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THE K-ENERGY ON HYPERSURFACES AND STABILITY

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Introduction

The notion of stability for a polarized projective variety was introduced by D. Mumford for the study of the moduli problem of projective varieties. The stability has been verified by Mumford for smooth algebraic curves, D. Gieseker for algebraic surfaces and Viehweg for algebraic manifolds, which are polarized by m-pluri-canonical bundles for m sufficiently large ([Md], [Gi], [Vi]). However, it still seems to be a challenging problem to check the stability for a given polarized variety, even if the variety is a singular hypersurface in some projective space. The purpose of this paper is to give a sufficient and intrinsic condition for a hypersurface to be stable or semistable. The condition is given in terms of the properness or lower buundedness of a generalized K-energy, which was introduced by T. Mubachi for Kähler manifolds. In particular, we will prove that any hypersurface is semistable if it has only orbifold singularities of codimension at least two and admits a Kähler-Einstein orbifold metric.

We denote by $R_{n,d}$ the space of all homogeneous polynomials on $\mathbb{C}^{n+2}$ of degree d, and B the projective space $PR_{n,d}$. Any point $[f]$ in B determinates a unique hypersurface Σ_f in $\mathbb{C}P^{n+1}$ of degree d. The special linear group $G = SL(n+2, \mathbb{C})$ induces an action on the vector space $R_{n,d}$ by assigning f to $f \circ \sigma^{-1}$ for any σ in G. Then we say that Σ_f is stable if the orbit Gf is closed and the stablier of f in G is finite; we say that Σ_f is semistable if the zero in $R_{n,d}$ is not contained in the closure of the orbit Gf. It is well-known that any smooth hypersurface Σ_f is stable. However, if Σ_f has only one isolated

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singularity, then it may not be semistable. The simplest examples are those cubic surfaces in $\mathbb{C}P^3$ with one singularity of type other than A_1 or A_2.

The K-energy is a functional on the space of admissible Kähler metrics in the Kähler class given by the polarization. It is in fact a Donaldson functional on a "virtual" holomorphic bundle and is defined in terms of the Bott-Chern class associated to the invariant polynomial Ch_{n+1} defining the (n+1)-th Chern Charactor (cf. section 3). Here is our first theorem,

Theorem 0.1. *Let Σ_f be a hypersurface in $\mathbb{C}P^{n+1}$, then Σ_f is stable if the K-energy is proper* (see Definition 5.1)*, and Σ_f is semistable if the K-energy is bounded from below.*

In fact, the proof here yields a stronger result which can be described as follows: let g_{FS} be the Fubini-Study metric on $\mathbb{C}P^{n+1}$, for any σ in G, $\sigma^* g_{FS}$ restricts to an admissible Kähler metric on Σ_f, we denote by Gg_{FS} the set of such admissible Kähler metrics, then Σ_f is stable if the K-energy is proper on Gg_{FS}, and Σ_f is semistable if the K-energy is bounded from below on Gg_{FS}.

An easy corollary of Theorem 0.1 is the following

Theorem 0.2. *Let Σ_f be a normal hypersurface in $\mathbb{C}P^{n+1}$ of degree d. Then*

(1) *if $3 \le d \le n+1$ and Σ_f has only orbifold singularities and admits a Kähler-Einstein orbifold metric, then Σ_f is semistable;*
(2) *if $d \ge n+2$ and Σ_f has only log-terminal singularities, then Σ_f is stable.*

The definition of a log-terminal singularity will be given in section 5. After some preparations, we give the proof for Theorem 0.1 in section 4 and for Theorem 0.2 in section 5. In section 6, we will briefly discuss some generalizations of above theorems without proof. In particular, Theorem 0.1, 0.2 will still be true for complete intersections in projective spaces. The details of these generalizations will appear in a forthcoming paper [T1], where we will deal with subvarieties in $\mathbb{C}P^N$ with canonical or anti-canonical polarization and higher codimensions.

It was S.T.Yau who brought the stability to my attention more than six years ago. I would like to thank him for sharing his insight that there must be a

connection between the existence of Kähler-Einstein metrics and the stability, analogous to the celebrated theorem that the existence of a Hermitian-Yang-Mills metric is equivalent to the stability on an irreducible holomorphic vector bundle (cf. [Do], [UY]).

1. Singular hermitian metrics on relative canonical bundles

As defined in last section, $R_{n,d}$ is the space of all homogeneous polynomials on $\mathbb{C}^{n+2}$ of degree d, and $B = PR_{n,d}$. We put

$$\mathcal{X} = B \times \mathbb{C}P^{n+1}$$

and

$$\Sigma = \{([f];\, x) \in \mathcal{X} \mid f(x) = 0\}$$

It is easy to show that Σ is a smooth hypersurface in $\mathcal{X}$. We also define a subvariety

$$\Sigma_s = \left\{([f];\, x) \in \mathcal{X} | \frac{\partial f}{\partial z_i}(x) = 0\,,\ i = 0, 1, \dots, n+1\right\}$$

where $z_0, \cdots z_{n+1}$ are homogeneous coordinates of $\mathbb{C}P^{n+1}$. Clearly, $\Sigma_s \subset \Sigma$.

Lemma 1.1. *Σ_s is smooth and irreducible,* $\dim_{\mathbb{C}} \Sigma_s = \dim_{\mathbb{C}} \Sigma - (n+1)$.

Proof. Let $([f];\, x) \subset \Sigma_s$. The group $G = SL(n+2, \mathbb{C})$ acts on $\mathcal{X}$ as follows: $\forall \sigma \in G$,

$$\sigma([f];\, x) = ([f \circ \sigma^{-1}];\, \sigma(x))\ .$$

Clearly, G preserves both Σ_s and Σ. Therefore, by a linear transformation, we may assume that $x = [1, 0, \dots, 0]$. Let us write

$$f = \sum_{i_0 + \cdots + i_{n+1} = d} a_{i_0 \cdots i_{n+1}} z_0^{i_0} \cdots z_{n+1}^{i_{n+1}}$$

Since $\frac{\partial f}{\partial z_j}(x) = 0$ for $0 \le j \le n+1$, we have

$$a_{d0\cdots 0} = a_{d-1\,10\cdots 0} = \cdots = a_{d-1\,0\cdots \underset{(i)}{1} 0\cdots 0} = \cdots = a_{d-1\cdots 1} = 0\ .$$

Suppose that $a_{i'_0\cdots i'_{n+1}} \neq 0$, so we may take $\{a_{i_0\cdots i_{n+1}}\}_{(i_0\cdots i_{n+1})\neq(i'_0,\cdots,i'_{n+1})}$ to be local coordinates of $\mathcal{X}$ near $([f];x)$. Then

$$d(\frac{\partial f}{\partial z_i})(x) = da_{d-1\cdots 1\cdots 0} + \cdots$$

Therefore, Σ_s is smooth at $([f];\, x)$. Moreover,

$$\dim_{\mathbb{C}} \Sigma_s = \dim_{\mathbb{C}} \mathcal{X} - (n+2)$$

The irreducibility of Σ_s follows from $H^0(\Sigma_s, \mathbb{C}) = \mathbb{C}$, which can be easily proved by using the Lefschetz theorem. □

Corollary 1.1. *$B_s = \pi_1(\Sigma_s)$ is irreducible, where $\pi_i \colon \mathcal{X} \to B$ or $\mathbb{C}P^{n+1}$ is the projection onto its i-th factor.*

In fact, B_s is the hypersurface in B classifying all singular hypersurfaces in $\mathbb{C}P^{n+1}$.

Define B_{ss} to be the subvariety of B_s consisting of all polynomials $[f]$ of the form $f_1^2 f_2$ with $\deg(f_1)$ positive. One can easily compute

$$\dim_{\mathbb{C}} B_{ss} = \binom{n+d-1}{d-2} + n + 1$$

Therefore, $\operatorname{codim}_{\mathbb{C}} B_{ss} \geq 2$.

Let $\mathcal{K}$ be the relative canonical bundle $K_\Sigma \otimes \pi_1^* K_B^{-1}$ over Σ. We can define a canonical metric $\|\cdot\|_{\mathcal{K}}$ on $\mathcal{K}|_{\Sigma\setminus\Sigma_s}$ as follows: at each point $([f];\, x) \in \Sigma \setminus \Sigma_s$, we choose a local coordinate system $(y_1, \cdots, y_b; z_1, \cdots, z_{n+1})$ of $\mathcal{X}$ at $([f];\, x)$ such that $y_1, \ldots, y_b$ is a local coordinate system of B at $[f]$ and $z_1, \cdots, z_n$ are tangent to $\pi_1^{-1}([f])$ and $\{z_{n+1} = 0\}$ defines $\pi_1^{-1}([f])$ locally, then $\mathcal{K}$ has a trivialization at $([f];\, x)$ given by

$$a\,(dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_b) \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)^{-1} \to a \in \mathbb{C}$$

we define

$$\|a(dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_b) \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)^{-1}\|_{\mathcal{K}}^2 = |a|^2 \left(\det(g_{i\bar{j}})\right)^{-1}$$

where $\{g_{i\bar{j}}\}_{1\leq i,j\leq n}$ is the metric tensor of the restriction of Fubini-Study metric $\omega_{FS}|_{\pi_1^{-1}([f])}$ on $\mathbb{C}P^{n+1}$, i.e.,

$$\omega_{FS}|_{\pi_1^{-1}([f])} = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

We need to check that it is well-defined. To see this, let $(y_1', \cdots, y_b'; z_1', \dots z_{n+1}')$ be another local coordinate system with the properties stated above, then we have transition functions

$$y_i' = F_i(y)\ , \qquad 1 \le i \le b$$
$$z_\alpha' = H_\alpha(z_1, \cdots, z_{n+1}; y_1, \cdots, y_b)\ , \quad 1 \le \alpha \le n+1$$

satisfying:

$$z_{n+1}' = H_{n+1}(z_1, \cdots, z_n, 0; 0, \cdots, 0) \equiv 0$$

therefore,

$$dy_1' \wedge \cdots \wedge dy_b' = \det(DF)\, dy_1 \wedge \cdots \wedge dy_b$$
$$dz_1' \wedge \cdots \wedge dz_n'\big|_{\pi_1^{-1}([f])} = \det(\left\{\frac{\partial H_\alpha}{\partial z_\beta}\right\}_{1 \le \alpha, b \le n})\, dz_1 \wedge \cdots \wedge dz_n\big|_{\pi_1^{-1}([f])}$$
$$\omega_{FS}\big|_{\pi_1^{-1}([f])} = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar\beta}'\, dz_\alpha' \cdots d\bar z_\beta'$$
$$= \frac{\sqrt{-1}}{2\pi} \sum_{\gamma,\delta=1}^{n} g_{\alpha\bar\beta}' \frac{\partial H_\alpha}{\partial z_\gamma} \overline{\frac{\partial H_\beta}{\partial z_\delta}}\, dz_\gamma \wedge d\bar z_\delta$$

i.e.,

$$g_{\gamma\bar\delta} = \sum_{\alpha,\beta=1}^{n+1} g_{\alpha\bar\beta}' \frac{\partial H_\alpha}{\partial z_\gamma} \overline{\frac{\partial H_\beta}{\partial z_\delta}}$$
$$= \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar\beta}' \frac{\partial H_\alpha}{\partial z_\gamma} \overline{\frac{\partial H_\beta}{\partial z_\delta}} \quad (\text{since } \frac{\partial H_{n+1}}{\partial z_\gamma} = 0 \text{ for } \gamma \le n)$$

it follows

$$\det(g_{\gamma\bar\delta})_{1 \le \gamma,\delta \le n} = \det(g_{\alpha\bar\beta}')_{1 \le \alpha,\beta \le n} |\det(\frac{\partial H_\alpha}{\partial z_\gamma})_{1 \le \alpha,\gamma \le n}|^2$$

Now $\mathcal{K}$ has a new trivialization at $([f]; x)$, i.e.

$$a'(dz_1' \wedge \cdots \wedge dz_n' \wedge dy_1' \wedge \cdots \wedge dy_b') \otimes \pi_1^*(dy_1' \wedge \cdots \wedge dy_b')^{-1} \to a' \in \mathbb{C}$$

The local section a in the first trivialization becomes

$$a' = a \det(\frac{\partial H_\alpha}{\partial z_\gamma})^{-1}_{1 \le \alpha,\gamma \le n}$$

in the second trivialization, so

$$|a'|^2 \det(g_{\alpha\bar\beta}')^{-1}_{1 \le \alpha,\beta \le n} = |a|^2 \det(g_{\gamma\bar\delta})^{-1}_{1 \le \gamma,\delta \le n}$$

i.e., $\| \ \|_{\mathcal{K}}$ is a well-defined hermitian norm on $K|_{\Sigma \setminus \Sigma_s}$. The group G acts on the total space of $\mathcal{K}$ and the subgroup $SU(n+2)$ acts by isometries of $\| \ \|_{\mathcal{K}}$ on $\mathcal{K}|_{\Sigma \setminus \Sigma_s}$. Introduce a smooth function ψ on $\Sigma \setminus \Sigma_s$ by

$$\psi([f];x) = \log \left(\frac{\sum_{\alpha=0}^{n+1} |\frac{\partial F}{\partial z_\alpha}|^2}{\sum_{d=i_0+\cdots+i_{n+1}} |a_{i_0\cdots i_{n+1}}|^2 \cdot \left(\sum_{\alpha=0}^{n+1} |z_\alpha|^2\right)^{d-1}} \right)$$

where F is the defining polynomial of Σ. Clearly, ψ diverges to $-\infty$ as $([f];x)$ tends to Σ_s.

We define a new hermitian metric $\| \cdot \|$ on $\mathcal{K}|_{\Sigma \setminus \Sigma_s}$ by

$$\| \cdot \|^2 = e^{\psi} \| \cdot \|_{\mathcal{K}}^2$$

We will prove that this hermitian metric can be smoothly extended across Σ_s and compute its curvature form. Clearly, $SU(n+2)$ still acts on $\mathcal{K}|_{\Sigma \setminus \Sigma_s}$ by isometries of $\| \cdot \|$.

Theorem 1.1. *The hermitian metric $\| \cdot \|$ can be extended to be a smooth metric on $\mathcal{K}$ over Σ. Moreover, the curvature form of this metric is*

$$\pi_1^* \omega_B + (d-n-2)\pi_2^* \omega_{FS}$$

where ω_B is the Kähler form of the Fubini-Study metric on $B = PR_{n,d}$.

Proof. Let $([f];x)$ be any point in Σ_s and U be a neighborhood of $([f];x)$ in $\mathcal{X}$. We want to extend $\| \cdot \|$ on $\mathcal{K}|_{U \cap \Sigma \setminus \Sigma_s}$ to $\mathcal{K}|_{U \cap \Sigma}$. By using the isometries from $SU(n+2)$, we may assume that

$$x = [1, 0, \dots, 0]$$

Since Σ is smooth and $\pi_1^*([f])$ is singular at x, there is a local coordinate system

$\{y_1, \dots, y_b; z_1, \cdots, z_{n+1}\}$ of $\mathcal{X}$ at $([f];x)$ such that $\{y_1, \cdots, y_{b-1}; z_1, \cdots, z_{n+1}\}$ restricts to a local coordinate sytem of Σ near $([f];x)$, in particular,

$$\frac{\partial F}{\partial y_b} \neq 0 \quad \text{in} \quad U ,$$

then $\mathcal{K}$ has a local holomorphic section

$$S_U = \left.\frac{dz_1 \wedge \cdots \wedge dz_{n+1} \wedge dy_1 \wedge \cdots \wedge dy_{b-1}}{\dfrac{\partial F}{\partial y_b}}\right|_{\Sigma} \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)$$

At any point $([\tilde{f}]; y) \in U \cap \Sigma \setminus \Sigma_s$, for some i, $z_1, \cdots, \hat{z}_i, \cdots, z_{n+1}$ give local coordinates of $\pi_1^{-1}([\tilde{f}])$ at y, for simplicity, we assume $i = n+1$. Since $F \equiv 0$ on Σ

$$\frac{\partial F}{\partial z_1} dz_1 + \cdots + \frac{\partial F}{\partial z_{n+1}} dz_{n+1} + \cdots + \frac{\partial F}{\partial y_b} dy_b \equiv 0 \quad \text{on} \quad \Sigma .$$

Multiplying both sides by $dz_1 \wedge \cdots \wedge dz_n$, we obtain

$$S_U = \pm \frac{dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_b}{\frac{\partial F}{\partial z_{n+1}}} \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)^{-1}$$

Let $\{g_{i\bar{j}}\}$ be the corresponding hermitian metrics of ω_{FS} in local coordinates z_1, ..., z_{n+1}, i.e.,

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n+1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

Then

$$\begin{aligned}
\omega_{FS}^n &= n! \left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_{i,j=1}^{n+1} (-1)^{i+j+n+1} \det(g_{\alpha\bar{\beta}})_{\substack{1 \le \alpha, \beta \le n+1 \\ \alpha \ne i,\, \beta \ne j}} \\
&\quad dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_{n+1} \ \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \\
&= n! (\frac{\sqrt{-1}}{2\pi})^n \Bigg\{ \det(g_{\alpha\bar{\beta}})_{1 \le \alpha, \beta \le n} - 2\mathrm{Re}(\sum_{i=1}^{n} (-1)^{n+i+1} \det(g_{\alpha\bar{\beta}})_{\substack{1 \le \alpha \le n+1 \\ \alpha \ne i \\ 1 \le \beta \le n}} \\
&\quad \cdot \frac{\partial z_{n+1}}{\partial z_i}) + \sum_{i,j=1}^{n} (-1)^{i+j} \det(g_{\alpha\bar{\beta}})_{\substack{1 \le \alpha, \beta \le n+1 \\ \alpha \ne i,\, \beta \ne j}} \frac{\partial z_{n+1}}{\partial z_i} \overline{\frac{\partial z_{n+1}}{\partial z_j}} \Bigg\} \\
&\quad dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n
\end{aligned}$$

(since $\frac{\partial z_{n+1}}{\partial z_i} = -\frac{\partial F}{\partial z_i} / \frac{\partial F}{\partial z_{n+1}}$)

$$
\begin{aligned}
&= n!(\frac{\sqrt{-1}}{2\pi})^n \left\{ \det(g_{\alpha\bar\beta})_{1\le\alpha,\beta\le n} |\frac{\partial F}{\partial z_{n+1}}|^2 + 2\mathrm{Re}\left(\sum_{i=1}^{n} (-1)^{n+i+1} \det(g_{\alpha\bar\beta})_{\substack{1\le\alpha\le n+1\\ \alpha\neq i\\ 1\le\beta\le n}} \right.\right. \\
&\quad \left.\left. \cdot \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_{n+1}}} \right) + \sum_{i,j=1}^{n} (-1)^{i+j} \det(g_{\alpha\bar\beta})_{\substack{1\le\alpha,\beta\le n+1\\ \alpha\neq i,\,\beta\neq j}} \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_j}} \right\} \\
&\quad \cdot |\frac{\partial F}{\partial z_{n+1}}|^{-2} \, dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n \\
&= n!(\frac{\sqrt{-1}}{2\pi})^n |\frac{\partial F}{\partial z_{n+1}}|^{-2} \det(g_{i\bar j})_{1\le i,j\le n+1} \sum_{i,j=1}^{n} g^{i\bar j} \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_j}} \, dz_1 \wedge d\bar z_1 \wedge \cdots \wedge d\bar z_n
\end{aligned}
$$

where $\{g^{i\bar j}\}$ is the inverse matrix of $\{g_{i\bar j}\}$. Therefore

$$
\|S_U\|_{\mathcal{K}}^2 = \left\{ \det(g_{i\bar j})_{1\le i,j\le n+1} \sum_{i,j=1}^{n+1} g^{i\bar j} \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_j}} \right\}^{-1}
$$

Now we may assume that $z_1, \dots, z_{n+1}$ are coordinates of $\mathbb{C}^{n+1} \subset \mathbb{C}P^{n+1}$. Then

$$
\begin{aligned}
g_{i\bar j} &= \frac{1}{1 + \sum_{i=1}^{n+1} |z_i|^2} \left(\delta_{ij} - \frac{\bar z_i z_j}{1 + \sum_{i=1}^{n+1} |z_i|^2} \right) \\
g^{i\bar j} &= (1 + |z|^2)\,(\delta_{ij} + \bar z_j z_i)
\end{aligned}
$$

it follows

$$
\sum_{i,j=1}^{n+1} g^{i\bar j} \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_j}} = (1+|z|^2) \left(\sum_{i=1}^{n+1} |\frac{\partial F}{\partial z_i}|^2 + \sum_{i,j=1}^{n+1} z_i \frac{\partial F}{\partial z_i} \cdot \overline{z_j \frac{\partial F}{\partial z_j}} \right)
$$

but $\sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial z_i} + \frac{\partial F}{\partial z_0} = \sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial z_i} + \frac{\partial F}{\partial z_0} = dF \equiv 0$ on Σ, so

$$
\sum_{i,j=1}^{n+1} g^{i\bar j} \frac{\partial F}{\partial z_i} \overline{\frac{\partial F}{\partial z_j}} = (1+|z|^2) \sum_{i=0}^{n+1} |\frac{\partial F}{\partial z_i}|^2
$$

On the other hand,

$$
\det(g_{i\bar j})_{1\le i,j\le n+1} = \frac{1}{(1+|z|^2)^{n+2}}
$$

Therefore,

$$\|S_U\|^2 = e^{\psi} \frac{(1+|z|^2)^{n+1}}{\sum_{i=0}^{n+1} |\frac{\partial F}{\partial z_i}|^2}$$

$$= \left(\sum_{i_0+\cdots+i_{n+1}=d} |a_{i_0\cdots i_{n+1}}|^2 \right)^{-1} (1+|z|^2)^{n+2-d}$$

This shows that the hermitian metric extends smoothly to a metric on $\mathcal{K}$ over Σ. The curvature form of this metric follows easily from the above formula for $\|S_U\|^2$. □

By the definition of the hermitian metric $\|\cdot\|$, one can easily deduce the following from Theorem 1.1,

Corollary 1.2. *The curvature form of the metric $\|\cdot\|_{\mathcal{K}}$ is given by*

$$R(\|\cdot\|_{\mathcal{K}}) = \pi_1^*\omega_B + (d-n-2)\pi_2^*\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi.$$

2. Extension of Quillen Metrics

In this section, we first introduce a hermitian metric on the tautological line bundle $\mathcal{O}_B(-1)$ over B. This metric will be related to the intrinsic structure of the underlying variety through the K-energy in next section. We will also show that this metric is actually the Quillen metric on the determinant line bundle of certain holomorphic "virtual" bundle over Σ. This explains how we arrived at this metric.

Define a current u as follows: for any smooth $(2b-2)$-form φ on B, where b is the complex dimension of B, we have

$$\int_B \varphi \wedge u \;=\; \int_\Sigma \pi_1^*\varphi \wedge \pi_2^*\omega_{FS}^{n+1} \tag{2.1}$$

It is easy to see that u is positive, closed and smooth in $B \setminus B_s$. To determine the cohomology class $[u]$ in $H^2(B,\mathbb{R})$, we simply plug ω^{b-1} into (2.1) and find

$$\begin{aligned} \int_B \omega_B^{b-1} \wedge u \;&=\; \int_\Sigma \pi_1^*\omega_B^{b-1} \wedge \pi_2^*\omega_{FS}^{n+1} \\ &= \int_{\mathcal{X}} \pi_1^*\omega_B^{b-1} \wedge \pi_2^*\omega_{FS}^{n+1} \wedge (\pi_1^*\omega_B + d\cdot\pi_2^*\omega_{FS}) \\ &= \int_{\mathcal{X}} \pi_1^*\omega_B^{b-1} \wedge \pi_2^*\omega_{FS}^{n+1} \wedge \pi_1^*\omega_B \;=\; 1 \end{aligned} \tag{2.2}$$

Therefore, the cohomology class $[u]$ coincides with the Kähler class of the Fubini-Study metric, which is just the first Chern class of $\mathcal{O}_B(1)$ over B.

Proposition 2.1. *There is a $\frac{1}{d}$-Hölder continuous hermitian metric $\|\cdot\|_u$ on $\mathcal{O}_B(-1)$ over B such that $-u$ is its curvature form in the sense of distribution, namely, for any smooth $(2b-2)$-form φ on B, we have*

$$\begin{aligned} &\int_B \varphi \wedge u \\ &= \int_B \varphi \wedge \left(\omega_B + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\frac{\|\cdot\|_u^2}{\|\cdot\|_B^2}\right)\right) \end{aligned} \tag{2.3}$$

where $\|\cdot\|_B$ denotes the standard metric on $\mathcal{O}_B(-1)$ whose curvature form is ω_B.

This is a special case of Theorem 5.2 in [T1]. The readers can find the proof there. We note that such a metric $\|\cdot\|_u$ is unique up to multiplications by positive constants.

Since ψ has only logarithmic singularity along Σ_s, the integral

$$\int_\Sigma \psi\, \pi_1^*\varphi \wedge \pi_2^*\omega_{FS}^n$$

is finite for any smooth $2b$-form φ on B. We can define a generalized function ψ_B by

$$\int_B \psi_B\varphi = \int_\Sigma \psi\pi_1^*\varphi \wedge \pi_2^*\omega_{FS}^n \tag{2.4}$$

where φ is a smooth $2b$-form on B.

Lemma 2.1. *The function ψ_B is smooth outside B_s, continuous outside B_{ss} and bounded from above.*

Proof. The smoothness of ψ_B outside B_s follows from the fact that the fibration $\pi_1 : \Sigma \setminus \Sigma_s \mapsto B \setminus B_s$ is smooth. Since ψ is bounded from above, so is ψ_B. Moreover,

$$\psi_B(y) = \int_{\pi_1^{-1}(y)} \psi\, \omega_{FS}^n \tag{2.5}$$

for any point y in $B \setminus B_s$. Given any point y in $B_s \setminus B_{ss}$, using the fact that the singular set of $\pi_1^{-1}(y)$ is of complex dimension at most $n-1$, it is easy to show

$$\lim_{z \to y, z \in B \setminus B_s} \psi_B(z) = \int_{\pi_1^{-1}(y)} \psi \omega_{FS}^n < \infty$$

Clearly, it follows the continuity of ψ_B at y. In fact, the value of ψ_B is given by the integral on the right in (2.5). The lemma is proved. □

By Proposition 2.1, the closed form $(d-n-2)u + d(n+1)\omega_B$ represents the same cohomology class as $(n+2)(d-1)\omega_B$ does.

DEFINITION 2.1. Fix $d > 1$, we define a singular hermitian metric on $\mathcal{O}_B(-1)$ by

$$\| \cdot \|_Q = e^{\frac{(n+1)}{2(n+2)(d-1)}\psi_B} \| \cdot \|_u^{\frac{d-n-2}{(n+2)(d-1)}} \| \cdot \|_B^{\frac{d(n+1)}{(n+2)(d-1)}} \tag{2.6}$$

where $\| \cdot \|_u$ and $\| \cdot \|_B$ are given as in Proposition 2.1.

Later on we will see how this metric is related to the K-energy.

In the following we will show that this metric is nothing but the Quillen metric on certain determinant line bundle over $B \setminus B_s$.

Consider a "virtual" holomorphic bundle $\mathcal{E}$ over Σ,

$$\mathcal{E} = (n+1)(\mathcal{K}^{-1} - \mathcal{K}) \otimes (\mathcal{L} - \mathcal{L}^{-1})^n - n(n+2-d)(\mathcal{L} - \mathcal{L}^{-1})^{n+1} \tag{2.7}$$

where

$$\mathcal{L} = \pi_2^* \mathcal{O}_{\mathbb{C}P^{n+1}}(1) \tag{2.8}$$

We denote by $\det(\mathcal{E}, \pi_1)$ the determinant line bundle over B through the fibration $\pi_1 : \Sigma \mapsto B$. It is defined to be

$$\prod_{j=0}^{n} \left(\det(\mathcal{K}^{-1} \otimes \mathcal{L}^{n-2j}, \pi_1) \otimes \det(\mathcal{K} \otimes \mathcal{L}^{n-2j}, \pi_1)^{-1} \right)^{p\binom{n}{j}} \otimes \prod_{j=0}^{n+1} \det(\mathcal{L}^{n+1-2j}, \pi_1)^{-q\binom{n+1}{j}}, \tag{2.9}$$

where $p = n+1$ and $q = n(n+2-d)$. For any coherent sheaf $\mathcal{F}$ over Σ, the determinant line bundle $\det(\mathcal{F}, \pi_1)$ is defined as follows (cf. [KH]): we first define a presheaf by assigning any open subset $V \subset B$ to the cohomolgy

$H^i(\pi^{-1}(V), \mathcal{F}|_{\pi^{-1}(V)})$, then the direct image $R^i\pi_{1*}\mathcal{F}$ is simply the sheaf generated by this presheaf, and define $\det(\mathcal{F}, \pi_1)$ to be the double dual of the sheaf

$$\bigotimes_{i=0}^{\infty} \det(R^i\pi_{1*}\mathcal{F})^{(-1)^i} \tag{2.10}$$

By the Grothendieck-Riemann-Roch Theorem, one can compute the first Chern class

$$C_1(\det(\mathcal{E}, \pi_1)) \;=\; -2^{n+1}(n+2)(d-1)\omega_B, \tag{2.11}$$

therefore, the determinant line bundle $\det(\mathcal{E}, \pi_1)$ coincides with $\mathcal{O}_B(-(n+2)(d-1))$.

Let us recall definition of the Quillen metric on $\det(\mathcal{E}, \pi_1)$. Let h be the hermitian metric on $\mathcal{L} = \pi_2^*\mathcal{O}_{\mathbb{C}P^{n+1}}(1)$ with curvature form $\pi_2^*\omega_{FS}$. Fix a smooth hypersurface X in $\mathbb{C}P^{n+1}$, the Fubini-Study metric restricts to a Kähler metric ω_X on X and consequently induces a hermitian metric h_X on $\mathcal{K}|_X$. Let $\mathcal{F}$ be one of bundles $\mathcal{K}^{\pm} \otimes \mathcal{L}^{n-2j}$ or $\mathcal{L}^{n+1-2j}$. Then the fiber of the determinant line bundle $\det(\mathcal{F}, \pi_1)$ over X can be naturally identified with the space

$$\bigotimes_{i=1}^{n} \det(H^i(X, \mathcal{F}|_X))^{(-1)^i} \tag{2.12}$$

The metrics h and h_X yield a hermitian metric $h_{\mathcal{F}}$ on $\mathcal{F}$. Together with the Kähler metric ω_X, this $h_{\mathcal{F}}$ defines an inner product $\langle\cdot,\cdot\rangle$ on $H^i(X, \mathcal{F}|_X)$, which is identified with the space of harmonic $\mathcal{F}$-valued (0,i)-forms,

$$\langle s_1, s_2\rangle = \int_X (s_1(x), s_2(x))_{\mathcal{F}}\omega_X^n \,, \qquad s_1, s_2 \in H^i(X, \mathcal{F}|_X) \tag{2.13}$$

where $(\cdot,\cdot)_{\mathcal{F}}$ denotes the inner product on the space of $\mathcal{F}$-valued $(0,i)$-forms on X induced by $h_{\mathcal{F}}$ and ω_X. In a local coordinate system $\{z_1, \cdots, z_n\}$ of X and a local holomorphic frame e of $\mathcal{F}$, we can express

$$s_1 = \sum \varphi_{\bar\alpha_1\cdots\bar\alpha_i} d\bar z_{\alpha_1} \wedge \cdots \wedge d\bar z_{\alpha_i} \tag{2.14}$$

$$s_2 = \sum \psi_{\bar\alpha_1\cdots\bar\alpha_i} d\bar z_{\alpha_1} \wedge \cdots \wedge d\bar z_{\alpha_i} \tag{2.15}$$

then

$$(s_1, s_2)_{\mathcal{F}} = a\, g^{\alpha_1\bar\beta_1} \cdots g^{\alpha_i\bar\beta_i} \varphi_{\bar\beta_1\cdots\bar\beta_i} \overline{\psi}_{\bar\alpha_1\cdots\bar\alpha_i} \tag{2.16}$$

where a is a local positive function representing $h_{\mathcal{F}}$ in the frame e and $\{g^{\alpha\bar{\beta}}\}$ is the inverse of the metric ω_X in the local coordinates. The inner product induces a L^2-metric on $\det H^i(X, \mathcal{F}|_X)$. Unfortunately, this L^2-metric does not vary continuously as X varies in B. A discontinuous point in B is where the dimension of $H^i(X, \mathcal{F}|_X)$ jumps for some i. The Quillen metric is introduced to eliminate this discontinuity. A Quillen metric is defined in terms of the L^2-metric and the Ray-Singer analytic torsion.

Let $P_{\mathcal{F}}$ be the harmonic projection from the space $\Gamma^i(X, \mathcal{F}|_X)$ of $\mathcal{F}$-valued (0,i)-forms onto $H^i(X, \mathcal{F}|_X)$. We define an Zeta function

(2.17)
$$\zeta_{\mathcal{F}}(s) = -\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \sum_{i=0}^{n} (-1)^i i \ \mathrm{tr}\left(e^{-u\,\Box^i_{\mathcal{F}}} - P_{\mathcal{F}}\right) du\ , \quad \Re e(s) > n$$

where

$$\Gamma(s) = \int_0^\infty u^{s-1} e^{-u}\, du \tag{2.18}$$

and $\Box^i_{\mathcal{F}}$ denotes the Laplacian on $\Gamma^i(X, \mathcal{F}|_X)$ induced by $h_{\mathcal{F}}$ and ω_X, i.e., if $\bar{\partial}^*$ denotes the adjoint operator of $\bar{\partial}$ on $\Gamma^i(X, \mathcal{F}|_X)$ with respect to h and ω, then

$$\Box^i_{\mathcal{F}} = \bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^* \tag{2.19}$$

It is now well-known that $\zeta_{\mathcal{F}}(s)$ extends to be a meromorphic function on $\mathbb{C}$ and the extension, still denoted by $\zeta_{\mathcal{F}}$, is holomorphic at the origin.

Definition 2.2. (Ray and Singer [RS]). The analytic torsion $\tau(\mathcal{F}|_X)$ is defined to be $-\zeta'_{\mathcal{F}}(0)$.

Definition 2.3. The Quillen metric $\|\cdot\|_{Q,\mathcal{F}}$ on the determinant line bundle $\det(\mathcal{F}, \pi_1)_X$ is the metric obtained by multiplying the L^2-metric by $e^{\tau(\mathcal{F}|_X)}$.

It can be proved that this Quillen metric depends on X smoothly. Using (2.12) and $\|\cdot\|_{Q,\mathcal{F}}$ for $\mathcal{F}$ being either $\mathcal{K}^\pm \otimes \mathcal{L}^{n-2j}$ or $\mathcal{L}^{n+1-2j}$, we can construct a Quillen metric $\|\cdot\|_{Q,\mathcal{E}}$ on $\det(\mathcal{E}, \pi_1)$.

Theorem 2.2. *There is a positive constant c such that the Quillen metric $\|\cdot\|_{Q,\mathcal{E}}$ is the same as $c\,\|\cdot\|_Q^{2^{n+1}(n+2)(d-1)}$.*

This theorem can be proved by the same arguments as in the proof of Theorem 4.1 and using two transgression formulas of Bismut, Gillet and Soule in [BGS] (see also [BF]). We omit the proof, since it is not used in the proof of our theorems.

Corollary 2.1. *The Quillen metric $\|\cdot\|_{Q,\mathcal{E}}$ over $B \setminus B_s$ extends to be a continuous metric over $B \setminus B_{ss}$.*

Clearly, it follows from Theorem 2.1 and Lemma 2.1.

3. The Generalized K-energy

In this section, we define the K-energy on singular varieties in $\mathbb{C}P^N$, and compute its complex Hessian. The K-energy was first introduced by T. Mabuchi [Ma] on compact Kähler manifolds. Our K-energy here is nothing else but the limit of the K-energy of T. Mabuchi, in case the singular variety is a limit of smooth Kähler manifolds.

Let X be an irreducible subvariety in $\mathbb{C}P^N$. We denote by $S(X)$ the subvariety of all singular points in X. We assume that the complex codimension of $S(X)$ is at least two, or equivalently, X is normal. Let L be the restriction to X of the hyperplane line bundle over $\mathbb{C}P^N$. A hermitian metric h on L is admissible if it extends to be a smooth metric over $\mathbb{C}P^N$ with positive curvature, in particular, for any σ in $G = SL(N+1,\mathbb{C})$, the restriction of $\sigma^* h_{FS}$ to X is an admissible metric on L, where h_{FS} is the standard metric on the hyperplane bundle whose curvature gives the Fubini-Study metric. We denote by $\mathcal{M}_X^+$ the set of all admissible metrics. Note that any metric h in $\mathcal{M}_X^+$ is of the form $e^{-\varphi} h_{FS}$ for some function φ.

Lemma 3.1. *Let $\{h_t\}_{0\le t\le 1}$ be a loop in $\mathcal{M}_X^+$. Then*

$$\int_0^1 \int_X \dot{\varphi}_t (s(\omega_t) - c)\omega_t^n \wedge dt \;=\; 0 \tag{3.1}$$

where $\{h_t = e^{-\varphi_t} h_0\}$, ω_t is the curvature $R(h_t)$ of h_t, $s(\omega_t)$ denotes the scalar curvature of the Kähler metric ω_t on X and c is any constant.

Proof. First we note that ψ, $|\nabla\psi|$ and $\Delta\psi$ are bounded for $\psi = \varphi_t$ or $\frac{\partial \varphi_t}{\partial t}$, where $|\nabla\psi|$ is the pointwise norm of the gradient of ψ with respect to the metric induced by ω_t. Let $\pi : \tilde{X} \mapsto X$ be a smooth resolution. Fix a Kähler form $\tilde{\omega}$ on $\tilde{X}$. If $\mathrm{Ric}(\omega)$ denotes the Ricci curvature of the metric induced by a Kähler form ω, then

$$\mathrm{Ric}(\tilde{\omega}) \; - \; \pi^* \mathrm{Ric}(\omega_t) \; = \; \partial\bar{\partial} \log \left(\frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \right) \tag{3.2}$$

Using the assumption that $S(X)$ is of at least codimension two and integrating by parts, one can easily prove

(3.3)

$$\begin{aligned}
&\int_0^1 \int_X \dot{\varphi}_t (s(\omega_t) - c)\, \omega_t^n \wedge dt \\
&= \int_0^1 \int_{\tilde{X}} \pi^* \dot{\varphi}_t \Big(-n\, \partial\bar{\partial} \log \Big(\frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \Big) \wedge \pi^* \omega_t^{n-1} + n \,\mathrm{Ric}(\tilde{\omega}) \wedge \pi^* \omega^{n-1} - c\, \pi^* \omega_t^n \Big) \wedge dt \\
&= - \int_0^1 \int_{\tilde{X}} \Big(n \log \Big(\frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \Big) \partial\bar{\partial} \pi^* \dot{\varphi}_t - \pi^* \dot{\varphi}_t \,(n \,\mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_t) \Big) \wedge \pi^* \omega_t^{n-1} \wedge dt \\
&= - \int_0^1 \int_{\tilde{X}} \Big(\log \Big(\frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \Big) \frac{\partial}{\partial t} (\pi^* \omega_t^n) - \pi^* \dot{\varphi}_t \,(n \,\mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_t) \wedge \pi^* \omega_t^{n-1} \Big) \wedge dt \\
&= - \int_{\tilde{X}} \Big[\log \Big(\frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \Big) - \frac{\pi^* \omega_t^n}{\tilde{\omega}^n} \Big]_0^1 \tilde{\omega}^n \\
&\qquad + \int_0^1 \int_{\tilde{X}} \pi^* \dot{\varphi}_t \,(n \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_t) \wedge \pi^* \omega_t^{n-1} \wedge dt \\
&= \int_0^1 \int_{\tilde{X}} \pi^* \dot{\varphi}_t \,(n \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_t) \wedge \pi^* \omega_t^{n-1} \wedge dt
\end{aligned}$$

Define $h_{t,s} = e^{-s\,\varphi_t} h_0$ and

$$D(s) = \int_0^1 \int_{\tilde{X}} s\, \pi^* \dot{\varphi}_t \,(n \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_t) \wedge \pi^* \omega_t^{n-1} \wedge dt \tag{3.4}$$

where $\omega_{t,s} = \omega_0 + s \partial\bar{\partial} \varphi_t$. It suffices to show that $D' \equiv 0$. Write $\varphi_{t,s} = s\, \varphi_t$, and we compute

$$\begin{aligned}
D'(s) \;&=\; \int_0^1 \int_{\tilde{X}} \frac{\partial^2 \varphi_{t,s}}{\partial t \partial s} (n \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_{t,s}) \wedge \pi^* \omega_{t,s}^{n-1} \wedge dt \\
&\quad + n \int_0^1 \int_{\tilde{X}} \frac{\partial \varphi_{t,s}}{\partial t} \partial\bar{\partial} \left(\frac{\partial \varphi_{t,s}}{\partial s} \right) \wedge ((n-1) \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_{t,s}) \wedge \omega_{t,s}^{n-1} \wedge dt \\
&=\; \int_0^1 \int_{\tilde{X}} \frac{\partial}{\partial t} \left(\frac{\partial \varphi_{t,s}}{\partial s} (n \; \mathrm{Ric}(\tilde{\omega}) - c\, \pi^* \omega_{t,s}) \wedge \pi^* \omega_{t,s}^{n-1} \right) \wedge dt \\
&=\; 0
\end{aligned}$$

The lemma is proved. □

By this lemma, we can define the generalized K-energy $D(h_0, h_1)$ by

$$D(h_0,h_1) \;=\; -\int_0^1\int_X \dot{\varphi}_t\,(s(\omega_t)-\mu)\omega_t^n\wedge dt \tag{3.5}$$

where $h_0, h_1 \in \mathcal{M}_X^+$, $\{h_t\}$ is any path from h_0 to h_1, and μ is defined to be the ratio

$$\frac{\int_X\, s(\omega_t)\omega_t^n}{\int_X \omega_t^n} \tag{3.6}$$

It is known that μ is independent of t and any choice of admissible metrics in $\mathcal{M}_X^+$. Obviously, by Lemma 3.1, we have

$$D(h_0,h_1) = -D(h_1,h_0)$$
$$D(h_0,h_2) = D(h_0,h_1) \;+\; D(h_1,h_2)$$

The following proposition can be proved by the same arguments in the proof of Lemma 3.1.

Proposition 3.1. *Assume that there is smooth function f_0 on $X\backslash S(X)$ satisfying:*

$$\mathrm{Ric}(\omega_0) - \frac{\mu}{n}\,\omega_0 = \partial\bar{\partial} f_0 \qquad \textit{on } X\backslash S(X) \tag{3.7}$$

Then we have

$$D(h_0,h_1) = \int_X \left(\log\left(\frac{\omega_1^n}{\omega_0^n}\right) - f_0\right)\omega_1^n + \int_X f_0\omega_0^n - \frac{\mu}{n}\,(I(h_0,h_1) - J(h_0,h_1)) \tag{3.8}$$

where $I(h_0,h_1)$ and $J(h_0,h_1)$ are defined as follows: write $h_1 = e^{-\varphi}h_0$, then $\omega_1 = \omega_0 + \partial\bar{\partial}\varphi$, and

$$I(h_0,h_1) = \int_X \varphi\,(\omega_0^n - \omega_1^n) \tag{3.9}$$

$$J(h_0,h_1) = \int_0^1 \frac{I(h_0, e^{-t\varphi}h_0)}{t}dt \tag{3.10}$$

In case X is a complete intersection in $\mathbb{C}P^N$, the assumption (3.7) is automatically true.

Next, we will compute the complex Hessian of the K-energy. Let Y be a complex manifold of dimension m and $h : Y \mapsto \mathcal{M}_X^+$ be a smooth map. Fix a hermitian metric h_0 in $\mathcal{M}_X^+$, the pull-back $h^*D(h_0,\cdot)$ is a real function on Y.

We want to compute its complex Hessian. We define a hermitian metric $h_{L,Y}$ on the bundle $\pi_2^* L$ over $Y \times X$, where π_j is the projection from $Y \times X$ onto its j-th factor,

$$h_{L,Y}\big|_{\{y\}\times X} = h(y)\big|_X \tag{3.11}$$

Note that each $h(y)$ is a metric on L with positive curvature. We denote by $R(h_{L,Y})$ the curvature form of $h_{L,Y}$ on $Y \times X$. It is easy to see that as a 2-form, $R(h_{L,Y})$ restricts to $R(h_y)$ on $\{y\} \times X$. The Kähler metrics defined by $R(h_y)$, where $y \in Y$, induce a hermitian metric k_Y on the relative canonical bundle $\pi_2^* K_X$ over $Y \times X$. We denote by $\mathrm{Ric}_{X|Y}$ the curvature of this hermitian metric. Then

$$\mathrm{Ric}_{X|Y} = \pi_1^* \mathrm{Ric}(\omega_0) - \partial\overline{\partial} \log \left(\frac{{\omega_y}^n}{\omega_0^n} \right) \tag{3.12}$$

where $\omega_y = R(h_y)$ and $\omega_0 = R(h_0)$. It follows from admissibility of h_y that the second form in (3.12) is bounded. Moreover, $\mathrm{Ric}_{X|Y}$ restricts to $\mathrm{Ric}(\omega_y)$ on $\{y\} \times X$.

Proposition 3.2. *Let X be an irreducible, normal projective variety, and Y, h, etc. be as above. Then the complex Hessian $D(h_0, h(\cdot))$ is given weakly as follows: for any smooth $(2m-2)$-form ϕ with compact support in Y,*

(3.13)
$$- \int_Y D(h_0, h(y))\, \partial_Y \overline{\partial}_Y \phi = \int_{Y\times X} \left(\mathrm{Ric}_{X|Y} - \frac{\mu}{n+1}\, \omega_{L,Y} \right) \wedge \omega_{L,Y}^n \wedge \pi_1^* \phi$$

Proof. We may assume that Y is an open ball in $\mathbb{C}^m$. Then for any y in Y, $\{h(ty)\}_{0\le t\le 1}$ is a path from $h(0)$ to $h(y)$. We define $\varphi_y = -\dot{h}(y)h(y)^{-1}$, $\omega_y = R(h_y)$ as above, and μ to be as in (3.6). Then by the definition, we have

(3.14)
$$\begin{aligned} D(h_0, h(y)) &= D(h_0, h(0)) + D(h(0), h(y)) \\ &= D(h_0, h(0)) - \int_0^1 \int_X \dot{\varphi}_{ty} (n \,\mathrm{Ric}(\omega_{ty}) - \mu\, \omega_{ty}) \wedge \omega_{ty}^{n-1} \wedge dt \\ &= D(h_0, h(0)) + \int_0^1 \int_{\{y\}\times X} \dot{\varphi}_{ty} \,(n \,\mathrm{Ric}_{X|Y} - \mu\, \omega_{L,Y}) \wedge \omega_{L,Y}^{n-1} \wedge dt \end{aligned}$$

where $\omega_{L,Y} = R(h_{L,Y})$. Let $\eta : R^1 \mapsto R^1$ be a smooth map satisfying: $0 \le \eta \le 1$, $\eta(t) = 0$ for $t \le 1$ and $\eta(t) = 1$ for $t \ge 2$. We denote by ρ the distance function from the singular set $S(X)$. Define dilation maps $\tau_t : Y\times \mapsto Y\times X$ by putting $\tau_t(y,x) = (ty,x)$. Then for any $(2m-2)$-form ϕ with compact support in Y, using (3.14) and (3.12), we have

(3.15)

$$\begin{aligned}
&-\int_Y D(h_0,h(y))\,\partial_Y\overline{\partial}_Y\phi\\
&=\lim_{\epsilon\to 0+}\left(\int_Y\left(\int_0^1\int_{\{ty\}\times X}\eta(\frac{\rho}{\epsilon})\,\dot\varphi_{ty}(n\ \mathrm{Ric}_{X|Y}-\mu\,\omega_{L,Y})\wedge\omega_{L,Y}^{n-1}\right)\wedge\partial_Y\overline{\partial}_Y\phi\wedge dt\right)\\
&=\lim_{\epsilon\to 0+}\left(\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\partial\overline{\partial}\left(\eta(\frac{\rho}{\epsilon})\,\dot\varphi_{ty}\tau_t^*(n\ \mathrm{Ric}_{X|Y}-\mu\,\omega_{L,Y})\wedge\tau_t^*\omega_{L,Y}^{n-1}\right)\wedge dt\right)\\
&=\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\partial\overline{\partial}\dot\varphi_{ty}\wedge\tau_t^*\left((n\ \mathrm{Ric}_{X|Y}-\mu\omega_{L,Y})\wedge\omega_{L,Y}^{n-1}\right)\wedge dt\\
&\quad+\lim_{\epsilon\to 0+}\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\left(\dot\varphi_{ty}\partial\overline{\partial}\eta(\frac{\rho}{\epsilon})+2\,\mathrm{Re}(\partial\eta(\frac{\rho}{\epsilon})\wedge\overline{\partial}\dot\varphi_{ty})\right)\wedge\\
&\quad\wedge\tau_t^*\left(n\ \mathrm{Ric}_{X|Y}-\mu\omega_{L,Y}\right)\wedge\tau_t^*\omega_{L,Y}^{n-1}\wedge dt
\end{aligned}$$

It follows from (3.12) and the remark after it that the integrand in last integral is bounded by $C\,\rho^{-2}$ for some constant C. Since the singular set is of codimension at least two, the last limit is zero. Therefore,

(3.16)

$$\begin{aligned}
&-\int_Y D(h_0,h(y))\,\partial_Y\overline{\partial}_Y\phi\\
&=\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\partial\overline{\partial}-\dot\varphi_{ty}\wedge\tau_t^*\left(n\ \mathrm{Ric}_{X|Y}-\mu\omega_{L,Y}\right)\wedge\tau_t^*\omega_{L,Y}^{n-1}\wedge dt\\
&=\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\left(\tau_t^*(\mathrm{Ric}_{X|Y})\wedge\frac{\partial}{\partial t}\tau_t^*\omega_{L,Y}^n-\frac{\mu}{n+1}\frac{\partial}{\partial t}\tau_t^*\omega_{L,Y}^{n+1}\right)\wedge dt\\
&=\int_{Y\times X}\pi_1^*\phi\wedge\left(\mathrm{Ric}_{X|Y}\wedge\omega_{L,Y}^n-\frac{\mu}{n+1}\omega_{L,Y}^{n+1}\right)\\
&\quad-\int_0^1\int_{Y\times X}\pi_1^*\phi\wedge\frac{\partial}{\partial t}\left(\tau_t^*(\mathrm{Ric}_{X|Y})\right)\wedge\tau_t^*\omega_{L,Y}^n\wedge dt
\end{aligned}$$

It remains to show that the last integral in (3.16) vanishes. By (3.12),

$$\frac{\partial}{\partial t}\left(\tau_t^*(\mathrm{Ric}_{X|Y})\right)=-\partial\overline{\partial}\left(\Delta_{ty}\varphi_y\right)\qquad(3.17)$$

where Δ_{ty} is the laplacian of the metric with Kähler form ω_{ty}. Therefore,

$$\int_0^1 \int_{Y\times X} \pi_1^*\phi \wedge \frac{\partial}{\partial t}\left(\tau_t^*(\mathrm{Ric}_{X|Y})\right) \wedge \tau_t^*\omega_{L,Y}^n \wedge dt$$

$$= -\int_Y \partial_Y \overline{\partial}_Y \phi \cdot \int_0^1 \int_X \Delta_{ty}\varphi_y\, \omega_{ty}^n \wedge dt = 0$$

Then (3.13) follows. □

Corollary 3.1. *Let X be in $\mathbb{C}P^N$ and G be the linear group* $\mathrm{SL}(N+1,\mathbb{C})$ *acting naturally on $\mathbb{C}P^N$. Define GX to be the variety $\{(\sigma,x)\,|\,x\in\sigma(X)\}$ in $G\times\mathbb{C}P^N$. Then for any smooth $(2\dim_{\mathbb{C}} G-2)$-form ϕ with compact support in G,*

(3.18)
$$-\int_G D(h_{FS},\sigma^* h_{FS})\,\partial_G\overline{\partial}_G\phi = \int_{GX}\left(\mathrm{Ric}_{GX|G} - \frac{\mu}{n+1}\sigma^*\omega_{FS}\right)\wedge(\sigma^*\omega_{FS})^n\wedge\pi_1^*\phi$$

where ω_{FS} is the curvature form of h_{FS}, and $\mathrm{Ric}_{GX|G}$ is the curvature of the metric on the relative canonical bundle $\mathcal{K}_{GX|G}$ induced by Kähler metrics $\sigma^*\omega_{FS}$.

Proof. Note that GX is biholomorphic to $G\times X$ by identifying (σ,x) with $(\sigma,\sigma^{-1}(x))$. Therefore, (3.18) follows from Proposition 3.2. □

Remark 3.1. Proposition 3.2 is a special case of a general theorem which can be roughly described as follows: Let E be a holomorphic "virtual" bundle over X and $\mathcal{M}_E$ denotes the space of all hermitian metrics on E. To avoid the complexity caused by singularity, we assume that X is smooth. Let Φ be any symmetric $GL_{\mathbb{C}}$-invariant multi-linear function of $n+1$ variables, such as, the function Ch_{n+1} defining the (n+1)-th Chern charactor. Then it follows from [BC] that, modulo $\mathrm{Im}(\partial)$ and $\mathrm{Im}(\overline{\partial})$, the (n,n)-form

$$\mathrm{BC}(\Phi;h_0,h_1) = \int_0^1 \Phi(R(h_t),\cdots,R(h_t);\dot{h}_t h_t^{-1})dt \tag{3.19}$$

is independent of the path $\{h_t\}$ from h_0 to h_1. This is the Bott-Chern class associated to the invariant function Φ. Integrating this Bott-Chern class over X, we obtain the Donaldson functional

$$D(\Phi;h_0,h_1) = \int_X \mathrm{BC}(\Phi;h_0,h_1) \tag{3.20}$$

The K-energy coincides, up to multiplication by constants, the Donaldson functional $D(\mathrm{Ch}_{n+1};\cdot,\cdot)$ restricted to $\mathcal{M}_X^+$, which is a subset in $\mathcal{M}_E$, in case

$$E = (n+1)(K_X^{-1} - K_X) \otimes (L - L^{-1})^n - \mu(L - L^{-1})^{n+1}$$

Let Y be a complex manifold and $h : Y \mapsto \mathcal{M}_E$ be a map as in Proposition 3.2. Then for any $(2m-2)$-form ψ on Y with compact support,

$$-\int_Y D(\Phi; h_0, h(y))\partial_Y\overline{\partial}_Y\psi = \int_{Y\times X} \Phi(R(h(y)), \cdots, R(h(y))) \wedge \pi_1^*\psi$$

4. The proof of Theorem 0.1

We will adopt the notations in sections 1 and 2. Let $X = \Sigma_f$ be a hypersurface of degree d in $\mathbb{C}P^{n+1}$, and G be the linear group $\mathrm{SL}(n+2,\mathbb{C})$. As before, h_{FS} denotes the standard metric on the hyperplane bundle over $\mathbb{C}P^{n+1}$. Its curvature ω_{FS} is the Kähler form of the Fubini-Study metric.

Theorem 4.1. *Let $B = PR_{n,d}$ be the space of all hypersurfaces of degree d, and $\|\cdot\|_Q$ be the hermitian metric on $\mathcal{O}_B(-1)$ defined in (2.6). Then for any σ in G, we have*

$$D(h_{FS}, \sigma^* h_{FS}) = \frac{(n+2)(d-1)}{n+1}\log\left(\frac{\|\cdot\|_Q(f\circ\sigma^{-1})}{\|\cdot\|_Q(f)}\right) \tag{4.1}$$

Proof. Let us denote by $\tilde{D}$ the function on G defined by the right side of (4.1). It is obviously continuous on G. By using the definition of $\|\cdot\|_Q$ in (2.6) and Corollary 1.2, we have that for any smooth $(2\dim_{\mathbb{C}} G - 2)$-form ϕ with compact support in G,

(4.2)
$$\begin{aligned}
&-\int_G \tilde{D}\cdot\partial_G\overline{\partial}_G\phi \\
&= \int_{G\Sigma_f}\left(\frac{n+2-d}{n+1}\pi_2^*\omega_{FS} - \pi_1^*\circ T_f^*\omega_B - \partial\overline{\partial}\psi\right)\wedge\pi_2^*\omega_{FS}^n\wedge\pi_1^*\phi \\
&= \int_{G\Sigma_f}\left(-(T_f\times\mathrm{id})^*R(\|\cdot\|_{\mathcal{K}}) - \frac{n(n+2-d)}{n+1}\pi_2^*\omega_{FS}\right)\wedge\pi_2^*\omega_{FS}^n\wedge\pi_1^*\phi
\end{aligned}$$

where $T_f : G \mapsto B$ is defined by $T_f(\sigma) = [f\circ\sigma^{-1}]$. Notice that $(T_f\times\mathrm{id})^*R(\|\cdot\|_{\mathcal{K}})$ restricts to $\mathrm{Ric}_{GX|G}$ on $G\Sigma_f$ in (3.18). Therefore, the difference $D(h_{FS},\sigma^*h_{FS}) - \tilde{D}(\sigma)$ is a pluriharmonic function on G. Since G is simply-connected, we can write

$$D(h_{FS}, \sigma^* h_{FS}) - \tilde{D}(\sigma) \;=\; (\log |w|^2)\,(\sigma)$$

for some holomorphic function w on G. On the other hand, using the definitions of $D(\cdot,\cdot)$ and $\|\cdot\|_Q$, it is not hard to prove that w has at most polynomial growth at the infinity of G, so w extends to be a meromorphic function on any compactification of G. However, G has a normal compactification $\overline{G}$ in $\mathbb{C}P^{n^2+2n}$ with irreducible divisor at infinity. Therefore, w has to be a nonzero constant. Then the theorem follows. □

The tautological line bundle $\mathcal{O}_B(-1)$ over B can be written

$$\mathcal{O}_B(-1) \;=\; \{(x,v) \,|\, x \in B,\, v \in x\} \tag{4.3}$$

The hermitian metric $\|\cdot\|_B$ on $\mathcal{O}_B(-1)$ over B is defined as follows: Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $R_{n,d}$, then for any (x,v) in $\mathcal{O}_B(-1)$,

$$\|\,(x,v)\,\|_B \;=\; \sqrt{\langle v,v\rangle} \tag{4.4}$$

Let us recall a simple fact.

Lemma 4.1. *Let Σ_f be a hypersurface in $\mathbb{C}P^{n+1}$ defined by f. We define a function F_0 on G by*

$$F_0(\sigma) = \log \|([f\circ\sigma^{-1}], f\circ\sigma^{-1})\|_B, \qquad \sigma \in G$$

Then

i) *X is stable if and only if F_0 is proper on G;*

ii) *X is semistable if and only if F_0 is bounded from below on G.*

Proof. Let us just show (i) here. The proof for (ii) is identical and omitted. First we assume that f is stable. If the function F_0 is not proper on the orbit G, then there is a sequence $\{\sigma_i\}$ in G which has no limit point in G, such that $F_0(\sigma_i)$ are uniformly bounded. By the definition of F_0 and $\|\cdot\|_B$, the square norms $\langle\sigma_i(f), \sigma_i(f)\rangle$ are uniformly bounded, i.e., all points $\sigma_i(f)$ lie in a compact subset of $R_{n,d}$. Without loss of generality, we may assume that $\lim_{i\to\infty} \sigma_i(f)$ exists in $R_{n,d}$. By the definition of the stability, the orbit $G\cdot f$ is closed, so we may assume that $\lim_{i\to\infty} \sigma_i(f) = \sigma(f)$ for some σ in G. Therefore, the stablizer of f in G is not finite, which contradicts to the assumption that f

is stable. Conversely, if F_0 is proper on G, it is clear that the stablizer of f in G is finite. Take any sequence σ_i in G such that $\sigma_i(f)$ has a limit in $R_{n,d}$, then $F_0(\sigma_i)$ is uniformly bounded independent of i. By the properness, it implies that σ_i has a limit σ in G. This shows that the orbit is closed, consequently, f is stable. □

Let $\|\cdot\|_Q$ be the Quillen metric defined in (2.6). Then

$$\frac{\|\cdot\|_B}{\|\cdot\|_Q} = e^{-\frac{(n+1)}{2(n+2)(d-1)}\psi_B} \frac{\|\cdot\|_B}{\|\cdot\|_u}^{\frac{n+2-d}{(n+2)(d-1)}} \tag{4.5}$$

where $\|\cdot\|_u$ is defined in Proposition 2.1. Then by Proposition 2.1 and Lemma 2.1, there is a positive constant c such that $\|\cdot\|_B \geq c\|\cdot\|_Q$, therefore, we have

Lemma 4.2. *Let Σ_f be a hypersurface in $\mathbb{C}P^{n+1}$ defined by f. Define a function F_f on G by*

$$F_f(\sigma) = \log\left(\|f\circ\sigma^{-1}\|_Q\right)$$

Then

i) *f is stable if the function F_f is proper on G in the following sense: for any $C > 0$, there is a constant $\delta = \delta(C) > 0$, so that $F_f(\sigma) \geq C$ whenever $\inf_{z\in\Sigma_f}\frac{|\sigma(z)|}{|z|} \leq \delta$.*

ii) *f is semistable if the function F_f is bounded from below.*

Now Theorem 0.1 follows easily from Lemma 4.2 and Theorem 4.1.

5. The lower boundedness of the K-energy

In this section, we give some sufficient conditions for the K-energy to be bounded from below or proper on the space of admissible Kähler metrics.

We assume that X is a normal, irreducible variety satisfying (3.7). Then by (3.8), we can have

(5.1)

$$D(h_0,h_1) = \int_X \log\left(\frac{\omega_1^n}{e^{f_0}\omega_0^n}\right)\omega_1^n + \int_X f_0\omega_0^n - \frac{\mu}{n}\left(I(h_0,h_1) - J(h_0,h_1)\right)$$

Lemma 5.1. *For any h_0, h_1 in $\mathcal{M}_X^+$,*

$$\frac{n}{n+1}I(h_0,h_1) \geq I(h_0,h_1) - J(h_0,h_1) \geq \frac{1}{n+1}I(h_0,h_1).$$

It follows from a straightforward computation (cf. [T2]). We omitted the proof.

DEFINITION 5.1. We say the K-energy $D(\cdot, \cdot)$ proper if for any fixed h_0 and $c > 0$, there is a constant $C = C(h_0, c)$ such that $D(h_0, h) \leq c$ implies $I(h_0, h) \leq C$.

The above lemma, together with (5.1), implies that if $\mu \leq 0$, the K-energy is bounded from below or proper if the first integral in (5.1) is so. We recall that a variety X has only log-terminal (resp. canonical) singularities if it satisfies the following two conditions: (i) for some integer $r \geq 1$, the Weil divisor rK_X is Cartier; (ii) if $\pi : \tilde{X} \mapsto X$ is a resolution and $\{E_i\}$ the family of all exceptional prime divisors of π, then $r\,K_{\tilde{X}} = \pi^*(r\,K_X) + r\sum \alpha_i E_i$ with $\alpha_i > -1$ (resp. $\alpha_i \geq 0$)(cf. [Re]). Suppose that X is a normal variety satisfying (3.7). Then it is clear from (i) and (ii) that X has only log-terminal singularities if and only if

$$K_{\tilde{X}} = \frac{\mu}{n}\pi^*(\mathcal{O}_{\mathbb{C}P^N}(-1)) + \sum \alpha_i E_i \tag{5.2}$$

for some $\alpha_i > -1$. Let S_i be a section of $[E_i]$ defining E_i, and $\|\cdot\|_i$ be a fixed hermitian metric on $[E_i]$, then (3.7) and (5.2) imply that there is a smooth volume form $d\tilde{V}$ on $\tilde{X}$ such that

$$\pi^* f_0 = \log\left(\frac{d\tilde{V}}{\omega_0^n}\right) - \sum \alpha_i \log \|S_i\|_i^2 \tag{5.3}$$

consequently, we have

$$\int_X e^{f_0}\omega_0^n = \int_{\tilde{X}} \left(\prod \|S_i\|_i^{-2\alpha_i}\right) d\tilde{V} < \infty \tag{5.4}$$

Theorem 5.2. *If $\mu \leq 0$ and X is a normal variety with only log-terminal singularities and satisfying (3.7), the K-energy is proper in the sense of* Definition 5.1.

Proof. Using the assumption that all singular points are log-terminal, one can show (cf. [T2]) that there are C, $\epsilon > 0$, such that for any $h = e^{-\varphi}h_0$ in $\mathcal{M}_X^+$,

$$\int_X e^{f_0 - \epsilon(\varphi - \sup_X \varphi)}\omega_0^n = \int_{\tilde{X}} e^{-\epsilon(\varphi - \sup_X \varphi)} \left(\prod \|S_i\|_i^{-2\alpha_i}\right) d\tilde{V} \leq C \tag{5.5}$$

By the concavity of logarithmic function, we deduce from this

$$\int_X \log\left(\frac{(\omega_0+\partial\bar\partial\varphi)^n}{e^{f_0}\omega_0^n}\right) \geq \epsilon I(h_0,h) - C' \tag{5.6}$$

where C' is a constant independent of h. Therefore, if $\mu \leq 0$, the K-energy in (3.8) is proper. □

Remark 5.1. In case $\mu < 0$, there should be a much weaker condition for the K-energy to be proper in the sense of Definition 5.1.

The following theorem is taken from [DT], whose proof follows an idea in [BM].

Theorem 5.3. *If X is a Kähler-Einstein orbifold, then the K-energy is bounded from below.*

Remark 5.2. It is natural to ask when the K-energy is proper. Let $\alpha(X)$ be the holomorphic invariant defined in [T2]. It can be proved that the K-energy is proper if this invariant is greater than $\frac{n}{n+1}$, where n is the complex dimension of X. Using the fact that any smooth hypersurface of degree greater than 2 is stable, one can easily show that the K-energy is proper on the subset of Kähler metrics of the form $\sigma^*\omega_{FS}|_X$, where $X = \Sigma_f$, $\deg(f) \geq 3$, $\sigma \in SL(n+2,\mathbb{C})$ and ω_{FS} denotes the Fubini-Study metric. We conjecture that the K-energy is proper if X has a Kähler-Einstein metric and no holomorphic vector fields.

It is obvious that Theorem 0.2 follows from Theorem 5.1, 5.2 and the fact that $\mu = n+2-d$ in case of hypersurfaces.

6. Some generalizations

In this section, we give some generalizations of Theorem 0.1 and 0.2. They can be proved by the same method with some modifications. The details of the proof will appear in a forthcoming paper [T1].

Let $\pi : \mathcal{X} \mapsto \mathcal{H}$ be a holomorphic fibration between projective varieties satisfying:

(1) $\mathcal{X}$ is a subvariety in $\mathcal{H} \times \mathbb{C}P^N$, and π coincides with the projection π_1 onto the first factor;

(2) There is an Zariski open subset $\mathcal{H}_0$ in $\mathcal{H}$ such that $\pi^{-1}(h)$ is a smooth n-fold for each h in $\mathcal{H}_0$;

(3) the linear group $G = SL(N+1, \mathbb{C})$ acts on both $\mathcal{H}_0$ and $\pi^{-1}(\mathcal{H}_0)$, and the action commutes with π_1, π_2;

(4) either the hyperplane bundle on $\mathbb{C}P^N$ restricts to $rK^{\pm}$ on each $\pi^{-1}(h)$ for some nonzero rational number r, or the first Chern class of $\pi^{-1}(h)$ vanishes, where h is in $\mathcal{H}_0$.

The typical example of such a fibration is a universal family of subvarieties in $\mathbb{C}P^N$ over a Hilbert scheme.

Theorem 6.1. *Let $\pi : \mathcal{X} \mapsto \mathcal{H}$ be as above. We denote by $\mathcal{L}$ the pull-back of the hyperplane bundle over $\mathbb{C}P^N$ by π_2. Put*

$$\mathcal{E} = (n+1)(\mathcal{K}_{\mathcal{K}}^{-1}) \otimes (\mathcal{L} - \mathcal{L})^n - n\mu(\mathcal{L} - \mathcal{L})^{n+1} \tag{6.1}$$

where $\mathcal{K} = \mathcal{K}_{\mathcal{X}} \otimes \mathcal{K}_{\mathcal{H}}^{-1}$ be the relative canonical bundle and $\mathcal{K}^{-1}|_{\pi^{-1}(h)} = \mu\mathcal{L}|_{\pi^{-1}(h)}$ for some h in $\mathcal{H}_0$. Assume that the inverse of the determinant line bundle $L_{\mathcal{H}} = \det(\mathcal{E}, \pi)$ is ample on $\mathcal{H}$. Then for any h in $\mathcal{H}_0$, we have

(i) *$X = \pi^{-1}(h)$ is stable with respect to G and $L_{\mathcal{H}}$ if the K-energy on X is proper;*

(ii) *X is semistable with respect to G and $L_{\mathcal{H}}$ if the K-energy on X is bounded from below.*

In particular, such an X is semistable if it admits a Kähler-Einstein metric.

An example of such a fibration in Theorem 6.1 is provided by complete intersections in $\mathbb{C}P^{n+k}$. Let $R_{n+k,d}$ be the space of all homogeneous polynomials on $\mathbb{C}^{n+k+1}$ of degree d. Put

$$B = B_1 \times \cdots \times B_k, \ B_i = PR_{n+k,d_i} \tag{6.2}$$

where d_i are integers greater than one. Then B classifies all complete intersections in $\mathbb{C}P^{n+k}$ defined by k homogeneous polynomials of degree $d_1, \ldots, d_k$. Some of these intersections may be of dimension higher than n. Let $\mathcal{X}$ be the universal family of those intersections, and $\mathcal{H} = B$. Then one can show that the determinant line bundle L_B in Theorem 6.1 is isomorphic to

$$2^{n+1}d_1\cdots d_k \bigotimes_{i=1}^{k} \mathcal{O}_{B_i}\left(\frac{n+k+1-\sum_{j=1}^{k} d_j}{d_i} - (n+1)\right)$$

Since each $d_i \geq 2$, $(n+1)d_i$ is greater than $n+k+1-\sum_{j=1}^{k} d_j$. This follows that L_B^{-1} is ample on B. Therefore, we have

Theorem 6.2. *Let X be a normal complete intersection in $\mathbb{C}P^{n+k}$. Then X is stable if the K-energy is proper, and X is semistable if the K-energy is bounded from below. In particular, X is semistable if it admits a Kähler-Einstein orbifold metric.*

There are more applications of Theorem 6.1 in addition to Theorem 6.2 on complete intersections.

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