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CONVEXITY PROPERTIES OF THE MOMENT MAP FOR CERTAIN NON-COMPACT MANIFOLDS

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ABSTRACT. Let M be a connected symplectic manifold on which a torus T acts in a Hamiltonian fashion and denote by Φ the corresponding moment map. If M is compact, the Atiyah, Guillemin-Sternberg theorem says that the set $\Phi(M)$ is convex; if M is non-compact this may not be the case. We show that if M is suitably convex, then $\Phi(M)$ can be explicitly described as the convex hull of a finite number of affine rays. As an application we obtain a natural symplectic proof of the Paneitz convexity theorem for the positive Kaehler elliptic co-adjoint orbits.

INTRODUCTION

Let M be a $2m$-dimensional connected symplectic manifold with fundamental 2-form ω. Consider a torus T with Lie algebra $\mathfrak{t}$. Assume that T acts on M in a Hamiltonian fashion and denote by $\Phi : M \to \mathfrak{t}^*$ the corresponding moment map; the pair (M, T) is a Hamiltonian T-space. The convexity theorem, proven independently by Atiyah [A] and Guillemin-Sternberg [GS], says that, if M is compact, the image set $\Phi(M)$ is the convex hull of the image via Φ of the T-fixed point set. If the manifold M is not compact, $\Phi(M)$ may not be convex.

COUNTEREXAMPLE 0.1. Consider $\mathbb{C}^2$ with the canonical symplectic form. The two dimensional Cartan, T, of $U(2)$ acts on $\mathbb{C}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{-i\theta_1} z_1, e^{-i\theta_2} z_2)$; this action is Hamiltonian, the moment map being given

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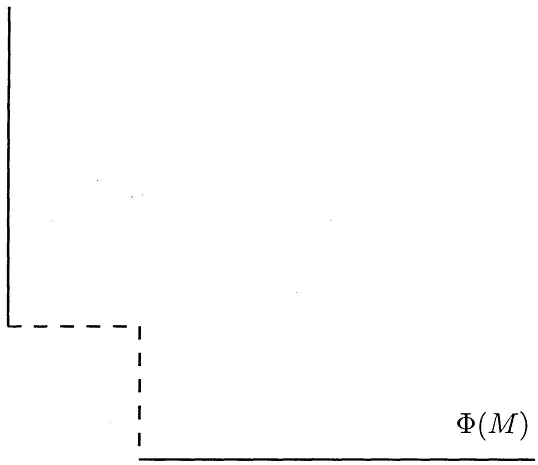

FIGURE 1. $\Phi(\mathbb{C}^2 - (D^1 \times D^1))$ is not convex

by:

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$$
$$(z_1, z_2) \longrightarrow \frac{1}{2}(|z_1|^2, |z_2|^2).$$

Let now $M = \mathbb{C}^2 - (D^1 \times D^1)$, D^1 denoting the closed unit disc in $\mathbb{C}$. M is a connected symplectic manifold. The action of T on $\mathbb{C}^2$ restricts to a Hamiltonian action on M. The corresponding moment map is the restriction of Φ to M, which we will still denote by Φ. The set $\Phi(M) = \mathbb{R}^2_+ - \{(x, y) \text{ s.t. } x \leq \frac{1}{2} \text{ and } y \leq \frac{1}{2}\}$ is not convex (Figure 1).

However, in many interesting situations (see examples below) the image set is still convex. From now on we will make an additional assumption. For each $X \in \mathfrak{t}$ denote by Φ_X the X-component of the moment map: $\Phi_X(p) = \langle \Phi(p), X \rangle$, $p \in M$.

ASSUMPTION 0.2. There exists an integral element $X_0 \in \mathfrak{t}$ such that Φ_{X_0} is a proper function having a minimum as its unique critical value.

REMARK 0.3. Since X_0 is integral, it generates a circle S^1 in T whose induced action on M is Hamiltonian with moment map Φ_{X_0}. Notice that the requirement that X_0 is integral is not at all restrictive, since this can always be achieved by a small perturbation.

The goal of this article is to show that, under this assumption, the image set $\Phi(M)$ can be explicitly described as the convex hull of a finite number of affine rays in $\mathfrak{t}^*$ stemming from the images of T-fixed points (see Theorem 1.4 below.)

EXAMPLE 0.4. Consider again $\mathbb{C}^2$ with the canonical symplectic form and the action of the Cartan, T, of $U(2)$. Take now the vector $X_0 = (1,1) \in \mathfrak{t}$ generating the diagonal circle in T; the corresponding component of the moment map is

$$\begin{aligned} \Phi_{X_0}: \quad \mathbb{C}^2 &\longrightarrow \mathbb{R} \\ (z_1, z_2) &\longrightarrow \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right). \end{aligned}$$

It is immediately seen that Φ_{X_0} is a proper function whose unique critical value is the minimum 0. Here $\Phi(M) = \mathbb{R}^2_+$.

EXAMPLE 0.5. Consider the upper sheet of the hyperboloid of two sheets:

$$\mathcal{H} = \left\{ (x, y, z) \in \mathbb{R}^3 |\ z = \sqrt{1 + x^2 + y^2} \right\}.$$

$\mathcal{H}$ is a connected symplectic manifold; in hyperbolic coordinates

$$\begin{cases} x = & \sinh t \, \cos s \\ y = & \sinh t \, \sin s \\ z = & \cosh t \end{cases}$$

and the symplectic form is given by $\omega = \sinh t \, ds \wedge dt$. Here $T = S^1$ acts on $\mathcal{H}$ by rotation around the z-axis. It is easy to check that this action is Hamiltonian with moment map given by the projection onto the z-axis; this function is indeed proper and the minimum 1 is its unique critical value. Here $\Phi(\mathcal{H}) = [1, \infty)$. This last assertion is just a fancy way of saying that the anti-symmetric part, $\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$, of all traceless 2×2 matrices with determinant 1 satisfies $|z| \geq 1$. In fact $\mathcal{H}$ is a positive Kaehler elliptic co-adjoint orbit of $Sl\,(2, \mathbb{R})$ and Φ is the projection onto the dual of the compact Cartan subalgebra. We will discuss the projections of positive Kaehler elliptic orbits of more general non-compact Lie groups in Section 2.

Acknowledgement. Our initial motivation came from [P], where Stephen Paneitz proves a convexity theorem for the positive Kaehler elliptic co-adjoint orbits. I am grateful to Victor Guillemin for his invaluable guidance. Finally, I wish to thank Regina Souza and Siye Wu for many helpful comments.

1. Convexity

1.1. Preliminaries. We recall that the moment mapping, Φ, is a T-equivariant mapping satisfying the equations:

$$d\Phi_X = \imath(\widehat{X})\omega \qquad X \in \mathfrak{t}, \tag{1}$$

where $\widehat{X}$ is the vector field on M with infinitesimal generator X. Let $p \in M$; denote by T_p the stabilizer group of p in T and by $\mathfrak{t}_p$ the stabilizer algebra. One of the basic consequences of (1) is

Lemma 1.1. [GS] *The image of $d\Phi_p$ is the annihilator space, $\mathfrak{t}_p^\perp$, of $\mathfrak{t}_p$ in $\mathfrak{t}^*$.*

If we apply Lemma 1.1 to the induced action of S^1 we get that, if p is regular for Φ_{X_0} X_0 is not in $\mathfrak{t}_p$, and $\dim \mathfrak{t}_p^\perp \geq 1$. Consider now

$$M^1 = \{\, p \in M \ :\ \Phi_{X_0}(p) \text{ regular}, \ \ \dim \mathfrak{t}_p^\perp = 1 \,\}.$$

M^1 is a highly disconnected submanifold of M ([GS], Theorem 3.5.). Consider, for each $p \in M^1$, the vector v_p in $\mathfrak{t}_p^\perp$ satisfying

$$\langle v_p, X_0 \rangle = 1. \tag{2}$$

It will be crucial for us to notice that for points in the same connected component of M^1, $\mathfrak{t}_p$, and therefore v_p, remain constant. Choose now a T-invariant metric on M, denote by X_0 the gradient of Φ_{X_0} and by f_t its flow; it is not restrictive to assume that away from the critical value we have $d\Phi_{X_0}(X_0) = 1$. Therefore, again by Lemma 1.1, we may assume

$$d\Phi_p(X_0) = v_p, \qquad p \in M^1. \tag{3}$$

Since the metric is T-invariant, f_t commutes with the action of T and therefore preserves each connected component of M^1.

Lemma 1.2.

$$\Phi(f_t(p)) = \Phi(p) + t\, v_p, \qquad\qquad p \in M^1.$$

Proof. From (3)

$$\frac{d}{dt}\Phi(f_t(p)) = [d\Phi]_{f_t(p)}(X_0) = v_{f_t(p)} = v_p. \quad \square$$

1.2. Statement and proof of the main result. Assume that Y is a *compact* symplectic manifold and that a torus H acts on it in a Hamiltonian fashion; denote by Ψ the corresponding moment map. The image set $\Psi(Y)$ is beautifully characterized in the Atiyah, Guillemin-Sternberg convexity theorem, which we recall. Denote by Y^H the set of points that are fixed by H.

Theorem 1.3. [A, GS] *Y^H is non-empty, $\Psi(Y^H)$ is finite, and $\Psi(Y) = \mathcal{CH}\ \Psi(Y^H)$.*

Here and in the following $\mathcal{CH}$ stands for *convex hull.* Consider now our original manifold M and denote by a the minimum value of Φ_{X_0}; since Φ_{X_0} is proper, for each $s \geq a$ the T-invariant space $M_s = \Phi_{X_0}^{-1}(s)$ is compact. Assume for a moment that S^1 acts freely on M_s, $s > a$. Then the Mardsen-Weinstein reduced space, $Y_s = M_s/S^1$, is a compact symplectic manifold, and the induced action of $H = T/S^1$ is Hamiltonian. The corresponding moment map, Ψ, is characterized by $\Psi \circ \pi = q \circ \Phi$, where $\pi : M_s \to Y_s$ and $q : \mathfrak{t}^* \to \mathfrak{h}^*$ denote the projections. By Theorem 1.3 we then have that the H-fixed point set Y_s^H is non-empty, $\Psi(Y_s^H)$ is finite, and

$$q(\Phi(M_s)) = \Psi(Y_s) = \mathcal{CH}\ \Psi(Y_s^H).$$

But if we consider the set $M_s^1 = M_s \cap M^1$ we have $\pi(M_s^1) = Y_s^H$ and in conclusion

$$\Phi(M_s) = \mathcal{CH}\ \Phi(M_s^1).$$

Notice, incidentally, that we have also shown that the set M^1 is non-empty. If S^1 does not act freely on M_s, Y_s is a V-manifold or orbifold. Such manifolds, though not necessarily smooth, do carry enough differentiable structures and a well defined symplectic form (see [W]); in addition, all of the properties that were needed above still hold. Fix now $u \in (a, \infty)$. Consider, for $p \in M_u^1$, the affine rays in $\mathbf{t}^*$

$$\alpha_p = \{\Phi(p) + t\, v_p \ : \ t \in [a-u, \infty)\ \}.$$

The choice of a regular u here is completely arbitrary since v_p does not depend on the choice of p on the same gradient trajectory. Notice that there are only finitely many distinct α_p's, each of which stems from the image of a fixed point for the action of T; in particular the T-fixed point set in M is non-empty. We are now ready to prove:

Theorem 1.4.

$$\Phi(M) = \mathcal{CH}\ \{\ \alpha_p \ : \ p \in M_u^1\ \}.$$

Proof. Begin by noticing that

$$\Phi(M) = \bigcup_{t=a-u}^{\infty} \Phi(M_{u+t}).$$

For a moment let us restrict our attention to $u + t > a$ (regular). By our previous remarks, $\Phi(M_{u+t}) = \mathcal{CH}\ \Phi(M_{u+t}^1)$. However, if we consider the gradient flow f_t, we have

$$M_{u+t} = f_t(M_u) \qquad \text{and} \qquad M_{u+t}^1 = f_t(M_u^1).$$

Therefore, by Lemma 1.2,

$$\begin{aligned} \Phi(M_{u+t}) &= \mathcal{CH}\ \{\ \Phi(f_t(p)) \ : \ p \in M_u^1\ \} \\ &= \mathcal{CH}\ \{\ \Phi(p) + t\, v_p \ : \ p \in M_u^1\ \}. \end{aligned}$$

Continuity then forces

$$\Phi(M_a) = \mathcal{CH}\ \{\ \Phi(p) + (a-u)\, v_p \ : \ p \in M_u^1\ \},$$

and in conclusion:

$$\Phi(M) = \bigcup_{t=a-u}^{\infty} \mathcal{CH}\ \{\Phi(p) + t\, v_p \ : \ p \in M_u^1\} = \mathcal{CH}\ \{\ \alpha_p \ : \ p \in M_u^1\ \}. \quad \square$$

2. The Kaehler elliptic co-adjoint orbits

Let (G, K) be an irreducible Hermitian symmetric pair. G is a non-compact, simple, connected Lie group with Lie algebra $\mathfrak{g}$ and K a maximal compact subgroup with Lie algebra $\mathfrak{k}$; K is connected and its center, S^1, is a circle. Let T be a Cartan subgroup of K and let W denote the Weyl group for the pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. It is always possible to choose a set of positive roots, Δ^+, for the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that the positive non-compact roots, Δ_n^+, are larger than the compact ones, Δ_c^+. With this choice, the elements of Δ_n^+ agree on vectors of the one-dimensional center of $\mathfrak{k}$; as a consequence we have that Δ_n^+ is W-invariant and that $\langle \alpha, \beta \rangle \geq 0$ for each $\alpha, \beta \in \Delta_n^+$. Consider now the convex, proper, W-symmetric cone in $\mathfrak{t}$:

$$\mathcal{C} = \{ \nu \in \mathfrak{t} \ : \ \alpha(\sqrt{-1}\nu) > 0 \text{ for } \alpha \in \Delta_n^+ \}.$$

It will be convenient for us to denote (-1) times the Killing form by $\langle \ , \ \rangle$. We will use $\langle \ , \ \rangle$ to identify $\mathfrak{g}$ with its dual, $\mathfrak{g}^*$; then $\mathfrak{k}$ and $\mathfrak{t}$ will be identified with $\mathfrak{k}^*$ and $\mathfrak{t}^*$, respectively, and adjoint orbits with the corresponding co-adjoint orbits. An elliptic orbit is, by definition, an orbit that intersects $\mathfrak{t}$. Let λ be an element of $\mathfrak{t}$ that lies in the cone $\mathcal{C}$ and let $\mathcal{O}_\lambda$ be the elliptic orbit $G \cdot \lambda$. The Kostant-Kirillov symplectic form gives $\mathcal{O}_\lambda$ the structure of a positive definite Kaehler manifold (see [Pr].)

The natural action of T on $\mathcal{O}_\lambda$ is Hamiltonian with moment map, $\Phi : \mathcal{O}_\lambda \to \mathfrak{t}$, given by the restriction to $\mathcal{O}_\lambda$ of the T-invariant projection of $\mathfrak{g}$ onto $\mathfrak{t}$. Let X_0 be the unique vector in the center of $\mathfrak{k}$ that satisfies $\alpha(\sqrt{-1}X_0) = 1$ for $\alpha \in \Delta_n^+$ (recall that the $\alpha's$ agree on the center); X_0 generates the center S^1. Our goal is to show that S^1 is the special circle of Assumption 0.2, and therefore that the projection $\Phi(\mathcal{O}_\lambda)$ is convex. An explicit description of this set, matching the one first given by Paneitz [P], is given in Theorem 2.4. First we need to recall certain basic facts from the structure theory of Hermitian symmetric spaces; we refer to [H, K1, K2] for a more extensive treatment.

2.1. A review of Hermitian symmetric spaces. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle \ , \ \rangle$; $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$. The diffeomorphism of $K \times \mathfrak{p}$ onto G, given by $(k, X) \to e^X k$ induces a

K-equivariant diffeomorphism

$$\mathcal{O}_\lambda \simeq K \cdot \lambda \times \mathfrak{p}. \tag{4}$$

For each $\alpha \in \Delta_n^+$ define $X_\alpha \in \mathfrak{t}$ by

$$\langle X_\alpha, \nu \rangle = \frac{\alpha(\sqrt{-1}\nu)}{\langle \alpha, \alpha \rangle}, \qquad \nu \in \mathfrak{t}.$$

Notice that $\langle X_\alpha, X_\beta \rangle \geq 0$, $\alpha, \beta \in \Delta_n^+$, and that the convex hull of all rays $\{ \, tX_\alpha \; : \; t \geq 0\}$, $\alpha \in \Delta_n^+$, is the closed proper cone, $\mathcal{C}^*$, *dual* to $\mathcal{C}$. Moreover, it is always possible to choose basis vectors E_α, F_α for $\mathfrak{p}$ with the following properties

$$[E_\alpha, X_\alpha] = 2F_\alpha, \quad [F_\alpha, X_\alpha] = -2E_\alpha, \quad [E_\alpha, F_\alpha] = 2X_\alpha, \tag{5}$$

$$[E_\alpha, X_0] = F_\alpha, \quad [F_\alpha, X_0] = -E_\alpha. \tag{6}$$

Note that from (6) follows easily that S^1 acts freely on $\mathfrak{p} - 0$. Also, there exists a maximal subset, Ψ, of strongly orthogonal positive non-compact roots with the property that $\mathfrak{a} = \sum_{\alpha\in\Psi} \mathbb{R}E_\alpha$ is a maximal abelian subalgebra of $\mathfrak{p}$. If we consider the subgroup $A = \exp(\mathfrak{a})$, then there is another very useful decomposition: $G = KAK$.

Denote now by π the orthogonal projection $\mathfrak{g} \to \mathfrak{k}$. Then we have the following crucial lemma:

Lemma 2.1.

$$\pi(a \cdot X_0) = X_0 + \frac{1}{2} \sum_{\alpha\in\Psi} (\cosh 2e_\alpha - 1) \, X_\alpha,$$

where $a = e^H$, $H = \sum_{\alpha\in\Psi} e_\alpha E_\alpha$.

Proof.

$$\begin{aligned} \pi(a \cdot X_0) &= \pi\left(e^{adH} \cdot X_0\right) \\ &= \pi\left(\sum_{k=0}^{\infty} \frac{ad^k(H)X_0}{k!}\right) \\ &= X_0 + \sum_{k=1}^{\infty} \frac{ad^{2k}(H)X_0}{(2k)!} \end{aligned}$$

$$= X_0 + \frac{1}{2}\sum_{\alpha\in\Psi}(\cosh 2e_\alpha - 1)\, X_\alpha,$$

where the last equality follows from an easy computation using (5), (6). □

2.2. Projecting the orbits. At this point we have all the information we need to show that $\mathcal{O}_\lambda$ satisfies Assumption 0.2. We start by proving

Proposition 2.2. *The component Φ_{X_0} is a proper function.*

Proof. Since X_0 is in the center of $\mathfrak{k}$ we have, using the KAK decomposition of G:

$$\Phi_{X_0}(\mathcal{O}_\lambda) = \Phi_{X_0}(KAK\cdot\lambda) = \langle KAK\cdot\lambda, X_0\rangle = \langle K\cdot\lambda, A\cdot X_0\rangle = \langle K\cdot\lambda, \pi(A\cdot X_0)\rangle.$$

By Lemma 2.1 $\pi(A \cdot X_0) \subseteq \mathcal{C}^*$. Since $\mathcal{C}$ lies entirely in $\mathfrak{t}$, one may replace $K \cdot \lambda$ with its projection onto $\mathfrak{t}$, $\Phi(K \cdot \lambda)$. It will now be enough to show that this last set is also contained in $\mathcal{C}$. By the Kostant convexity theorem, [Ko], $\Phi(K \cdot \lambda)$ is given by the convex hull of the set $W \cdot \lambda$; but $\mathcal{C}$ is convex and W-symmetric, and since λ is in $\mathcal{C}$ so is all of $\Phi(K \cdot \lambda)$. □

Proposition 2.3. *$\Phi_{X_0}(\lambda)$ is the unique critical value of Φ_{X_0} and a minimum.*

Proof. By Lemma 1.1 a point is critical for Φ_{X_0} if and only if it is fixed by the center of K, S^1. Therefore, in view of the K-equivariant diffeomorphism (4), and since S^1 acts freely on $\mathfrak{p} - 0$, the critical set of Φ_{X_0} consists of the orbit $K \cdot \lambda$; this set maps to the unique value $\langle\lambda, X_0\rangle$. To show that this value is a minimum we argue as in Proposition 2.2 and we get, for $g = kah$ ($a \in A$, $h, k \in K$)

$$\Phi_{X_0}(g\cdot\lambda) = \langle\Phi(h\cdot\lambda), \pi(a^{-1}\cdot X_0)\rangle.$$

Finally, by Lemma 2.1 we have

$$\Phi_{X_0}(g\cdot\lambda) = \langle\lambda, X_0\rangle + \frac{1}{2}\sum_{\alpha\in\Psi}(\cosh 2e_\alpha - 1)\langle\Phi(h\cdot\lambda), X_\alpha\rangle,$$

where the second term on the left hand-side is non-negative since $\Phi(h \cdot \lambda) \in \mathcal{C}$. □

We can finally apply Theorem 1.4 to get

Theorem 2.4.

$$\Phi(\mathcal{O}_\lambda) = \mathcal{C}H\,\{\bigcup_{w\in W}(w\cdot\lambda + \mathcal{C}^*)\}.$$

Proof. A direct computation using (5) shows that, for our orbits, the set of regular points with codimension-one stabilizer is

$$M^1 = \{\, e^{V_\alpha} w\cdot\lambda : \qquad 0 \neq V_\alpha \in \operatorname{span}(E_\alpha, F_\alpha), \quad \alpha\in\Delta_n^+, \quad w\in W\}.$$

In addition, for $p = e^{V_\alpha} w\cdot\lambda$, $\mathfrak{t}_p^\perp = \operatorname{span}(X_\alpha)$; therefore $v_p = X_\alpha$ and the rays are given by $\alpha_p = \{\, w\cdot\lambda + t\,X_\alpha : \quad t\geq 0\,\}$, $w\in W$, $\alpha\in\Delta_n^+$. Now apply Theorem 1.4, using the fact that the vectors X_α, $\alpha\in\Delta_n^+$, generate $\mathcal{C}^*$. □

We conclude with two pictures of the image set, each relative to a generic orbit of a rank-two group. The difference in the image set amounts to a different structure of the corresponding root system.

Example 2.5. Consider

$$G = Sp\,(2,\mathbb{R}) = \left\{ g\in Gl\,(4,\mathbb{R}) \;:\; g^{tr}\begin{pmatrix} 0 & I_2 \\ -I_2 & 0\end{pmatrix} g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0\end{pmatrix}\right\},$$

then

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A\end{pmatrix} \in Gl\,(4;\mathbb{R}) \;:\; A + \sqrt{-1}B \in U(2)\right\} \simeq U(2),$$

and

$$\mathfrak{t} = \left\{ \nu = \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ -x & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \end{pmatrix} \;:\; (x,y)\in\mathbb{R}^2 \right\} \simeq \mathbb{R}^2.$$

With this last identification in mind, the one-dimensional center of $\mathfrak{k}$ is the $x = y$ line, and the compact Weyl group, W, acts by symmetry with respect to this axis. It is not difficult to check that the positive non-compact roots are $\alpha_1(\sqrt{-1}\nu) = 2x$, $\alpha_2(\sqrt{-1}\nu) = 2y$, and $\alpha_3(\sqrt{-1}\nu) = x+y$, and that $X_{\alpha_1} = (1,0)$, $X_{\alpha_2} = (0,1)$, and $X_{\alpha_3} = (1/2,1/2)$. Therefore we have that $\mathcal{C}^* = \overline{\mathcal{C}}$ is the first quadrant. In Figure 2 we have drawn the image set for an orbit through a generic $\lambda\in\mathcal{C}$, according to Theorem 2.4.

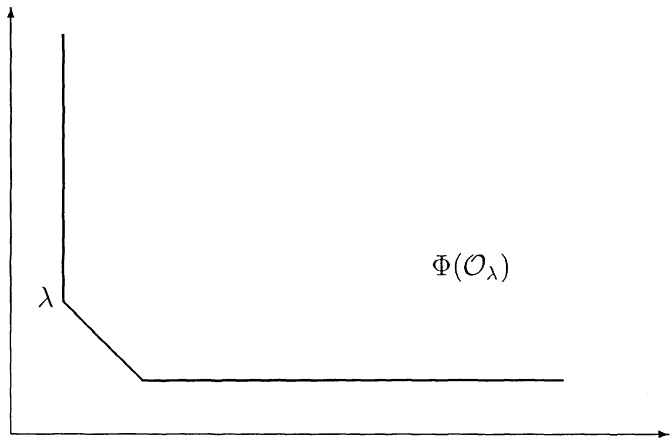

FIGURE 2. The $Sp(2,\mathbb{R})$ case.

EXAMPLE 2.6. We get a slightly different picture in the case that

$$G = SU(2,1) = \left\{ g \in Sl(3;\mathbb{C}) \ : \ \overline{g^{tr}} \begin{pmatrix} I_2 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} I_2 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$K = S(U(2) \times U(1)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix} \ : \ A \in U(2), \ z \in U(1), \ z \det(A) = 1 \right\},$$

and

$$\mathfrak{t} = \left\{ \nu = \begin{pmatrix} -\sqrt{-1}x & 0 & 0 \\ 0 & -\sqrt{-1}y & 0 \\ 0 & 0 & \sqrt{-1}(x+y) \end{pmatrix} \ : \ (x,y) \in \mathbb{R}^2 \right\} \simeq \mathbb{R}^2.$$

As in the previous example, the one-dimensional center of $\mathfrak{k}$ is the $x = y$ line, and W acts by symmetry about this axis. Here there are only two positive non-compact roots, $\alpha_1(\sqrt{-1}\nu) = x + 2y$ and $\alpha_2(\sqrt{-1}\nu) = y + 2x$; with our choice of coordinates, $X_{\alpha_1} = (1,0)$ and $X_{\alpha_2} = (0,1)$. Again $\mathcal{C}^*$ is the first quadrant. The difference here is that $\mathcal{C} \supset \mathcal{C}^*$; this gives us a greater freedom of choice for λ. In Figure 3 we have deliberately chosen λ outside of $\mathcal{C}^*$.

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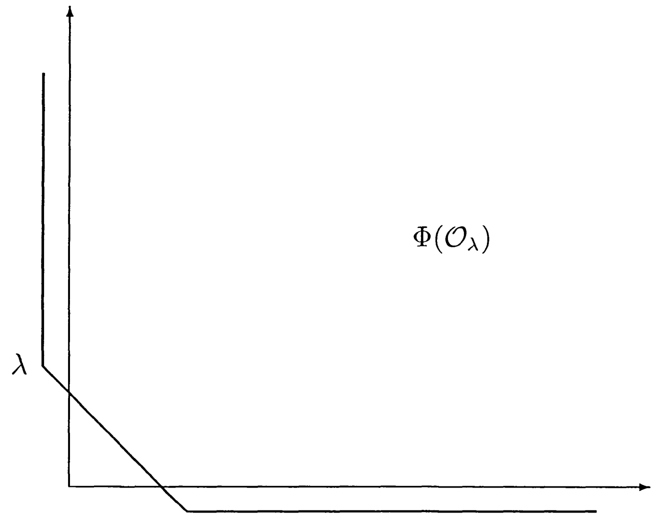

FIGURE 3. The $SU(2,1)$ case.

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