $r = 0$. Namely, the standard flat metric, which is stationary, and the cigar soliton. Let $\{u, v\}$ be the standard flat coordinates on $\mathbb{R}^2$, then the cigar soliton is a metric expressed as $ds^2 = \dfrac{du^2 + dv^2}{1 + u^2 + v^2}$, and has scalar curvature $R = \dfrac{4}{1 + u^2 + v^2}$. As the distance goes to infinity from any fixed point p_0, the cigar soliton approaches a flat cylinder with the same C_∞. Note that on a flat soliton, we have $C_\infty = \infty$, while on a cigar soliton, $C_\infty < \infty$. Furthermore, in the case discussed in this paper, any 2-soliton is a gradient soliton.

Theorem 6.1. *There are only two types of complete gradient solitons on R^2 with $r = 0$. They are the cigar solitons and the flat solitons.*

See also [H-1] and [W-1].

Proof. Here we are going to use the notation introduced in [H-1] and [W-1]. Let (u, v) be the standard rectangular coordinates on R^2, and (x, y) be the coordinates on the cylinder with $-\infty < x < \infty$, and $0 < y \leq 2\pi$, such that

$$u = e^x \cos y$$
$$v = e^x \sin y.$$

Any soliton metric $ds^2 = g(u,v)(du^2 + dv^2)$ on R^2 induces a soliton metric $ds^2 = g(x,y)(dx^2 + dy^2)$ on the cylinder where $g(u,v) = g(x,y)e^{-2x}$. From [H-1], a gradient soliton yields $g(x,y) = g(x)$. Thus, $g(x)e^{-2x}$ is a smooth function of $u^2 + v^2 = e^{2x}$ as $x \to -\infty$. The scalar curvature is given by $R = -\frac{1}{g}(\frac{g_x}{g})_x$. Since $r = 0$, the metric $g(x,t) = g(x + Ct)$ is a soliton on the cylinder if and only if $Cg_x = (\frac{g_x}{g})_x$. There are two cases for the constant C:

1. If $C = 0$, then $R = 0$. This soliton is the flat metric.

2. If $C \neq 0$, then $\frac{g_x}{g} = Cg + C_0$, and $\frac{dg}{g(Cg + C_0)} = dx$. That is,

$$\left(\frac{C}{Cg + C_0} - \frac{1}{g}\right)dg = -C_0 dx,$$

and

$$\ln|Cg + C_0| - \ln|g| = -C_0 x + C_1.$$

This implies $\frac{|Cg + C_0|}{g} = C_2 e^{-C_0 x}, C_2 > 0$. Furthermore, we have

$$g = \frac{C_0 e^{C_0 x}}{C_2 \pm C e^{C_0 x}}.$$

Rescaling the parameters (i.e., let $C_2 = 1, C = 1$, and $C_0 = B$) and rescaling the metric yield

$$g = \frac{e^{Bx}}{1 \pm e^{Bx}}.$$

Since ge^{-2x} is a smooth function of e^{2x} as $x \to -\infty$, and $g > 0$ as $x \to +\infty$; we have $g(x) = \frac{e^{2x}}{1 + e^{2x}}$ and $ds^2 = \frac{du^2 + dv^2}{1 + u^2 + v^2}$. Therefore the metric is a cigar soliton. □

Lemma 6.2. *For any complete soliton on $\mathbb{R}^2$, if the curvature R is decaying to zero at distance infinity, then the soliton has either positive curvature or constant zero curvature.*

Proof. On a complete soliton with curvature R decaying to zero at distance infinity, $\frac{d}{dt}R_{\text{inf}} = 0$ and $R_{\text{inf}} \leq 0$. If $R_{\text{inf}} = 0$, by the strong maximum principle in the interior, we have either $R \equiv 0$ or $R > 0$. If $R_{\text{inf}} < 0$, then R_{inf} occurs in the interior and $\Delta R_{\text{inf}} > 0$. This contradicts with

$$0 = \frac{\partial}{\partial t} R_{\text{inf}} = \Delta R_{\text{inf}} + R_{\text{inf}}^2 > 0. \tag{6.2}$$

Hence $R_{inf} = 0$ and $R > 0$. $\square$

Lemma 6.3. *On a complete soliton on $(\mathbb{R}^2, ds^2 = g_{ij}dx^i dx^j)$ with positive curvature, if $M_{ij} = D_i D_j u - \frac{1}{2} R g_{ij}$ decays off to zero at distance infinity, then ds^2 is a gradient soliton.*

Proof. Let $ds^2(t)$ be the solution of the Ricci flow with $ds^2(0) = ds^2$. Then, for any time t, there exists a diffeomorphism $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$|M_{ij}|^2(\phi_t(x), t) = |M_{ij}|^2(x, 0), \qquad \forall x \in \mathbb{R}^2.$$

In particular, if at time $t = 0$, $\max_{x \in \mathbb{R}^2} |M_{ij}|^2(x, 0) = |M_{ij}|^2(\overline{x}, 0)$, then for all t,

$$\max_{x \in \mathbb{R}^2} |M_{ij}|^2(x, t) = |M_{ij}|^2(\phi_t(\overline{x}), t)$$
$$R(\phi_t(\overline{x}), t) = R(\overline{x}, 0) > 0.$$

The evolution equation of $|M_{ij}|^2$ is $\frac{\partial}{\partial t}|M_{ij}|^2 = \Delta |M_{ij}|^2 - |D_k M_{ij}|^2 - 2R|M_{ij}|^2$. The maximum principle implies $\lim_{t \to \infty} \max_{x \in \mathbb{R}^2} |M_{ij}|^2(x, t) = 0$. On the other hand, $\frac{\partial}{\partial t} \max_{x \in \mathbb{R}^2} |M_{ij}|^2(x, t) = 0$ on a soliton, hence $|M_{ij}|^2 = 0$. That is , $\frac{\partial}{\partial t} g_{ij} = -R g_{ij} = 2 D_i D_j u = L_{\nabla u} g_{ij}$. Therefore ds^2 is also a gradient soliton. $\square$

6.1. The Expanding Ricci gradient Solitons. The expanding gradient Ricci solitons are solutions which satisfies

$$(\frac{1}{t} + R) ds^2 = -L_{\nabla f} ds^2 \tag{6.3}$$

for some function f for all time $t > 0$. For the purpose of this paper, we will only consider the following α-shrinking Ricci flow

$$\frac{\partial}{\partial t} ds^2 = -(\alpha + R) ds^2, \tag{6.4}$$

where α is a positive constant. We will show the existence of the α-expanding solitons, that is, there exists a function f such that

$$\frac{\partial}{\partial t} ds^2 = -(\alpha + R) ds^2 = L_{\nabla f} ds^2. \tag{6.5}$$

By the arguments in [H-1], the induced metric of gradient solitons on a cylinder may be written as $ds^2 = g(x)(dx^2 + dy^2)$ with coordinates $-\infty < x < \infty$, and $0 < y \leq 2\pi$. In particular, there exists a constant C such that $g(x,t) = g(x - Ct)$ and $ds^2 = g(x,t)(dx^2 + dy^2)$ is the α-expanding solitons on a cylinder. Since the scalar curvature $R = -\frac{1}{g}(\frac{g'}{g})'$, the α-shrinking Ricci flow may be rewritten as

$$Cg' = (\alpha + R) g = \Big(\alpha - \frac{1}{g}(\frac{g'}{g})'\Big) g. \tag{6.6}$$

Let Σ^α be an α-expanding gradient solition on the plane, then the aperture can be expressed in terms of the function g and

$$\begin{aligned} A(\Sigma^\alpha) &= \lim_{r \to \infty} \frac{\sqrt{g(x(r))}}{r} \qquad \text{where } r = \int_0^{x(r)} \sqrt{g(\xi)}\, d\xi \\ &= \frac{1}{2} \lim_{x \to \infty} \frac{g'}{g}. \end{aligned} \tag{6.7}$$

Geometrically the expanding gradient soliton Σ^α asymptotically approaches a rotational symmetric cone with aperture $A(\Sigma^\alpha)$. That is, there exist coordinates $\{\ell, y\}$ with $\ell = \ell(x)$ on the cylinder such that

$$\lim_{x \to +\infty} g(x)(dx^2 + dy^2) = d\ell^2 + A^2 \ell^2 dy^2. \tag{6.8}$$

When x approaches ∞, let $\ell = e^{Ax}$, then $g(x) = A^2 e^{2Ax}$ and $\lim_{x \to +\infty} \frac{g'}{g} = 2A$.

On Σ^α with positive curvature, theorem 4.1 and the rotational symmetry implies that $\lim_{x \to \infty} R = 0$. Therefore, $\alpha = C \lim_{x \to +\infty} \frac{g'}{g} = 2AC$ and (6.6) is equivalent to

$$\frac{\alpha}{2A}\frac{g'}{g} = \alpha + R. \tag{6.9}$$

To recover a smooth metric on $\mathbb{R}^2$ from $g(x)$, we need $g(x)e^{-2x}$ to be a smooth function in e^{2x} as $x \to -\infty$; i.e., $g(x) = ae^{2x} + be^{4x} + \dots$ as $x \to -\infty$. Hence $\lim_{x\to-\infty} \frac{g_x}{g} = 2$. The positivity of the curvature implies $\left(\frac{g_x}{g}\right)_x = -gR < 0$, and therefore the maximum of the curvature occurs as $x \to -\infty$ and

$$\begin{aligned} R_{\max} &= \left(-\alpha + \frac{\alpha}{2A}\frac{g_x}{g}\right) \quad (x \to -\infty) \\ &= \alpha(-1 + \frac{1}{A}) \end{aligned} \tag{6.10}$$

We have showed the following theorem which was conjectured by Richard Hamilton.

Theorem 6.4. *For any given positive constants α and $A \in (0,1)$, there exists a unique rotationally symmetric, α-expanding gradient solitons Σ^α on $\mathbb{R}^2$ with positive curvature and aperture A. Furthermore, if we define function φ as $\varphi(A) = \frac{1}{\alpha}R_{\max}(\Sigma^\alpha)$, then φ is a decreasing function in A.*

7. Integral estimates for $\mathbb{R}^2$ with finite circumference

In this section , we will derive some time-independent integral estimates.

Lemma 7.1. *On $(\mathbb{R}^2, ds^2(t))$ with (*1) and $C_\infty < \infty$, if $M_{ij} = D_iD_ju + \frac{1}{2}Rg_{ij}$, then under the Ricci Flow $\forall\tau > 0$, we have a uniform bound of $\int_{\mathbb{R}^2} |M_{ij}|^2 d\mu, \forall t \in [\tau, \infty)$.*

Proof. At any given time $t \in [\tau, \infty)$, let $\{D_n\}$ be a sequence of open sets such that $\bigcup_{n=1}^{\infty} D_n = \mathbb{R}^2, D_n \subset D_{n+1}$, and $\lim_{n\to\infty} L(\partial D_n) = C_\infty$. Then we may compute

$$\int_{D_n} |M_{ij}|^2 d\mu = \int_{D_n} |D_iD_ju|^2 d\mu - \int_{D_n} \frac{R^2}{2} d\mu,$$

and

$$\int_{D_n} |D_iD_ju|^2 d\mu = -\int_{D_n} D_iD_iD_juD_jud\mu + \int_{\partial D_n} D_\nu(D_ju)D_juds.$$

Since

$$D_iD_iD_ju = D_jD_iD_iu + R_{ijil}D_lu = -D_jR + \frac{R}{2}D_ju,$$

and

$$\int_{\mathrm{D}_n} D_jRD_ju d\mu = -\int_{\mathrm{D}_n} R\Delta u d\mu + \int_{\partial \mathrm{D}_n} RD_\nu(u)ds = \int_{\mathrm{D}_n} R^2 d\mu + \int_{\partial \mathrm{D}_n} RD_\nu(u)ds,$$

one obtains

$$\int_{\mathrm{D}_n} |M_{ij}|^2 d\mu = \int_{\mathrm{D}_n} \frac{R^2}{2} d\mu - \int_{\mathrm{D}_n} \frac{R}{2}|Du|^2 d\mu + \int_{\partial \mathrm{D}_n} RD_\nu(u)ds + \int_{\partial \mathrm{D}_n} D_\nu(D_ju)D_ju ds.$$

By corollary 2.5 and theorem 5.8, there exist constants $C(\tau)$ and $C_0 = C_0(ds^2(0))$ such that $|R| \leq C(\tau), |D^k u| \leq C(\tau), |D^k R| \leq C(\tau)$, and $\int_{\mathbb{R}^2} R d\mu = C_0$, $\forall$ $1 \leq k \leq 2$, $\forall t \in [\tau, \infty)$. Then we have

$$\int_{\mathrm{D}_n} \frac{R^2}{2} d\mu \leq \frac{C(\tau)}{2} \int_{\mathrm{D}_n} R d\mu \leq \frac{C(\tau)C_0}{2},$$

$$\int_{\mathrm{D}_n} \frac{R}{2}|Du|^2 d\mu \leq \frac{C(\tau)}{2} \int_{\mathrm{D}_n} R d\mu \leq \frac{C(\tau)C_0}{2}.$$

Since $C_\infty < \infty$, then

$$\left| \int_{\partial \mathrm{D}_n} RD_\nu u ds \right| \leq C(\tau)^2 L(\partial \mathrm{D}_n) \to C(\tau)^2 C_\infty, \text{ as } n \to \infty.$$

$$\left| \int_{\partial \mathrm{D}_n} D_\nu D_j u D_j u ds \right| \leq C(\tau)^2 L(\partial \mathrm{D}_n) \to C(\tau)^2 C_\infty, \text{ as } n \to \infty.$$

Hence after a short time, $\int_{\mathbb{R}^2} |M_{ij}|^2 d\mu$ has a uniform bound for all time. □

The following theorem is obtained from looking at $\frac{d}{dt} \int_{\mathbb{R}^2} |M_{ij}|^2 d\mu$, in the case when $C_\infty < \infty$.

Lemma 7.2. *On $M = (\mathbb{R}^2, g_{ij})$ with (*1) and $C_\infty < 0$ at $t = 0$, under the Ricci Flow after a short time $\tau > 0$, we have*

$$\int_{t=\tau}^{\infty} \int_{\mathbb{R}^2} 2|D_k M_{ij}|^2 d\mu + \int_{t=\tau}^{\infty} \int_{\mathbb{R}^2} 3R|M_{ij}|^2 d\mu \leq C(\tau).$$

Proof. Let D_n be chosen as before, then we may compute

$$\frac{d}{dt}\int_{\mathrm{D}_n}|M_{ij}|^2 d\mu = \int_{\mathrm{D}_n}(\Delta|M_{ij}|^2 - 2|D_k M_{ij}|^2)d\mu - \int_{\mathrm{D}_n} 3R|M_{ij}|^2 d\mu.$$

$$\int_{\mathrm{D}_n}\Delta|M_{ij}|^2 d\mu = \int_{\partial\mathrm{D}_n} 2M_{ij}D_\nu(M_{ij})ds.$$

Hence we only need to show $\lim_{n\to\infty}\int_{\partial\mathrm{D}_n} D_\nu(M_{ij})M_{ij}ds = 0.$

After any short time $\tau > 0$, $\int|M_{ij}|^2 dA$ and $|D_k M_{ij}|^2$ are bounded, and $i(M_\tau) > 0$, using the same arguments as in Proposition 5.5, we have $|M_{ij}|^2$ going to zero as the distance approaches the infinity. This yields

$$\lim_{r\to\infty}\left|\int_{\partial\mathrm{D}_n} D_\nu(M_{ij})M_{ij}\right| ds \le \lim_{r\to\infty}\sup_{\partial\mathrm{D}_n}|M_{ij}|\sup_{\partial\mathrm{D}_n}|D_\nu(M_{ij})|L[\partial\mathrm{D}_n] = 0.$$

Consequently,

$$\frac{d}{dt}\int_{\mathbb{R}^2}|M_{ij}|^2 d\mu = -2\int_{\mathbb{R}^2}|D_k M_{ij}|^2 d\mu - \int_{\mathbb{R}^2} 3R|M_{ij}|^2 d\mu$$

the theorem follows. □

Lemma 7.3. *On* $M = (\mathbb{R}^2, ds^2)$ *with* (*1) *and* $C_\infty < \infty$, *a soliton is also a gradient soliton.*

Proof. Lemma 7.1, bounds for $|D_k M_{ij}|^2$ and $i(M) > 0$ imply that $|M_{ij}|^2$ decays to zero as the distance goes to infinity. Combining with Proposition 5.5 and lemmas 6.2–6.3, any soliton is a gradient soliton. □

Now we will use the above lemmas to classify the limit of the metric at time infinity as obtained in Theorem 3.2. Since ds^2 and $\widehat{ds^2}$ are the same metrics which only differ by a diffeomorphism, they induce the same covariant derivatives. If we define $\widehat{M_{ij}}(A,B,t) = D_i D_j \hat{u} + \frac{1}{2}\hat{R}\widehat{g_{ij}}$, then it is easy to see that

$$\widehat{M_{ij}}(A,B,t) = M_{ij}(x,y,t),$$

and

$$\int_{\mathbb{R}^2}(2|D_k M_{ij}|^2 + 3R|M_{ij}|^2)(x,y,t)d\mu = \int_{\mathbb{R}^2}(2|D_k M_{ij}|^2 + 3R|M_{ij}|^2)(\phi_t(A,B),t)\widehat{d\mu}.$$

If $C_\infty(0) < \infty$, by lemma 7.2, the uniform bounds of $|D^k u|$ for all $k \geq 1$ and the positive curvature imply

$$\lim_{l\to\infty} |D_k \widehat{M_{ij}}|^2_{\hat{t}_l}(A, B, t_l) = \lim_{l\to\infty} |D_k M_{ij}|^2(\phi_{t_l}(A, B), t_l) = 0$$

and

$$\lim_{l\to\infty} \widehat{R}|\widehat{M_{ij}}|^2_{\hat{t}_l}(A, B, t_l) = \lim_{l\to\infty} R|M_{ij}|^2(\phi_{t_l}(A, B), t_l) = 0.$$

In particular, $\lim_{l\to\infty} |D_k \widehat{M_{ij}}|^2_{\hat{t}_l}(A, B, t_l) = 0$ implies $\lim_{l\to\infty} |\widehat{M_{ij}}|_{\hat{t}_l}(A, B, t_l)$ is a constant. That is, either $\widehat{R} \equiv 0$ or $\lim_{l\to\infty} |\widehat{M_{ij}}|^2_{\hat{t}_l} \equiv 0$. Combining this with theorem 3.2, $\widehat{ds^2}(\infty)$ is a cigar soliton and $C_\infty(\widehat{ds^2}(\infty)) \leq C_\infty(ds^2(0))$.

Theorem 7.4. *On $(\mathbb{R}^2, ds^2)$ with (*1) and $C_\infty < \infty$, the limit of the modified subsequence convergence at time infinity is a cigar soliton with circumference at infinity no bigger than that at time zero.*

8. The Harnack inequality for $\mathbb{R}^2$ with positive aperture

In this section, we will further discuss the behavior of the limiting solution at time infinity in the case when $A > 0$ by using the Harnack inequality and by comparing the limiting solutions to the expanding solitons. The results and the proofs in this section are mainly due to Richard Hamilton. Recently Richard Hamilton proved a matrix form of the Harnack inequality for complete manifolds with nonnegative curvature operator under the Ricci Flow ([H-2]). In dimension 2 we may state the inequality as the following:

Theorem 8.1 (Hamilton). *On a complete surface with nonnegative curvature, under the Ricci Flow, as long as the solution exists, we have*

$$(8.1)\quad \tilde{Q}_{ij} = D_i D_j R + \frac{1}{2} R^2 g_{ij} + V_i D_j R + V_j D_i R + R V_i V_j + \frac{1}{2t} R g_{ij} \geq 0$$

for any vector field V.

Corollary 8.2. *On a complete surface with positive curvature, under the Ricci flow*

$$(8.2)\qquad Q_{ij} = D_i D_j R + \frac{1}{2} R^2 g_{ij} + \frac{1}{2t} R g_{ij} - \frac{D_i R D_j R}{R} \geq 0.$$

Furthermore, the equality holds if and only if the metric is an expanding gradient soliton.

Proof. When the minimum of $\tilde{Q}_{ij}$ occurs, we have $D_{V_i}\tilde{Q}_{ij} = D_jR + RV_j = 0$, hence $Q_{ij} \geq 0$. If $R > 0$ and $Q_{ij} = 0$, then

$$\frac{D_iD_jR}{R} - \frac{D_iRD_jR}{R^2} + \frac{1}{2}Rg_{ij} + \frac{1}{2t}g_{ij} = 0. \tag{8.3}$$

Let $f = \log R$, then(8.3) can be expressed as

$$D_iD_jf + \frac{1}{2}Rg_{ij} + \frac{1}{2t}g_{ij} = 0. \tag{8.4}$$

Furthermore, we have

$$-(R + \frac{1}{t})g_{ij} = 2D_iD_jf = L_{\nabla f}\, g_{ij}. \tag{8.5}$$

This implies that g_{ij} is an expanding soliton.

Conversely, if g_{ij} is an expanding gradient soliton, there exists a function f (from [H-2]), such that

$$-(R + \frac{1}{t})g_{ij} = L_{\nabla f}g_{ij} = 2D_iD_jf, \tag{8.6}$$

Differentiating (8.6) in the j-th component and contracting (8.6) by g_{ij} yield

$$-D_iR = 2D_iD_jD_jf + RD_if = 2D_i\Delta f + RD_if;$$

and

$$-(R + \frac{1}{t}) = \Delta f.$$

Therefore $D_iR = RD_if$, that is,$\nabla f = \nabla \log R$. Hence $Q_{ij} = 0$. □

Corollary 8.3. *On a complete surface with positive curvature, under the Ricci Flow, as long as the solution exists, we have*

$$\frac{\partial}{\partial t}(tR) \geq 0.$$

On an expanding gradient soliton, $t \cdot R_{\max}(t)$ is a constant under the Ricci Flow.

Proof. Let $Q = g^{ij}Q_{ij} = \Delta R + R^2 + \frac{1}{t}R - \frac{|DR|^2}{R} \geq 0$, from (8.2). Then $\frac{\partial}{\partial t}R + \frac{1}{t}R = (\Delta R + R^2) + \frac{1}{t}R \geq \frac{|DR|^2}{R} \geq 0$; which implies $\frac{\partial}{\partial t}(tR) \geq 0$.

On an expanding gradient soliton, $Q = 0$ and $\Delta R + R^2 + \frac{1}{t}R = \frac{|DR|^2}{R}$. This implies $\Delta R_{\max} + R^2_{\max} + \frac{1}{t}R_{\max} = 0$, thus $\frac{\partial}{\partial t}(tR_{\max}) = 0$. Consequently, $tR_{\max}(t)$ is a constant in time. □

Now we will use the existence of the expanding gradient soliton on $\mathbb{R}^2$ and $Q_{ij} \geq 0$, to prove the following result.

Theorem 8.4 (Hamilton). *On a complete $(\mathbb{R}^2, g_{ij}(t))$ with (*1) and $A(ds^2(0)) > 0$, then under the Ricci Flow we have*

$$tR_{\max} \leq C,$$

for all time $t \geq 0$, where C only depends on $A(g_{ij})$.

Corollary 8.5. *On a complete $(\mathbb{R}^2, g_{ij}(t))$ with (*1) and $A(ds^2(0)) > 0$, the limit of the modified subsequence convergence at time infinity is the flat metric.*

Lemma 8.6. *For any given time $T \geq 1$, let $g_{ij}(T)$ be the solution of a complete metric on $(\mathbb{R}^2, g_{ij})$ under the Ricci Flow at time T. Let q be a point on $\mathbb{R}^2$ and $R(q,T) = R_{\max}(T)$. If $R > 0$ at $t = T$, then*

$$\frac{1}{R^2}\Big(\frac{dR}{ds}\Big)^2 + R + \frac{1}{T}\log R \leq R_{\max}(T) + \frac{1}{T}\log R_{\max}(T) \qquad t = T$$

where s is the arc length along the minimizing geodesic connects any point p and the point q.

Proof. Along any geodesic γ out of q, let $\frac{d\gamma^i}{ds} = V^i$. Then $\frac{d^2\gamma^i}{ds^2} = 0$, and $|V|^2 = 1$.

For any given function F, $\frac{dF}{ds} = V^i D_i F$ and $\frac{d^2F}{ds^2} = V^i V^j D_i D_j F$. Multiply Q_{ij} by $V^i V^j$, then

$$V^i V^j D_i D_j R + \frac{1}{2}R^2|V|^2 + \frac{1}{2T}R|V|^2 \geq \frac{V^i D_i R V^j D_j R}{R}.$$

That is

$$\frac{d^2R}{ds^2} + \frac{1}{2}R^2 + \frac{R}{2T} \geq \frac{1}{R}\Big(\frac{dR}{ds}\Big)^2.$$

Let

$$X_T = \frac{d}{ds}\Big[\frac{1}{R^2}(\frac{dR}{ds})^2 + R + \frac{1}{T}\ln R\Big]$$
$$= \frac{2}{R^2}\frac{dR}{ds}\Big[\frac{d^2R}{ds^2} - \frac{1}{R}(\frac{dR}{ds})^2 + \frac{1}{2}R^2 + \frac{1}{2T}R\Big].$$

For any given point p on the geodesic γ, we can always assume that $\frac{dR}{ds} \leq 0$, by comparing $R(p,T)$ to some local maximum of the curvature $R^{\text{loc}}_{\max}(T)$ along γ at time T. Then $X_T \leq 0$, which implies

$$\frac{1}{R^2}(\frac{dR}{ds})^2 + R + \frac{1}{T}\ln R \leq R^{\text{loc}}_{\max}(T) + \frac{1}{T}\ln R^{\text{loc}}_{\max}(T)$$
$$\leq R_{\max}(T) + \frac{1}{T}\ln R_{\max}(T). \quad \square$$

Proof of Theorem 8.4. From the above lemma and the positivity of the curvature, we have

$$\Big|\frac{dR}{ds}\Big| \leq R\sqrt{R_{\max}(T) + \frac{1}{T}\ln R_{\max}(T) - R - \frac{1}{T}\ln R}.$$

Using the assumption that $\frac{dR}{ds} \leq 0$, we have

$$0 \leq -\frac{dR}{ds} \leq R\sqrt{R_{\max}(T) + \frac{1}{T}\ln R_{\max}(T) - R - \frac{1}{T}\ln R}.$$

On the other hand, Theorem 6.4 and (6.10) provide a rotationally symmetric $\frac{1}{T}$-expanding Ricci gradient solition solution $\Sigma^{\frac{1}{T}}$ with

$$R^{\Sigma^{\frac{1}{T}}}_{\max} = R_{\max}(T).$$

Let $R^{\Sigma^{\frac{1}{T}}}_{\max}$ at time T occurs at point $\pi \in \Sigma^{\frac{1}{T}}$. Since $\Sigma^{\frac{1}{T}}$ is rotationally symmetric, we may define

$$\phi = \phi(s) = R^{\Sigma^{\frac{1}{T}}}(\xi(s), T),$$

where s is the arc length of the geodesic ξ starting from π. In particular, $\phi(0) = R^{\Sigma^{\frac{1}{T}}}(\gamma(0), T) = R_{\max}(T)$, and

$$0 \leq -\frac{d\phi}{ds} = \phi\sqrt{(R_{\max}(T) + \frac{1}{T}\ln R_{\max}(T) - \phi - \frac{1}{T}\ln\phi)} = G(\phi).$$

From straightforward computation, we have

$$R(\gamma(s), T) \geq \phi(s) = R^{\Sigma^{\frac{1}{T}}}(\xi(s), T).$$

Combining this with the Rauch comparison theorem, we have

$$L_{M_T}(\partial B_r) \leq L_{\Sigma^{\frac{1}{T}}}(\partial B_r) \qquad \forall r \geq 0$$

so

$$A(g_{ij}(T)) = \lim_{r\to\infty} \frac{L_{M_T}(\partial B_r)}{r} \leq \lim_{r\to\infty} \frac{L_{\Sigma^{\frac{1}{T}}}(\partial B_r)}{r} = A(\Sigma^{\frac{1}{T}}) = \varphi^{-1}(T \cdot R_{\max}(T)).$$

Since $A(g_{ij})$ is preserved under the Ricci Flow and φ is a decreasing function, then for any given time T

$$T \cdot R_{max}(T) \leq \varphi(A(ds^2)) = C$$

where C is a constant depends only on $A(ds^2)$. □

Theorem 8.7. *On* $(\mathbb{R}^2, ds^2)$ *with* (*1) *and* $A > 0$, *the limit of the modified subsequence convergence at time infinity is the flat metric.*

Appendix A. Ricci Flow on Hyperbolic Surfaces

Sigurd B. Angenent

Let $\Omega \subset \mathbb{R}^2$ be an open subset, and consider a family of Riemannian metrics g^t on Ω given by

$$g^t = (ds)^2 = e^{2u(x,y,t)}\left((dx)^2 + (dy)^2\right) \tag{A.1}$$

i.e. by $g_{ij} = e^{2u}\delta_{ij}$. The Ricci–curvature of g^t is given by

$$\mathrm{Ric}_{ij}(x,y,t) = -\Delta u(x,y,t)\delta_{ij} = -\left(u_{xx} + u_{yy}\right)\delta_{ij} \tag{A.2}$$

where Δ stands for the ordinary Euclidean Laplace operator. The family of metrics g^t therefore evolves by the Ricci Flow if and only if $(e^{2u})_t = \Delta u$. If we define $v = e^{2u}, w = e^{-2u}$, then the Ricci Flow for the metrics g^t is equivalent with either of the following two PDEs:

$$\frac{\partial v}{\partial t} = \Delta\left(\log v\right) \tag{A.3}$$

$$\frac{\partial w}{\partial t} = w\Delta w - (\nabla w)^2. \tag{A.4}$$

This first of these equations is note worthy since it can be interpreted as the formal limit case of the "Porous Medium Equation,"

$$\text{(PME)} \qquad \frac{\partial U}{\partial t} = \Delta U^m,$$

as the constant $m > 0$ tends to 0. Indeed, after rescaling the time by $t = \tau/m$, the PME is equivalent with

$$\text{(PME')} \qquad \frac{\partial U}{\partial \tau} = \Delta\left(\frac{U^m - 1}{m}\right);$$

if one lets $m \downarrow 0$, then one obtains equation (A.3).

There is a vast literature on the PME, and we refer the reader to Aronson's survey paper [Ar] for more background.

The PME has a well-known similarity solution, the Barenblatt solution, which for $0 < m < 1$, and in two space dimensions is given by

$$U(x, y, t) = \left(\frac{4m}{1-m} \cdot \frac{mt}{A(mt)^{1/m} + r^2}\right)^{1/(1-m)}$$

where $r = \sqrt{x^2 + y^2}$, and $A > 0$ is any constant. If one chooses $A = R^2 \left(R^2/4m\right)^{1/m}$, then upon talking the limit $m \downarrow 0$ the Barenblatt solution shifted in time converges to the "soliton solution" which Lang Fang Wu has considered in the main body of this paper:

$$\text{(A.5)} \qquad V(x, y, t) = \lim_{m \downarrow 0} U\left(x, y, \frac{R^2 + 4t}{4m}\right) = \left(e^{4t/R^2} + \left(\frac{r}{R}\right)^2\right)^{-1}$$

(with $R > 0$ any constant.)

Some of Lang Fang Wu's results may therefore be interpreted as sufficient conditions on the initial data $v(x, y, 0)$ which guarantee that the corresponding solution v of (A.3) on the entire plane asymptotically behaves like the "soliton–solution" (A.5).

If one looks for other radially symmetric similarity solutions of the Ricci Flow equations (A.3) or (A.4)), then one soon finds that, in addition to (A.5), there is indeed another one, namely

$$\text{(A.6)} \qquad W(x, y, t) = \frac{1}{8t}\left(1 - r^2\right)^2.$$

This solution, when restricted to the unit disk $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, defines a well–known one parameter family of metrics on the unit disk: $g^t = \left((dx)^2 + (dy)^2\right)/W(x,y,t)$ is precisely the familiar constant (negative) curvature metric on Ω, with scalar curvature $R = g^{ij} Ric_{ij} = -1/2t$.

As we shall show below, one can easily prove some general statements about the asymptotic behaviour of a large class of solutions of (A.4), just by using the maximum principle and the special solution (A.6).

From here on we consider a classical solution $w(x,y,t)$ of (A.4), i.e. we assume that w is a smooth solution of (A.4) on the region $\Omega \times (0,\infty)$ which extends to a continuous function on $\bar{\Omega} \times [0,\infty)$. We shall also assume that

$$w(x,y,t) \begin{cases} > 0 & \text{if } r < 1, t > 0, \\ = 0 & \text{if } r = 1 \text{ and } t \geq 0. \end{cases}$$

Then we shall prove that the special solution W is the largest possible solution, i.e. that

$$w(x,y,t) \leq \frac{1}{8t}\left(1 - r^2\right)^2 \tag{A.7}$$

for all $(x,y) \in \Omega, t > 0$. We shall also prove that

$$\lim_{t\to\infty} 8tw(x,y,t) = \left(1 - r^2\right)^2, \tag{A.8}$$

uniformly in $(x,y) \in \Omega$. Thus all classical solutions behave asymptotically like the special solution W.

If we also assume that $w(x,y,0) \geq \delta(1 - r^2)^2$ for some $\delta > 0$, then we can improve the statement (A.8) about asymptotic behaviour to the effect that

$$w(x,y,t) = \frac{1}{8t}\left(1 - r^2\right)^2\left(1 + \mathcal{O}\left(t^{-1}\right)\right). \tag{A.9}$$

To prove the upper bound (A.7) we observe that

$$w_*(x,y,t) = \frac{1}{8A(t+\varepsilon)}\left(A - r^2\right)^2$$

is also a solution of (A.4), and that this solution is strictly positive on $\bar{\Omega} \times [0,\infty)$ if one chooses $A > 1, \varepsilon > 0$. It follows that $z = w - w_*$ satisfies

$$\frac{\partial z}{\partial t} = w\Delta z - \nabla\left(w + w_*\right) \cdot \nabla z + \left(\Delta w_*\right) z,$$

while $z(x,y,t) < 0$ if r is sufficiently close to 1. If one also chooses $\varepsilon > 0$ small enough, then one has $z < 0$ whenever $t > 0$ is sufficiently small. The maximum principle for parabolic equations then implies that $z < 0$ everywhere, and (A.7) follows by letting $\varepsilon \downarrow 0, A \downarrow 1$.

To prove convergence, we observe that for any $0 < R < 1$ and $T > 0$

$$W_{R,T}(x,y,t) = \frac{1}{8R^2(t+T)}\left(R^2 - r^2\right)^2$$

is a solution of (A.4) which is strictly positive on $\Omega_R \times [0,\infty)$, where $\Omega_R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$.

Since the given classical solution w is strictly positive on Ω, there is a $T_R < \infty$ for each $R < 1$ such that $w(x,y,0) \geq R^2/8T_R$ for $(x,y) \in \Omega_R$. Hence $w(x,y,0) \geq W_{R,T_R}(x,y,0)$ on Ω_R; furthermore, W_{R,T_R} vanishes on $\partial\Omega_R \times [0,\infty)$, while w is positive there, so we may apply the maximum principle, and conclude that $w \geq W_{R,T_R}$ on $\Omega_R \times [0,\infty)$.

Consider $z(x,y,t) = \left(1-r^2\right)^2 - 8tw(x,y,t)$; it follows from (A.7) that $z \geq 0$, and we must prove that $\sup_{(x,y)\in\Omega} z(x,y,t) \to 0$ as $t \to \infty$.

If $r \leq R$, then we have just shown that

$$z(x,y,t) \leq \left(1-r^2\right)^2 - \frac{1}{(T_R+t)R^2}\left(R^2-r^2\right)^2 \leq 1 - R^2 + \frac{T_R}{T_R+t}R^4.$$

If $R \leq r < 1$ then positivity of w implies that $z \leq \left(1-r^2\right)^2 \leq \left(1-R^2\right)^2 \leq 1 - R^2$ so that we have $z \leq 1 - R^2 + T_R R^4/(T_R+t)$ on Ω. By choosing t large enough, and R close enough to 1, we can make $\sup_\Omega z(\cdot,t)$ as small as we like, and this shows that (A.8) holds uniformly on Ω.

The proof of the asymptotic result (A.9) runs along the same lines, but is shorter; we note that

$$\tilde{w} = \frac{\delta}{1+8\delta t}\left(1-r^2\right)^2$$

is again a solution of (A.4). The same kind of argument involving the maximum principle shows that $w \geq \tilde{w}$, so that

$$0 \leq 1 - \frac{8tw(x,y,t)}{\left(1-r^2\right)^2} \leq \frac{1}{1+8\delta t} \to 0.$$

which clearly implies (A.9).

Geometrically, the condition $w(x,y,0) \geq \delta\left(1-r^2\right)^2$ is quite natural since it is invariant under the group of conformal automorphisms

$$f(z) = e^{i\theta} z + \overset{\alpha}{-}1 + \bar{\alpha} z, \qquad \theta \in \mathbb{R}, |\alpha| < 1,$$

of the unit disk. The condition is certainly satisfied by any metric which one obtains by lifting the metric of a compact surface of genus $g \geq 2$ to its universal cover, the unit disk Ω.

On the other hand, a quick computation shows that if $k > 2$ is any constant, then

$$W_+(x,y,t) = \lambda\left(1-r^2\right)^k, \qquad W_-(x,y,t) = \frac{1}{4kt}\left(1-r^2\right)^k$$

are super- and sub–solutions of (A.4), respectively ($\lambda > 0$ is a constant.) Using the maximum principle one can show that if w is a classical solution as above for which

$$\delta\left(1-r^2\right)^k \leq w(x,y,0) \leq \frac{1}{\delta}\left(1-r^2\right)^k$$

holds for some $\delta > 0$ and $k > 2$, then $w(x,y,t)/\left(1-r^2\right)^k$ will be bounded both from above and away from zero for any $t > 0$. In this case the asymptotic result (A.8) still holds, but the sharper estimate (A.9) is definitely untrue.

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