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THE GEOMETRIC SOBOLEV EMBEDDING FOR VECTOR FIELDS AND THE ISOPERIMETRIC INEQUALITY

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1. Introduction and Statement of the Results

The purpose of this paper is to prove an optimal embedding theorem for the Sobolev spaces associated to some general families of vector fields in $\mathbb{R}^n$. A priori, we do not require C^∞ smoothness of the vector fields. In fact, the main point of the paper is to show that the geometric Sobolev embedding, of which all others with optimal exponents are trivial corollaries, is a consequence of two general facts: (i) The doubling condition for the balls in a metric naturally associated to the given fields; (ii) The possibility of representing a function compactly supported in a ball, in terms of a metric fractional integral which involves the (degenerate) gradient associated to the fields. We do not use any interpolation in the course of the proofs, but just a Vitali covering lemma for spaces of homogeneous type which follows from (i). In this sense, the approach in this paper gives a new proof of the embeddings even for the ordinary Sobolev spaces (the constants, however, may not be optimal, for this see [T] or [A]).

Notable examples of systems of vector fields to which our results apply are:

(I) C^∞ vector fields $X_1, \ldots, X_m$ in $\mathbb{R}^n$ satisfying Hörmander's condition on the Lie algebra [H]

$$\operatorname{rank} \operatorname{Lie}[X_1, \ldots, X_m](x) = n \text{ at every } x \in \mathbb{R}^n.$$

(II) Generalized Baouendi–Grushin [B], [Gr] type families

$$X_j = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, k, \quad X_j = (x_1^2 + \cdots + x_k^2)^{\frac{\alpha}{2}} \frac{\partial}{\partial x_j}, \quad j = k+1, \ldots, n.$$

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Here, $1 \le k < n$ and $\alpha > 0$. We emphasize that in this second example the fields may not even be C^1 globally, although they are so outside of a set of n–dimensional measure zero.

We now list our general assumptions. Suppose that a system $X = \{X_1, \ldots, X_m\}$ of continuous vector fields is given in $\mathbb{R}^n$. Assume that associated to X there exists a metric $d_X : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ such that, if $B(x,R) = \{y \in \mathbb{R}^n | d_X(x,y) < R\}$, the following is true:

Assumption A. *Given a bounded set $U \subset \mathbb{R}^n$ there exist positive numbers C_1 and R_0 such that for any $x \in U$ and $0 < R \le R_0$*

$$|B(x, 2R)| \le C_1 |B(x,R)|.$$

In order to state our second hypothesis we consider for a function u its X–gradient given by

$$D_X u = (X_1 u, \ldots, X_m u).$$

Assumption B. *Given a bounded set $U \subset \mathbb{R}^n$ there exist positive numbers C_2 and R_0 such that for any $x \in U$, $B(x,R)$ with $0 < R \le R_0$, and any compactly supported Lipschitz function in $B(x,R)$ one has*

$$|u(y)| \le C_2 \int_{B(x,R)} |D_X u(\xi)| \frac{d_X(y,\xi)}{|B(y, d_X(y,\xi))|} d\xi,$$

for every $y \in B(x,R)$.

We now introduce the relevant family of Sobolev spaces. Given an open set $\Omega \subset \mathbb{R}^n$ and $1 \le p < \infty$ consider the functional

$$J_{p,\Omega}(u) = \int_\Omega |D_X u|^p dx = \int_\Omega \left[\sum_{j=1}^m (X_j u)^2 \right]^{\frac{p}{2}} dx.$$

We denote by $\overset{\circ}{S}{}^{1,p}(\Omega)$ the completion of $C_0^1(\Omega)$ in the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left[J_{p,\Omega}(u) + \|u\|^p_{L^p(\Omega)} \right]^{\frac{1}{p}}.$$

One of the main results in this paper is the following

Theorem 1.1. *Let $U \subset \mathbb{R}^n$ be a bounded set and, with C_1 as in Assumption A, set $Q = \frac{\log C_1}{\log 2}$. There exists a constant $C_3 > 0$ such that for any $x_0 \in U$, $B_R = B(x_0, R)$ with $0 < R \le R_0$, and any $u \in \overset{\circ}{S}{}^{1,1}(B_R)$ one has*

$$\left(\frac{1}{|B_R|}\int_{B_R} |u|^k dx\right)^{\frac{1}{k}} \le C_3 R\left(\frac{1}{|B_R|}\int_{B_R} |D_X u| dx\right),$$

where $1 \le k \le \frac{Q}{Q-1}$.

If one applies Theorem 1.1 to u^α, with $u \in C_0^1(B_R)$, $u \ge 0$, and $\alpha = p(\frac{Q-1}{Q-p})$, one immediately obtains the following

Theorem 1.2. *With U, Q and B_R as in Theorem 1.1, for any $1 \le p < Q$ and $u \in \overset{\circ}{S}{}^{1,p}(B_R)$ one has*

$$\left(\frac{1}{|B_R|}\int_{B_R} |u|^{kp} dx\right)^{\frac{1}{kp}} \le \alpha C_3 R\left(\frac{1}{|B_R|}\int_{B_R} |D_X u|^p dx\right)^{\frac{1}{p}},$$

where $1 \le k \le \frac{Q}{Q-p}$.

Remark. The exponents in Theorem 1.1 and Theorem 1.2 determined by the number Q are best possible. This can be seen by considering the case of left–invariant vector fields on a nilpotent homogeneous group G, see [Fr] and [FS2]. If N denotes the homogeneous dimension of G, then Assumption A holds with $C_1 = 2^N$. One has then $Q = N$. In this case Theorem 1.2 is a scale invariant version of a theorem in [Fr], see also the earlier paper [FS1] concerned with the Heisenberg group. Theorem 1.1, however, is new even in the context of nilpotent, homogeneous Lie groups.

We recall that the classical embedding of Sobolev [S]: $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, with $q = \frac{np}{n-p}$, was originally proved only for $1 < p < n$. It was only in the late fifties that Gagliardo [Ga] and Nirenberg [N] independently proved that $W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ by means of an elegant integral inequality. Meanwhile, in his fundamental paper [DG] De Giorgi had developed his theory of generalized perimeters based on the notion of variation of an L^1 distribution. A few years after De Giorgi's paper, Fleming and Rishel [FR] gave a beautiful geometric new proof of Gagliardo and Nirenberg's result based on Federer's co–area formula and the celebrated isoperimetric inequality [Fe]

$$P(E) \geq n\Gamma(1/2)\Gamma(n/2+1)^{-1/n}|E|^{(n-1)/n},$$

where $P(E)$ denotes the perimeter of E according to De Giorgi.

In our setting, the idea of Gagliardo and Nirenberg does not easily extend.

Our approach has been to first establish the following weak version of Theorem 1.1, whose proof only uses Assumptions A and B.

Theorem 1.3. *Let U, Q and B_R be as in Theorem 1.1. Then, there exists a constant $C_3 > 0$ such that for every Lipschitz function u compactly supported in B_R one has for $\lambda > 0$*

$$\Big|\{x \in B_R|\, |u(x)| > \lambda\}\Big|^{(Q-1)/Q} \leq C_3\lambda^{-1}R|B_R|^{-1/Q}\int_{B_R}|D_X u|dx.$$

We then use Theorem 1.3 to prove the following remarkable

Theorem 1.4 (Isoperimetric inequality). *With U, Q and B_R as in Theorem 1.1, and C_3 as in Theorem 1.3, for every C^1 open set $E \subset \overline{E} \subset B_R$, one has*

$$|E|^{\frac{Q-1}{Q}} \leq C_3 R|B_R|^{-\frac{1}{Q}}P_X(E;B_R),$$

where $P_X(E;B_R)$ denotes the X-perimeter of E in B_R (see Definition 3.3).

Finally, we use Theorem 1.4 to prove Theorem 1.1. In this, we adapt to our setting the beautiful idea of Fleming and Rishel [FR], who deduced the embedding $W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{n/n-1}(\mathbb{R}^n)$ from the classical isoperimetric inequality. The latter, of course, is a direct consequence of the Brunn–Minkowski inequality, see [Fe], but we could not find an analogous direct proof of Theorem 1.4 in our setting. We mention that, for the special case of the Heisenberg group $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$, Pansu [P] had proved an isoperimetric inequality. His method, however, is quite different from that of this paper.

It is worth emphasizing that our approach shows, in particular, that Theorem 1.1 is, in fact, equivalent to either Theorem 1.3 or Theorem 1.4. We mention that in an earlier paper [CDG] we gave a different, direct proof of Theorem 1.2 in the case $1 < p < Q$.

We close this section with some comments concerning Assumptions A and B above. For vector fields of Hörmander type, such as in example (I), Assumption A follows from the important works [FP], [NSW]. Assumption B, instead, was proved in [D], see also [CDG], using the size estimates of the fundamental solution obtained in [NSW], [SC]. For Baouendi–Grushin type vector fields such as in example (II), Assumption A was proved in [FrL], whereas Assumption B follows from [Fr].

2. Proof of Theorem 1.3

This section is devoted to proving the weak estimate in Theorem 1.3. We start with a result about fractional integration in spaces of homogeneous type. In what follows, we assume that a metric $d(x,y)$ is given such that $(\mathbb{R}^n, d, dx)$, where dx is Lebesgue measure, satisfies Assumption A. That is to say, $(\mathbb{R}^n, d, dx)$ is a space of homogeneous type, see [CW]. Given U, C_1 and R_0 as in Assumption A, denote by Q the number $\frac{\log C_1}{\log 2}$. For $0<\alpha<Q$ we define

$$I_\alpha f(x) = \int_{B_R} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi, \quad 0<\alpha<Q.$$

Here $B_R = B(x_0, R)$, with $x_0 \in U$ and $R \le R_0$. We then have the following

Theorem 2.1. *I_α maps continuously $L^1(B_R)$ into $L^{q,\infty}(B_R)$ with $q = \frac{Q}{Q-\alpha}$. Moreover, there exists $C>0$ such that for any $f \in L^1(B_R)$ and $\lambda > 0$*

$$\Big|\{x \in B_R | I_\alpha f(x) > \lambda\}\Big| \le C\lambda^{-q} R^{\alpha q} |B_R|^{1-q} \|f\|^q_{L^1(B_R)}.$$

Proof. We set $f \equiv 0$ in B_R^C. For $x \in B_R$ and $0 < \varepsilon < R$ we write

$$I_\alpha f(x) = I_\alpha^1 f(x) + I_\alpha^2 f(x),$$

where

$$I_\alpha^1 f(x) = \int_{B(x,\varepsilon)} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi,$$
$$I_\alpha^2 f(x) = \int_{B(x,\varepsilon)^C \cap B_R} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi.$$

We claim the existence of constants C_4, $C_5 > 0$ such that

$$I^1_\alpha f(x) \le C_4 Mf(x)\varepsilon^\alpha \tag{2.2}$$

$$I^2_\alpha f(x) \le C_5 R^Q |B_R|^{-1} \varepsilon^{\alpha-Q} \|f\|_{L^1(B_R)}, \tag{2.3}$$

where in (2.2) we have let

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(\xi)| d\xi,$$

the Hardy–Littlewood maximal operator with respect to the metric balls. Assuming for the moment the estimates (2.2), (2.3), we have for any $\lambda > 0$

$$\Big|\{x \in B_R | I_\alpha f(x) > 2\lambda\}\Big| \le \Big|\{x \in B_R | I^1_\alpha f(x) > \lambda\}\Big| + \Big|\{x \in B_R | I^2_\alpha f(x) > \lambda\}\Big|. \tag{2.4}$$

From (2.2) we obtain

$$\Big|\{x \in B_R | I^1_\alpha f(x) > \lambda\}\Big| \le \Big|\Big\{x \in B_R | Mf(x) > \frac{\lambda}{C_4 \varepsilon^\alpha}\Big\}\Big|. \tag{2.5}$$

At this point we observe that, because of Assumption A, a Vitali type covering lemma holds for the metric balls $B(x,R)$, see [CW]. By means of the latter one can easily prove, following the argument in [St] for the ordinary Hardy–Littlewood maximal function, the weak–L^1 continuity of Mf. One thus has for (2.5)

$$\Big|\{x \in B_R | I^1_\alpha f(x) > \lambda\}\Big| \le \frac{C\varepsilon^\alpha}{\lambda} \|f\|_{L^1(B_R)}. \tag{2.6}$$

To estimate the second term in the right–hand side of (2.4) we distinguish two cases. Suppose first that

$$\lambda > C_5 R^\alpha |B_R|^{-1} \|f\|_{L^1(B_R)},$$

with C_5 as in (2.3). If we choose

$$\varepsilon = \Big[C_5 R^Q |B_R|^{-1} \|f\|_{L^1(B_R)} \lambda^{-1}\Big]^{1/(Q-\alpha)}, \tag{2.7}$$

then we have $0 < \varepsilon < R$ and (2.3) implies $\|I_\alpha^2 f\|_{L^\infty(B_R)} \leq \lambda$. We conclude

$$\Big|\{x \in B_R | I_\alpha^2 f(x) > \lambda\}\Big| = 0.$$

From (2.4), (2.6), (2.7) we infer

$$\begin{aligned}\Big|\{x \in B_R | I_\alpha f(x) > 2\lambda\Big| &= \Big|\{x \in B_R | I_\alpha^1 f(x) > \lambda\}\Big| \\ &\leq C\lambda^{-1}\|f\|_{L^1(B_R)}\Big[C_5 R^Q |B_R|^{-1}\|f\|_{L^1(B_R)}\lambda^{-1}\Big]^{1/(Q-\alpha)} \\ &= C\lambda^{-q}R^{\alpha q}|B_R|^{1-q}\|f\|^q_{L^1(B_R)},\end{aligned}$$

with $q = Q/(Q-\alpha)$. If, instead,

$$0 < \lambda \leq C_5 R^\alpha |B_R|^{-1}\|f\|_{L^1(B_R)},$$

the desired estimate follows trivially, since

$$\Big|\{x \in B_R | I_\alpha f(x) > 2\lambda\}\Big| \leq |B_R| \leq C_5^q \lambda^{-q}|B_R|^{1-q}R^{\alpha q}\|f\|^q_{L^1(B_R)}.$$

In order to complete the proof of the theorem we are thus left with proving (2.2), (2.3). Using Assumption A we obtain

$$\begin{aligned}I_\alpha^1 f(x) &= \sum_{k=0}^\infty \int_{\{2^{-(k+1)}\varepsilon \leq d(x,\xi) < 2^{-k}\varepsilon\}} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi \\ &\leq \sum_{k=0}^\infty (2^{-k}\varepsilon)^\alpha |B(x, 2^{-(k+1)}\varepsilon)|^{-1} \int_{B(x,2^{-k}\varepsilon)} |f(\xi)| d\xi \\ &\leq C\sum_{k=0}^\infty (2^{-k}\varepsilon)^\alpha |B(x, 2^{-k}\varepsilon)|^{-1} \int_{B(x,2^{-k}\varepsilon)} |f(\xi)| d\xi \\ &\leq C_4 Mf(x)\varepsilon^\alpha.\end{aligned}$$

This proves (2.2). Next, for $x \in B_R$ the triangle inequality gives $B_R \subset B(x, 2R)$. Then

$$I_\alpha^2 f(x) \leq \int_{B(x,\varepsilon)^C \cap B(x,2R)} |f(\xi)| \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} d\xi. \tag{2.8}$$

Assumption A implies for any $\xi \in B(x, 2R)$

$$|B(x, d(x,\xi))| \geq C\Big(\frac{d(x,\xi)}{R}\Big)^Q |B(x, 2R)|.$$

From this estimate we infer for some $C_5 > 0$

$$\sup_{\xi\in B(x,\varepsilon)^C\cap B(x,2R)} \frac{d(x,\xi)^\alpha}{|B(x,d(x,\xi))|} \leq C_5\frac{R^Q}{|B(x,2R)|}\varepsilon^{\alpha-Q} \leq C_5\frac{R^Q}{|B_R|}\varepsilon^{\alpha-Q}.$$

Substitution in (2.8) immediately yields (2.3). This completes the proof. □

Remark. In [CDG] we proved that when $1 < p \leq \infty$ and $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{Q}$, then $I_\alpha : L^p(B_R) \to L^q(B_R)$ continuously.

Next, we give the

Proof of Theorem 1.3. By Assumption B there exists a constant C_2 such that for any compactly supported, Lipschitz function in $B_R = B(x_0, R)$ one has

$$|u(x)| \leq C_2 I_1(|D_X u|)(x), \qquad x \in B_R.$$

Applying Theorem 2.1 with $\alpha = 1$ we immediately reach the desired conclusion. □

Remark. For smooth vector fields satisfying Hörmander's condition, Assumption B follows from the following identity, see [D], valid for any bounded open set $\Omega \subset \mathbb{R}^n$ and any $u \in \mathrm{Lip}(\Omega)$ with compact support in Ω

$$u(x) = \int_\Omega \langle D_X u(\xi), D_X\Gamma(x,\xi)\rangle d\xi, \qquad x \in \Omega. \tag{2.9}$$

Here, $\Gamma(x,\xi)$ is the fundamental solution of the operator $\mathcal{L} = \sum_{j=1}^m X_j^* X_j$, where X_j^* denotes the formal adjoint of X_j. The identity (2.9) and the estimate in [NSW], [SC]

$$|D_X\Gamma(x,\xi)| \leq C\frac{d(x,\xi)}{|B(x,d(x,\xi))|},$$

where now $d(x,\xi) = d_X(x,\xi)$ is the Carnot–Carathéodory distance associated to the system $X = (X_1, \ldots, X_m)$, yield Assumption B. For non–smooth vector fields of the type (II), assumption B was proved in [Fr] using the results in [FrL].

3. The Isoperimetric Inequality

The aim of this section is to prove Theorem 1.4. We begin by introducing a suitable notion of variation of an L^1 distribution which generalizes that originally due to De Giorgi [DG], see also [G] and the references therein. In this section we assume that the system $X = (X_1, \dots, X_m)$ is constituted by continuous vector fields, which, furthermore, are C^1 outside of a set $Z \subset \mathbb{R}^n$ of zero n–dimensional Lebesgue measure. This assumption is trivially satisfied by systems of Hörmander type (I), or of type (II). Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $\mathcal{F}$ the class of functions $\varphi = (\varphi_1, \dots, \varphi_m) \in C_0^1(\Omega)^m$ such that $\|\varphi\|_\infty = \sup_{x\in\Omega}(\sum_{j=1}^m |\varphi_j(x)|^2)^{1/2} \leq 1$.

Definition 3.1. Let $u \in L^1(\Omega)$. We define the *X–variation* of u in Ω as

$$\mathrm{Var}_X(u;\Omega) = \sup_{\varphi\in\mathcal{F}} \int_\Omega u \sum_{j=1}^m X_j^* \varphi_j dx,$$

where X_j^* denotes the formal adjoint of X_j.

Remark. When $X_j = \frac{\partial}{\partial x_j}$ and $m = n$, the X–variation of u coincides with that in [DG], [FR].

Remark. When $u \in C^1(\Omega)$ we have

$$\mathrm{Var}_X(u;\Omega) = \int_\Omega |D_X u| dx.$$

This follows by an integration by parts in the integral defining $\mathrm{Var}_X(u;\Omega)$.

We denote by $BV_X(\Omega)$ the space of all functions $u \in L^1(\Omega)$ such that $\mathrm{Var}_X(u;\Omega) < \infty$. $BV_X(\Omega)$ becomes a Banach space when endowed with its natural norm

$$\|u\|_{BV_X(\Omega)} = \|u\|_{L^1(\Omega)} + \mathrm{Var}_X(u;\Omega).$$

Remark. Let $E \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Denote by χ_E the characteristic function of E. Then one has

$$\mathrm{Var}_X(\chi_E;\Omega) = \int_{\partial E\cap\Omega} \left[\sum_{j=1}^m \langle X_j, \eta\rangle^2\right]^{1/2} dH_{n-1}, \tag{3.2}$$

where η denotes the outward unit normal to ∂E and dH_{n-1} the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. To prove (3.2) let $X_j = \sum_{k=1}^n b_{jk} \frac{\partial}{\partial x_k}$, for $j = 1, \ldots, m$, so that $X_j^* = -\sum_{k=1}^m \frac{\partial}{\partial x_k}(b_{jk} \cdot)$. For $\varphi \in \mathcal{F}$ we thus have

$$\begin{aligned}\int_\Omega \chi_E \sum_{j=1}^m X_j^* \varphi_j dx &= -\int_{E\cap\Omega} \sum_{j=1}^m \sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{jk}\varphi_j) dx \\ &= -\int_{\partial E\cap\Omega} \sum_{j=1}^m \sum_{k=1}^n b_{jk}\eta_k \varphi_j dH_{n-1} \\ &= -\int_{\partial E\cap\Omega} \sum_{j=1}^m \langle X_j, \eta\rangle \varphi_j dH_{n-1}.\end{aligned}$$

Taking the supremum in $\varphi \in \mathcal{F}$ in the above chain of equalities we obtain (3.2).

DEFINITION 3.3. Given a measurable set $E \subset \mathbb{R}^n$ we define the *X–perimeter* of E in Ω, and denote it by $P_X(E;\Omega)$, as follows

$$P_X(E;\Omega) = \mathrm{Var}_X(\chi_E;\Omega).$$

We are now ready to give the

Proof of Theorem 1.4. Let $r(x) = \inf_{y\in\overline{E}} |x-y| = \mathrm{dist}(x, \overline{E})$. Set $\delta_0 = \frac{1}{2}\mathrm{dist}(\overline{E}, \partial B_R)$ and for $0 < \delta \le \delta_0$ consider the function $u_\delta(x) = (1 - \frac{r(x)}{\delta})^+$. Clearly, u_δ is a Lipschitz function with compact support in B_R, and $u_\delta \equiv 1$ on $\overline{E}$. For $\lambda \in (0,1)$ we apply Theorem 1.3 to u_δ obtaining

$$\begin{aligned}|E|^{Q-1/Q} &\le \Big|\{x \in B_R | u_\delta(x) > \lambda\}\Big|^{Q-1/Q} \\ &\le C_3 \lambda^{-1} R |B_R|^{-1/Q} \int_{B_R} |D_X u_\delta| dx \\ &= C_3 \lambda^{-1} R |B_R|^{-1/Q} \delta^{-1} \int_{A_\delta} |D_X r| dx,\end{aligned}$$

where we have let $A_\delta = \{x \in B_R | 0 < r(x) < \delta\}$. Letting $\lambda \to 1$ in the above inequality and applying Federer's co–area formula [Fe, Theorem 3.2.3] to the integral on A_δ we obtain

$$(3.4)\ |E|^{Q-1/Q} \le C_3 R |B_R|^{-1/Q} \delta^{-1} \int_0^\delta \int_{\{x\in B_R | r(x) = t\}} |D_X r|\, |\nabla r|^{-1} dH_{n-1} dt,$$

where $|\nabla r|$ denotes the Euclidean length of the ordinary gradient of the function r. We remark that the application of the co–area formula is justified by the fact that for any bounded set $U \subset \mathbb{R}^n$ there exists a constant $C > 0$, depending on $X = (X_1, \ldots, X_m)$ and U, such that for any Lipschitz function on U one has: $|D_X\varphi| \leq C|\nabla\varphi|$. At this point we observe that on the set $\{x \in B_R | r(x) = t\}$ we have

$$|D_X r|\,|\nabla r|^{-1} = \left[\sum_{j=1}^{m}\langle X_j, \nabla r|\nabla r|^{-1}\rangle^2\right]^{1/2} = \left[\sum_{j=1}^{m}\langle X_j, \eta\rangle^2\right]^{1/2},$$

where η is the outward unit normal to the level $\{x \in B_R | r(x) = t\}$. If we let $\delta \to 0$ in (3.4) we thus obtain

$$|E|^{Q-1/Q} \leq C_3 R|B_R|^{-1/Q} \int_{\partial E}\left[\sum_{j=1}^{m}\langle X_j, \eta\rangle^2\right]^{1/2} dH_{n-1}.$$

Since E has C^1 boundary by (3.2) and Definition 3.3 we reach the conclusion. □

4. Proof of Theorem 1.1

In this section we prove that Theorem 1.4, which we deduced from the weak embedding Theorem 1.3, in fact implies Theorem 1.1, the strong embedding. In this last part we closely follow the beautiful approach of Fleming and Rishel [FR]. We start with $u \in C_0^1(B_R)$, $u \geq 0$. By Sard's theorem for a.e. $t > 0$, $t \leq \max_{B_R} u$, the set $E_t = \{x \in B_R | u(x) > t\}$ is a compact subdomain of B_R with C^1 boundary. Federer's co–area formula gives

$$\begin{aligned}
\text{(4.1)} \qquad \int_{B_R} |D_X u| dx &= \int_0^\infty \int_{\partial E_t} |D_X u|\,|\nabla u|^{-1} dH_{n-1} dt \\
&= \int_0^\infty \int_{\partial E_t} \left[\sum_{j=1}^{m}\langle X_j, \eta\rangle^2\right]^{1/2} dH_{n-1} dt \\
&= \int_0^\infty \mathrm{Var}_X(\chi_{E_t}; B_R) dt = \int_0^\infty P_X(E_t; B_R) dt,
\end{aligned}$$

where in the second to the last equality we have used (3.2). Theorem 1.4 now gives

$$P_X(E_t; B_R) \geq C_3^{-1} R^{-1} |B_R|^{1/Q} |E_t|^{(Q-1)/Q}.$$

Inserting this inequality in (4.1) we obtain

$$\int_{B_R} |D_X u| dx \geq C_3^{-1} R^{-1} |B_R|^{1/Q} \int_0^\infty |E_t|^{(Q-1)/Q} dt. \tag{4.2}$$

On the other hand, one has

$$\int_{B_R} |u|^{Q/(Q-1)} dx = (Q(Q-1) \int_0^\infty t^{1/(Q-1)} |E_t| dt. \tag{4.3}$$

We now observe that

$$(Q/(Q-1)) \int_0^\infty t^{1/(Q-1)} |E_t| dt \leq \left(\int_0^\infty |E_t|^{(Q-1)/Q} dt \right)^{Q(Q-1)} \tag{4.4}$$

The proof of (4.4) easily follows from the fact that $V(t) = |E_t|$ is a decreasing function. Putting (4.2)–(4.4) together we finally obtain for any $u \in C_0^1(B_R)$

$$\left(\frac{1}{|B_R|} \int_{B_R} |u|^{Q/(Q-1)} dx \right)^{(Q-1)/Q} \leq C_3 R \left(\frac{1}{|B_R|} \int_{B_R} |D_X u| dx \right).$$

By density, the same estimate holds for $u \in \overset{\circ}{S}{}^{1,1}(B_R)$. This completes the proof of Theorem 1.1. □

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