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RIGIDITY THEOREMS FOR PRIMITIVE FANO 3-FOLDS

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Introduction

A fundamental problem in the classification theory of algebraic manifolds is how many different projective structures can exist on a given manifold X_0. The answer may vary from only few structures to the existence of moduli spaces.

In case X_0 is the projective space $\mathbf{P}_n$, it is known by Hirzebruch-Kodaira [HK] and Yau [Y] that any projective manifold homeomorphic to X_0 is again $\mathbf{P}_n$. For n even this requires the existence of a Kähler-Einstein metric on the potential candidate X homeomorphic to $\mathbf{P}_n$. But already for the quadric Q_n the analogous result is known only in case n is odd (Brieskorn [Br]). Even the surface case is unsettled : there might be a surface of general type which is homeomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$. The projective structures on $\mathbf{P}_1 \times \mathbf{P}_1$ of Kodaira dimension $\neq 2$ are just the ruled surfaces $\mathbf{P}(\mathcal{O}_{\mathbf{P}_1} \oplus \mathcal{O}_{\mathbf{P}_1}(-n))$, $n \in \mathbf{N}$ even.

Unknown are also the possible projective structures on $\mathbf{P}(\mathcal{O}_{\mathbf{P}_1} \oplus \mathcal{O}_{\mathbf{P}_1}(-1))$ different from $\mathbf{P}(\mathcal{O}_{\mathbf{P}_1} \oplus \mathcal{O}_{\mathbf{P}_1}(-n))$, $n \in \mathbf{N}$ odd, which again are suspected not to exist.

The next interesting surfaces to look at would be Fano surfaces X_0 (i.e. $-K_{X_0}$ is ample), which are classically called del Pezzo surfaces. It is well known that Barlow's surface (which is of general type) is homeomorphic to $\mathbf{P}_2$ blown up in 8 points. But for instance it is unknown whether there is a surface of general type homeomorphic to $\mathbf{P}_2$ blown up in, say, 2 points.

The aim of this paper is the study of projective structures on certain Fano 3-folds X_0. As we already saw in the surface case, difficulties arise to exclude possible X with K_X ample, or K_X nef ($(K_X.C) \geq 0$ for every curve C). In the 3-fold case this can be excluded if we know that $\chi(\mathcal{O}_X) > 0$ using a result of Miyaoka. Of course, $\chi(\mathcal{O}_{X_0}) = 1$, so we ask whether $\chi(\mathcal{O}_X)$ is a topological invariant for projective 3-folds.

Clearly $\dim H^i(X, \mathcal{O}_X)$ are topological invariants for $i = 1, 2$ if $b_2 \leq 2$ but whether $\dim H^3(X, \mathcal{O}_X)$ is also invariant is a deep unsolved problem. We can force $H^3(X, \mathcal{O}_X)$ to vanish by requiring $b_3(X_0) = 0$. So we deal only with Fano 3-folds with vanishing b_3. In case $b_2(X_0) = 1$ those X_0 are well understood and easy to deal with : X_0 is $\mathbf{P}_3, Q_3$, one 3-fold of index 2 and a family of index 1 ; any X homeomorphic to X_0 is again of the same type.

So we turn to the case $b_2 \geq 2$; we will restrict ourselves here only to $b_2 = 2$, Fano 3-folds with $b_2 \geq 2$ are classified by Mori-Mukai [MM 1,2], the most interesting case being $b_2 = 2$ or 3. Such a X_0 is called primitive if it is not the blow-up of another 3-fold along a smooth curve. In order not to overload the paper we will also restrict ourselves to primitive X_0 ; but certainly similar results can be proved also in the imprimitive case using the same methods. Our result is now :

Theorem. *Let X_0 be a primitive Fano 3-fold with $b_2 = 2, b_3 = 0$. Let X be a projective smooth 3-fold homeomorphic to X. Then either $X \simeq X_0$, or $X \simeq \mathbf{P}(E)$ with a rank 2-vector bundle E on $\mathbf{P}_2$ whose Chern classes (c_1, c_2) belong to the following set : $\{(0,0), (-1,1), (-1,0), (0,-1), (0,3)\}$ or $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}_1}(a) \oplus \mathcal{O}_{\mathbf{P}_1}(b) \oplus \mathcal{O}_{\mathbf{P}_1}(c))$ with $a + b + c \equiv 0(3)$.*

In fact, X_0 is by the Mori-Mukai classification of the form $\mathbf{P}(V)$ with V a 2-bundle on $\mathbf{P}_2$ of the form :

$$\mathcal{O} \oplus \mathcal{O}(-n) \text{ with } 0 \leq n \leq 2, \quad T_{\mathbf{P}_2}, \text{ or } V \text{ is given by an extension :}$$

$$0 \to \mathcal{O}_{\mathbf{P}_2}(-2) \to \mathcal{O}^3_{\mathbf{P}_2} \to V \to 0.$$

Now E is just a bundle topologically isomorphic to V, i.e. with the same Chern classes.

Using analogous methods, we are able in § 7 to answer a question asked in [C2] : if Z_0 is a Moishezon non-projective twistor space, does there exist a projective threefold Z which is homeomorphic to Z_0 ? The answer is no, at least when b_2 is odd. Let us recall that such a Z_0 is the first known example of a manifold of class $\mathcal{C}$ (i.e. : bimeromorphic to a compact Kähler one) admitting arbitrarily small deformations which are not in the class $\mathcal{C}$. This exhibits another pathology of these Z_0. Howeover, it would be interesting to have an example of a Moishezon manifold Z_0, diffeomorphic to some projective Z, but admitting arbitrarily small deformations which are not in $\mathcal{C}$.

The relationship with the other investigations of this paper is that Z_0 is nearly Fano in the sense that the Kodaira dimension of its anticanonical bundle is $3 = \dim_{\mathbf{C}}(Z_0)$.

1. Basic material on Fano 3-folds

Let X be a projective manifold with canonical bundle K_X. X is called Fano if $-K_X$ is ample. Fano manifolds are simply connected and satisfy

$$H^q(X, \mathcal{O}_X) = 0, q \geq 1$$

by Kodaira's vanishing theorem.

1.1. In case $b_2(X) = 1$ all Fano 3-folds are classified by Iskovskih, Shokurov and also Mukai [Is 1,2], [Mu]. Those with $b_3(X) = 0$ can be listed as follows :

(a) $X = \mathbf{P}_3$,

(b) $X = Q_3$, the 3-dimensional smooth quadric,

(c) X is of index 2, i.e. $-K_X = 2L$ with $L \in Pic(X)$ the ample generator of $Pic(X) \simeq \mathbf{Z}$, and $L^3 = 5$. X is unique by these properties and usually called V_5.

(d) X is of index one, i.e. $-K_X = L$; $L^3 = 22$. These build up a family and we write $X = A_{22}$.

1.2. Fano 3-folds X with $b_2 \geq 2$ are classified in [MM 1,2], we will only consider those with $b_2 = 2$. First recall that X is called primitive if it is not the blow-up of a 3-fold Y with $b_2 = 1$ along a smooth curve. It is obvious that this is equivalent to saying that X is not the blow up of any 3-fold along a smooth curve. The classification heavily depends on Mori's theory of extremal rays, cone theorem etc. We will make freely use of this and refer e.g. to [KMM]. X being Fano with $b_2 = 2$ we have exactly two extremal maps R_i on X giving rise to contractions

$$\varphi_i : X \to Y_i.$$

Then $Pic(Y_i) \simeq \mathbf{Z}$, in fact Y_i are Fano with only terminal singularities with $b_2 = 1$, so fix ample generators L_i' on Y_i and put

$$L_i = \varphi_i^*(L_i').$$

Lemma 1.3. $Pic(X) = \mathbf{Z}.L_1 \oplus \mathbf{Z}.L_2$.

Proof. [MM 1] □

1.4. We now give a table of all primitive (five) Fano 3-folds X with $b_2(X) = 2$, $b_3(X) = 0$ and their relevant numerical properties needed in this paper, according to [MM 1,2]. As to notations, let $D_{2,1}$ denote a smooth divisor of bidegree $(2,1)$ in $\mathbf{P}_2 \times \mathbf{P}_2$ and let W_4 be the Veronese cone in $\mathbf{P}_6$.

The last column means the following : (a, b) is the pair determined by the equation (observe (1.3) !) $-K_X = aL_1 + bL_2$.

X	Y_1	Y_2	$-K_X^3$	L_1^3	$L_1^2L_2$	$L_1L_2^2$	L_2^3	(a,b)
$\mathbf{P}_1 \times \mathbf{P}_2$	$\mathbf{P}_1$	$\mathbf{P}_2$	54	0	0	1	0	$(2,3)$
$\mathbf{P}(T_{\mathbf{P}_2})$	$\mathbf{P}_2$	$\mathbf{P}_2$	48	0	1	1	0	$(2,2)$
$\mathbf{P}(\mathcal{O}_{\mathbf{P}_2} \oplus \mathcal{O}_{\mathbf{P}_2}(-1))$	$\mathbf{P}_2$	$\mathbf{P}_3$	56	0	1	1	1	$(2,2)$
$\mathbf{P}(\mathcal{O}_{\mathbf{P}_2} \oplus \mathcal{O}_{\mathbf{P}_2}(-2))$	$\mathbf{P}_2$	W_4	62	0	1	2	4	$(1,2)$
$D_{2,1}$	$\mathbf{P}_2$	$\mathbf{P}_2$	30	0	1	2	0	$(1,2)$

1.5. The structure of a Mori contraction $\varphi : X \to Y$ of an extremal ray on a smooth 3-fold X is completely determined by [Mo] and given in the following list :

(a) φ is a modification. Then either φ is the blow-up of a smooth curve in the smooth 3-fold Y. Or there is an unique irreducible divisor $E \subset X$ contracted by φ to a point and either
  (a1) $E \simeq \mathbf{P}_2$ with normal bundle $N_E = \mathcal{O}(a), a = -1, -2$
  (a2) $E \simeq \mathbf{P}_1 \times \mathbf{P}_1$ with $N_E = \mathcal{O}(-1,-1)$
  (a3) E is a (singular) quadric cone with $N_E = \mathcal{O}(-1)$.

(b) $\dim Y = 2$. Then φ is a $\mathbf{P}_1$-bundle or a conic bundle.

(c) $\dim Y = 1$. Then φ is a $\mathbf{P}_2$ -bundle, a quadric bundle, or the general fibre F of φ is a del Pezzo surface with $1 \leq K_F^2 \leq 6$.

(d) $\dim Y = 0$ and X is Fano with $b_2 = 1$.

1.6. We now describe the structures of φ_i in the table (1.4) according to (1.5) ; see again [MM 1,2].

In case $X = \mathbf{P}_1 \times \mathbf{P}_2$ this is obvious ; for $X = \mathbf{P}(T_{\mathbf{P}_2})$ we have two $\mathbf{P}_1$-bundle structures. $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ is a $\mathbf{P}_1$-bundle over $\mathbf{P}_2$ and the blow up of a point in $\mathbf{P}_3$. $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ is a $\mathbf{P}_1$-bundle over $\mathbf{P}_2$ and also the blow-up of the unique singular (quadruple) point on W_4 ; the exceptional divisor D is $\mathbf{P}_2$ with normal bundle $\mathcal{O}(-2)$. Finally $D_{2,1}$ is a $\mathbf{P}_1$-bundle over $Y_1 = \mathbf{P}_2$ via φ_1 and a conic bundle over $Y_2 = \mathbf{P}_2$ via φ_2 (by our choice of (a, b) !) with φ_i being the restriction of the projection pr_i to $\mathbf{P}_2$.

The $\mathbf{P}_1$-bundle structure is given as $\mathbf{P}(F)$ with F a 2-bundle on $\mathbf{P}_2$ defined by an extension

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^3 \to F \to 0.$$

2. Topological invariants

Let X_0 be a smooth projective 3-fold with $b_1 = 0, b_2 \leq 2$ and assume X to be another smooth projective 3-fold homeomorphic to X_0. By Hodge decomposition :

$$H^q(X, \mathcal{O}_X) = H^q(X_0, \mathcal{O}_{X_0}) = 0$$

for $q = 1, 2$.

Although $b_3(X) = b_3(X_0)$, the Hodge decomposition of H^3 might a priori be quite different, so let us formulate :

PROBLEM 2.1. Is $h^3(X, \mathcal{O}_X)$ a topological invariant for projective 3-folds ? (Equivalently, we could ask for $h^{2,1}$, and the same can be asked also in general for $h^{2,0}$).

Because of the unsolved problem (2.1) we will always assume $b_3(X_0) = 0$. Then clearly

$$H^3(X, \mathcal{O}_X) = H^3(X_0, \mathcal{O}_{X_0}) = 0,$$

and hence :

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) = 1.$$

This vanishing has far-reaching consequences by the following result of Miyaoka [Mi], which is an immediate consequence of his inequality $c_1^2 \leq 3c_2$.

Theorem 2.2. *Let X be a projective 3-fold with K_X nef. Then $\chi(\mathcal{O}_X) \leq 0$.*

Corollary 2.3. *Let X_0 be a Fano 3-fold with $b_3 = 0, X$ a projective 3-fold homeomorphic to X_0. Then K_X is not nef.*

In particular, X carries an extremal ray by [Mo] and we can use Mori theory to examine the structure of X (if $b_2 \geq 2$). This will be done in § 4. If we don't assume $b_3 = 0$ in (2.3) then there is no apparent reason why K_X could not be ample for instance.

We now come to in important method to determine K_X going back to Hirzebruch-Kodaira [HK]. Here let us suppose X_0 to be a Fano 3 fold with $b_2 \leq 2$ for simplicity. In case $b_2 = 1$ we fix an ample generator L_0 on X_0. In case $b_2 = 2$ we let L_1, L_2 be as in (1.3). Then if $b_2 = 1$ we can write

$$c_1(X) = c_1(X_0) + 2sc_1(L), \quad s \in \mathbf{Z}$$

and for $b_2 = 2$:

$$c_1(X) = c_1(X_0) + 2\Big(s_1 c_1(L_1) + s_2 c_1(L_2)\Big).$$

Observe that the factor 2 comes from the invariance of the Stiefel-Whitney class $w_2(X)$ which is the residue class of $c_1(X)$ in $H^2(X, \mathbf{Z}_2)$. Then we have :

Proposition 2.4. *Let $\mathcal{G}$ be a holomorphic line bundle on X_0, $\widetilde{\mathcal{G}}$ the corresponding one on X. Then*

(a) $\chi(X, \widetilde{\mathcal{G}}) = \chi(X_0, \mathcal{G} \otimes L^s)$ *if* $b_2(X_0) = 1$
(b) $\chi(X, \widetilde{\mathcal{G}}) = \chi(X_0, \mathcal{G} \otimes L_1^{s_1} \otimes L_2^{s_2})$ *if* $b_2(X_0) = 2$.

The line bundle $\widetilde{\mathcal{G}}$ corresponding to $\mathcal{G}$ means the following : $\mathcal{G}$ can be viewed as a topological line bundle on X and since $Pic(X) \simeq H^2(X, \mathbf{Z})$ by $H^q(X, \mathcal{O}_X) = 0, q = 1, 2$, its carries a unique holomorphic structure, namely $\widetilde{\mathcal{G}}$.

Proof. We prove only (a), (b) being completely the same. By Riemann-Roch (see e.g. [Hi])

$$\chi(X, \widetilde{\mathcal{G}}) = \left[e^{\frac{1}{2}c_1(X)+c_1(\mathcal{G})}.\sum_{i=0}^{\infty} \hat{A}_i(p_1, p_2, ...)\right]_3,$$

where p_i are the Pontrjagin classes of X and $\hat{A}_i$ certain universal functions. Since $p_i(X) = p_i(X_0)$ (Novikov) and since $c_1(X) = c_1(X_0) + 2sc_1(L)$ by assumption, we obtain :

$$\begin{aligned}\chi(X, \widetilde{\mathcal{G}}) &= \left[e^{\frac{1}{2}c_1(X_0)+c_1(L^s)+c_1(\mathcal{G})} . \sum \hat{A}_i(p_1(X_0), p_2(X_0), ...)\right]_3 \\ &= \chi(X_0, \mathcal{G} \otimes L^s),\end{aligned}$$

again by Riemann-Roch. □

Remark 2.5. Of course the arguments above are independant of dimension 3 and of the Fano property of X_0. The only requirements we need are that $c_1(X) - c_1(X_0)$ contains a holomorphic line bundle on X, that $\mathcal{G}$ has a holomorphic structure $\widetilde{\mathcal{G}}$ on X, and that, moreover : $Pic(X_0) = \mathbf{Z}$ or $\mathbf{Z}^2$.We finish this section by stating for later use the following well-known result :

Proposition 2.6. *Let S be an algebraic surface with $\pi_1(S)$ finite and $b_2(S) = 1$. Then $S \simeq \mathbf{P}_2$.*

A proof can be found in [BPV, p. 135].

3. Fano 3-folds with $b_2 = 1$

We are going to study 3-folds homeomorphic to Fano 3-folds with $b_2 = 1$. From (2.3) we immediately obtain :

Theorem 3.1. *If X is a projective 3-fold homeomorphic to the Fano 3-fold X_0 with $b_2 = 1, b_3 = 0$, then X is again Fano and in fact $X \simeq X_0$ resp. is of type A_{22} if X_0 is of type A_{22}.*

Proof. By (2.3) K_X is not nef. Since $Pic(X) \simeq \mathbf{Z}$, $-K_X$ must be ample, so X is Fano. By the classification of Fano 3-folds it suffices now to prove $c_1(X) = c_1(X_0)$. Writing $c_1(X) = c_1(X_0) + 2sc_1(L)$ $(s \geq -\frac{1}{2}$ index $(X_0))$, L the ample generator, we obtain from (2.4) :

$$\chi(L^s) = \chi(\mathcal{O}_X) = 1.$$

Using Riemann-Roch for instance it is easy to solve this equation to obtain $s = 0$. □

Of course (3.1) is known by [HK] for $\mathbf{P}_3$, by [Br] for Q_3 and in the other cases by [LS]. We should mention that the use of (2.3) can be avoided by solving

$$\chi(L^s) = \chi(\mathcal{O}_X) = 1$$

also for all $s < 0$. In fact $\chi(L^s) = -h^3(\mathcal{O}_X)$ for $s < 0$, hence $\chi(L^s) \neq 1$.

This arguments works in all odd dimensions, on the other hand it is not known whether there is a projective n-fold X, n even, homeomorphic to a quadric Q_n, with K_X ample.

Remark 3.2. If we don't assume $b_3 = 0$ in (3.1) we cannot conclude $\chi(\mathcal{O}_X) > 0$ and hence K_X could be ample. If K_X is known not to be ample or trivial, then clearly X is Fano and one can apply Iskovshih's classification to X. We exclude the case $K_X = \mathcal{O}_X$ as follows. Assume $K_X = \mathcal{O}_X$. By the invariance of w_2, X_0 is a Fano 3-fold of index 2 or 4. Since $X_0 \neq \mathbf{P}_3$, X_0 has in fact index 2. Hence in the equation

$$0 = c_1(X) = c_1(X_0) + 2sc_1(L)$$

we hace $s = -1$.

Let $\tilde{L} \in Pic(X)$ be the ample generator. By (2.4) we have

$$\chi(X, \tilde{L}^t) = \chi(X_0, L^{t-1}),$$

in particular

(1) $$\chi(X, \tilde{L}) = \chi(\mathcal{O}_{X_0}) = 1.$$

By Riemann-Roch we get

(2) $$\chi(X, \tilde{L}) = \frac{c_1(\tilde{L})^3}{6} + \frac{1}{12} c_1(\tilde{L}) \cdot c_2(X).$$

Miyaoka's inequality $c_1^2(X) \leq 3c_2(X)$ ([Mi]) yields $c_1(\tilde{L}) \cdot c_2(X) \geq 0$. We even must have strict inequality; if $c_1(\tilde{L}) \cdot c_2(X) = 0$ we would get (by $b_2(X) = b_4(X) = 1$) $c_2(X) = 0$, so X would be covered by a torus [Y], contradiction.

Thus it is possible, using (1) and (2), to compute the pair $(c_1(\tilde{L})^3, c_2(X))$, since by Iskovskih, $1 \leq c_2(\tilde{L})^3 = c_1(L)^3 \leq 4$ (observe $b_3(X_0) > 0$).

Identitfying $H^2(X_0, \mathbf{Z})$ and $H^4(X_0, \mathbf{Z})$ with $\mathbf{Z}$, the intersection product is just multiplication, and we obtain: $(c_1(L)^3, c_2(X)) = (1,10), (2,8), (3,6), (4,4)$. Now consider the Pontrjagin class

$$p_1(X) = c_1^2(X) = c_2(X).$$

$p_1(X)$ is a topological invariant. We compute easily in the four cases: $p_1(X_0) = -8, -4, 0, 4$. On the other hand $p_1(X) = -c_2(X) = -10, -8, -6, -4$, contradiction.

We can try to determine the type of K_X by (2.4). In fact, (2.4) gives, if we write $c_1(X) = c_1(X_0) + 2sc_1(L)$ as in (2.4),

$$\chi(X, \mathcal{O}_X) = \chi(X_0, L^s).$$

Since $\chi(X, \mathcal{O}_X) = 1 - h^3(\mathcal{O}_X)$ and $h^3(\mathcal{O}_X) \leq \frac{b_3(X)}{2} = \frac{b_3(X_0)}{2}$, we obtain :

$$\chi(X_0, L^s) \geq 1 - \frac{b_3(X_0)}{2}.$$

Observe that we may assume $s < 0$, otherwise X is already Fano. Now we can go to the list of Fano 3-folds X_0 with $b_2 = 1, b_3 > 0$ (of index 1 or 2) ; b_3 being known, we can try to solve the above inequality using Riemann-Roch on X_0. Then we obtain setting $c_1(X) = \mu c_1(L) = (2s + \tau)$, τ the index of X_0 :

(3.3)

index	L^3	$\frac{b_3}{2}$	s	μ
2	1	21	$-2 \geq s \geq -5$	$-2, -4, -6, -8$
2	2	10	$-2, -3$	$-2, -4$
2	3	5	-2	-2
2	4	2	-2	-2
1	2	52	$-1 \geq s \geq -5$	$-1, -3, \cdots, -9$
1	4	30	$-1 \geq s \geq -3$	$-1, -3, -5$
1	6	20	$-1, -2$	$-1, -3$
1	8	14	$-1, -2$	$-1, -3$
1	$8 < L^3 \leq 18$	...	-1	-1

In any case there are only finitely many possibilities for K_X ; in a lot of cases only the "dual" possibility $c_1(X) = -c_1(X_0)$. At least we can conclude that all the X homeomorphic to a given Fano 3-fold X_0 with $b_2 = 1$ form a bounded family.

4. Structure of Mori contractions on topological primitive Fano 3-folds with $b_2 = 2, b_3 = 0$ and the main result

Let X_0 always denote a Fano 3-fold with $b_2 = 2, b_3 = 0$. We assume that X_0 is primitive, i.e. X_0 is not the blow-up of a (Fano) 3-fold along a smooth curve. Let X be a projective smooth 3-fold homeomorphic to X_0. By (2.3) we know that K_X is not nef, so there is a contraction $\varphi : X \to Y$ of an extremal ray on X.

We let $\varphi_i : X_0 \to Y_i$ be the two contractions on X_0 as on (1.2) and let L_i be as in (1.2) :

$$L_i = \varphi_i^*(L_i')$$

for ample generators L_i' on Y_i.

The list of all possible X_0 together with $\varphi_i : X_0 \to Y_i$ is given in (1.4) and (1.5). In order to determine K_X we will make the following ansatz as in Section. 2 :

$$c_1(X) = c_1(X_0) + 2s_1c_1(L_1) + 2s_2c_1(L_2)$$

and we know that for any line bundle $\mathcal{G}$ on X_0, with corresponding bundle $\widetilde{\mathcal{G}}$ on X (2.4 (b)) :

$$\chi(X, \widetilde{\mathcal{G}}) = \chi(X_0, \mathcal{G} \otimes L_1^{s_1} \otimes L_2^{s_2}), \tag{4.1.1}$$

in particular

$$1 = \chi(X, \mathcal{O}_X) = \chi(X_0, L_1^{s_1} \otimes L_2^{s_2}); \tag{4.1.2}$$

often we will abbreviate $L_1^a \otimes L_2^b$ by $\mathcal{O}_{X_0}(a, b)$.

Proposition 4.2. *Assume that φ contracts a divisor E to a point.*
Then either $X \simeq X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ or $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ and $E^3 = 4$.

Proof. According to (1.3) write :

$$E = a_1L_1 + a_2L_2, \quad a_i \in \mathbf{Z}.$$

So $E^3 = a_1^3L_1^3 + 3a_1^2a_2L_1^2L_2 + 3a_1a_2^2L_1L_2^2 + a_2^3L_2^3$. On the other hand : $E^3 = 1, 2$ or 4 by (1.5).

If $X_0 = \mathbf{P}_1 \times \mathbf{P}_2$, $\mathbf{P}(T_{\mathbf{P}_2})$ or $D_{2,1}$ in (1.4), we conclude :

$$3(a_1^2 a_2 L_1^2 L_2 + a_1 a_1^2 L_1 L_2^2) = 1, 2, 4$$

which is impossible.

Hence $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\alpha)), \alpha = -1, -2$ (1.4).

(a) First assume $\alpha = -1$. Then we obtain :

$$3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3 = 1, 2 \text{ or } 4.$$

Trivial calculations show that $E^3 = 2$ or 4 are not possible, so $E^3 = 1$ and φ is the blow-up of a simple point. In particular Y is smooth with $Pic(Y) = \mathbf{Z}$,

$$K_X = \varphi^*(K_Y) + E,$$

and obviously Y is Fano. In order to determine it, we solve :

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1; \tag{4.1.2}$$

it is an easy exercise to see e.g. via Riemann-Roch : $s_1 = s_2 = 0$. Hence $c_1(X) = c_1(X_0)$. So $K_X^3 = K_{X_0}^3 = -56$, hence $-56 = K_Y^3 + 8E^3$ yields $K_Y^3 = -64$ and by the classification we conclude $Y \simeq \mathbf{P}_3$; so $X \simeq X_0$.

(b) Finally let $\alpha = -2$.

Then our equation reads :

$$3a_1^2 a_2 + 6a_1 a_2^2 + 4a_2^3 = 1, 2 \text{ or } 4.$$

The only solution for $E^3 = 1$ is $(a_1, a_2) = (-1, 1)$, $E^3 = 2$ being impossible. So it is sufficient to exclude $E^3 = 1$. (In this case Y is Fano with $b_2 = 1$, $b_3 = 0$, so $Y = \mathbf{P}_3, Q_3, V_5$ or A_{22}).

Using $L = \varphi^*(\mathcal{O}_Y(1))$ we have $L^2.E = 0$, on the other hand writing $c_1(L) = \alpha_1 c_1(L_1) + \alpha_2 c_1(L_2)$:

$$\begin{aligned} L^2.E &= (\alpha_1 L_1 + \alpha_2 L_2)^2.(-L_1 + L_2) \\ &= (\alpha_1 + \alpha_2)^2 + 2(\alpha_1 - 2^2 - \alpha_2). \end{aligned}$$

Both equations imply $\alpha_1 = \alpha_2 = 0$, a contradiction. $\square$

Remark 4.3. If $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ in (4.2) then we will show in (4.8) that in this case $X \simeq X_0$, too.

Proposition 4.4. *φ is never the blow-up of a smooth curve in a smooth 3-fold Y.*

Proof. Assume that φ is the blow-up of the smooth curve C in Y. Since $b_3(X) = 0$, we conclude $b_3(Y) = 0$ and $C \simeq \mathbf{P}_1$. Y being Fano with $b_2 = 1$, we have $Y = \mathbf{P}_3, Q_3, V_5$ or A_{22}.

Let $\mathcal{O}_Y(1)$ be the ample generator and $L = \varphi^*(\mathcal{O}_Y(1))$. Then $L^3 = 1, 2, 5$ or 22, respectively. On the other hand, write again :

$$c_1(L) = a_1 c_1(L_1) + a_2 c_1(L_2).$$

Then we have the equation

$$3a_1^2 a_2 L_1^2 L_2 + 3a_1 a_2^2 L_1 L_2^2 + a_2^3 L_2^3 = 1, 2, 5 \text{ or } 22.$$

From table (1.4) we conclude that necessarily $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\alpha))$ with $\alpha = -1, -2$, because otherwise the left hand side would be divisible by 3.

(a) $\alpha = -1$.

Then the only solutions are $(0, 1)$ (with $L^3 = 1$) and $(-1, 2)$ (with $L^3 = 2$). If $(a_1, a_2) = (0, 1)$ then $c_1(L) = c_1(L_2)$.

Let F, F_2 be a general non-trivial fiber of φ resp. φ_2.

Then $(-K_X.F) = 1$, $(-K_{X_0}.F_2) = 2$. Since $c_1(X) = c_1(X_0) = 0$ (proof of (4.2)), it follows via $c_1(L) = c_1(L_2)$ that $[F_2]$ is an even multiple of $[F_1]$ in $H^4(X_0, \mathbf{Z})$, i.e. $[F]$ is divisible by 2 in $H^4(X_0, \mathbf{Z})$ which is clearly false. So assume now $(a_1, a_2) = (-1, 2)$. Write $E = \alpha_1 c_1(L_1) + \alpha_2 c_1(L_2)$.

Then the equation $L^2 E = 0$ yields $\alpha_2 = 0$, so $E^3 = 0$. On the other hand $E^3 = -c_1(N_{C|Y})$ which is absurd since $Y = Q_3$.

(b) $\alpha = -2$.

Now the only solution is $(a_1, a_2) = (-1, 1)$ with $L^3 = 1$, so $Y \simeq \mathbf{P}_3$. With $E = \alpha_1 c_1(L_1) + \alpha_2 c_2(L_2)$ we obtain as in (a) :

$$0 = L^2 E = -3\alpha_2, \text{ hence } E^3 = 0$$

and we conclude $c_1(N_{C|Y}) = 0$, contradiction. $\square$

From now on we may assume that φ is not a modification, hence $\dim Y = 1$ or 2 and Y is smooth.

Proposition 4.5. *Assume* $\dim Y = 2$. *Then* $Y \simeq \mathbf{P}_2$ *and either :*

(c1) *φ is a $\mathbf{P}_1$-bundle, or*

(c2) *φ is a proper conic bundle over $\mathbf{P}_2$ and $X_0 = D_{2,1}$, a divisor of bidegree $(2,1)$ in $\mathbf{P}_2 \times \mathbf{P}_2$, moreover $c_1(X) = c_1(X_0)$ and $c_1(L) = c_1(L_2)$.*

Proof. Since $\pi_1(Y) = 0$, X being simply connected, and since $b_2(Y) = 1$, we conclude $Y \simeq \mathbf{P}_2$ by (2.6). So X is a $\mathbf{P}_1$-bundle or a conic bundle over $\mathbf{P}_2$. Let $L = \varphi^*(\mathcal{O}_{\mathbf{P}_2}(1))$ and write

$$c_1(L) = a_1 c_1(L_1) + a_2 c_1(L_2).$$

We are going to solve the equation

$$(*) \qquad 0 = L^3 = 3a_1^2 a_2 L_1^2 L_2 + 3a_1 a_2^2 L_1 L_2^2 + a_2^3 L_2^3.$$

But first we claim :

(a) if $a_2 = 0$ in case of $X_0 \neq \mathbf{P}_1 \times \mathbf{P}_2$, X is a $\mathbf{P}_1$-bundle over Y.

So assume for the proof : $a_2 = 0$. If $a_1 \neq \pm 1$, then L would be divisible by some line bundle L' which necessarily has to be of the form $\varphi^*(\mathcal{O}_{\mathbf{P}_2}(m))$, which is absurd. So $|a_1| = 1$.

Assume φ is not a $\mathbf{P}_1$-bundle. Then let F a component of a reducible fiber of φ. We have

$$(-K_X.F) = 1.$$

Now let F_1 be a fiber of φ_1. Then $c_1(L) = \pm c_1(L_1)$ yields $[F] = \pm[F_1]$ in $H^4(X_0, \mathbf{Z})$.

Hence $(-K_{X_0}.F_1) = \pm(K_{X_0}.F) \equiv 1(2)$ by the invariance of the Stiefel-Whitney class w_2. On the other hand, φ_1 is a $\mathbf{P}_1$-bundle if $X_0 \neq \mathbf{P}_1 \times \mathbf{P}_2$, so

$$(-K_{X_0}.F_1) = 2,$$

contradiction.

(b) Now let $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(+\alpha)), \alpha = -1, -2$. Then $(*)$ gives immediately $a_2 = 0$, so we are done by (a).

(c) If $X_0 = \mathbf{P}_1 \times \mathbf{P}_2$ then $(*)$ reads

$$3a_1 a_2^2 = 0, \text{ so } a_1 = 0 \text{ or } a_2 = 0.$$

If $a_2 = 0$ we would have $L^2 = 0$ which is impossible. So we can apply (a).

(d) For $\mathbf{P}(T_{\mathbf{P}_2})$, $(*)$gives :

$$3(a_1^2a_2 + a_1a_2^2) = 0$$

so $a_1 = 0$ or $a_2 = 0$ or $a_1 = -a_2$ and it is sufficient to exclude the latter possibility. But if $a_1 = -a_2$, then

$$L^2 = a_1^2(L_1 - L_2)^2.$$

Since $L^2 = F$, a fiber of φ, we obtain :

$$(-K_X.F) = -2a_1^2$$

if we suppose $c_1(X) = c_1(X_0)$. Since $-(K_X.F) > 0$, we have a contradiction. In order to verify : $c_1(X) = c_1(X_0)$, we write as usual :

$$c_1(X) = c_1(X_0) + 2s_1c_1(L_1) + 2s_2c_1(L_2)$$

and have (4.1.2) to solve the equation

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1.$$

But $X_0 = \mathbf{P}(T_{\mathbf{P}_2})$ can be viewed as divisor of bidegree (1,1) in $\mathbf{P}_2 \times \mathbf{P}_2$, hence

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = \chi(\mathcal{O}_{\mathbf{P}_2\times\mathbf{P}_2}(s_1, s_2)) - \chi(\mathcal{O}_{\mathbf{P}_2\times\mathbf{P}_2}(s_1 - 1, s_2 - 1)) = 1.$$

Now compute, using :

$$\chi(\mathcal{O}_{\mathbf{P}_2}(t)) = \frac{(t+1)(t+2)}{2}$$

to get $s_1 = s_2 = 0$.

(e) It remains to treat $X_0 = D_{2,1}$.

In this case $(*)$ reads

$$3a_1^2a_2 + 6a_1a_2^2 = 0.$$

If $a_2 = 0$, then φ is a $\mathbf{P}_1$-bundle by (a) and we are done. So either $a_1 = 0$ or $a_1 = -2a_2$.

First we want to exclude the later possibility. So assume $a_1 = -2a_2$. Using

$$c_1(X) = (1 + 2s_1)c_1(L_1) + (2 + 2s_2)c_1(L_2)$$

and

$$L^2.(-K_X) = F.(-K_X) = 2,$$

we obtain : $a_2^2 = 1$; moreover $s_1 = -2s_2 - 3$.

In order to determine (s_1, s_2), we use :

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1.$$

In fact,

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = \chi(\mathcal{O}_{\mathbf{P}_2 \times \mathbf{P}_2}(s_1, s_2)) - \chi(\mathcal{O}_{\mathbf{P}_2 \times \mathbf{P}_2}(s_1 - 2, s_2 - 1))$$

is an explicit polynomial, and via the relation between s_1 and s_2, we easily obtain :

$$s_1 = 0, s_2 = -3.$$

Now consider the equation (4.1.1)

$$\chi(X, L^t) = \chi(\mathcal{O}_{X_0}(-2t, t-3)).$$

Clearly $\chi(X, L^t) = \frac{(t+1)(t+2)}{2}$. The right hand side is also easily computed (go again to $\mathbf{P}_2 \times \mathbf{P}_2$), and it turns out that both polynomials are different, contradiction.

So we are left with the case $a_1 = 0$. Then we want to show that φ is a conic bundle, that $c_1(X) = c_1(X_0)$ and $c_1(L) = c_1(L_2)$.

As before, by a divisibility argument we get $|a_2| = 1$, so $c_1(L) = \pm c_1(L_2)$ also it is easy to see that φ_2 cannot be a $\mathbf{P}_1$-bundle, hence must be a proper conic bundle. We have

$$c_1(X) = (1 + s_1)L_1 + (2 + s_2)L_2.$$

Since (general) fiber of φ and φ_2 have the same cohomology class, we obtain by intersecting $-K_X$ with a general fiber easily : $s_1 = 0$.

So by (4.1.1)

$$\chi(X, L^t) = \chi(X_0, \mathcal{O}_{X_0}(0, t + s_2)), \quad (\text{resp. } \chi(X_0, \mathcal{O}_{X_0}(-t + s_2)),$$

hence

$$\frac{(t+1)(t+2)}{2} = \frac{(t+s_2+1)(t+s_2+2)}{2} \quad (\text{resp. } \frac{(-t+s_2+1)(-t+s_2+2)}{2})$$

which gives $s_2 = 0$.

This ends the proof of (4.5) $\square$

Remark 4.6. We will see in sect. 5 that in fact if $X_0 = D_{1,2}$ and φ is a conic bundle then $X \simeq X_0$.

Proposition 4.7. *Assume* $\dim Y = 1$. *Then* $Y \simeq \mathbf{P}_1$, X *is a* $\mathbf{P}_2$*-bundle over* $\mathbf{P}_1$ *and* $X_0 \simeq \mathbf{P}_1 \times \mathbf{P}_2$. X *is of the form* $\mathbf{P}(E)$ *with* $E = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$ *with* $a + b + c \equiv 0(3)$.

Proof. Obviously Y is rational. Write again :

$$c_1(L) = a_1 c_1(L_1) + a_2 c_1(L_2).$$

Then from $L_1.L^2 = 0$ and $L^3 = 0$ we obtain $a_2 = 0$ and hence

$$X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$$

and also easily : X is a $\mathbf{P}_1$-bundle, or the two equations :

$$\begin{aligned} &2a_1 L_1^2 L_2 + a_2 L_1 L_2^2 = 0 \\ &3a_1^2 L_1^2 L_2 + 3a_1 a_2 L_1 L_2^2 + a_2^2 L_2^3 = 0, \end{aligned} \qquad \text{are satisfied.}$$

Now using table (1.4) it is trivial to obtain a contradiction in all cases but $a_2 = 0$. If $a_2 = 0$ we proceed as above. So $X = \mathbf{P}(E) \to \mathbf{P}_1$, and the 3-bundle E has obviously the form as stated above. $\square$

We are coming now back to a special situation to be still treated (see (4.3)).

Proposition 4.8. *Assume that* φ *contracts a divisor* $E \simeq \mathbf{P}_2$ *with* $E^3 = 4$ *to a point and assume* $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. *Then* $X \simeq X_0$.

Proof. Write

$$c_1(\mathcal{O}_X(E)) = \alpha_1 c_1(L_1) + \alpha_2 c_1(L_2).$$

Then solve the equation

$$4 = E^3 = \alpha_2(3\alpha_1^2 + 6\alpha_1\alpha_2 + 4\alpha_2^2) :$$

the solutions are $(\alpha_1, \alpha_2) = (0, 1), (-1, 4)$ and $(-2, 1)$. Now put $c_1(L) = a_1 c_1(L_1) + a_2 c_1(L_2)$ in to $L^2.E = 0$. Then this rules already $(\alpha_1, \alpha_2) = (0, 1)$ resp. $(-1, 4)$.

So $(\alpha_1, \alpha_2) = (-2, 1)$. This gives by $L^2.E = 0$: $(a_1, a_2) = (0, a_2)$, hence by divisibility as usual : $a_2 = 1$. Moreover we see that lines in E and lines in the exceptional divisor of φ_2 have the same cohomology class. This implies by intersecting

$$c_1(X) = (1+s_1)c_1(L_1) + (2+s_2)c_1(L_2)$$

with such a line :

$$s_1 = 0, \quad \text{resp. } s_1 = 1 \text{ if } c_2 = -1.$$

The case $s_1 = -1$, $a_2 = -1$ is excluded as follows. From $1 = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}(-1, s_2))$ we first see $s_2 > 0$. By Serre duality we obtain $\chi(X_0, L_2^{-s_2-2}) = -1$. Computing on the Veroese cone $Y_2 = W_4$ we easily derive a contradiction. Now by (4.1.2) :

$$1 = \chi(X, \mathcal{O}_X) = \chi(X_0, \mathcal{O}_{X_0}(0, s_2))$$

and consequently $s_2 = 0$. So $c_1(X) = c_1(X_0)$. Let $L' \in Pic(Y)$ with $\varphi^*(L') = L$.

We want to compute Fujita's Δ-invariant :

$$\Delta(L') = 3 + L'^3 - h^\circ(L').$$

First note : $L'^3 = L^3 = L_2^3 = 4$.

In order to compute $h^\circ(L') = h^\circ(L)$ we notice that because of $c_1(X) = c_1(X_0)$ and because of the invariance of $p_1(X) = c_1^2 - 2c_2$, we have $c_2(X) = c_2(X_0)$, too, and hence by Riemann-Roch :

$$\chi(L) = \chi(L_2).$$

This $\chi(L') = \chi(L) = 7$.

Now Y is 2-Gorenstein (see [Mo]), $\rho(Y) = 1$ and L' is the ample generator of $Pic(Y) \simeq \mathbf{Z}$. Moreover we compute easily :

$$-K_Y = \frac{3}{2}L'.$$

Hence we get

$$H^q(Y, L') = 0$$

by the vanishing theorem of Kawamata-Viehweg (see e.g. [KMM]), since $L' - K_Y$ is ample. Consequently $h^\circ(L') = 7$ and $\Delta(L') = 0$. By [Fj], the linear system $|L'|$ is base point free and in fact defines an embedding :

$$Y \hookrightarrow \mathbf{P}_6.$$

Now the unique singular point $y_0 \in Y$ is a quadruple point by [Mo], hence if $l \subset \mathbf{P}_6$ is a line through y_0, then either $l \cap Y = \{y_0\}$, or $l \subset Y$.

This Y is the cone over the Veronese $\mathbf{P}_2 \hookrightarrow \mathbf{P}_5$ with vertex y_0. But this is also exactly the description of $Y_2 = W_4$, then $X \simeq X_0$. $\square$

Taking the results of sect. 5 for granted (see remark 4.6) we can rephrase the results of the section as follows.

Theorem 4.9. *Let X_0 be a primitive Fano 3-fold with $b_2 = 2$, $b_3 = 0$. Let X be a projective 3-fold homeomorphic to X_0. Then either $X \simeq X_0$ or $X = \mathbf{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ with $a+b+c \equiv 0(3)$. $X \simeq \mathbf{P}(E)$ with E a rank 2-bundle on $\mathbf{P}_2$ given in the following table (we normalise E such that $c_1(E) = -1$ or 0).*

In fact, every X_0 has the form $\mathbf{P}(V)$ (unique up to $\mathbf{P}(T_{\mathbf{P}_2})$) over $\mathbf{P}_2$ and $c_i(E) = c_i(V)$ (i.e. E and V are topologically the same).

(4.9)

X_0	$c_1(E)$	$c_2(E)$
$\mathbf{P}_1 \times \mathbf{P}_2$	0	0
$\mathbf{P}(T_{\mathbf{P}_2})$	0	1
$\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$	-1	0
$\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$	0	-1
$D_{2,1}$	0	3

Proof. We consider our extremal contraction $\varphi : X \to Y$.

(1) If φ is a modification, then by (4.2), (4.4) and (4.8) : $X \simeq X_0$.

(2) If $\dim Y = 2$, then by (4.5) : $Y \simeq \mathbf{P}_2$ and either X is a $\mathbf{P}_1$-bundle over $\mathbf{P}_2$ or $X_0 \simeq D_{2,1}$ and X is a conic bundle. In the latter case, $X \simeq X_0$ by (5. 1).

So assume $X \simeq \mathbf{P}(E) \to \mathbf{P}_2$.

Now write $X_0 = \mathbf{P}(V)$ with $V = \mathcal{O}_{\mathbf{P}_2} \oplus \mathcal{O}_{\mathbf{P}_2}(-n)$, $n = 0, 1, 2$ or $V = T_{\mathbf{P}_2}$ or $c_1(V) = 0, c_2(V) = 3$ (in case $X_0 = D_{2,1}$).

Then $p_1(\mathbf{P}(E)) = p_1(\mathbf{P}(V))$.

Since $\varphi(p_1(\mathbf{P}(E))) = (c_1^2(E) - 4c_2(E))$ for the projection $\varphi : X \to Y$ and since we know $\varphi_* = \varphi_{i*}$ for $i = 1$ or 2, we conclude

$$c_1^2(E) - 4c_2(E) = c_1^2(V) - 4c_2(V).$$

Since E is normalized and V is explicitly known we obtain our table.

(3) If $\dim Y = 1$, then apply (4.7). □

Remark 4.10. Of course if $X = \mathbf{P}(E)$ as in the table, then $X \simeq X_0$ topologically, since two rank 2-bundle on $\mathbf{P}_2$ with the same Chern classes are topologically equivalent (see [OSS]).

Some words to the existence of E with $c_i(E)$ as given on the table. There are always a lot of instable 2-bundles E which can be constructed by the Serre correspondence (see [OSS]). But a semi-stable E (different from the original bundle) exists only in $X_0 = D_{2,1}$; they are described by a moduli space of dimension 9.

5. The proper conic bundle case

After 4.5 and 4.6, the last remaining case is the following :

Proposition 5.1. *Let X be a threefold homeomorphic to $X_0 = D_{2,1}$ (see 1.4 for notations).*

Assume $\varphi : X \longrightarrow \mathbf{P}_2$ is a proper conic bundle, that $c_1(X) = c_1(X_0) = L_1 + 2L_2$, and $L_2 = \varphi^*(\mathcal{O}_{\mathbf{P}_2}(1))$, the identifications being obtained from the equalities : $Pic(X) = H^2(X, \mathbf{Z}) = H^2(X_0, \mathbf{Z}) = Pic(X_0)$.

Then X is analytically isomorphic to X_0.

The proof of (5.1) will be prepared by several lemmata. We denote by l a general line in $\mathbf{P}_2$ meeting the discriminant locus Δ of the conic bundle transversally. Then $S = S_l = \Phi^{-1}(l)$ is a smooth surface.

Lemma 5.2. *S is the blow-up of a ruled surface $\mathbf{P}(\mathcal{O}_{P_1} \oplus \mathcal{O}_{P_1}(k))$ in three points.*

Proof. Clearly S is the blow-up of a ruled surface $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(k))$ in say d points (with $d = deg\ \Delta$). Now $K_S^2 = (K_X + L_2)^2.L_2 = (L_1 + L_2)^2.L_2 = 5$ (c.f. 1.4). Hence $d = 3$. □

Lemma 5.3.

(1) $-K_S = L_1 + L_2$

(2) $\chi(S, L_1) = 3$

(3) $H^2(S, L_1) = 0, h^0(S, L_1) \geq 3$

(4) $L_1|S$ *is generated by global sections.*

(5) $h^0(S, L_1) = 3, H^0(S, L - L_2) = 0$ *and* $v : H^0(S, L_1) \to H^0(F, L_1)$ *is an isomorphism, (F a fiber of the conic bundle).*

Proof. (1) follows from $K_X = -L_1 - 2L_2$ by adjunction.

(2) is clear by Riemann-Roch.

(3) $H^2(S, L_1) = H^0(S, -2L_1 - L_2) = 0$, since $L_1|F = \mathcal{O}(2)$, where F is a general fiber of $\Phi|S$. So by (2) : $h^0(S, L_1) \geq 3$.

(4) Here we use the results and notations of Sect. 7. By (7.2) the instability of the conic bundle X fulfills $n(X) \leq \deg \Delta - 2$, hence $n(X) \leq 1$. Thus $n(X) = 1$. Consequently S is $\mathbf{F}_0$ or $\mathbf{F}_1$ blown up in 3 points. In other words, S is $\mathbf{P}_2$ blownup in 4 points. No 3 of them can be collinear, otherwise we would have a section C with $C^2 = -2$. So S is a del Pezzo surface and it follows easily that $L_1|_S = -K_S - L_2$ is nef. The global generatedness can be deduced either directly or by computing Fujita's Δ-genus: $\Delta(S, L_1) \leq 0$ and by applying Fujita's fundamental results. Note that always $\Delta \geq 0$, hence $\Delta(S, L_1) = 0$, which gives already the first claim of (5).

(5) Use the exact sequence

$$0 \to H^0(S, L_1 - L_2) \to H^0(S, L_1) \to H^0(F, L_1)$$

with F a general fiber of Φ. Then (4) together with $L_1|F = \mathcal{O}(2)$ gives the claim. □

Lemma 5.4.

(1) $H^2(S, L_1 + \mu L_2) = 0$ *for all* $\mu \in \mathbf{Z}$.

(2) $H^1(S, L_1 + \mu L_2) = 0$ *for all* $\mu \geq -1$.

(3) $H^1(X, L_1 + \mu L_2) = 0$ *for all* $\mu \geq -2$.

Proof. (1) $H^2(S, L_1 + \mu L_2) = H^0(S, -2L_1 - (\mu + 1)L_2) = 0$ for all μ, since $L_1|F$ is positive for a general fiber F of Φ.

(2) Now let $\mu \geq -1$. From the exact sequence

$$0 \to (L_1 + \mu L_2)|S \to (L_1 + (\mu + 1)L_2)|S \to L_1|F \to 0$$

we see that it is sufficient to show surjectivity of $H^0(S, L_1) \to H^0(L_1|F)$. But this was already proved in 5.3 (5).

(3) Now use the exact sequence on X

$$0 \to L_1 + \mu L_2 \to L_1 + (\mu+1)L_2 \to (L_1 + (\mu+1)L_2)|S \to 0.$$

By (1) and (2) we get for $\mu \geq -2$

$$H^1(X, L_1 + \mu L_2) \simeq H^1(X, L_1 + (\mu+1)L_2). \quad \square$$

Since $H^1(X, L_1 + \mu L_2) \cong H^1(\mathbf{P}_2, \Phi_*(L_1) \otimes \mathcal{O}_{P_2}(\mu)) = 0$ for $\mu \gg 0$, we conclude.

Proof of Proposition 5.1. By Riemann-Roch and our assumptions : $\chi(X, L_1) = \chi(X_0, L_1) = 3$, so from 5.4 (3) we obtain $h^0(X, L_1) \geq 3$. Since $h^0(S, L_1 - L_2) = 0$, we conclude :

$$H^0(X, L_1 - L_2) = 0,$$

hence the restriction $H^0(X, L_1) \xrightarrow{r} H^0(S, L_1)$ is injective. Since $h^0(S, L_1) = 3$, we conclude $h^0(X, L_1) = 3$, so r is an isomorphism. But this implies that L_1 is nef : assume that there is a curve $C \subset X$ with $(L_1.C) < 0$. Then for generic $l \subset \mathbf{P}_2 : C \cap S_l = \emptyset$, since otherwise we would find $s \in H^0(X, L_1)$ such that $s|C \neq 0$ (use 5.3 (4) and the fact that r is an isomorphism). Thus $\Phi(C) \cap l = \emptyset$ which is absurd. Now L_1 being nef, $-K_X = L_1 + 2L_2$ is ample as sum of two nef line bundles generating $Pic(X)$. So X is Fano and consequently $X \simeq X_0$ by Iskovskih's classification. $\square$

6. Moishezon twistor spaces are not topologically projective

For X a compact complex manifold, let $w_2(X) \in H^2(X, \mathbf{Z}/2\mathbf{Z})$ be its second Stiefel-Whitney class, whose vanishing means that K_X is divisible by two in $Pic(X)$.

Theorem 6.1. *Let X be a projective threefold. Then : $b_1(X) = b_3(X) = w_2(X) = 0$ iff X is one of the following :*

i) *Fano with $b_2 = 1$, of index $r = 2$ or 4 (in this last case, $X = \mathbf{P}_3$),*
ii) *a $\mathbf{P}_1$-bundle $\mathbf{P}(V)$ over a surface S with $b_1(S) = 0$, with V a 2-bundle over S such that $(detV + K_S)$ is divisible by 2 in $Pic(S)$.*
iii) *obtained from the above manifolds by blowing-up finitely many points.*

Remarks. **1.** It is obvious that the conditions $b_1 = b_3 = w_2 = 0$ are necessary to belong to the above classes.

2. If one only assumes that X has at most terminal singularities, and that $b_1 = b_3 = 0$, it is still true that X is uniruled.

Proof. We have : $h^{1,0} = h^{3,0} = 0$, hence : $\chi(\mathcal{O}_X) = 1 + h^{2,0} \geq 1$. Thus : K_X is not nef (2.2). Let $\varphi : X \to Y$ be the contraction of an extremal ray in X. By Mori's list and because $K_X = 2L, L \in Pic(X)$, we see that if φ is a modification, it has to be the contraction of a smooth divisor E of X, E isomorphic to $\mathbf{P}_2$, with normal bundle $E_{|E} \cong \mathcal{O}_E(1)$ (because in all other cases, a curve $C \subset E$ exists such that : $(-K_X.C) = 1$, contradicting $w_2 = 0$). Thus : Y is smooth and satisfies the same conditions : $b_1 = b_3 = w_2 = 0$ as X. We can thus assume that $\dim(Y) \leq 2$.

Assume first that $Y = S$ is a surface ; then Y is smooth, and φ can't be a conic bundle, otherwise a curve C exists, which is contained in a fiber of φ such that $(-K_X.C) = 1$, again contradicting $w_2 = 0$. Hence φ is a $\mathbf{P}_1$-bundle, and $b_1(S) = b_1(X) = 0$. Moreover, $K_X = \mathcal{O}_{\mathbf{P}(V)}(-2) + \varphi^*(detV + K_S)$, if $X = \mathbf{P}(V)$ for V a rank 2 bundle over S, so we are in case (ii). Assume now that $Y = C$ is a curve. Let F be a smooth fiber of φ ; then F is a minimal Del Pezzo surface, otherwise, an exeptional curve of the first kind C_0 on F would satisfy: $1 = (-K_F.C_0) = (-K_X.C_0)$, contradicting : $w_2 = 0$. Thus F is either $\mathbf{P}_2$ or $\mathbf{P}_1 \times \mathbf{P}_1$. The case $F = \mathbf{P}_2$ is again excluded, since : $-K_{X|F} = -K_F = \mathcal{O}_{\mathbf{P}}(3)$ in this case. The case $F = \mathbf{P}_1 \times \mathbf{P}_1$ is also excluded by the proposition below.

The last possible case is : $\dim Y = 0$, so X is Fano with $b_2(X) = 1$, and $r = 2, 4$ since $w_2 = 0$. □

Proposition 6.2. *There is no quadric bundle* $\varphi : X \to C \cong \mathbf{P}_1(\mathbf{C})$ *with* $2 = b_2(X)$; $b_3(X) = w_2(X) = 0$.

Proof. If φ were smooth, we would have $b_2(X) = 3$. The set Δ of singular fibers of φ, which are isomorphic to the quadric cone in $\mathbf{P}_3$ after [Mo] is thus nonempty.

Since :

$$\chi(X) = \chi(C).\chi(F) + \sum_{c\in\Delta}(\chi(X_c) - \chi(F))$$

where χ is the topological Euler-Poincaré characteristic, $F = \mathbf{P}_1 \times \mathbf{P}_1$, and $X_c := \varphi^{-1}(c)$, we get from

$$\chi(X_c) = 3, \chi(F) = 4, \chi(X) = 6,$$

that δ consists of exactly two points. □

On the other hand, we can embed X in a $\mathbf{P}_3$-bundle $P := \mathbf{P}(E^*)$, where E^* is a 4-bundle on C normalised in such a way that $X \in |2L|$, with $L = \mathcal{O}_P(1)$.

Let $c_1 \in \mathbf{Z}$ be the degree of E. We have a quadrilinear symmetric map $\Psi : S^2(E) \to S^2(detE)$ which sends any quadratic form B on E to its discriminant. X is the zero locus of some $s \in H^\circ(P, 2L)$, and let $\sigma := \Psi \circ s \in H^\circ(C, S^2(detE)) = H^\circ(C, \mathcal{L})$, where $\mathcal{L}$ has degree $2c_1$. Then we conclude $c_1 = 1$ since $\{\sigma = 0\} = \Delta$. We now compute : $K_X = (K_p + 2L)_{|X} = (-4L + \varphi^*(c_1 - 2))_{|X}$, and so $w_2(X) \neq 0$ since c_1 is odd. (Here : $Pic(C)$ is identified with $\mathbf{Z}$ in the usual way).

Corollary 6.3. *Let M^4 be a compact connected anti-self dual Riemannian fourfold, and let $\tau : Z \to M^4$ be its twistor space ([AHS]).*

Assume that Z is Moishezon, but not projective. Then there is no projective threefold Z_0 which is homeomorphic to Z if $n \geq 3$ is even, with $n = b_2(Z) - 1$.

Probably this remains true if n is odd, too. This answers a question (3.15) asked in [C2].

Remarks. Recall that $\tau : Z \to M^4$ is a differentiable (non holomorphic) submersion whose fibers are holomorphic rational curves on Z with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and that $w_2(Z) = 0$. Recall that if Z is Moishezon, it is "almost Fano", ie : the Kodaira dimension of K_Z^{-1} is 3. ([P],[V]).

It is shown in [C] that M^4 is homeomorphic to either S^4 or the connected sum $\sharp n\mathbf{P}_2(\mathbf{C})$ of n copies of $\mathbf{P}_2(\mathbf{C})$ if Z is Moishezon. It is shown in [H] that if Z is projective, it is either $\mathbf{P}_3(\mathbf{C})$ or $\mathbf{P}(T_{\mathbf{P}_2(\mathbf{C})})$, with M^4 respectively S^4 or $\mathbf{P}_2(\mathbf{C})$ with metrics conformal to the usual ones. Examples with arbitrary n are known to exist ([*P2*] : $n = 2$; [*K*] : $n = 3$; [*L*] all n). It is shown in [C2], [L2] that small generic deformations of Kurke-Lebrun's examples are not in the class $\mathcal{C}$, thus showing that Kodaira-Spencer stability theorem is not true in the class of compact manifolds bimeromorphic to Kähler ones. The above corollary thus exhibits another difference between these Z and projective manifolds.

Proof. Let $M = M^4$, thus M is topologically $\sharp n\mathbf{P}_2(\mathbf{C})$, with $n \geq 2$. We describe $H^2(X, \mathbf{Z})$ together with its bilinear intersection form. Let $(\alpha_1, ..., \alpha_n)$ be an orthogonal basis of $H^2(M, \mathbf{Z})$ (ie : $\alpha_i\alpha_j = 0$ if $i \neq j$, $\alpha_i^2 = 1$). We identify α_i and $\tau^*\alpha_i$. Let $\tilde{c} = \frac{1}{2}c_1(Z)$. A $\mathbf{Z}$-basis of $H^2(Z, \mathbf{Z})$ is then : $(c, \alpha_1, ..., \alpha_n)$ where : $c = \frac{1}{2}(\tilde{c} + \alpha_1 + ... + \alpha_n)$, which is integral (see [P3]).

The intersection form is defined by :

$$\tilde{c}^3 = 2(4-n); \tilde{c}^2.\alpha_i = 0; \tilde{c}.\alpha_i^2 = -2 \quad \text{for all } i, \text{ so}$$
$$c^3 = 1-n; c^2.\alpha_i = -1; c.\alpha_i^2 = -1 \quad \text{for all } i. \quad \square$$

We now assume that Z_0 is a projective threefold homeomorphic to Z.

Lemma 6.4. *Z_0 is not blow-up in a point of any smooth projective threefold Z_1.*

Proof. Otherwise there would exist E and $L \neq 0$ in $Pic(Z_0)$ such that : $E^3 = 1, E^2.L = E.L^2 = 0$ (just take the class E of the exceptional divisor of the blow-up, and the class L of the lifting of any ample line bundle on Z_1).

However, a direct computation shows that the equations :

$$(\epsilon c + \epsilon_1\alpha_1 + ... + \epsilon_n\alpha_n)^3 = 1 = \epsilon[\epsilon^2(1-n) - 3(\sum \epsilon_i^2 + \epsilon(\sum \epsilon_i))]$$

have no integer solutions $(\epsilon, \epsilon_i), (\lambda, \lambda_i)$ if $n \geq 3$. $\square$

Lemma 6.5. *Z_0 is not a $\mathbf{P}_1$-bundle over any algebraic surface S.*

Proof. Let $\varphi_0 : Z_0 \to S$ be any such $\mathbf{P}_1$-bundle structure. Then $(\varphi_0)^*(H^2(S, \mathbf{Z}))$ generates a sublattice of rank n in $H^2(Z_0, \mathbf{Z})$ (which has rank $(n+1)$), and consisting of classes L such that : $L^3 = 0$. Now, if

$$L = \lambda c_1 + \lambda_1\alpha_1 + ... + \lambda_n\alpha_n,$$

one has :

$$L^3 = \lambda[\lambda^2(1-n) - 3(\sum \lambda_i^2 + \lambda\lambda_i)] = \lambda.Q(\lambda, \lambda_i),$$

where Q is a definite negative quadratic form on $\mathbf{R}^{n+1}$. Thus $\varphi_0^*(H^2(S, \mathbf{Z})) = \tau^* H^2(M^4, \mathbf{Z})$. But this shows that the intersection form on S would be definite of rank $n \geq 2$, which is impossible if n is even by Hodge index theorem (which forces $h^{1,1}(S) = 1$). □

(6.4) and (6.5) imply now together with theorem (6.1) that Z_0 has $b_2 = 1$, contradiction.

7. A bound for the degree of instability of a conic bundle

Definition and Construction 7.1 (1) Let S be a smooth rational surface with a surjective holomorphic map $\phi : S \to \mathbf{P}_1$. Let $C \subset S$ be a section of ϕ. C is said to be **minimal** if its selfintersection number C^2 is minimal with respect to all sections of ϕ. We call

$$n(\phi) = -C^2,$$

where C is minimal, the **degree** of ϕ. Loosely speaking, when it is clear which map ϕ is meant, we put $n(S) = n(\phi)$.

(2) Let $\Phi : X \to \mathbf{P}_2$ be a proper conic bundle, i.e. the discriminant locus $\Delta \subset \mathbf{P}_2$ is not empty. Let d be the degree of Δ which number we also call the degree of the conic bundle Φ. Let $G = \mathbf{P}_2^*$ be the variety of lines in $\mathbf{P}_2$. Let G^* be the Zariski open set in G consisting of those lines which meet Δ in d distinct points tranversely. Then for $l \in G^*$, the surface $S_l = \Phi^{-1}(l)$ is a smooth surface and in fact a Hirzebruch surface $\mathbf{F}_k = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ blown up in d points. We denote by $n(l) = n(\Phi|S_l)$ its degree of instability. Finally let $n(X) = n(\Phi)$ be the minimum of all $n(l), l \in G^*$. We call $n(X)$, or better $n(\Phi)$, the **degree of instability** of the conic bundle X.

Our main result in this section is

Theorem 7.2. *Assume that the conic bundle $\Phi : X \to \mathbf{P}_2$ is standard (i.e. $Pic(X) = \mathbf{Z}K_X + \Phi^*(Pic(\mathbf{P}_2))$) and assume moreover that the degree of Φ is d. Then*

$$n(X) \leq d - 2,$$

in particular $n(X)$ is finite.

First let us show the following

Proposition 7.3. *Let $\pi : S_0 \to \mathbf{P}_1$ be a ruled surface, i.e. a $\mathbf{P}_1$-bundle over $\mathbf{P}_1$. Let $\sigma : S \to S_0$ be the blow-up of $b \geq 3$ distinct points on S_0. Let n be the degree of instability of $S \to \mathbf{P}_1$. Assume that $n \geq b - 1$. Then there exists a unique minimal section of S.*

Proof. Write $S_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-\nu))$ with $\nu \geq 0$. Then ν is the degree of instability of S_0. Let C_0 be a minimal section of S_0; so $C_0^2 = -\nu$. If $\nu > 0$, then C_0 is unique.

Assume first $\nu \geq 2$. Then we claim that the strict transform $\overline{C_0}$ of C_0 in S is the unique minimal section of S. In fact, take a section C of S_0 such that the strict transform $\overline{C}$ is minimal and assume of course that $C \neq C_0$, if also C_0 is minimal. Since

$$C^2 \geq \nu$$

by the elementary theory of ruled surfaces, we have for the strict transform

$$\overline{C} = -n \geq \nu - b,$$

hence $n \leq b - \nu \leq n + 1 - \nu$ by our assumption. This contradicts $\nu \geq 2$ and settles the proposition in this case.

In case $\nu \geq 1$ we see by the same construction, that we must have

$$C^2 = 1, \overline{C}^2 = \nu - b = 1 - b,$$

i.e. all b points have to be on C, if $C \neq C_0$. Observe that here we must have $C \cap C_0 = \emptyset$, hence none of the points to be blown up is on C_0. Now let $\overline{C'}$ be another minimal section. Then by the same reasoning as for C all points to be blown up are on C',too. But this contradicts $C.C' = C^2 = 1$.

It remains to settle the case $\nu = 0$. But this is an obvious exercise. □

Coming back to our conic bundle $\Phi : X \to \mathbf{P}_2$ and to the proof of (7.2), we assume that $n(X) \geq d - 1$. Then by (7.3) there exists for every $l \in G^*$ a unique minimal section C_l of $\Phi_l : S_l \to l$ (observe $d \geq 3$). We want to show that the curves C_l form an algebraic family.

Proposition 7.4. *There exists a unique component T of the Chow scheme of curves in X and a bimeromorphic map $\Phi_* : T \to G$ together with Zariski open sets $T^* \subset T, G^{**} \subset G^*$ such that for all $t \in T^*$:*

$$\{t\} = C_l \text{ with } l = \Phi_*(t),$$

where $\{t\}$ denotes the curve parametrised by t.

Before giving the proof of (7.4) let us first show how (7.2) is proved by means of (7.4). Assume as before that $n(X) \geq d-1$. Fix $a \in \mathbf{P}_2 \setminus \Delta$ and let

$$P_a = \{l \in G | a \in l\}$$

be the pencil of lines through a. Let D be the Zariski closure of $\bigcup C_l$, where l runs over $P_a \cap G^*$.

By (7.4) D is a prime divisor in X such that $\Phi|D : D \to \mathbf{P}_2$ is bimeromorphic. But this divisor is not a linear combination of K_X and $\Phi^*(\mathcal{O}(1))$: intersect with a general fiber of Φ to obtain the contradiction. Hence Φ is not a standard conic bundle, contradicting our assumption.

It remains to give the

Proof of 7.4 . (1) First we compute $(-K_X.C_l)$ for $l \in G^*$. We have an exact sequence, namely the normal bundle sequence for the embeddings $C_l \subset S_l \subset X$:

$$0 \to \mathcal{O}(-n(l)) \to N_{C_l|X} \to \mathcal{O}(1) \to 0.$$

Here $N = N_{C_l|X}$ is the normal bundle of C_l in X. We conclude $c_1(N) = 1 - n(l)$, hence

$$(-K_X.C_l) = 3 - n(l) \qquad (*).$$

(2) Thus the curves C_l form a bounded family and therefore there exists a component T of the Chow scheme containing all C_l for l in some nonempty Zariski open subset U of G^*. We have

$$dimT \leq h^0(N) \leq 2,$$

thus $\dim T = 2$.

(3) For $t \in T$ generic, we let

$$\Phi_*(t) = \Phi(C),$$

where C is the section determined by t. Clearly Φ_* extends to a meromorphic map $T \to G$.

By construction there exists a Zariski open set $G^{**} \subset G^+$ such that $C_l \subset \Phi_*^{-1}(l)$ for $l \in G^{**}$. We have even $C_l = \Phi_*^{-1}(l)$: otherwise we would have some $t \in T$ such that the curve B_t corresponding to t is contained in S_l. But $B_t^2 = -n(l)$ by $(*)$, and because of the fact that $(-K_X.B_t)$ does not depend on t. Hence Φ_* is bimeromorphic. $\square$

Note that $C_l^2 = -n(X)$ for all $l \in G^{**}$.

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